CHAPTER 5

Time-Domain Analysis of Control Systems

In this chapter, we depend on the background material discussed in Chapters 1–4 to arrive at the time response of simple control systems. In order to find the time response of a control system, we first need to model the overall system dynamics and find its equation of motion. The system could be composed of mechanical, electrical, or other sub-systems. Each sub-system may have sensors and actuators to sense the environment and to interact with it. Next, using Laplace transforms, we can find the transfer function of all the sub-components and use the block diagram approach or signal flow diagrams to find the interactions among the system components. Depending on our objectives, we can manipulate the system final response by adding feedback or poles and zeros to the system block diagram. Finally, we can find the overall transfer function of the system and, using inverse Laplace transforms, obtain the time response of the system to a test input—normally a step input.

Also in this chapter, we look at more details of the time response analysis, discuss transient and steady state time response of a simple control system, and develop simple design criteria for manipulating the time response. In the end, we look at the effects of adding a simple gain or poles and zeros to the system transfer function and relate them to the concept of control. We finally look at simple proportional, derivative, and integral controller design concepts in time domain. Throughout the chapter, we utilize MATLAB in simple toolboxes to help with our development.

5-1 TIME RESPONSE OF CONTINUOUS-DATA SYSTEMS: INTRODUCTION

Because time is used as an independent variable in most control systems, it is usually of interest to evaluate the state and output responses with respect to time or, simply, the time response. In the analysis problem, a reference input signal is applied to a system, and the performance of the system is evaluated by studying the system response in the time domain. For instance, if the objective of the control system is to have the output variable track the input signal, starting at some initial time and initial condition, it is necessary to compare the input and output responses as functions of time. Therefore, in most control-system problems, the final evaluation of the performance of the system is based on the time responses.

The time response of a control system is usually divided into two parts: the transient response and the steady-state response. Let $y(t)$ denote the time response of a continuous-data system; then, in general, it can be written as

$$y(t) = y_t(t) + y_{ss}(t) \tag{5-1}$$

where $y_t(t)$ denotes the transient response and $y_{ss}(t)$ denotes the steady-state response. In control systems, *transient response* is defined as the part of the time response that goes to zero as time becomes very large. Thus, $y_t(t)$ has the property

$$\lim_{t \to \infty} y_t(t) = 0 \tag{5-2}$$

The steady-state response is simply the part of the total response that remains after the transient has died out. Thus, the steady-state response can still vary in a fixed pattern, such as a sine wave, or a ramp function that increases with time.

All real, stable control systems exhibit transient phenomena to some extent before the steady state is reached. Because inertia, mass, and inductance are unavoidable in physical systems, the response of a typical control system cannot follow sudden changes in the input instantaneously, and transients are usually observed. Therefore, the control of the transient response is necessarily important, because it is a significant part of the dynamic behavior of the system, and the deviation between the output response and the input or the desired response, before the steady state is reached, must be closely controlled.

The steady-state response of a control system is also very important, because it indicates where the system output ends up when time becomes large. For a position-control system, the steady-state response when compared with the desired reference position gives an indication of the final accuracy of the system. In general, if the steady-state response of the output does not agree with the desired reference exactly, the system is said to have a steady-state error.

The study of a control system in the time domain essentially involves the evaluation of the transient and the steady-state responses of the system. In the design problem, specifications are usually given in terms of the transient and the steady-state performances, and controllers are designed so that the specifications are all met by the designed system.

5-2 TYPICAL TEST SIGNALS FOR THE TIME RESPONSE OF CONTROL SYSTEMS

Unlike electric networks and communication systems, the inputs to many practical control systems are not exactly known ahead of time. In many cases, the actual inputs of a control system may vary in random fashion with respect to time. For instance, in a radar-tracking system for antiaircraft missiles, the position and speed of the target to be tracked may vary in an unpredictable manner, so that they cannot be predetermined. This poses a problem for the designer, because it is difficult to design a control system so that it will perform satisfactorily to all possible forms of input signals. For the purpose of analysis and design, it is necessary to assume some basic types of test inputs so that the performance of a system can be evaluated. By selecting these basic test signals properly, not only is the mathematical treatment of the problem systematized, but the response due to these inputs allows the prediction of the system's performance to other more complex inputs. In the design problem, performance criteria may be specified with respect to these test signals so that the system may be designed to meet the criteria. This approach is particularly useful for linear systems, since the response to complex signals can be determined by superposing those due to simple test signals.

When the response of a linear time-invariant system is analyzed in the frequency domain, a sinusoidal input with variable frequency is used. When the input frequency is swept from zero to beyond the significant range of the system characteristics, curves in terms of the amplitude ratio and phase between the input and the output are drawn as functions of frequency. It is possible to predict the time-domain behavior of the system from its frequency-domain characteristics.

To facilitate the time-domain analysis, the following deterministic test signals are used.

Step-Function Input: The step-function input represents an instantaneous change in the reference input. For example, if the input is an angular position of a mechanical shaft, a step input represents the sudden rotation of the shaft. The mathematical representation of a step function or magnitude R is

$$\begin{aligned} r(t) &= R \quad t \geq 0 \\ &= 0 \quad t < 0 \end{aligned} \tag{5-3}$$

where R is a real constant. Or

$$r(t) = Ru_s(t) \tag{5-4}$$

where $u_s(t)$ is the unit-step function. The step function as a function of time is shown in Fig. 5-1(a). The step function is very useful as a test signal because its initial instantaneous jump in amplitude reveals a great deal about a system's quickness in responding to inputs with abrupt changes. Also, because the step function contains, in principle, a wide band of frequencies in its spectrum, as a result of the jump discontinuity, it is equivalent to the application of numerous sinusoidal signals with a wide range of frequencies.

Ramp-Function Input: The ramp function is a signal that changes constantly with time. Mathematically, a ramp function is represented by

$$r(t) = Rtu_s(t) \tag{5-5}$$

where R is a real constant. The ramp function is shown in Fig. 5-1(b). If the input variable represents the angular displacement of a shaft, the ramp input denotes the constant-speed

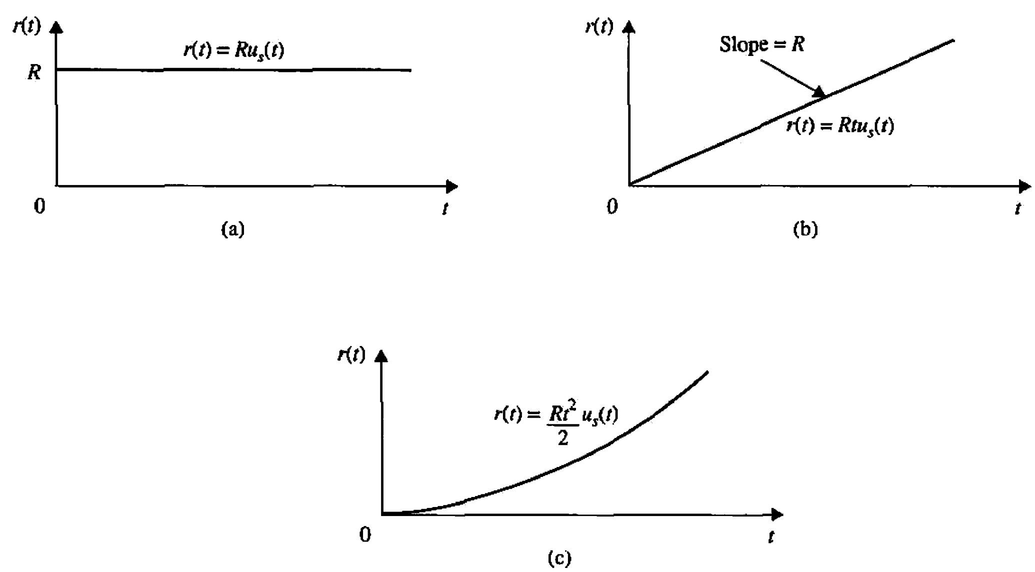


Figure 5-1 Basic time-domain test signals for control systems. (a) Step function. (b) Ramp function. (c) Parabolic function.

rotation of the shaft. The ramp function has the ability to test how the system would respond to a signal that changes linearly with time.

Parabolic-Function Input: The parabolic function represents a signal that is one order faster than the ramp function. Mathematically, it is represented as

$$r(t) = \frac{Rt^2}{2} u_s(t) \tag{5-6}$$

where R is a real constant and the factor $1/2$ is added for mathematical convenience because the Laplace transform of $r(t)$ is simply R/s^3. The graphical representation of the parabolic function is shown in Fig. 5-1(c).

These signals all have the common feature that they are simple to describe mathematically. From the step function to the parabolic function, the signals become progressively faster with respect to time. In theory, we can define signals with still higher rates, such as t^3, which is called the *jerk function*, and so forth. However, in reality, we seldom find it necessary or feasible to use a test signal faster than a parabolic function. This is because, as we shall see later, in order to track a high-order input accurately, the system must have high-order integrations in the loop, which usually leads to serious stability problems.

5-3 THE UNIT-STEP RESPONSE AND TIME-DOMAIN SPECIFICATIONS

As defined earlier, the transient portion of the time response is the part that goes to zero as time becomes large. Nevertheless, the transient response of a control system is necessarily important, because both the amplitude and the time duration of the transient response must be kept within tolerable or prescribed limits. For example, in the automobile idle-speed control system described in Chapter 1, in addition to striving for a desirable idle speed in the steady state, the transient drop in engine speed must not be excessive, and the recovery in speed should be made as quickly as possible. For linear control systems, the characterization of the transient response is often done by use of the unit-step function $u_s(t)$ as the input. The response of a control system when the input is a unit-step function is called the unit-step response. Fig. 5-2 illustrates a typical unit-step response of a linear control system. With reference to the unit-step response, performance criteria commonly used for the characterization of linear control systems in the time domain are defined as follows:

1. ***Maximum overshoot.*** Let $y(t)$ be the unit-step response. Let y_{max} denote the maximum value of $y(t)$; y_{ss}, the steady-state value of $y(t)$; and $y_{max} \geq y_{ss}$. The maximum overshoot of $y(t)$ is defined as

$$\text{maximum overshoot} = y_{max} - y_{ss} \tag{5-7}$$

The maximum overshoot is often represented as a percentage of the final value of the step response; that is,

$$\text{percent maximum overshoot} = \frac{\text{maximum overshoot}}{y_{ss}} \times 100\% \tag{5-8}$$

The maximum overshoot is often used to measure the relative stability of a control system. A system with a large overshoot is usually undesirable. For design

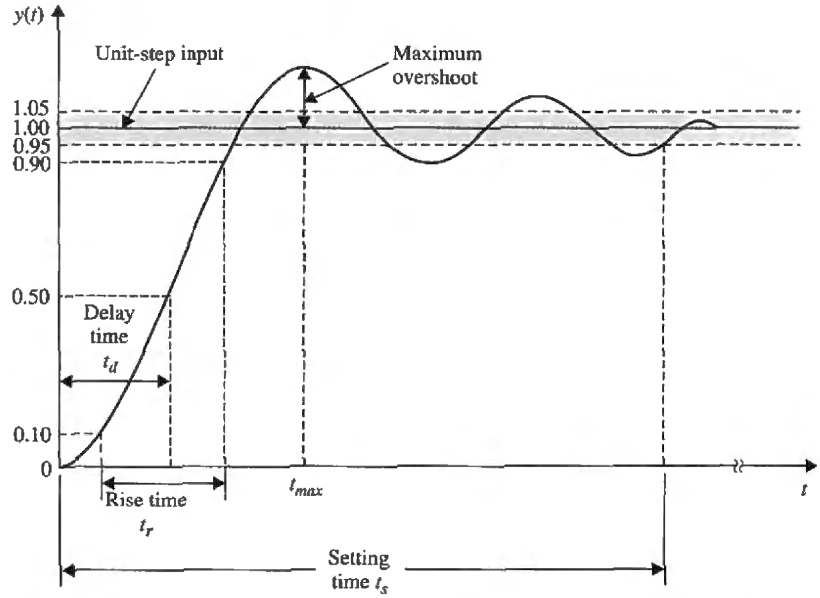


Figure 5-2 Typical unit-step response of a control system illustrating the time-domain specifications.

purposes, the maximum overshoot is often given as a time-domain specification. The unit-step illustrated in Fig. 5-2 shows that the maximum overshoot occurs at the first overshoot. For some systems, the maximum overshoot may occur at a later peak, and, if the system transfer function has an odd number of zeros in the right-half s-plane, a negative undershoot may even occur [4, 5] (Problem 5-23).

2. ***Delay time.*** The delay time t_d is defined as the time required for the step response to reach 50% of its final value. This is shown in Fig. 5-2.
3. ***Rise time.*** The rise time t_r is defined as the time required for the step response to rise from 10 to 90% of its final value, as shown in Fig. 5-2. An alternative measure is to represent the rise time as the reciprocal of the slope of the step response at the instant that the response is equal to 50% of its final value.
4. ***Settling time.*** The settling time t_s is defined as the time required for the step response to decrease and stay within a specified percentage of its final value. A frequently used figure is 5%.

   The four quantities just defined give a direct measure of the transient characteristics of a control system in terms of the unit-step response. These time-domain specifications are relatively easy to measure when the step response is well defined, as shown in Fig. 5-2. Analytically, these quantities are difficult to establish, except for simple systems lower than the third order.
5. ***Steady-state error.*** The steady-state error of a system response is defined as the discrepancy between the output and the reference input when the steady state ($t \to \infty$) is reached.

   It should be pointed out that the steady-state error may be defined for any test signal such as a step-function, ramp-function, parabolic-function, or even a sinusoidal input, although Fig. 5-2 only shows the error for a step input.

5-4 STEADY-STATE ERROR

One of the objectives of most control systems is that the system output response follows a specific reference signal accurately in the steady state. The difference between the output and the reference in the steady state was defined earlier as the steady-state error. In the real world, because of friction and other imperfections and the natural composition of the system, the steady state of the output response seldom agrees exactly with the reference. Therefore, steady-state errors in control systems are almost unavoidable. In a design problem, one of the objectives is to keep the steady-state error to a minimum, or below a certain tolerable value, and at the same time the transient response must satisfy a certain set of specifications.

The accuracy requirement on control systems depends to a great extent on the control objectives of the system. For instance, the final position accuracy of an elevator would be far less stringent than the pointing accuracy on the control of the Large Space Telescope, which is a telescope mounted onboard a space shuttle. The accuracy of position control of such a system is often measured in microradians.

5-4-1 Steady-State Error of Linear Continuous-Data Control Systems

Linear control systems are subject to steady-state errors for somewhat different causes than nonlinear systems, although the reason is still that the system no longer "sees" the error, and no corrective effort is exerted. In general, the steady-state errors of linear control systems depend on the type of the reference signal and the type of the system.

Definition of the Steady-State Error with Respect to System Configuration

Before embarking on the steady-state error analysis, we must first clarify what is meant by system error. In general, we can regard the error as a signal that should be quickly reduced to zero, if possible. Let us refer to the closed-loop system shown in Fig. 5-3, where $r(t)$ is the input; $u(t)$, the actuating signal; $b(t)$, the feedback signal; and $y(t)$, the output. The error of the system may be defined as

$$e(t) = \text{reference signal} - y(t) \tag{5-9}$$

where the reference signal is the signal that the output $y(t)$ is to track. When the system has unity feedback, that is, $H(s) = 1$, then the input $r(t)$ is the reference signal, and the error is simply

$$e(t) = r(t) - y(t) \tag{5-10}$$

The steady-state error is defined as

$$e_{ss} = \lim_{t\to\infty} e(t) \tag{5-11}$$

When $H(s)$ is not unity, the actuating signal $u(t)$ in Fig. 5-2 may or may not be the error, depending on the form and the purpose of $H(s)$. Let us assume that the objective of the

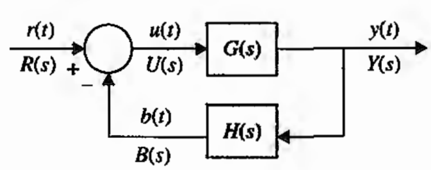


Figure 5-3 Nonunity feedback control system.

system in Fig. 5-3 is to have the output $y(t)$ track the input $r(t)$ as closely as possible, and the system transfer functions are

$$G(s) = \frac{1}{s^2(s+12)} \quad H(s) = \frac{5(s+1)}{(s+5)} \tag{5-12}$$

We can show that, if $H(s) = 1$, the characteristic equation is

$$s^3 + 12s^2 + 1 = 0 \tag{5-13}$$

which has roots in the right-half s-plane, and the closed-loop system is unstable. We can show that the $H(s)$ given in Eq. (5-12) stabilizes the system, and the characteristic equation becomes

$$s^4 + 17s^3 + 60s^2 + 5s + 5 = 0 \tag{5-14}$$

In this case, the system error may still be defined as in Eq. (5-10).

However, consider a velocity control system in which a step input is used to control the system output that contains a ramp in the steady state. The system transfer functions may be of the form

$$G(s) = \frac{1}{s^2(s+12)} \quad H(s) = K_t s \tag{5-15}$$

where $H(s)$ is the transfer function of an electromechanical or electronic tachometer, and K_t is the tachometer constant. The system error should be defined as in Eq. (5-9), where the reference signal is the *desired velocity* and not $r(t)$. In this case, because $r(t)$ and $y(t)$ are not of the same dimension, it would be meaningless to define the error as in Eq. (5-10). To illustrate the system further, let $K_t = 10$ volts/rad/sec. This means that, for a unit-step input of 1 volt, the desired velocity in the steady state is 1/10 or 0.1 rad/sec, because when this is achieved, the output voltage of the tachometer would be 1 volt, and the steady-state error would be zero. The closed-loop transfer function of the system is

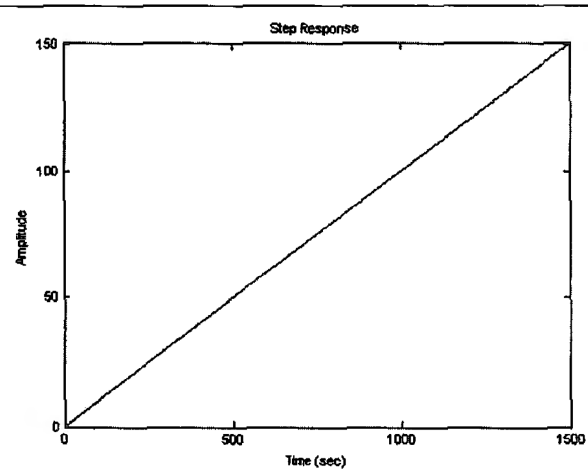
$$M(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} = \frac{1}{s(s^2 + 12s + 10)} \tag{5-16}$$

Toolbox 5-4-1

For the system in Eq. 5-15:

$$G(s) = \frac{1}{s^2(s+12)} \quad H(s) = K_t s$$

```
% use Kt=10
%Step input
Kt=10;
Gzpk=zpk([],[0 0 -12],1)
G=tf(Gzpk)
H=zpk(0,[],Kt)
cloop=feedback(G,H)
step(cloop)
xlabel('Time(sec)');
ylabel('Amplitude');
```



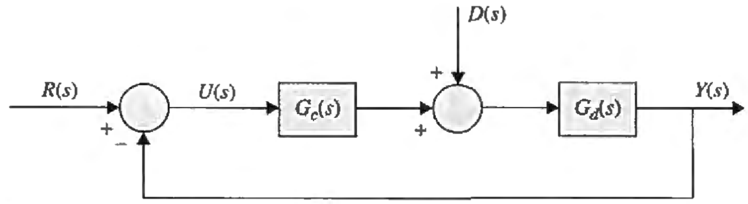


Figure 5-4 System with disturbance input.

For a unit-step function input, $R(s) = 1/s$. The output time response is

$$y(t) = 0.1t - 0.12 - 0.000796e^{-11.1t} + 0.1208e^{-0.901t} \quad t \geq 0 \tag{5-17}$$

Because the exponential terms of $y(t)$ in Eq. (5-17) all diminish as $t \to \infty$, the steady-state part of $y(t)$ is $0.1t - 0.12$. Thus, the steady-state error of the system is

$$e_{ss} = \lim_{t\to\infty} [0.1t - y(t)] = 0.12 \tag{5-18}$$

More explanations on how to define the reference signal when $H(s) \neq 1$ will be given later when the general discussion on the steady-state error of nonunity feedback systems is given.

Not all system errors are defined with respect to the response due to the input. Fig. 5-4 shows a system with a disturbance $d(t)$, in addition to the input $r(t)$. The output due to $d(t)$ acting alone may also be considered an error.

Because of these reasons, the definition of system error has not been unified in the literature. To establish a systematic study of the steady-state error for linear systems, we shall classify three types of systems and treat these separately.

1. Systems with unity feedback; $H(s) = 1$.
2. Systems with nonunity feedback, but $H(0) = K_H =$ constant.
3. Systems with nonunity feedback, and $H(s)$ has zeros at $s = 0$ of order N.

The objective here is to establish a definition of the error with respect to one basic system configuration so that some fundamental relationships can be determined between the steady-state error and the system parameters.

Type of Control Systems: Unity Feedback Systems

Consider that a control system with unity feedback can be represented by or simplified to the block diagram with $H(s) = 1$ in Fig. 5-3. The steady-state error of the system is written

$$\begin{aligned} e_{ss} &= \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) \\ &= \lim_{s\to 0} \frac{sR(s)}{1 + G(s)} \end{aligned} \tag{5-19}$$

Clearly, e_{ss} depends on the characteristics of $G(s)$. More specifically, we can show that e_{ss} depends on the number of poles $G(s)$ has at $s = 0$. This number is known as the type of the control system or, simply, system type.

We can show that the steady-state error e_{ss} depends on the type of the control system. Let us formalize the system type by referring to the form of the forward-path transfer function $G(s)$. In general, $G(s)$ can be expressed for convenience as

$$G(s) = \frac{K(1+T_1 s)(1+T_2 s)\cdots(1+T_{m1}s+T_{m2}s^2)}{s^j(1+T_a s)(1+T_b s)\cdots(1+T_{n1}s+T_{n2}s^2)}e^{-T_d s} \tag{5-20}$$

where K and all the T's are real constants. The system type refers to the order of the pole of $G(s)$ at $s=0$. Thus, the closed-loop system having the forward-path transfer function of Eq. (5-20) is type j, where $j=0, 1, 2, \ldots$ The total number of terms in the numerator and the denominator and the values of the coefficients are not important to the system type, as system type refers only to the number of poles $G(s)$ has at $s=0$. The following example illustrates the system type with reference to the form of $G(s)$.

EXAMPLE 5-4-1

$$G(s) = \frac{K(1+0.5s)}{s(1+s)(1+2s)(1+s+s^2)} \quad \text{type 1} \tag{5-21}$$

$$G(s) = \frac{K(1+2s)}{s^3} \quad \text{type 3} \tag{5-22}$$

Now let us investigate the effects of the types of inputs on the steady-state error. We shall consider only the step, ramp, and parabolic inputs.

Steady-State Error of System with a Step-Function Input

When the input $r(t)$ to the control system with $H(s)=1$ of Fig. 5-3 is a step function with magnitude R, $R(s)=R/s$, the steady-state error is written from Eq. (5-19),

$$e_{ss} = \lim_{s\to 0}\frac{sR(s)}{1+G(s)} = \lim_{s\to 0}\frac{R}{1+G(s)} = \frac{R}{1+\lim_{s\to 0}G(s)} \tag{5-23}$$

For convenience, we define

$$K_p = \lim_{s\to 0} G(s) \tag{5-24}$$

as the step-error constant. Then Eq. (5-23) becomes

$$e_{ss} = \frac{R}{1+K_p} \tag{5-25}$$

A typical e_{ss} due to a step input when K_p is finite and nonzero is shown in Fig. 5-5. We see from Eq. (5-25) that, for e_{ss} to be zero, when the input is a step function, K_p must be infinite. If $G(s)$ is described by Eq. (5-20), we see that, for K_p to be infinite, j must be at least equal to unity; that is, $G(s)$ must have at least one pole at $s=0$. Therefore, we can summarize the steady-state error due to a step function input as follows:

Type 0 system: $e_{ss} = \dfrac{R}{1+K_p} = \text{constant}$

Type 1 or higher system: $e_{ss} = 0$

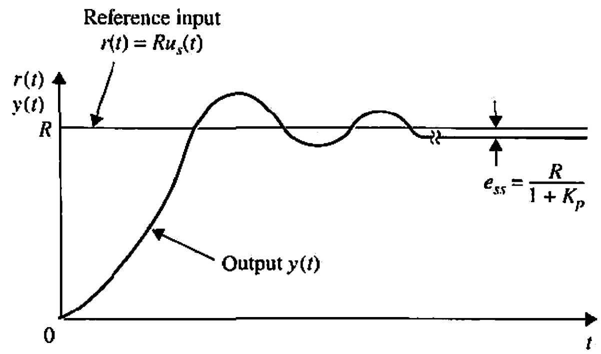


Figure 5-5 Typical steady-state error due to a step input.

Steady-State Error of System with a Ramp-Function Input

When the input to the control system $[H(s) = 1]$ of Fig. 5-3 is a ramp function with magnitude R,

$$r(t) = Rtu_s(t) \tag{5-26}$$

where R is a real constant, the Laplace transform of $r(t)$ is

$$R(s) = \frac{R}{s^2} \tag{5-27}$$

The steady-state error is written using Eq. (5-19),

$$e_{ss} = \lim_{s \to 0} \frac{R}{s + sG(s)} = \frac{R}{\lim_{s \to 0} sG(s)} \tag{5-28}$$

We define the ramp-error constant as

$$K_v = \lim_{s \to 0} sG(s) \tag{5-29}$$

Then, Eq. (5-26) becomes

$$e_{ss} = \frac{R}{K_v} \tag{5-30}$$

which is the steady-state error when the input is a ramp function. A typical e_{ss} due to a ramp input when K_v is finite and nonzero is illustrated in Fig. 5-6.

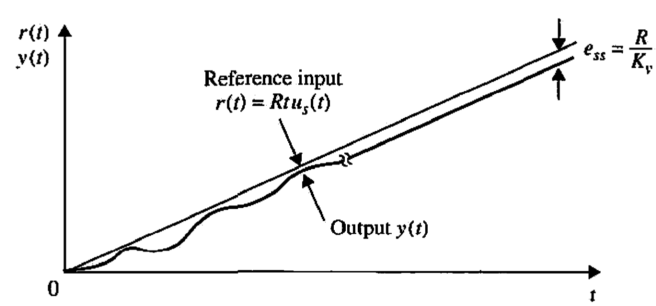


Figure 5-6 Typical steady-state error due to a ramp-function input.

Eq. (5-30) shows that, for e_{ss} to be zero when the input is a ramp function, K_v must be infinite. Using Eqs. (5-20) and (5-29), we obtain

$$K_v = \lim_{s \to 0} sG(s) = \lim_{s \to 0} \frac{K}{s^{j-1}} \quad j = 0, 1, 2, \ldots \tag{5-31}$$

Thus, for K_v to be infinite, j must be at least equal to 2, or the system must be of type 2 or higher. The following conclusions may be stated with regard to the steady-state error of a system with ramp input:

Type 0 system: $e_{ss} = \infty$

Type 1 system: $e_{ss} = \dfrac{R}{K_v} = \text{constant}$

Type 2 system: $e_{ss} = 0$

Steady-State Error of System with a Parabolic-Function Input

When the input is described by the standard parabolic form

$$r(t) = \frac{Rt^2}{2} u_s(t) \tag{5-32}$$

the Laplace transform of $r(t)$ is

$$R(s) = \frac{R}{s^3} \tag{5-33}$$

The steady-state error of the system in Fig. 5-3 with $H(s) = 1$ is

$$e_{ss} = \frac{R}{\lim_{s \to 0} s^2 G(s)} \tag{5-34}$$

A typical e_{ss} of a system with a nonzero and finite K_a due to a parabolic-function input is shown in Fig. 5-7.

Defining the parabolic-error constant as

$$K_a = \lim_{s \to 0} s^2 G(s) \tag{5-35}$$

the steady-state error becomes

$$e_{ss} = \frac{R}{K_a} \tag{5-36}$$

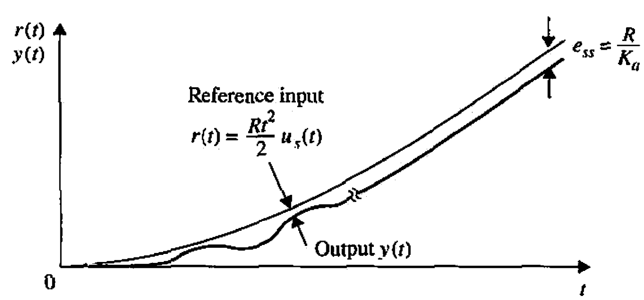


Figure 5-7 Typical steady-state error due to a parabolic-function input.

TABLE 5-1 Summary of the Steady-State Errors Due to Step-, Ramp-, and Parabolic-Function Inputs for Unity-Feedback Systems

				Steady-State Error e_{ss}		
Type of System	Error Constants			Step Input	Ramp Input	Parabolic
j	K_p	K_v	K_a	$\frac{R}{1+K_p}$	$\frac{R}{K_v}$	$\frac{R}{K_a}$
0	K	0	0	$\frac{R}{1+K}$	∞	∞
1	∞	K	0	0	$\frac{R}{K}$	∞
2	∞	∞	K	0	0	$\frac{R}{K}$
3	∞	∞	∞	0	0	0

Following the pattern set with the step and ramp inputs, the steady-state error due to the parabolic input is zero if the system is of type 3 or greater. The following conclusions are made with regard to the steady-state error of a system with parabolic input:

Type 0 system: $e_{ss} = \infty$

Type 1 system: $e_{ss} = \infty$

Type 2 system: $e_{ss} = \dfrac{R}{K_a} = \text{constant}$

Type 3 or higher system: $e_{ss} = 0$

We cannot emphasize often enough that, for these results to be valid, the closed-loop system must be stable.

By using the method described, the steady-state error of any linear closed-loop system subject to an input with order higher than the parabolic function can also be derived if necessary. As a summary of the error analysis, Table 5-1 shows the relations among the error constants, the types of systems with reference to Eq. (5-20), and the input types.

As a summary, the following points should be noted when applying the error-constant analysis just presented.

1. The step-, ramp-, or parabolic-error constants are significant for the error analysis only when the input signal is a step function, ramp function, or parabolic function, respectively.
2. Because the error constants are defined with respect to the forward-path transfer function $G(s)$, the method is applicable to only the system configuration shown in Fig. 5-3 with $H(s) = 1$. Because the error analysis relies on the use of the final-value theorem of the Laplace transform, it is important to check first to see if $sE(s)$ has any poles on the $j\omega$-axis or in the right-half s-plane.
3. The steady-state error properties summarized in Table 5-1 are for systems with unity feedback only.
4. The steady-state error of a system with an input that is a linear combination of the three basic types of inputs can be determined by superimposing the errors due to each input component.
5. When the system configuration differs from that of Fig. 5-3 with $H(s) = 1$, we can either simplify the system to the form of Fig. 5-3 or establish the error signal and apply the final-value theorem. The error constants defined here may or may not apply, depending on the individual situation.

When the steady-state error is infinite, that is, when the error increases continuously with time, the error-constant method does not indicate how the error varies with time. This is one of the disadvantages of the error-constant method. The error-constant method also does not apply to systems with inputs that are sinusoidal, since the final-value theorem cannot be applied. The following examples illustrate the utility of the error constants and their values in the determination of the steady-state errors of linear control systems with unity feedback.

EXAMPLE 5-4-2 Consider that the system shown in Fig. 5-3 with $H(s) = 1$ has the following transfer functions. The error constants and steady-state errors are calculated for the three basic types of inputs using the error constants.

a. $G(s) = \dfrac{K(s+3.15)}{s(s+1.5)(s+0.5)}$ $\quad H(s) = 1 \quad$ Type 1 system

Step input: Step-error constant $K_p = \infty$ $\quad e_{ss} = \dfrac{R}{1+K_p} = 0$

Ramp input: Ramp-error constant $K_v = 4.2K$ $\quad e_{ss} = \dfrac{R}{K_v} = \dfrac{R}{4.2K}$

Parabolic input: Parabolic-error constant $K_a = 0$ $\quad e_{ss} = \dfrac{R}{K_a} = \infty$

These results are valid only if the value of K stays within the range that corresponds to a stable closed-loop system, which is $0 < K < 1.304$.

b. $G(s) = \dfrac{K}{s^2(s+12)}$ $\quad H(s) = 1 \quad$ Type 2 system

The closed-loop system is unstable for all values of K, and error analysis is meaningless.

c. $G(s) = \dfrac{5(s+1)}{s^2(s+12)(s+5)}$ $\quad H(s) = 1 \quad$ Type 2 system

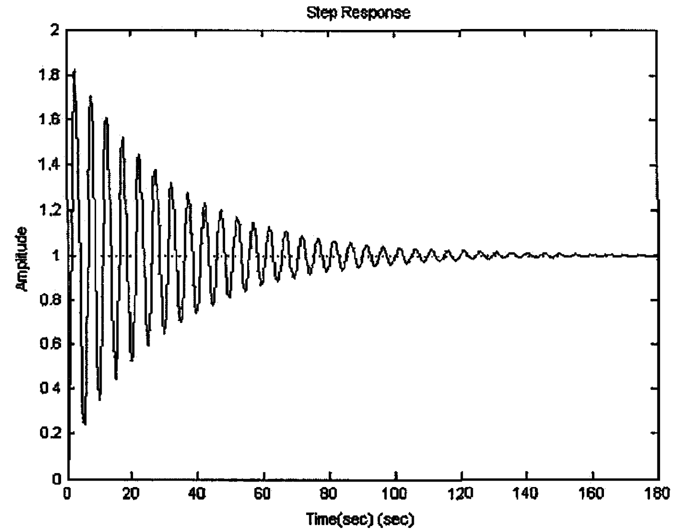
We can show that the closed-loop system is stable. The steady-state errors are calculated for the three basic types of inputs.

Toolbox 5-4-2

For the system in Example 5-4-2:

(a) $G(s) = \dfrac{K(s+3.15)}{s(s+1.5)(s+0.5)}$ $\quad H(s) = 1 \quad$ Type 1 system

```
% Step input
K=1; % Use K=1
Gzpk=zpk([-3.15],[0 -1.5 -0.5],1)
G=tf(Gzpk);
H=1;
clooptf=feedback(G,H)
step(clooptf)
xlabel('Time(sec)');
ylabel('Amplitude');
```



Similarly you may obtain the ramp and parabolic responses

```
%Ramp input
t=0:0.1:50;
u=t;
[y,x]=lsim(clooptf,u,t);
plot(t,y,t,u);
title('Closed-loop response for Ramp Input')
xlabel('Time(sec)')
ylabel('Amplitude')
%Parabolic input
t=0:0.1:50;
u=0.5*t.*t;
[y,x]=lsim(clooptf,u,t);
plot(t,y,t,u);
title('Closed-loop response for Parabolic Input')
xlabel('Time(sec')
ylabel('Amplitude')
```

Step input:	Step-error constant: $K_p = \infty$	$e_{ss} = \dfrac{R}{1+K_p} = 0$
Ramp input:	Ramp-error constant: $K_v = \infty$	$e_{ss} = \dfrac{R}{K_v} = 0$
Parabolic input:	Parabolic-error constant: $K_a = 1/12$	$e_{ss} = \dfrac{R}{K_a} = 12R$

Relationship between Steady-State Error and Closed-Loop Transfer Function

In the last section, the steady-state error of a closed-loop system was related to the forward-path transfer function $G(s)$ of the system, which is usually known. Often, the closed-loop transfer function is derived in the analysis process, and it would be of interest to establish the relationships between the steady-state error and the coefficients of the closed-loop transfer function. As it turns out, the closed-loop transfer function can be used to find the steady-state error of systems with unity as well as nonunity feedback. For the present discussion, let us impose the following condition:

$$\lim_{s\to 0} H(s) = H(0) = K_H = \text{constant} \tag{5-37}$$

which means that $H(s)$ cannot have poles at $s = 0$. Because the signal that is fed back to be compared with the input in the steady state is K_H times the steady-state output, when this feedback signal equals the input, the steady-state error would be zero. Thus, we can define the reference signal as $r(t)/K_H$ and the error signal as

$$e(t) = \frac{1}{K_H} r(t) - y(t) \tag{5-38}$$

or, in the transform domain,

$$E(s) = \frac{1}{K_H} R(s) - Y(s) = \frac{1}{K_H}[1 - K_H M(s)]R(s) \tag{5-39}$$

where $M(s)$ is the closed-loop transfer function, $Y(s)/R(s)$. Notice that the above development includes the unity-feedback case for which $K_H = 1$. Let us assume that $M(s)$ does not have any poles at $s = 0$ and is of the form

$$M(s) = \frac{Y(s)}{R(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \tag{5-40}$$

where $n > m$. We further require that all the poles of $M(s)$ are in the left-half s-plane, which means that the system is stable. The steady-state error of the system is written

$$e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{1}{K_H}[1 - K_H M(s)]sR(s) \tag{5-41}$$

Substituting Eq. (5-40) into the last equation and simplifying, we get

$$e_{ss} = \frac{1}{K_H} \lim_{s \to 0} \frac{s^n + \cdots + (a_1 - b_1 K_H)s + (a_0 - b_0 K_H)}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} sR(s) \tag{5-42}$$

We consider the three basic types of inputs for $r(t)$.

1. ***Step-function input.*** $R(s) = R/s$.

   For a step-function input, the steady-state error in Eq. (5-42) becomes

$$e_{ss} = \frac{1}{K_H}\left(\frac{a_0 - b_0 K_H}{a_0}\right)R \tag{5-43}$$

   Thus, the steady-state error due to a step input can be zero only if

$$a_0 - b_0 K_H = 0 \tag{5-44}$$

   or

$$M(0) = \frac{b_0}{a_0} = \frac{1}{K_H} \tag{5-45}$$

   This means that, *for a unity-feedback system $K_H = 1$, the constant terms of the numerator and the denominator of $M(s)$ must be equal, that is, $b_0 = a_0$, for the steady-state error to be zero.*

2. ***Ramp-function input.*** $R(s) = R/s^2$.
For a ramp-function input, the steady-state error in Eq. (5-42) becomes

$$e_{ss} = \frac{1}{K_H} \lim_{s \to 0} \frac{s^n + \cdots + (a_1 - b_1 K_H)s + (a_0 - b_0 K_H)}{s(s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0)} R \tag{5-46}$$

The following values of e_{ss} are possible:

$$e_{ss} = 0 \qquad \text{if} \quad a_0 - b_0 K_H = 0 \quad \text{and} \quad a_1 - b_1 K_H = 0 \tag{5-47}$$

$$e_{ss} = \frac{a_1 - b_1 K_H}{a_0 K_H} R = \text{constant} \quad \text{if} \quad a_0 - b_0 K_H = 0 \quad \text{and} \quad a_1 - b_1 K_H \neq 0 \tag{5-48}$$

$$e_{ss} = \infty \qquad \text{if} \quad a_0 - b_0 K_H \neq 0 \tag{5-49}$$

3. ***Parabolic-function input.*** $R(s) = R/s^3$.
For a parabolic input, the steady-state error in Eq. (5-42) becomes

$$e_{ss} = \frac{1}{K_H} \lim_{s \to 0} \frac{s^n + \cdots + (a_2 - b_2 K_H)s^2 + (a_1 - b_1 K_H)s + (a_0 - b_0 K_H)}{s^2(s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0)} R \tag{5-50}$$

The following values of e_{ss} are possible:

$$e_{ss} = 0 \qquad \text{if} \quad a_i - b_i K_H = 0 \quad \text{for} \quad i = 0,\ 1,\ \text{and}\ 2 \tag{5-51}$$

$$e_{ss} = \frac{a_2 - b_2 K_H}{a_0 K_H} R = \text{constant} \quad \text{if} \quad a_i - b_i K_H = 0 \quad \text{for} \quad i = 0 \quad \text{and}\ 1 \tag{5-52}$$

$$e_{ss} = \infty \qquad \text{if} \quad a_i - b_i K_H \neq 0 \quad \text{for} \quad i = 0 \quad \text{and}\ 1 \tag{5-53}$$

EXAMPLE 5-4-3 The forward-path and closed-loop transfer functions of the system shown in Fig. 5-3 are given next. The system is assumed to have unity feedback, so $H(s) = 1$, and thus $K_H = H(0) = 1$.

$$G(s) = \frac{5(s+1)}{s^2(s+12)(s+5)} \qquad M(s) = \frac{5(s+1)}{s^4 + 17s^3 + 60s^2 + 5s + 5} \tag{5-54}$$

The poles of $M(s)$ are all in the left-half s-plane. Thus, the system is stable. The steady-state errors due to the three basic types of inputs are evaluated as follows:

Step input: $e_{ss} = 0$ since $a_0 = b_0 (= 5)$

Ramp input: $e_{ss} = 0$ since $a_0 = b_0 (= 5)$ and $a_1 = b_1 (= 5)$

Parabolic input: $e_{ss} = \dfrac{a_2 - b_2 K_H}{a_0 K_H} R = \dfrac{60}{5} R = 12R$

Because this is a type 2 system with unity feedback, the same results are obtained with the error-constant method.

EXAMPLE 5-4-4 Consider the system shown in Fig. 5-3, which has the following transfer functions:

$$G(s) = \frac{1}{s^2(s+12)} \quad H(s) = \frac{5(s+1)}{s+5} \tag{5-55}$$

Then, $K_H = H(0) = 1$. The closed-loop transfer function is

$$M(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{s+5}{s^4+17s^3+60s^2+5s+5} \tag{5-56}$$

Comparing the last equation with Eq. (5-40), we have $a_0 = 5$, $a_1 = 5$, $a_2 = 60$, $b_0 = 5$, $b_1 = 1$, and $b_2 = 0$. The steady-state errors of the system are calculated for the three basic types of inputs.

Unit-step input, $r(t) = u_s(t)$: $\quad e_{ss} = \frac{a_0 - b_0 K_H}{a_0} = 0$

Unit-ramp input, $r(t) = tu_s(t)$: $\quad e_{ss} = \frac{a_1 - b_1 K_H}{a_0 K_H} = \frac{5-1}{5} = 0.8$

Unit-parabolic input, $r(t) = tu_s(t)/2$: $\quad e_{ss} = \infty \quad \text{since } a_1 - b_1 K_H \neq 0$

It would be illuminating to calculate the steady-state errors of the system from the difference between the input and the output and compare them with the results just obtained.

Applying the unit-step, unit-ramp, and unit-parabolic inputs to the system described by Eq. (5-56), and taking the inverse Laplace transform of $Y(s)$, the outputs are

Unit-step input:

$$\begin{aligned} y(t) = {} & 1 - 0.00056e^{-12.05t} - 0.0001381e^{-4.886t} \\ & - 0.9993e^{-0.0302t}\cos 0.2898t - 0.1301e^{-0.0302t}\sin 0.2898t \quad t \geq 0 \end{aligned} \tag{5-57}$$

Thus, the steady-state value of $y(t)$ is unity, and the steady-state error is zero.

Unit-ramp input:

$$\begin{aligned} y(t) = {} & t - 0.8 + 4.682 \times 10^{-5}e^{-12.05t} + 2.826 \times 10^{-5}e^{-4.886t} \\ & + 0.8e^{-0.0302t}\cos 0.2898t - 3.365e^{-0.0302t}\sin 0.2898t \quad t \geq 0 \end{aligned} \tag{5-58}$$

Thus, the steady-state portion of $y(t)$ is $t - 0.8$, and the steady-state error to a unit ramp is 0.8.

Unit-parabolic input:

$$\begin{aligned} y(t) = {} & 0.5t^2 - 0.8t - 11.2 - 3.8842 \times 10^{-6}e^{-12.05t} - 5.784 \times 10^{-6}e^{-4.886t} \\ & + 11.2e^{-0.0302t}\cos 0.2898t + 3.9289e^{-0.0302t}\sin 0.2898t \quad t \geq 0 \end{aligned} \tag{5-59}$$

The steady-state portion of $y(t)$ is $0.5t^2 - 0.8t - 11.2$. Thus, the steady-state error is $0.8t + 11.2$, which becomes infinite as time goes to infinity.

EXAMPLE 5-4-5 Consider that the system shown in Fig. 5-3 has the following transfer functions:

$$G(s) = \frac{1}{s^2(s+12)} \quad H(s) = \frac{10(s+1)}{s+5} \tag{5-60}$$

Thus,

$$K_H = \lim_{s \to 0} H(s) = 2 \tag{5-61}$$

The closed-loop transfer function is

$$M(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{s+5}{s^4+17s^3+60s^2+10s+10} \tag{5-62}$$

The steady-state errors of the system due to the three basic types of inputs are calculated as follows:

Unit-step input $r(t) = u_s(t)$:

$$e_{ss} = \frac{1}{K_H}\left(\frac{a_0 - b_0 K_H}{a_0}\right) = \frac{1}{2}\left(\frac{10 - 5 \times 2}{10}\right) = 0 \tag{5-63}$$

Solving for the output using the $M(s)$ in Eq. (5-62), we get

$$y(t) = 0.5u_s(t) + \text{transient terms} \tag{5-64}$$

Thus, the steady-state value of $y(t)$ is 0.5, and because $K_H = 2$, the steady-state error due to a unit-step input is zero.

Unit-ramp input $r(t) = tu_s(t)$:

$$e_{ss} = \frac{1}{K_H}\left(\frac{a_1 - b_1 K_H}{a_0}\right) = \frac{1}{2}\left(\frac{10 - 1 \times 2}{10}\right) = 0.4 \tag{5-65}$$

The unit-ramp response of the system is written

$$y(t) = [0.5t - 0.4]u_s(t) + \text{transient terms} \tag{5-66}$$

Thus, using Eq. (5-38), the steady-state error is calculated as

$$e(t) = \frac{1}{K_H} r(t) - y(t) = 0.4u_s(t) - \text{transient terms} \tag{5-67}$$

Because the transient terms will die out as t approaches infinity, the steady-state error due to a unit-ramp input is 0.4, as calculated in Eq. (5-66).

Unit-parabolic input $r(t) = t^2u_s(t)/2$:

$$e_{ss} = \infty \quad \text{since} \quad a_1 - b_1 K_H \neq 0$$

The unit-parabolic input is

$$y(t) = \left[0.25t^2 - 0.4t - 2.6\right]u_s(t) + \text{transient terms} \tag{5-68}$$

The error due to the unit-parabolic input is

$$e(t) = \frac{1}{K_H} r(t) - y(t) = (0.4t - 2.6)u_s(t) - \text{transient terms} \tag{5-69}$$

Thus, the steady-state error is $0.4t + 2.6$, which increases with time.

Steady-State Error of Nonunity Feedback: $H(s)$ Has Nth-Order Zero at $s = 0$

This case corresponds to desired output being proportional to the Nth-order derivative of the input in the steady state. In the real world, this corresponds to applying a tachometer or rate feedback. Thus, for the steady-state error analysis, the reference signal can be defined as $R(s)/K_H s^N$, and the error signal in the transform domain may be defined as

$$E(s) = \frac{1}{K_H s^N} R(s) - Y(s) \tag{5-70}$$

where

$$K_H = \lim_{s \to 0} \frac{H(s)}{s^N} \tag{5-71}$$

We shall derive only the results for $N = 1$ here. In this case, the transfer function of $M(s)$ in Eq. (5-40) will have a pole at $s = 0$, or $a_0 = 0$. The steady-state error is written from Eq. (5-70) as

$$e_{ss} = \frac{1}{K_H} \lim_{s \to 0} \left[\frac{s^{n-1} + \cdots + (a_2 - b_1 K_H)s + (a_1 - b_0 K_H)}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s}\right] sR(s) \tag{5-72}$$

For a step input of magnitude R, the last equation is written

$$e_{ss} = \frac{1}{K_H}\lim_{s\to 0}\left[\frac{s^{n-1} + \cdots + (a_2 - b_1K_H)s + (a_1 - b_0K_H)}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s}\right]R \tag{5-73}$$

Thus, the steady-state error is

$$e_{ss} = 0 \qquad \text{if} \quad a_2 - b_1K_H = 0 \quad \text{and} \quad a_1 - b_0K_H = 0 \tag{5-74}$$

$$e_{ss} = \frac{a_2 - b_1K_H}{a_1K_H}R = \text{constant} \qquad \text{if} \quad a_1 - b_0K_H = 0 \quad \text{but} \quad a_2 - b_1K_H \neq 0 \tag{5-75}$$

$$e_{ss} = \infty \qquad \text{if} \quad a_1 - b_0K_H \neq 0 \tag{5-76}$$

We shall use the following example to illustrate these results.

EXAMPLE 5-4-6 Consider that the system shown in Fig. 5-3 has the following transfer functions:

$$G(s) = \frac{1}{s^2(s+12)} \qquad H(s) = \frac{10s}{s+5} \tag{5-77}$$

Thus,

$$K_H = \lim_{s\to 0}\frac{H(s)}{s} = 2 \tag{5-78}$$

The closed-loop transfer function is

$$M(s) = \frac{Y(s)}{R(s)} = \frac{s+5}{s^4 + 17s^3 + 60s^2 + 10s} \tag{5-79}$$

The velocity control system is stable, although $M(s)$ has a pole at $s = 0$, because the objective is to control velocity with a step input. The coefficients are identified to be $a_0 = 0$, $a_1 = 10$, $a_2 = 60$, $b_0 = 5$, and $b_1 = 1$.

For a unit-step input, the steady-state error, from Eq. (5-75), is

$$e_{ss} = \frac{1}{K_H}\left(\frac{a_2 - b_1K_H}{a_1}\right) = \frac{1}{2}\left(\frac{60 - 1\times 2}{10}\right) = 2.9 \tag{5-80}$$

To verify this result, we find the unit-step response using the closed-loop transfer function in Eq. (5-79). The result is

$$y(t) = (0.5t - 2.9)u_s(t) + \text{transient terms} \tag{5-81}$$

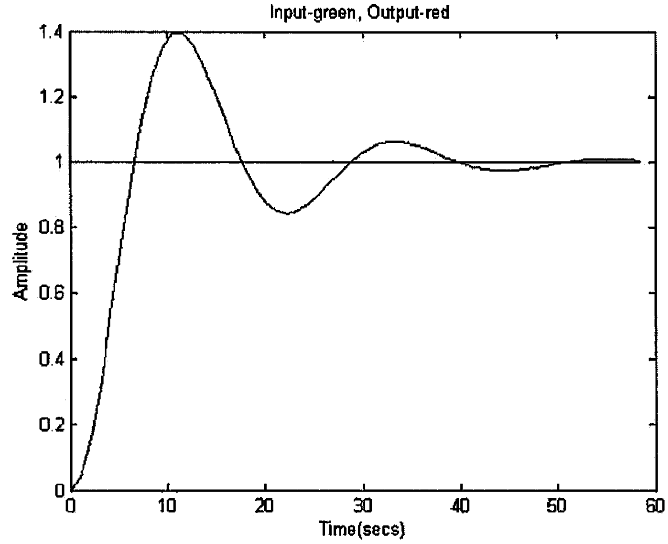
From the discussion that leads to Eq. (5-70), the reference signal is considered to be $tu_s(t)/K_H = 0.5tu_s(t)$ in the steady state; thus, the steady-state error is 2.9. Of course, it should be pointed out that if $H(s)$ were a constant for this type 2 system, the closed-loop system would be unstable. So, the derivative control in the feedback path also has a stabilizing effect.

Toolbox 5-4-3

The corresponding responses for Eq. 5-79 are obtained by the following sequence of MATLAB functions

```
t=0:0.1:50;
num = [1 5];
den = [1 17 60 10 0];
sys = tf(num,den);
sys_cl=feedback(sys,1);
[y,t]=step(sys_cl);
```

```
u=ones(size(t));
plot(t,y,'r',t,u,'g')
xlabel('Time(secs)')
ylabel('Amplitude')
title('Input-green, Output-red')
```



5-4-2 Steady-State Error Caused by Nonlinear System Elements

In many instances, steady-state errors of control systems are attributed to some nonlinear system characteristics such as nonlinear friction or dead zone. For instance, if an amplifier used in a control system has the input–output characteristics shown in Fig. 5-8, then, when the amplitude of the amplifier input signal falls within the dead zone, the output of the amplifier would be zero, and the control would not be able to correct the error if any exists. Dead-zone nonlinearity characteristics shown in Fig. 5-8 are not limited to amplifiers. The flux-to-current relation of the magnetic field of an electric motor may exhibit a similar characteristic. As the current of the motor falls below the dead zone D, no magnetic flux, and, thus, no torque will be produced by the motor to move the load.

The output signals of digital components used in control systems, such as a microprocessor, can take on only discrete or quantized levels. This property is illustrated by the quantization characteristics shown in Fig. 5-9. When the input to the quantizer is within $\pm q/2$, the output is zero, and the system may generate an error in the output whose

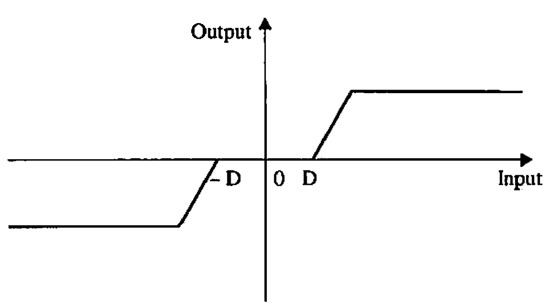


Figure 5-8 Typical input–output characteristics of an amplifier with dead zone and saturation.

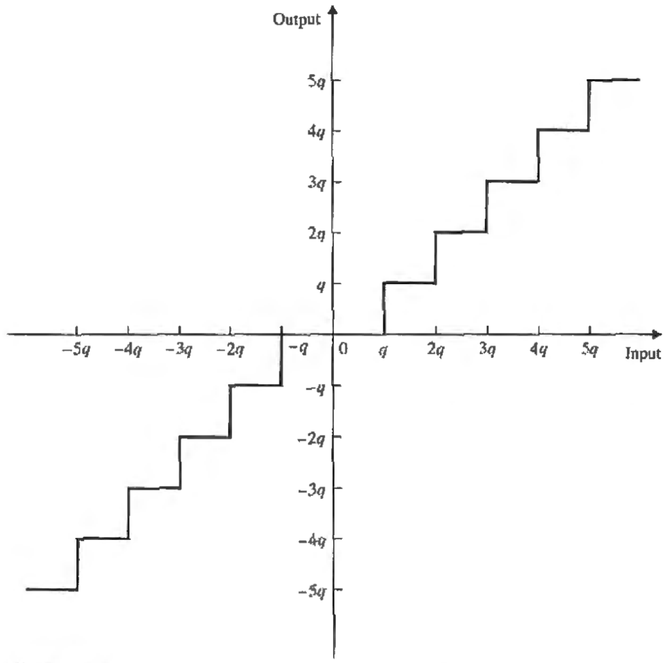


Figure 5-9 Typical input–output characteristics of a quantizer.

magnitude is related to $\pm q/2$. This type of error is also known as the quantization error in digital control systems.

When the control of physical objects is involved, friction is almost always present. Coulomb friction is a common cause of steady-state position errors in control systems. Fig. 5-10 shows a restoring-torque-versus-position curve of a control system. The torque curve typically could be generated by a step motor or a switched-reluctance motor or from a closed-loop system with a position encoder. Point 0 designates a stable equilibrium point on the torque curve, as well as the other periodic intersecting points along the axis where the slope on the torque curve is negative. The torque on either side of point 0 represents a restoring torque that tends to return the output to the equilibrium point when some angular-displacement disturbance takes place. When there is no friction, the position error should be zero, because there is always a restoring torque so long as the position is not at the stable

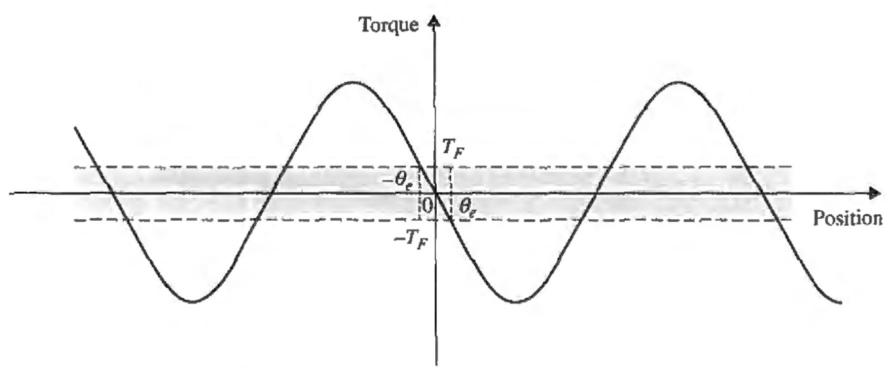


Figure 5-10 Torque-angle curve of a motor or closed-loop system with Coulomb friction.

equilibrium point. If the rotor of the motor sees a Coulomb friction torque T_F, then the motor torque must first overcome this frictional torque before producing any motion. Thus, as the motor torque falls below T_F as the rotor position approaches the stable equilibrium point, it may stop at any position inside the error band bounded by $\pm\theta_e$, as shown in Fig. 5-10.

Although it is relatively simple to comprehend the effects of nonlinearities on errors and to establish maximum upper bounds on the error magnitudes, it is difficult to establish general and closed-form solutions for nonlinear systems. Usually, exact and detailed analysis of errors in nonlinear control systems can be carried out only by computer simulations.

Therefore, we must realize that there are no error-free control systems in the real world, and, because all physical systems have nonlinear characteristics of one form or another, steady-state errors can be reduced but never completely eliminated.

5-5 TIME RESPONSE OF A PROTOTYPE FIRST-ORDER SYSTEM

Consider the **prototype first-order system** of form

$$\frac{dy(t)}{dt} + \frac{1}{\tau}y(t) = \frac{1}{\tau}f(t) \tag{5-82}$$

where τ is known as the **time constant** of the system, which is a measure of how fast the system responds to initial conditions of external excitations. Note that the input in Eq. (5-82) is scaled by $\frac{1}{\tau}$ for cosmetic reasons.

For a unit-step input

$$f(t) = u_s(t) = \begin{cases} 0, & t < 0, \\ 1, & t \geq 0 \end{cases} \tag{5-83}$$

If $y(0) = \dot{y}(0) = 0$, $\mathcal{L}(u_s(t)) = \dfrac{1}{s}$ and $\mathcal{L}(y(t)) = Y(s)$, then

$$Y(s) = \frac{1}{s}\frac{1/\tau}{s + 1/\tau} \tag{5-84}$$

Applying the inverse Laplace transform to Eq. (5-84), we get the time response of Eq. (5-82):

$$y(t) = 1 - e^{-t/\tau} \tag{5-85}$$

where τ is the time for $y(t)$ to reach 63% of its final value of $\lim_{t\to\infty} y(t) = 1$.

Fig. 5-11 shows typical unit-step responses of $y(t)$ for a general value of τ. As the value of time constant τ decreases, the system response approaches faster to the final value. The step response will not have any overshoot for any combination of system parameters.

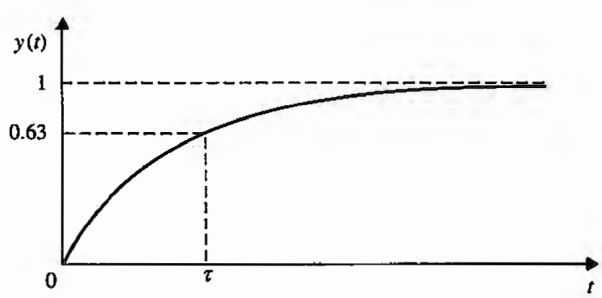


Figure 5-11 Unit-step response of a prototype first-order system.

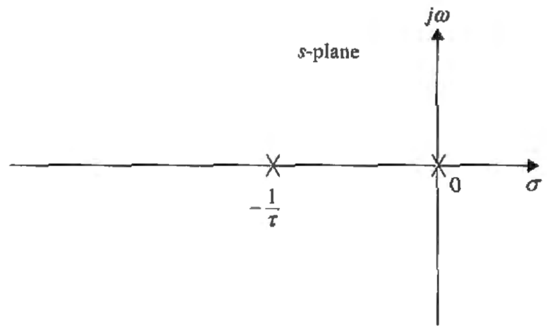


Figure 5-12 Pole configuration of the transfer function of a prototype first-order system.

Fig. 5-12 shows the location of the pole at $s = -\frac{1}{\tau}$ in the s-plane of the system transfer function. For positive τ, the pole at $s = -\frac{1}{\tau}$ will always stay in the left-half s-plane, and the system is always stable.

5-6 TRANSIENT RESPONSE OF A PROTOTYPE SECOND-ORDER SYSTEM

Although true second-order control systems are rare in practice, their analysis generally helps to form a basis for the understanding of analysis and design of higher-order systems, especially the ones that can be approximated by second-order systems.

Consider that a second-order control system with unity feedback is represented by the block diagram shown in Fig. 5-13. The open-loop transfer function of the system is

$$G(s) = \frac{Y(s)}{E(s)} = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)} \tag{5-86}$$

where ζ and ω_n are real constants. The closed-loop transfer function of the system is

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{5-87}$$

The system in Fig. 5-13 with the transfer functions given by Eqs. (5-86) and (5-87) is defined as the prototype second-order system.

The characteristic equation of the prototype second-order system is obtained by setting the denominator of Eq. (5-87) to zero:

$$\Delta(s) = s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \tag{5-88}$$

For a unit-step function input, $R(s) = 1/s$, the output response of the system is obtained by taking the inverse Laplace transform of the output transform:

$$Y(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} \tag{5-89}$$

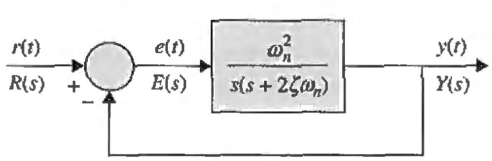


Figure 5-13 Prototype second-order control system.

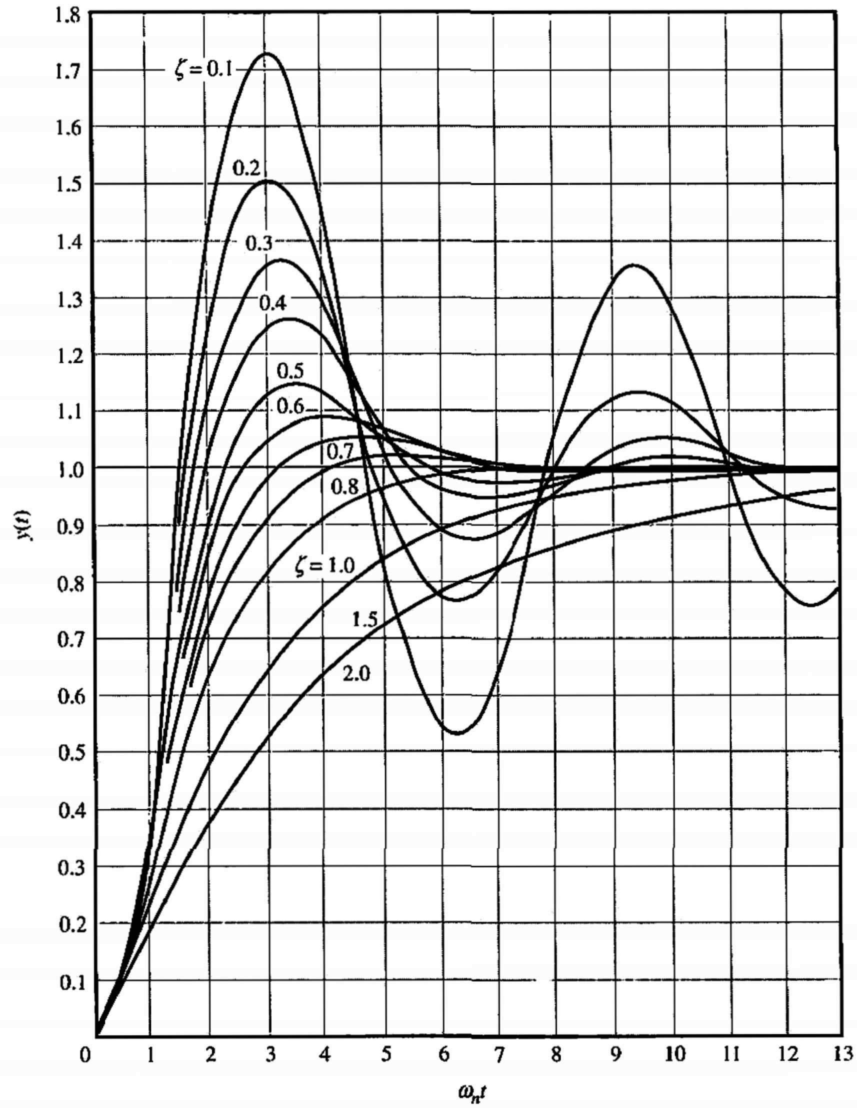


Figure 5-14 Unit-step responses of the prototype second-order system with various damping ratios.

This can be done by referring to the Laplace transform table in Appendix C. The result is

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_n \sqrt{1-\zeta^2} t + \cos^{-1}\zeta\right) \quad t \geq 0 \tag{5-90}$$

Fig. 5-14 shows the unit-step responses of Eq. (5-90) plotted as functions of the normalized time $\omega_n t$ for various values of ζ. As seen, the response becomes more oscillatory with larger overshoot as ζ decreases. When $\zeta \geq 1$, the step response does not exhibit any overshoot; that is, $y(t)$ never exceeds its final value during the transient. The responses also show that ω_n has a direct effect on the rise time, delay time, and settling time but does not affect the overshoot. These will be studied in more detail in the following sections.

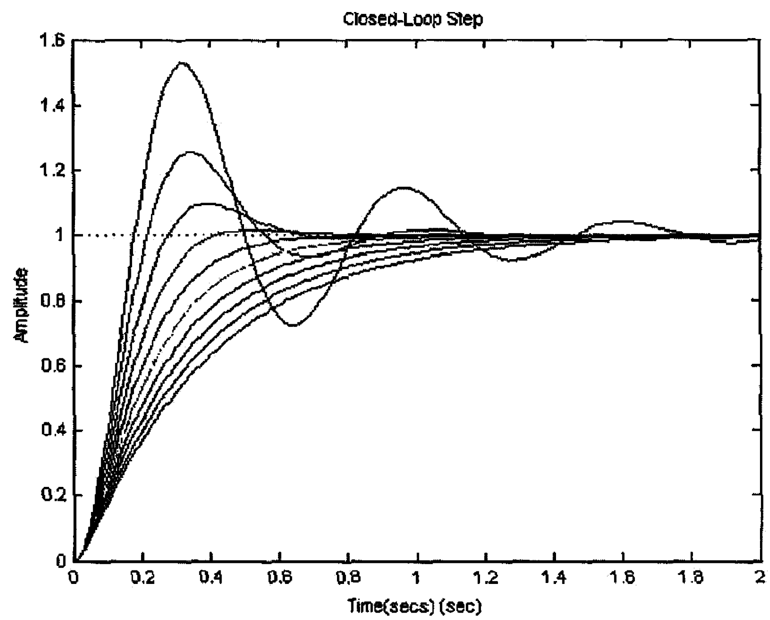
5-6-1 Damping Ratio and Damping Factor

The effects of the system parameters ζ and ω_n on the step response $y(t)$ of the prototype second-order system can be studied by referring to the roots of the characteristic equation in Eq. (5-88).

Toolbox 5-6-1

The corresponding time responses for Fig. 5-14 are obtained by the following sequence of MATLAB functions

```
clear all
w=10;
for l=[0.2 0.4 0.6 0.8 1 1.2 1.4 1.6 1.8 2]
t=0:0.1:50;
num = [w.^2];
den = [1 2*l*w w.^2];
t=0:0.01:2;
step(num,den,t) hold on;
end
xlabel('Time(secs)')
ylabel('Amplitude')
title('Closed-Loop Step')
```



The two roots can be expressed as

$$s_1, s_2 = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$
$$= -\alpha \pm j\omega \tag{5-91}$$

where

$$\alpha = \zeta\omega_n \tag{5-92}$$

and

$$\omega = \omega_n\sqrt{1-\zeta^2} \tag{5-93}$$

The physical significance of ζ and α is now investigated. As seen from Eqs. (5-90) and (5-92), α appears as the constant that is multiplied to t in the exponential term of $y(t)$.

Therefore, α controls the rate of rise or decay of the unit-step response $y(t)$. In other words, α controls the "damping" of the system and is called the damping factor, or the damping constant. The inverse of α, $1/\alpha$, is proportional to the time constant of the system.

When the two roots of the characteristic equation are real and equal, we called the system critically damped. From Eq. (5-91), we see that critical damping occurs when $\zeta = 1$. Under this condition, the damping factor is simply $\alpha = \omega_n$. Thus, we can regard ζ as the damping ratio; that is,

$$\zeta = \text{damping ratio} = \frac{\alpha}{\omega_n} = \frac{\text{actual damping factor}}{\text{damping factor at critical damping}} \tag{5-94}$$

5-6-2 Natural Undamped Frequency

The parameter ω_n is defined as the natural undamped frequency. As seen from Eq. (5-91), when $\zeta = 0$, the damping is zero, the roots of the characteristic equation are imaginary, and Eq. (5-90) shows that the unit-step response is purely sinusoidal. Therefore, ω_n corresponds to the frequency of the undamped sinusoidal response. Eq. (5-91) shows that, when $0 < \zeta < 1$, the imaginary part of the roots has the magnitude of ω. When $\zeta \neq 0$, the response of $y(t)$ is not a periodic function, and ϖ defined in Eq. (5-93) is not a frequency. For the purpose of reference, ω is sometimes defined as the conditional frequency, or the damped frequency.

Fig. 5-15 illustrates the relationships among the location of the characteristic equation roots and α, ζ, ω_n, and ω. For the complex-conjugate roots shown,

- ω_n is the radial distance from the roots to the origin of the s-plane.
- α is the real part of the roots.
- ω is the imaginary part of the roots.
- ζ is the cosine of the angle between the radial line to the roots and the negative axis when the roots are in the left-half s-plane, or

$$\zeta = \cos\theta \tag{5-95}$$

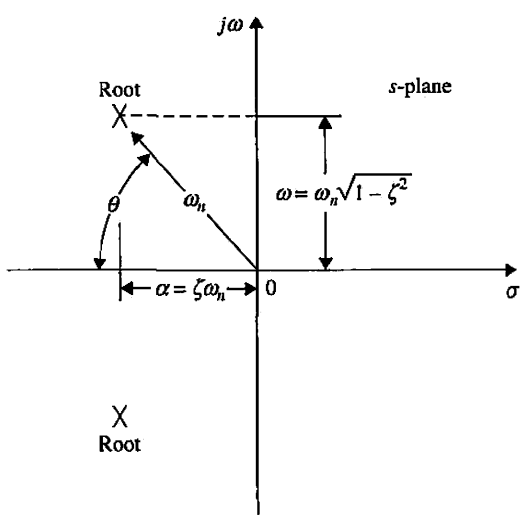


Figure 5-15 Relationships among the characteristic-equation roots of the prototype second-order system and α, ζ, ω_n, and ω.

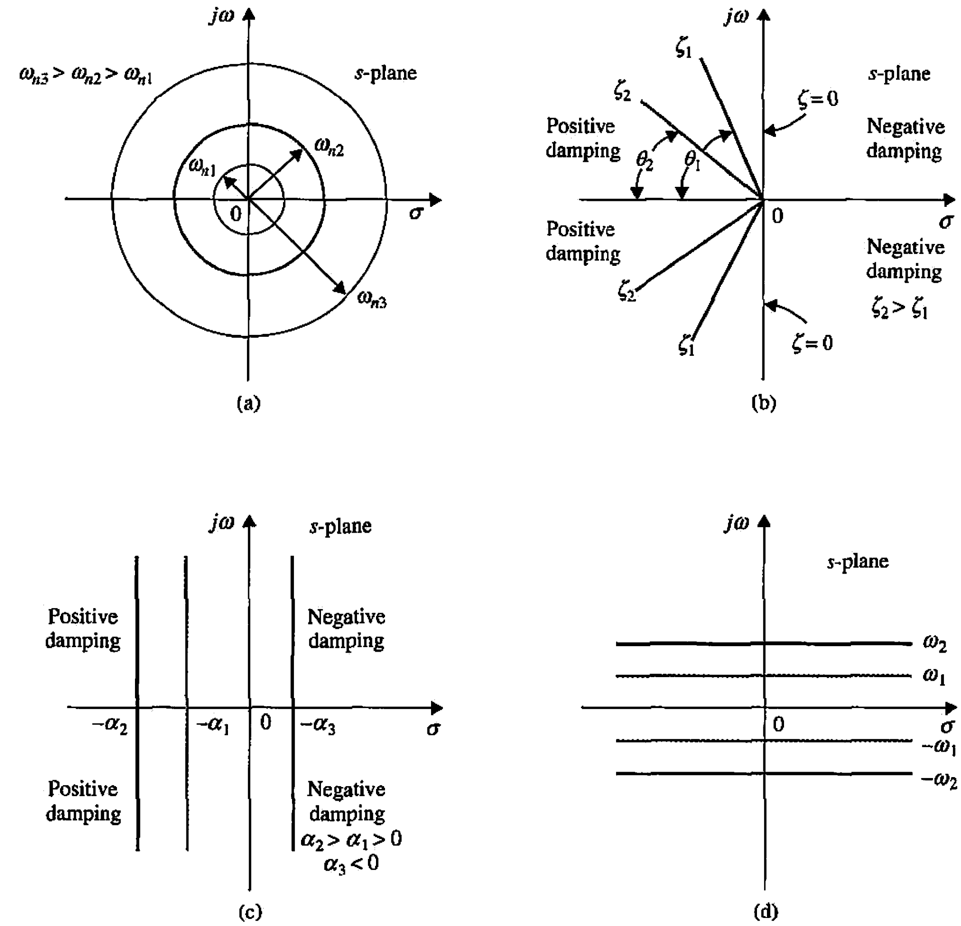


Figure 5-16 (a) Constant-natural-undamped-frequency loci. (b) Constant-damping-ratio loci. (c) Constant-damping-factor loci. (d) Constant-conditional-frequency loci.

Fig. 5-16 shows in the s-plane (a) the constant-ω_n loci, (b) the constant-ζ loci, (c) the constant-α loci, and (d) the constant-ω loci. Regions in the s-plane are identified with the system damping as follows:

- The left-half s-plane corresponds to positive damping; that is, the damping factor or damping ratio is positive. Positive damping causes the unit-step response to settle to a constant final value in the steady state due to the negative exponent of $\exp(-\zeta\omega_n t)$. The system is stable.
- The right-half s-plane corresponds to negative damping. Negative damping gives a response that grows in magnitude without bound, and the system is unstable.
- The imaginary axis corresponds to zero damping ($\alpha = 0$ or $\zeta = 0$). Zero damping results in a sustained oscillation response, and the system is marginally stable or marginally unstable.

Thus, we have demonstrated with the help of the simple prototype second-order system that the location of the characteristic equation roots plays an important role in the transient response of the system.

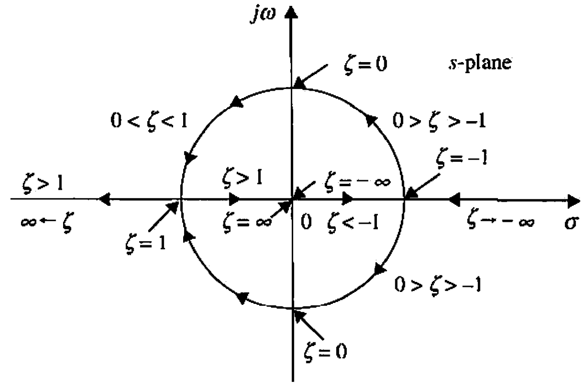


Figure 5-17 Locus of roots of the characteristic equation of the prototype second-order system.

The effect of the characteristic equation roots on the damping of the second-order system is further illustrated by Fig. 5-17 and Fig. 5-18. In Fig. 5-17, ω_n is held constant while the damping ratio ζ is varied from $-\infty$ to $+\infty$. The following classification of the system dynamics with respect to the value of ζ is made:

$0<\zeta<1$:	$s_1, s_2 = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}\quad(-\zeta\omega_n<0)$	*underdamped*
$\zeta=1$:	$s_1, s_2 = -\omega_n$	*critically damped*
$\zeta>1$:	$s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$	*overdamped*
$\zeta=0$:	$s_1, s_2 = \pm j\omega_n$	*undamped*
$\zeta<0$:	$s_1, s_2 = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}\quad(-\zeta\omega_n>0)$	*negatively damped*

Fig. 5-18 illustrates typical unit-step responses that correspond to the various root locations already shown.

In practical applications, only stable systems that correspond to $\zeta>0$ are of interest. Fig. 5-14 gives the unit-step responses of Eq. (5-90) plotted as functions of the normalized time $\omega_n t$ for various values of the damping ratio ζ. As seen, the response becomes more oscillatory as ζ decreases in value. When $\zeta\geq 1$, the step response does not exhibit any overshoot; that is, $y(t)$ never exceeds its final value during the transient.

5-6-3 Maximum Overshoot

The exact relation between the damping ratio and the amount of overshoot can be obtained by taking the derivative of Eq. (5-90) with respect to t and setting the result to zero. Thus,

$$\frac{dy(t)}{dt} = \frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\zeta\sin(\omega t+\theta) - \sqrt{1-\zeta^2}\cos(\omega t+\theta)\right]\quad t\geq 0 \tag{5-96}$$

where ω and θ are defined in Eqs. (5-93) and (5-95), respectively. We can show that the quantity inside the square bracket in Eq. (5-96) can be reduced to $\sin\omega t$. Thus, Eq. (5-96) is simplified to

$$\frac{dy(t)}{dt} = \frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\omega_n\sqrt{1-\zeta^2}t\quad t\geq 0 \tag{5-97}$$

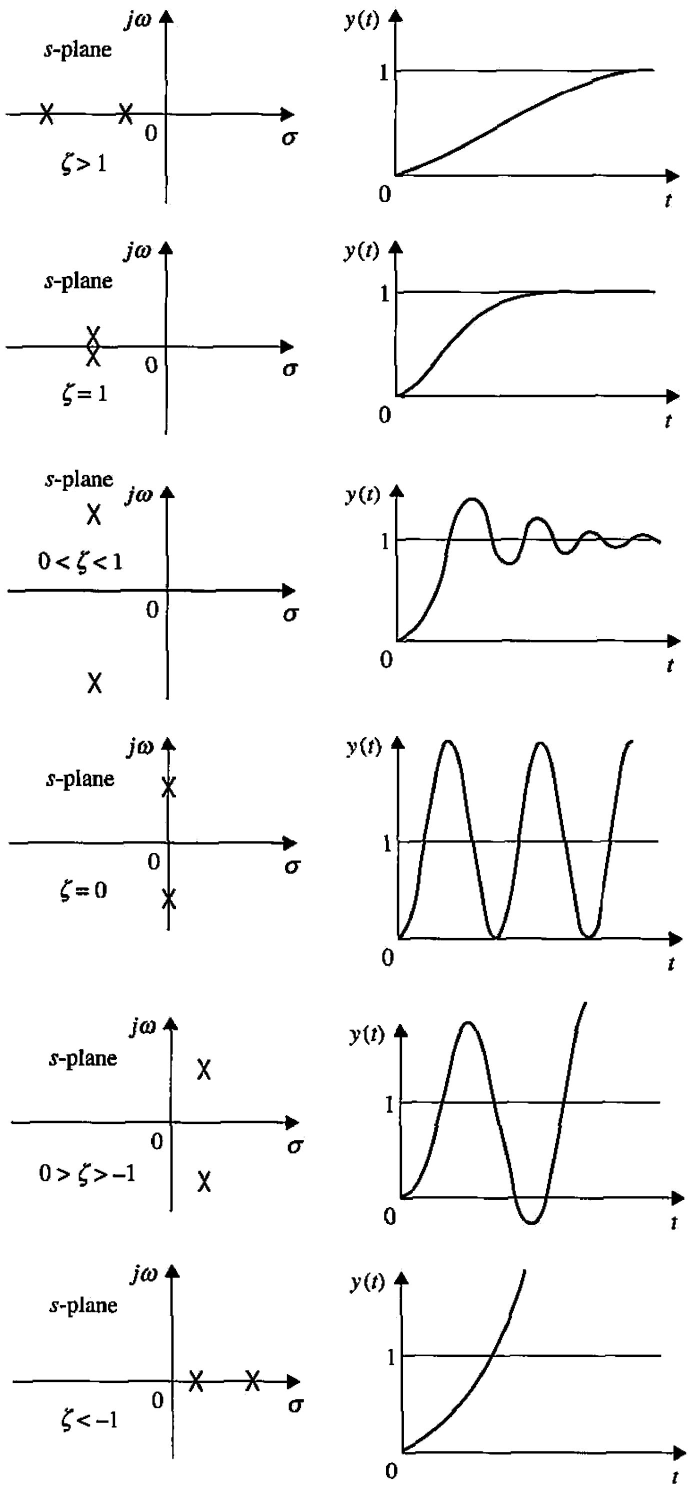


Figure 5-18 Step-response comparison for various characteristic-equation-root locations in the s-plane.

Setting $dy(t)/dt$ to zero, we have the solutions: $t = \infty$ and

$$\omega_n\sqrt{1-\zeta^2}t = n\pi \quad n = 0, 1, 2, \ldots \tag{5-98}$$

from which we get

$$t = \frac{n\pi}{\omega_n\sqrt{1-\zeta^2}} \quad n = 0, 1, 2, \ldots \tag{5-99}$$

The solution at $t = \infty$ is the maximum of $y(t)$ only when $\zeta \geq 1$. For the unit-step responses shown in Fig. 5-13, the first overshoot is the maximum overshoot. This corresponds to $n = 1$ in Eq. (5-99). Thus, the time at which the maximum overshoot occurs is

$$t_{\max} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \tag{5-100}$$

With reference to Fig. 5-13, the overshoots occur at odd values of n, that is, $n = 1, 3, 5, \ldots$, and the undershoots occur at even values of n. Whether the extremum is an overshoot or an undershoot, the time at which it occurs is given by Eq. (5-99). It should be noted that, *although the unit-step response for $\zeta \neq 0$ is not periodic, the overshoots and the undershoots of the response do occur at periodic intervals, as shown in Fig. 5-19.*

The magnitudes of the overshoots and the undershoots can be determined by substituting Eq. (5-99) into Eq. (5-90). The result is

$$y(t)|_{\max\ or\ \min} = 1 - \frac{e^{-n\pi\zeta/\sqrt{1-\zeta^2}}}{\sqrt{1-\zeta^2}}\sin(n\pi + \theta) \quad n = 1, 2, \ldots \tag{5-101}$$

or

$$y(t)|_{\max\ or\ \min} = 1 + (-1)^{n-1}e^{-n\pi\zeta/\sqrt{1-\zeta^2}} \quad n = 1, 2, \ldots \tag{5-102}$$

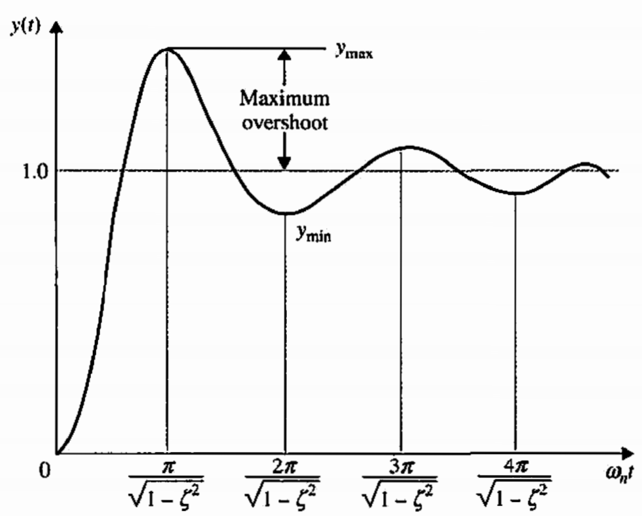


Figure 5-19 Unit-step response illustrating that the maxima and minima occur at periodic intervals.

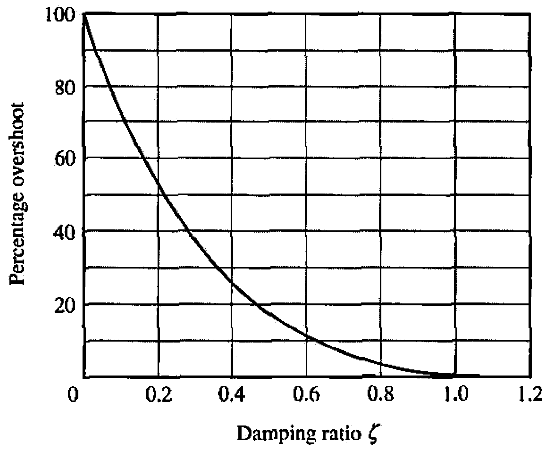


Figure 5-20 Percent overshoot as a function of damping ratio for the step response of the prototype second-order system.

The maximum overshoot is obtained by letting $n = 1$ in Eq. (5-102). Therefore,

$$\text{maximum overshoot} = y_{\max} - 1 = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \tag{5-103}$$

and

$$\text{percent maximum overshoot} = 100e^{-\pi\zeta/\sqrt{1-\zeta^2}} \tag{5-104}$$

Eq. (5-103) shows that the maximum overshoot of the step response of the prototype second-order system is a function of only the damping ratio ζ. The relationship between the percent maximum overshoot and the damping ratio given in Eq. (5-104) is plotted in Fig. 5-20. The time $t_{\max}$ in Eq. (5-100) is a function of both ζ and ω_n.

5-6-4 Delay Time and Rise Time

It is more difficult to determine the exact analytical expressions of the delay time t_d, rise time t_r, and settling time t_s, even for just the simple prototype second-order system. For instance, for the delay time, we would have to set $y(t) = 0.5$ in Eq. (5-90) and solve for t. An easier way would be to plot $\omega_n t_d$ versus ζ, as shown in Fig. 5-21, and then approximate the curve by a straight line or a curve over the range of $0 < \zeta < 1$. From Fig. 5-21, the delay time for the prototype second-order system is approximated as

$$t_d \cong \frac{1 + 0.7\zeta}{\omega_n} \quad 0 < \zeta < 1.0 \tag{5-105}$$

We can obtain a better approximation by using a second-order equation for t_d:

$$t_d \cong \frac{1.1 + 0.125\zeta + 0.469\zeta^2}{\omega_n} \quad 0 < \zeta < 1.0 \tag{5-106}$$

For the rise time t_r, which is the time for the step response to reach from 10 to 90% of its final value, the exact value can be determined directly from the responses of Fig. 5-14. The

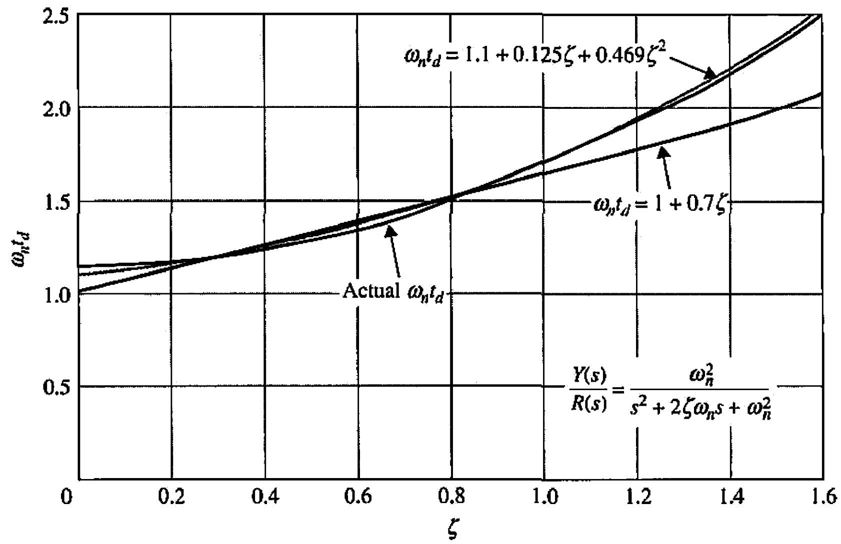


Figure 5-21 Normalized delay time versus ζ for the prototype second-order system.

plot of $\omega_n t_r$ versus ζ is shown in Fig. 5-22. In this case, the relation can again be approximated by a straight line over a limited range of ζ:

$$t_r = \frac{0.8 + 2.5\zeta}{\omega_n} \quad 0 < \zeta < 1 \tag{5-107}$$

A better approximation can be obtained by using a second-order equation:

$$t_r = \frac{1 - 0.4167\zeta + 2.917\zeta^2}{\omega_n} \quad 0 < \zeta < 1 \tag{5-108}$$

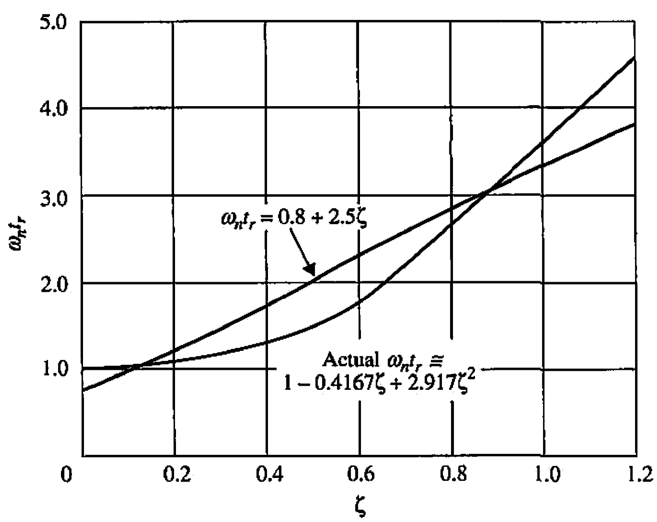


Figure 5-22 Normalized rise time versus ζ for the prototype second-order system.

From this discussion, the following conclusions can be made on the rise time and delay time of the prototype second-order system:

- t_r and t_d are proportional to ζ and inversely proportional to ω_n.
- Increasing (decreasing) the natural undamped frequency ω_n will reduce (increase) t_r and t_d.

5-6-5 Settling Time

From Fig. 5-14, we see that, when $0 < \zeta < 0.69$, the unit-step response has a maximum overshoot greater than 5%, and the response can enter the band between 0.95 and 1.05 for the last time from either the top or the bottom. When ζ is greater than 0.69, the overshoot is less than 5%, and the response can enter the band between 0.95 and 1.05 only from the bottom. Fig. 5-23(a) and (b) show the two different situations. Thus, the settling time has a

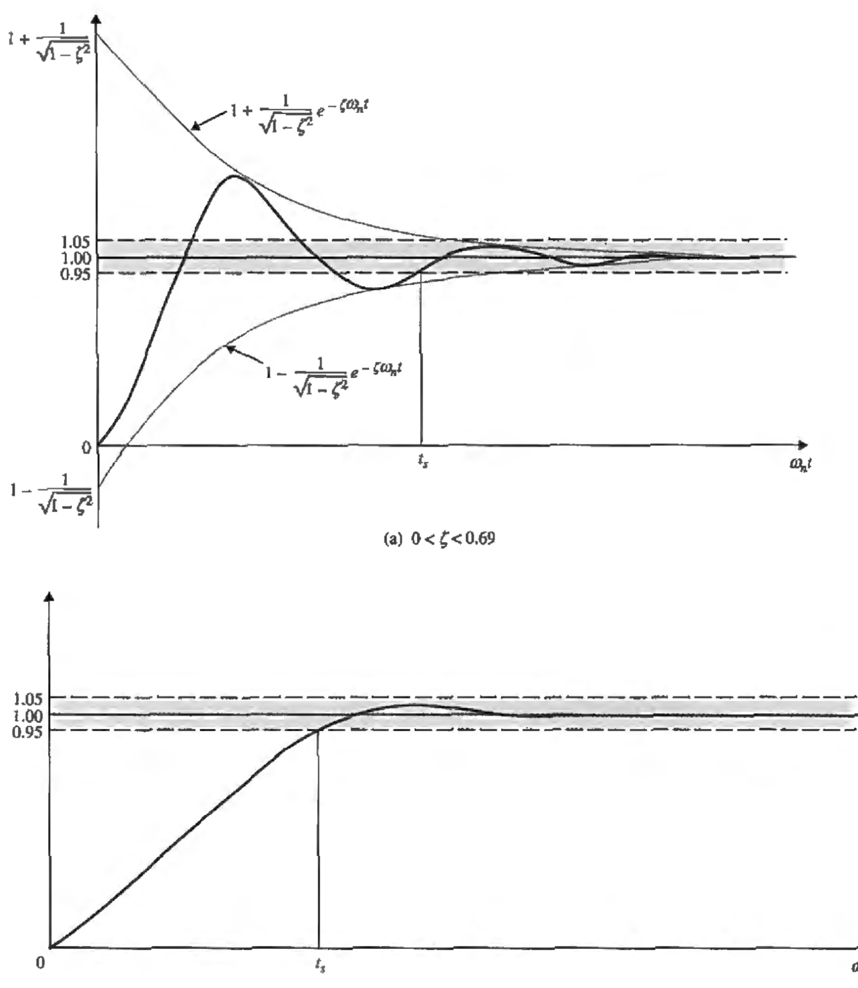


Figure 5-23 Settling time of the unit-step response.

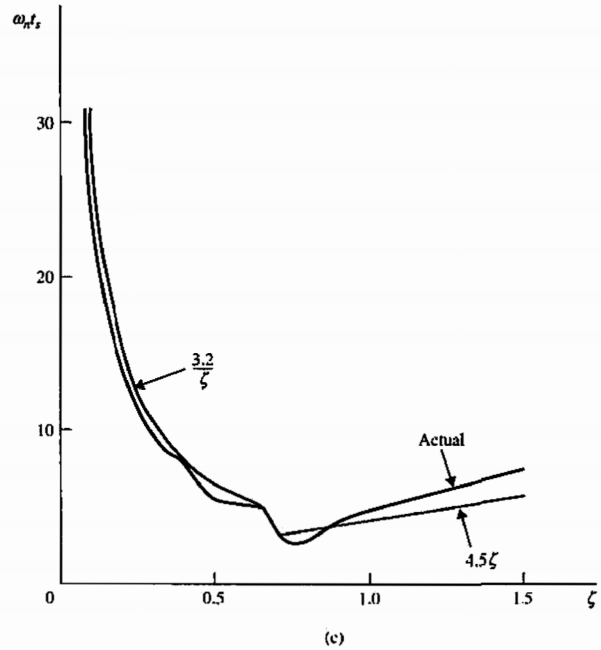


Figure 5-23 (*continued*)

discontinuity at $\zeta = 0.69$. The exact analytical description of the settling time t_s is difficult to obtain. We can obtain an approximation for t_s for $0 < \zeta < 0.69$ by using the envelope of the damped sinusoid of $y(t)$, as shown in Fig. 5-23(a) for a 5% requirement. In general, when the settling time corresponds to an intersection with the upper envelope of $y(t)$, the following relation is obtained:

$$1 + \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t_s} = \text{upper bound of unit-step response} \tag{5-109}$$

When the settling time corresponds to an intersection with the bottom envelope of $y(t)$, t_s must satisfy the following condition:

$$1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t_s} = \text{lower bound of unit-step response} \tag{5-110}$$

For the 5% requirement on settling time, the right-hand side of Eq. (5-109) would be 1.05, and that of Eq. (5-110) would be 0.95. It is easily verified that the same result for t_s is obtained using either Eq. (5-109) or Eq. (5-110).

Solving Eq. (5-109) for $\omega_n t_s$, we have

$$\omega_n t_s = -\frac{1}{\zeta} \ln\left(c_{ts}\sqrt{1-\zeta^2}\right) \tag{5-111}$$

where c_{ts} is the percentage set for the settling time. For example, if the threshold is 5 percent, the $c_{ts} = 0.05$. Thus, for a 5-percent settling time, the right-hand side of

Eq. (5-111) varies between 3.0 and 3.32 as ζ varies from 0 to 0.69. We can approximate the settling time for the prototype second-order system as

$$t_s \cong \frac{3.2}{\zeta\omega_n} \quad 0 < \zeta < 0.69 \tag{5-112}$$

The approximation will be poor for small values of $\zeta(< 0.3)$.

When the damping ratio ζ is greater than 0.69, the unit-step response will always enter the band between 0.95 and 1.05 from below. We can show by observing the responses in Fig. 5-14 that the value of $\omega_n t_s$ is almost directly proportional to ζ. The following approximation is used for t_s for $\zeta > 0.69$.

$$t_s = \frac{4.5\zeta}{\omega_n} \quad \zeta > 0.69 \tag{5-113}$$

Fig. 5-23(c) shows the actual values of $\omega_n t_s$ versus ζ for the prototype second-order system described by Eq. (5-87), along with the approximations using Eqs. (5-112) and (5-113) for their respective effective ranges. The numerical values are shown in Table 5-2.

We can summarize the relationships between t_s and the system parameters as follows:

- For $\zeta < 0.69$, the settling time is inversely proportional to ζ and ω_n. A practical way of reducing the settling time is to increase ω_n while holding ζ constant. Although

TABLE 5-2 Comparison of Settling Times of Prototype Second-Order System, $\omega_n t_s$

ζ	Actual	$\frac{3.2}{\zeta}$	4.5ζ
0.10	28.7	30.2	
0.20	13.7	16.0	
0.30	10.0	10.7	
0.40	7.5	8.0	
0.50	5.2	6.4	
0.60	5.2	5.3	
0.62	5.16	5.16	
0.64	5.00	5.00	
0.65	5.03	4.92	
0.68	4.71	4.71	
0.69	4.35	4.64	
0.70	2.86		3.15
0.80	3.33		3.60
0.90	4.00		4.05
1.00	4.73		4.50
1.10	5.50		4.95
1.20	6.21		5.40
1.50	8.20		6.75

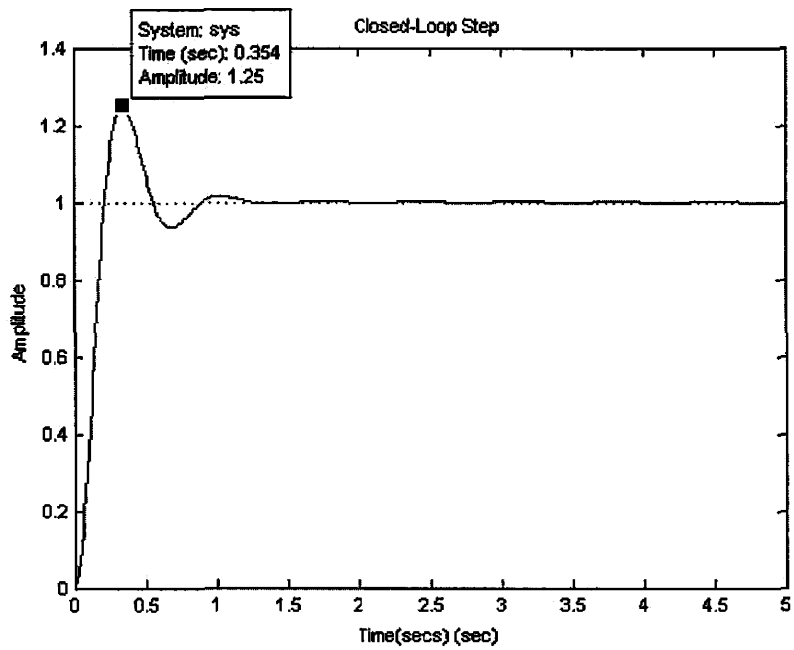
the response will be more oscillatory, the maximum overshoot depends only on ζ and can be controlled independently.

- For $\zeta > 0.69$, the settling time is proportional to ζ and inversely proportional to ω_n. Again, t_s can be reduced by increasing ω_n.

Toolbox 5-6-2

To find PO, rise time, and settling time using MATLAB, point at a desired location on the graph and right-click to display the x and y values. For example

```
clear all
w=10;l=0.4;
t=0:0.01:5;
num = [w.^2];
den = [1 2*l*w w.^2];
step(num,den,t)
xlabel('Time(secs)')
ylabel('Amplitude')
title('Closed-Loop Step')
```



It should be commented that the settling time for $\zeta > 0.69$ is truly a measure of how fast the step response rises to its final value. It seems that, for this case, the rise and delay times should be adequate to describe the response behavior. As the name implies, settling time should be used to measure how fast the step response settles to its final value. It should also be pointed out that the 5% threshold is by no means a number cast in stone. More stringent design problems may require the system response to settle in any number less than 5%.

Keep in mind that, while the definitions on y_{max}, t_{max}, t_d, t_r, and t_s apply to a system of any order, the damping ratio ζ and the natural undamped frequency ω_n strictly apply only to a second-order system whose closed-loop transfer function is given in Eq. (5-87). Naturally, the relationships among t_d, t_r, and t_s and ζ and ω_n are valid only for the same second-order system model. However, these relationships can be used to measure the performance of higher-order systems that can be approximated by second-order ones, under the stipulation that some of the higher-order poles can be neglected.

5-7 SPEED AND POSITION CONTROL OF A DC MOTOR

Servomechanisms are probably the most frequently encountered electromechanical control systems. Applications include robots (each joint in a robot requires a position servo), numerical control (NC) machines, and laser printers, to name but a few. The common characteristic of all such systems is that the variable to be controlled (usually position or velocity) is fed back to modify the command signal. The servomechanism that will be used in the experiments in this chapter comprises a dc motor and amplifier that are fed back the motor speed and position values.

One of the key challenges in the design and implementation of a successful controller is obtaining an accurate model of the system components, particularly the actuator. In Chapter 4, we discussed various issues associated with modeling of dc motors. We will briefly revisit the modeling aspects in this section.

5-7-1 Speed Response and the Effects of Inductance and Disturbance-Open Loop Response

Consider the armature-controlled dc motor shown in Fig. 5-24, where the field current is held constant in this system. The system parameters include

R_a = armature resistance, ohm
L_a = armature inductance, henry
v_a = applied armature voltage, volt
v_b = back emf, volt
θ = angular displacement of the motor shaft, radian
T = torque developed by the motor, *N-m*
J_L = moment of inertia of the load, *kg-m*^{2}
T_L = any external load torque considered as a disturbance, *N-m*
J_m = moment of inertia of the motor (motor shaft), *kg-m*^{2}
J = equivalent moment of inertia of the motor and load connected to the motor-shaft, $J = J_L/n^2 + J_m$, *kg-m*^{2} (refer to Chapter 4 for more details)
n = gear ratio
B = equivalent viscous-friction coefficient of the motor and load referred to the motor shaft, *N-m*/rad/sec (in the presence of gear ratio, B must be scaled by n; refer to Chapter 4 for more details)
K_t = speed sensor (usually a tachometer) gain

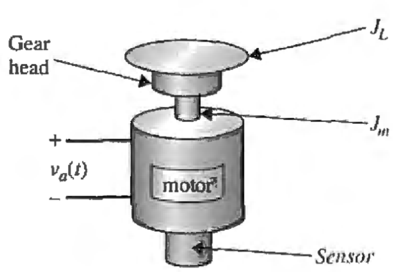


Figure 5-24 An armature-controlled dc motor with a gear head and a load inertia J_L.

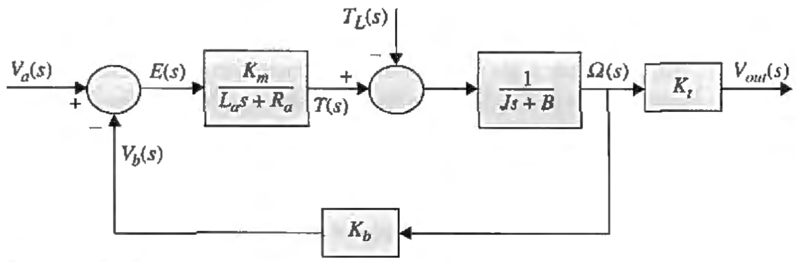


Figure 5-25 Block diagram of an armature-controlled dc motor.

As shown in Fig. 5-25, the armature-controlled dc motor is itself a feedback system, where back-emf voltage is proportional to the speed of the motor. In Fig. 5-25, we have included the effect of any possible external load (e.g., the load applied to a juice machine by the operator pushing in the fruit) as a disturbance torque T_L. The system may be arranged in input–output form such that $V_a(s)$ is the input and $\Omega(s)$ is the output:

$$\Omega(s) = \frac{\dfrac{K_m}{R_a J_m}}{\left(\dfrac{L_a}{R_a}\right)s^2 + \left(1 + \dfrac{BL_a}{R_a J_m}\right)s + \dfrac{K_m K_b + R_a B}{R_a J_m}} V_a(s) - \frac{\left\{1 + s\left(\dfrac{L_a}{R_a}\right)\right\}/J_m}{\left(\dfrac{L_a}{R_a}\right)s^2 + \left(1 + \dfrac{BL_a}{R_a J_m}\right)s + \dfrac{K_m K_b + R_a B}{R_a J_m}} T_L(s) \tag{5-114}$$

The ratio L_a/R_a is called the *motor electric-time constant*, which makes the system speed-response transfer function second order and is denoted by τ_e. Also, it introduces a zero to the disturbance-output transfer function. However, as discussed in Chapter 4, because L_a in the armature circuit is very small, τ_e is neglected, resulting in the simplified transfer functions and the block diagram of the system. Thus, the speed of the motor shaft may be simplified to

$$\Omega(s) = \frac{\dfrac{K_m}{R_a J_m}}{s + \dfrac{K_m K_b + R_a B}{R_a J_m}} V_a(s) - \frac{\dfrac{1}{J_m}}{s + \dfrac{K_m K_b + R_a B}{R_a J_m}} T_L(s) \tag{5-115}$$

or

$$\Omega(s) = \frac{K_{eff}}{\tau_m s + 1} V_a(s) - \frac{\dfrac{\tau_m}{J_m}}{\tau_m s + 1} T_L(s) \tag{5-116}$$

where $K_{eff} = K_m/(R_a B + K_m K_b)$ is the motor gain constant, and $\tau_m = R_a J_m/(R_a B + K_m K_b)$ is the motor mechanical time constant. If the load inertia and the gear ratio are incorporated into the system model, the inertia J_m in Eqs. (5-114) through (5-116) is replaced with J (total inertia).

Using superposition, we get

$$\Omega(s) = \Omega(s)|_{T_L(s)=0} + \Omega(s)|_{V_a(s)=0} \tag{5-117}$$

To find the response $\omega(t)$, we use superposition and find the response due to the individual inputs. For $T_L = 0$ (no disturbance and $B = 0$) and an applied voltage $V_a(t) = A$, such that $V_a(s) = A/s$,

$$\omega(t) = \frac{A}{K_b}(1 - e^{-t/\tau_m}) \tag{5-118}$$

In this case, note that the motor mechanical time constant τ_m is reflective of how fast the motor is capable of overcoming its own inertia J_m to reach a steady state or constant speed dictated by voltage V_a. From Eq. (5-118), the speed final value is $\omega(t) = A/K_b$. As τ_m increases, the approach to steady state takes longer.

If we apply a constant load torque of magnitude D to the system (i.e., $T_L = D/s$), the speed response from Eq. (5-118) will change to

$$\omega(t) = \frac{1}{K_b}\left(A - \frac{R_a D}{K_m}\right)(1 - e^{-t/\tau_m}) \tag{5-119}$$

which clearly indicates that the disturbance T_L affects the final speed of the motor. From Eq. (5-119), at steady state, the speed of the motor is $\omega_{fv} = \frac{1}{K_b}(A - \frac{R_a D}{K_m})$. Here the final value of $\omega(t)$ is reduced by $R_a D/K_m K_b$. A practical note is that the value of $T_L = D$ may never exceed the motor stall torque, and hence for the motor to turn, from Eq. (5-119), $AK_m/R_a > D$, which sets a limit on the magnitude of the torque T_L. For a given motor, the value of the stall torque can be found in the manufacturer's catalog.

If the load inertia is incorporated into the system model, the final speed value becomes $\omega_{fv} = A/K_b$. Does the stall torque of the motor affect the response and the steady-state response? In a realistic scenario, you must measure motor speed using a sensor. How would the sensor affect the equations of the system (see Fig. 5-25)?

5-7-2 Speed Control of DC Motors: Closed-Loop Response

As seen previously, the output speed of the motor is highly dependant on the value of torque T_L. We can improve the speed performance of the motor by using a proportional feedback controller. The controller is composed of a sensor (usually a tachometer for speed applications) to sense the speed and an amplifier with gain K (proportional control) in the configuration shown in Fig. 5-26. The block diagram of the system is also shown in Fig. 5-27.

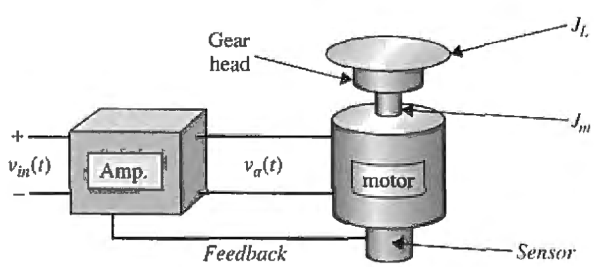


Figure 5-26 Feedback control of an armature-controlled dc motor with a load inertia.

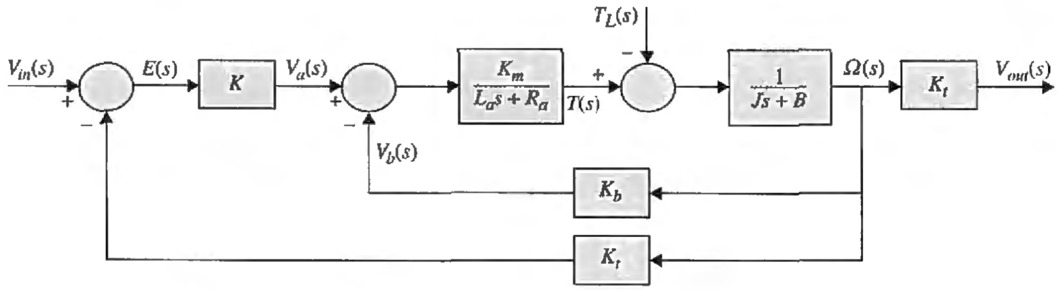


Figure 5-27 Block diagram of a speed-control, armature-controlled dc motor.

Note that the speed at the motor shaft is sensed by the tachometer with a gain K_t. For ease in comparison of input and output, the input to the control system is converted from voltage V_{in} to speed Ω_{in} using the tachometer gain K_t. Hence, assuming $L_a = 0$, we have

$$\Omega(s) = \frac{\dfrac{K_t K_m K}{R_a J_m}}{s + \left(\dfrac{K_m K_b + R_a B + K_t K_m K}{R_a J_m}\right)} \Omega_{in}(s) - \frac{\dfrac{1}{J_m}}{s + \left(\dfrac{K_m K_b + R_a B + K_t K_m K}{R_a J_m}\right)} T_L(s) \tag{5-120}$$

For a step input $\Omega_{in} = A/s$ and disturbance torque value $T_L = D/s$, the output becomes

$$\omega(t) = \frac{AKK_m K_t}{R_a J_m}\tau_c(1 - e^{-t/\tau_c}) - \frac{\tau_c D}{J_m}(1 - e^{-t/\tau_c}) \tag{5-121}$$

where $\tau_c = \frac{R_a J_m}{K_m K_b + R_a B + K_t K_m K}$ is the system mechanical-time constant. The steady-state response in this case is

$$\omega_{fv} = \left(\frac{AKK_m K_t}{K_m K_b + R_a B + K_t K_m K} - \frac{R_a D}{K_m K_b + R_a B + K_t K_m K}\right) \tag{5-122}$$

where $\omega_{fv} \to A$ as $K \to \infty$. So, speed control may reduce the effect of disturbance. As in Section 5-7-1, the reader should investigate what happens if the inertia J_L is included in this model. If the load inertia J_L is too large, will the motor be able to turn? Again, as in Section 5-7-1, you will have to read the speed-sensor voltage to measure speed. How will that affect your equations?

5-7-3 Position Control

The position response in the open-loop case may be obtained by integrating the speed response. Then, considering Fig. 5-25, we have $\Theta(s) = \Omega(s)/s$. The open-loop transfer function is therefore

$$\frac{\Theta(s)}{V_a(s)} = \frac{K_m}{s(L_a J_m s^2 + (L_a B + R_a J_m)s + R_a B + K_m K_b)} \tag{5-123}$$

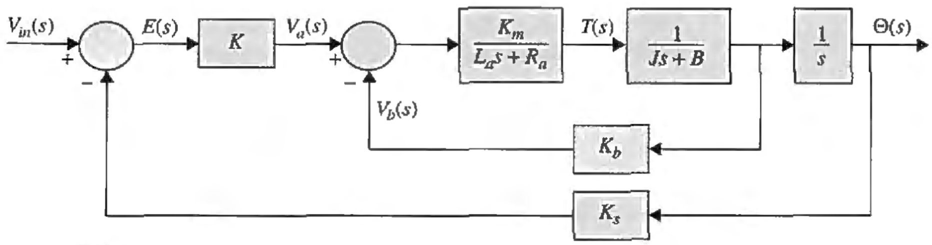


Figure 5-28 Block diagram of a position-control, armature-controlled dc motor.

where we have used the total inertia J. For small L_a, the time response in this case is

$$\theta(t) = \frac{A}{K_b}(t + \tau_m e^{-t/\tau_m} - \tau_m) \tag{5-124}$$

which implies that the motor shaft is turning without stop at a constant steady-state speed. To control the position of the motor shaft, the simplest strategy is to use a proportional controller with gain K. The block diagram of the closed-loop system is shown in Fig. 5-28. The system is composed of an angular position sensor (usually an encoder or a potentiometer for position applications). Note that, for simplicity, the input voltage can be scaled to a position input $\Theta_{in}(s)$ so that the input and output have the same units and scale. Alternatively, the output can be converted into voltage using the sensor gain value. The closed-loop transfer function in this case becomes

$$\frac{\Theta(s)}{\Theta_{in}(s)} = \frac{\dfrac{KK_mK_s}{R_a}}{(\tau_e s + 1)\left\{J s^2 + \left(B + \dfrac{K_bK_m}{R_a}\right)s + \dfrac{KK_mK_s}{R_a}\right\}} \tag{5-125}$$

where K_s is the sensor gain, and, as before, $\tau_e = (L_a/R_a)$ may be neglected for small L_a.

$$\frac{\Theta(s)}{\Theta_{in}(s)} = \frac{\dfrac{KK_mK_s}{R_aJ}}{s^2 + \left(\dfrac{R_aB + K_mK_b}{R_aJ}\right)s + \dfrac{KK_mK_s}{R_aJ}} \tag{5-126}$$

Later, in Chapter 6, we set up numerical and experimental case studies to test and verify the preceding concepts and learn more about other practical issues.

5-8 TIME-DOMAIN ANALYSIS OF A POSITION-CONTROL SYSTEM

In this section, we shall analyze the performance of a system using the time-domain criteria established in the preceding section. The purpose of the system considered here is to control the positions of the fins of an airplane as discussed in Example 4-11-1.

Recall from Chapter 4 that

$$G(s) = \frac{\Theta_y(s)}{\Theta_e(s)} = \frac{K_sK_1K_iKN}{s[L_aJ_ts^2 + (R_aJ_t + L_aB_t + K_1K_2J_t)s + R_aB_t + K_1K_2B_t + K_iK_b + KK_1K_tK_i]} \tag{5-127}$$

The system is of the third order, since the highest-order term in $G(s)$ is s^3. The electrical time constant of the amplifier-motor system is

$$\tau_a = \frac{L_a}{R_a + K_1K_2} = \frac{0.003}{5+5} = 0.0003\,\text{sec} \tag{5-128}$$

The mechanical time constant of the motor-load system is

$$\tau_t = \frac{J_t}{B_t} = \frac{0.0002}{0.015} = 0.01333\,\text{sec} \tag{5-129}$$

Because the electrical time constant is much smaller than the mechanical time constant, on account of the low inductance of the motor, we can perform an initial approximation by neglecting the armature inductance L_a. The result is a second-order approximation of the third-order system. Later we will show that this is not the best way of approximating a high-order system by a low-order one. The forward-path transfer function is now

$$\begin{aligned} G(s) &= \frac{K_sK_1K_iKN}{s[(R_aJ_t + K_1K_2J_t)s + R_aB_t + K_1K_2B_t + K_iK_b + KK_1K_iK_t]} \\ &= \frac{\dfrac{K_sK_1K_iKN}{R_aJ_t + K_1K_2J_t}}{s\left(s + \dfrac{R_aB_t + K_1K_2B_t + K_iK_b + KK_1K_iK_t}{R_aJ_t + K_1K_2J_t}\right)} \end{aligned} \tag{5-130}$$

Substituting the system parameters in the last equation, we get

$$G(s) = \frac{4500K}{s(s+361.2)} \tag{5-131}$$

Comparing Eq. (5-131) and (5-132) with the prototype second-order transfer function of Eq. (5-86), we have

$$\text{natural undamped frequency } \omega_n = \pm\sqrt{\frac{K_sK_1K_iKN}{R_aJ_t + K_1K_2J_t}} = \pm\sqrt{4500K}\ \text{rad/sec} \tag{5-132}$$

$$\text{damping ratio } \zeta = \frac{R_aB_t + K_1K_2B_t + K_iK_b + KK_1K_iK_t}{2\sqrt{K_sK_1K_iKN(R_aJ_t + K_1K_2J_t)}} = \frac{2.692}{\sqrt{K}} \tag{5-133}$$

Thus, we see that the natural undamped frequency ω_n is proportional to the square root of the amplifier gain K, whereas the damping ratio ζ is inversely proportional to $\sqrt{K}$.

The closed-loop transfer function of the unity-feedback control system is

$$\frac{\Theta_y(s)}{\Theta_e(s)} = \frac{4500K}{s^2 + 361.2s + 4500K} \tag{5-134}$$

5-8-1 Unit-Step Transient Response

For time-domain analysis, it is informative to analyze the system performance by applying the unit-step input with zero initial conditions. In this way, it is possible to characterize the system performance in terms of the maximum overshoot and some of the other measures, such as rise time, delay time, and settling time, if necessary.

Let the reference input be a unit-step function $\theta_r(t) = u_s(t)$ rad; then $\Theta(s) = 1/s$. The output of the system, with zero initial conditions, is

$$\theta_y(t) = \mathcal{L}^{-1}\left[\frac{4500K}{s(s^2 + 361.2s - 4500K)}\right] \tag{5-135}$$

The inverse Laplace transform of the right-hand side of the last equation is carried out using the Laplace transform table in Appendix D, or using Eq. (5-90) directly. The following results are obtained for the three values of K indicated.

***K* = 7.248(ζ ≅ 1.0):**

$$\theta_y(t) = \left(1 - 151e^{-180t} + 150e^{-181.2t}\right)u_s(t) \tag{5-136}$$

***K* = 14.5(ζ = 0.707):**

$$\theta_y(t) = \left(1 - e^{-180.6t}\cos 180.6t - 0.9997e^{-180.6t}\sin 180.6t\right)u_s(t) \tag{5-137}$$

***K* = 181.17(ζ = 0.2):**

$$\theta_y(t) = \left(1 - e^{-180.6t}\cos 884.7t - 0.2041e^{-180.6t}\sin 884.7t\right)u_s(t) \tag{5-138}$$

The three responses are plotted as shown in Fig. 5-29. Table 5-3 gives the comparison of the characteristics of the three unit-step responses for the three values of K used. When

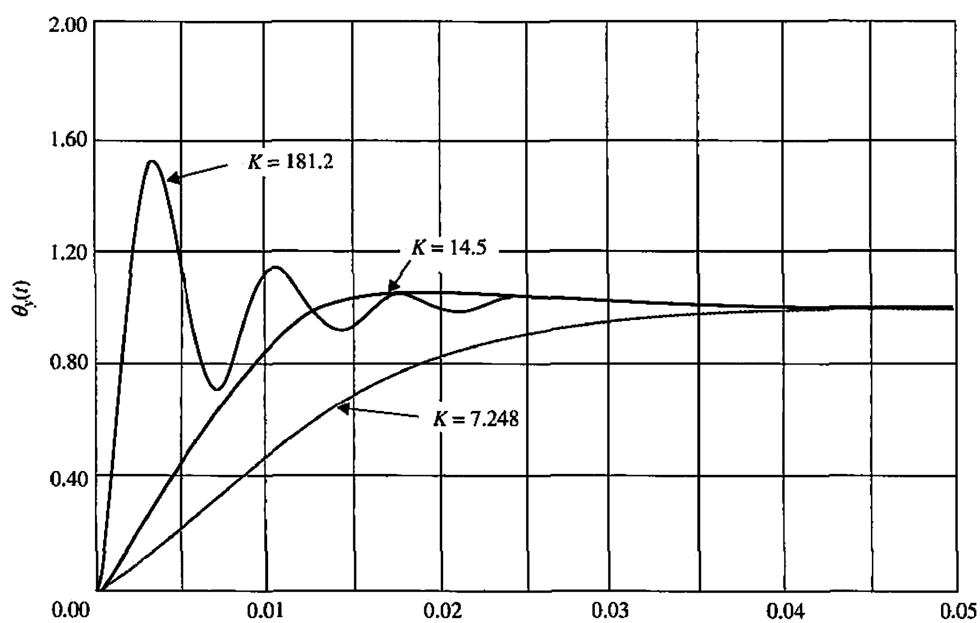


Figure 5-29 Unit-step responses of the attitude-control system in Fig. 4-78; $L_a = 0$.

TABLE 5-3 Comparison of the Performance of the Second-Order Position-Control System with the Gain K Values

Gain K	ζ	ω_n (rad/sec)	% Max overshoot	t_d (sec)	t_r (sec)	t_s (sec)	t_{max} (sec)
7.24808	1.000	180.62	0	0.00929	0.0186	0.0259	—
14.50	0.707	255.44	4.3	0.00560	0.0084	0.0114	0.01735
181.17	0.200	903.00	52.2	0.00125	0.00136	0.0150	0.00369

$K = 181.17$, $\zeta = 0.2$, the system is lightly damped, and the maximum overshoot is 52.7%, which is excessive. When the value of K is set at 7.248, ζ is very close to 1.0, and the system is almost critically damped. The unit-step response does not have any overshoot or oscillation. When K is set at 14.5, the damping ratio is 0.707, and the overshoot is 4.3%. It should be pointed out that, in practice, it would be time consuming, even with the aid of a computer, to compute the time response for each change of a system parameter for either analysis or design purposes. Indeed, one of the main objectives of studying control systems theory, using either the conventional or modern approach, is to establish methods so that the total reliance on computer simulation can be reduced. The motivation behind this discussion is to show that the performance of some control systems can be predicted by investigating the roots of the characteristic equation of the system. For the characteristic equation of Eq. (5-135), the roots are

$$s_1 = -180.6 + \sqrt{32616 - 4500K} \tag{5-139}$$

$$s_2 = -180.6 - \sqrt{32616 - 4500K} \tag{5-140}$$

Toolbox 5-8-1

The Fig. 5-29 responses may be obtained by the following sequence of MATLAB functions.

```
%Equation 5.136
%Unit-Step Transient Response

for k=[7.248 14.5 181.2]
num = [4500*k];
den = [1 361.2 4500*k];
step(num,den)
hold on;
end
xlabel('Time(secs)')
ylabel('Amplitude')
title('Closed-Loop Step')
```

For $K = 7.24808$, 14.5, and 181.2, the roots of the characteristic equation are tabulated as follows:

$K = 7.24808$:	$s_1 = s_2 = -180.6$	
$K = 14.5$:	$s_1 = -180.6 + j180.6$	$s_2 = -180.6 - j180.6$
$K = 181.2$:	$s_1 = -180.6 + j884.7$	$s_2 = -180.6 + j884.7$

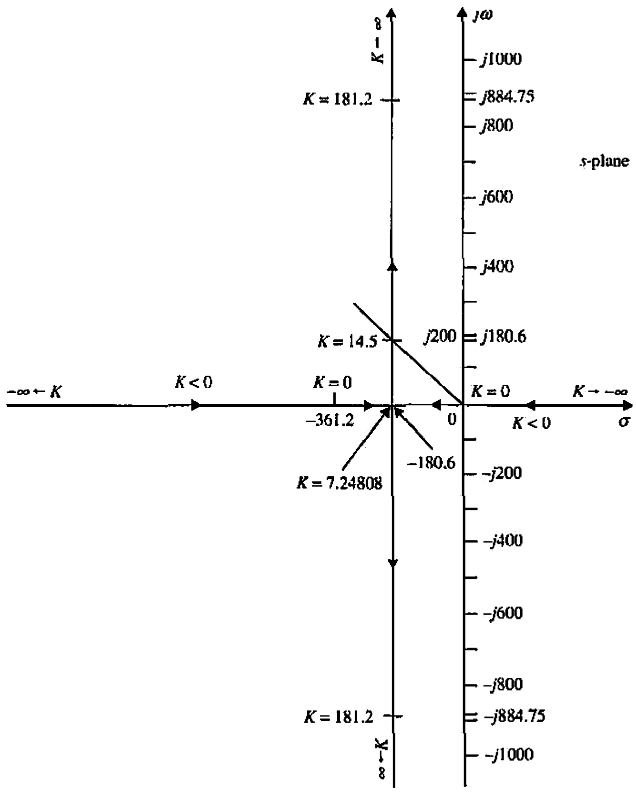


Figure 5-30 Root loci of the characteristic equation in Eq. (5-134) as K varies.

These roots are marked as shown in Fig. 5-30. The trajectories of the two characteristic equation roots when K varies continuously from $-\infty$ to ∞ are also shown in Fig. 5-30. These root trajectories are called the root loci (see Chapter 4) of Eq. (5-135) and are used extensively for the analysis and design of linear control systems.

From Eqs. (5-140) and (5-141), we see that the two roots are real and negative for values of K between 0 and 7.24808. This means that the system is overdamped, and the step response will have no overshoot for this range of K. For values of K greater than 7.24808, the natural undamped frequency will increase with $\sqrt{K}$. When K is negative, one of the roots is positive, which corresponds to a time response that increases monotonically with time, and the system is unstable. The dynamic characteristics of the transient step response as determined from the root loci of Fig. 5-30 are summarized as follows:

Amplifier Gain Dynamics	Characteristic Equation Roots	System
$0 < K < 7.24808$	Two negative distinct real roots	Overdamped ($\zeta > 1$)
$K = 7.24808$	Two negative equal real roots	Critically damped ($\zeta = 1$)
$7.24808 < K < \infty$	Two complex-conjugate roots with negative real parts	Underdamped ($\zeta < 1$)
$-\infty < K < 0$	Two distinct real roots, one positive and one negative	Unstable system ($\zeta < 0$)

5-8-2 The Steady-State Response

Because the forward-path transfer function in Eq. (5-132) has a simple pole at $s = 0$, the system is of type 1. This means that the steady-state error of the system is zero for all positive values of K when the input is a step function. Substituting Eq. (5-132) into Eq. (5-24), the step-error constant is

$$K_p = \lim_{s \to 0} \frac{4500K}{s(s + 361.2)} = \infty \tag{5-141}$$

Thus, the steady-state error of the system due to a step input, as given by Eq. (5-25), is zero. The unit-step responses in Fig. 5-29 verify this result. The zero-steady-state condition is achieved because only viscous friction is considered in the simplified system model. In the practical case, Coulomb friction is almost always present, so the steady-state positioning accuracy of the system can never be perfect.

5-8-3 Time Response to a Unit-Ramp Input

The control of position may be affected by the control of the profile of the output, rather than just by applying a step input. In other words, the system may be designed to follow a reference profile that represents the desired trajectory. It may be necessary to investigate the ability of the position-control system to follow a ramp-function input.

For a unit-ramp input, $\theta_r(t) = tu_s(t)$. The output response of the system in Fig. 4-79 is

$$\theta_y(t) = \mathcal{L}^{-1}\left[\frac{4500K}{s^2(s^2 + 361.2s + 4500K)}\right] \tag{5-142}$$

which can be solved by using the Laplace transform table in Appendix C. The result is

$$\theta_y(t) = \left[t - \frac{2\zeta}{\omega_n} + \frac{e^{-\zeta\omega_n t}}{\omega_n\sqrt{1-\zeta^2}} \sin\left(\omega_n\sqrt{1-\zeta^2}t + \theta\right)\right] u_s(t) \tag{5-143}$$

where

$$\theta = \cos^{-1}(2\zeta^2 - 1) \quad (\zeta < 1) \tag{5-144}$$

The values of ζ and ω_n are given in Eqs. (5-134) and (5-133), respectively. The ramp responses of the system for the three values of K are presented in the following equations.

K = 7.248:

$$\theta_y(t) = (t - 0.01107 - 0.8278e^{-181.2t} + 0.8389e^{-180t})u_s(t) \tag{5-145}$$

K = 14.5:

$$\begin{aligned}\theta_y(t) = (t &- 0.005536 + 0.005536e^{-180.6t}\cos 180.6t \\ &- 5.467 \times 10^{-7}e^{-180.6t}\sin 180.6t)u_s(t)\end{aligned} \tag{5-146}$$

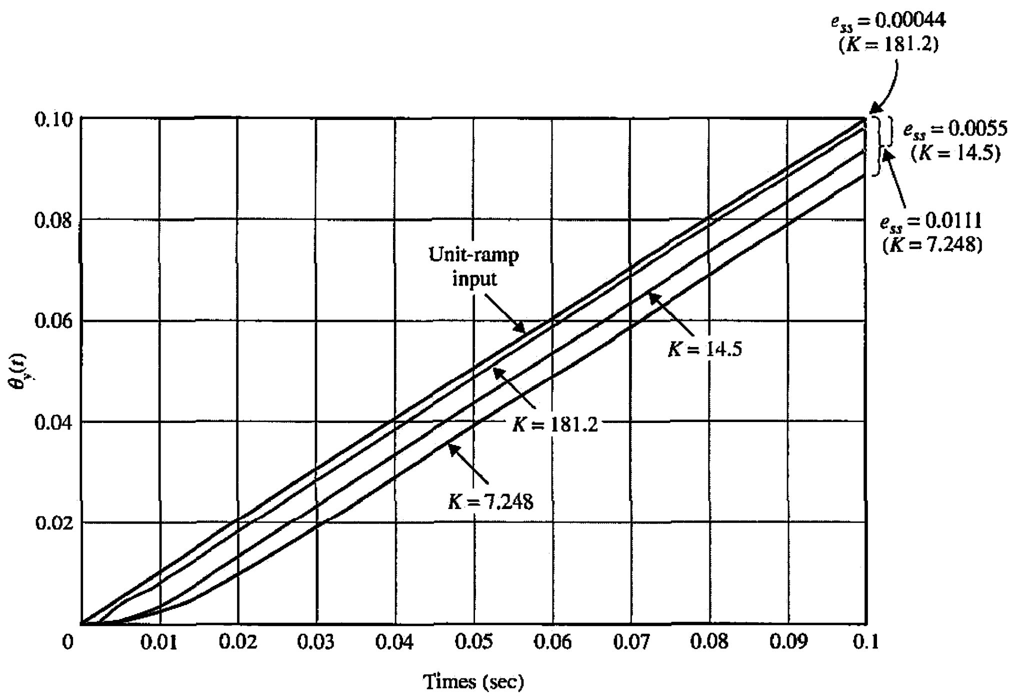


Figure 5-31 Unit-ramp responses of the attitude-control system in Fig. 4-78; $L_a = 0$.

K= 181.2:

$$\begin{aligned}\theta_y(t) = (&t - 0.000443 + 0.000443e^{-180.6t}\cos 884.7t \\ &- 0.00104e^{-180.6t}\sin 884.7t)u_s(t)\end{aligned} \tag{5-147}$$

These ramp responses are plotted as shown in Fig. 5-31. Notice that the steady-state error of the ramp response is not zero. The last term in Eq. (5-144) is the transient response. The steady-state portion of the unit-ramp response is

$$\lim_{t\to\infty}\theta_y(t) = \lim_{t\to\infty}\left[\left(t - \frac{2\zeta}{\omega_n}\right)u_s(t)\right] \tag{5-148}$$

Thus, the steady-state error of the system due to a unit-ramp input is

$$e_{ss} = \frac{2\zeta}{\omega_n} = \frac{0.0803}{K} \tag{5-149}$$

which is a constant.

A more direct method of determining the steady-state error due to a ramp input is to use the ramp-error constant K_v. From Eq. (5-31),

$$K_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0}\frac{4500K}{s + 361.2} = 12.46K \tag{5-150}$$

Thus, the steady-state error is

$$e_{ss} = \frac{1}{K_v} = \frac{0.0803}{K} \tag{5-151}$$

which agrees with the result in Eq. (5-149).

Toolbox 5-8-2

The Fig. 5-31 responses are obtained by the following sequence of MATLAB functions

```
for k=[7.248 14.5 181.2]
  clnum = [4500*k];
  clden = [1 361.2 4500*k];
  t=0:0.0001:0.3;
  u = t;
  [y,x]=lsim(clnum,clden,u,t);
  plot(t,y,t,u);
  hold on;
end

title('Unit-ramp responses')
xlabel('Time(sec)')
ylabel('Amplitude')
```

The result in Eq. (5-151) shows that the steady-state error is inversely proportional to K. For $K = 14.5$, which corresponds to a damping ratio of 0.707, the steady-state error is 0.0055 rad or, more appropriately, 0.55% of the ramp-input magnitude. Apparently, if we attempt to improve the steady-state accuracy of the system due to ramp inputs by increasing the value of K, the transient step response will become more oscillatory and have a higher overshoot. This phenomenon is rather typical in all control systems. For higher-order systems, if the loop gain of the system is too high, the system can become unstable. Thus, by using the controller in the system loop, the transient and the steady-state error can be improved simultaneously.

5-8-4 Time Response of a Third-Order System

In the preceding section, we have shown that the prototype second-order system, obtained by neglecting the armature inductance, is always stable for all positive values of K. It is not difficult to prove that, in general, all second-order systems with positive coefficients in the characteristic equations are stable.

Let us investigate the performance of the position-control system with the armature inductance $L_a = 0.003$ H. The forward-path transfer function of Eq. (5-128) becomes

$$G(s) = \frac{1.5 \times 10^7 K}{s(s^2 + 3408.3s + 1{,}204{,}000)} = \frac{1.5 \times 10^7 K}{s(s + 400.26)(s + 3008)} \tag{5-152}$$

The closed-loop transfer function is

$$\frac{\Theta_y(s)}{\Theta_r(s)} = \frac{1.5 \times 10^7 K}{s^3 + 3408.3s^2 + 1{,}204{,}000s + 1.5 \times 10^7 K} \tag{5-153}$$

The system is now of the third order, and the characteristic equation is

$$s^3 + 3408.3s^2 + 1{,}204{,}000s + 1.5 \times 10^7 K = 0 \tag{5-154}$$

Transient Response

The roots of the characteristic equation are tabulated for the three values of K used earlier for the second-order system:

$K = 7.248$:	$s_1 = -156.21$	$s_2 = -230.33$	$s_3 = -3021.8$
$K = 14.5$:	$s_1 = -186.53 + j192$	$s_2 = -186.53 - j192$	$s_3 = -3035.2$
$K = 181.2$:	$s_1 = -57.49 + j906.6$	$s_2 = -57.49 - j906.6$	$s_3 = -3293.3$

Comparing these results with those of the approximating second-order system, we see that, when $K = 7.428$, the second-order system is critically damped, whereas the third-order system has three distinct real roots, and the system is slightly overdamped. The root at -3021.8 corresponds to a time constant of 0.33 millisecond, which is more than 13 times faster than the next fastest time constant because of the pole at -230.33. Thus, the transient response due to the pole at -3021.8 decays rapidly, and the pole can be neglected from the transient standpoint. The output transient response is dominated by the two roots at -156.21 and -230.33. This analysis is verified by writing the transformed output response as

$$\Theta_y(s) = \frac{10.87 \times 10^7}{s(s + 156.21)(s + 230.33)(s + 3021.8)} \tag{5-155}$$

Taking the inverse Laplace transform of the last equation, we get

$$\theta_y(t) = \left(1 - 3.28e^{-156.21t} + 2.28e^{-230.33t} - 0.0045e^{-3021.8t}\right)u_s(t) \tag{5-156}$$

The last term in Eq. (5-156), which is due to the root at -3021.8, decays to zero very rapidly. Furthermore, the magnitude of the term at $t = 0$ is very small compared to the other two transient terms. This simply demonstrates that, in general, the contribution of roots that lie relatively far to the left in the s-plane to the transient response will be small. The roots that are closer to the imaginary axis will dominate the transient response, and these are defined as the **dominant roots** of the characteristic equation or of the system.

When $K = 14.5$, the second-order system has a damping ratio of 0.707, because the real and imaginary parts of the two characteristic equation roots are identical. For the third-order system, recall that the damping ratio is strictly not defined. However, because the effect on transient of the root at -3021.8 is negligible, the two roots that dominate the transient response correspond to a damping ratio of 0.697. Thus, for $K = 14.5$, the second-order approximation by setting L_a to zero is not a bad one. It should be noted, however, that the fact that the second-order approximation is justified for $K = 14.5$ does not mean that the approximation is valid for all values of K.

When $K = 181.2$, the two complex-conjugate roots of the third-order system again dominate the transient response, and the equivalent damping ratio due to the two roots is only 0.0633, which is much smaller than the value of 0.2 for the second-order system. Thus, we see that the justification and accuracy of the second-order approximation diminish as the value of K is increased. Fig. 5-32 illustrates the root loci of the third-order characteristic equation of Eq. (5-154) as K varies.

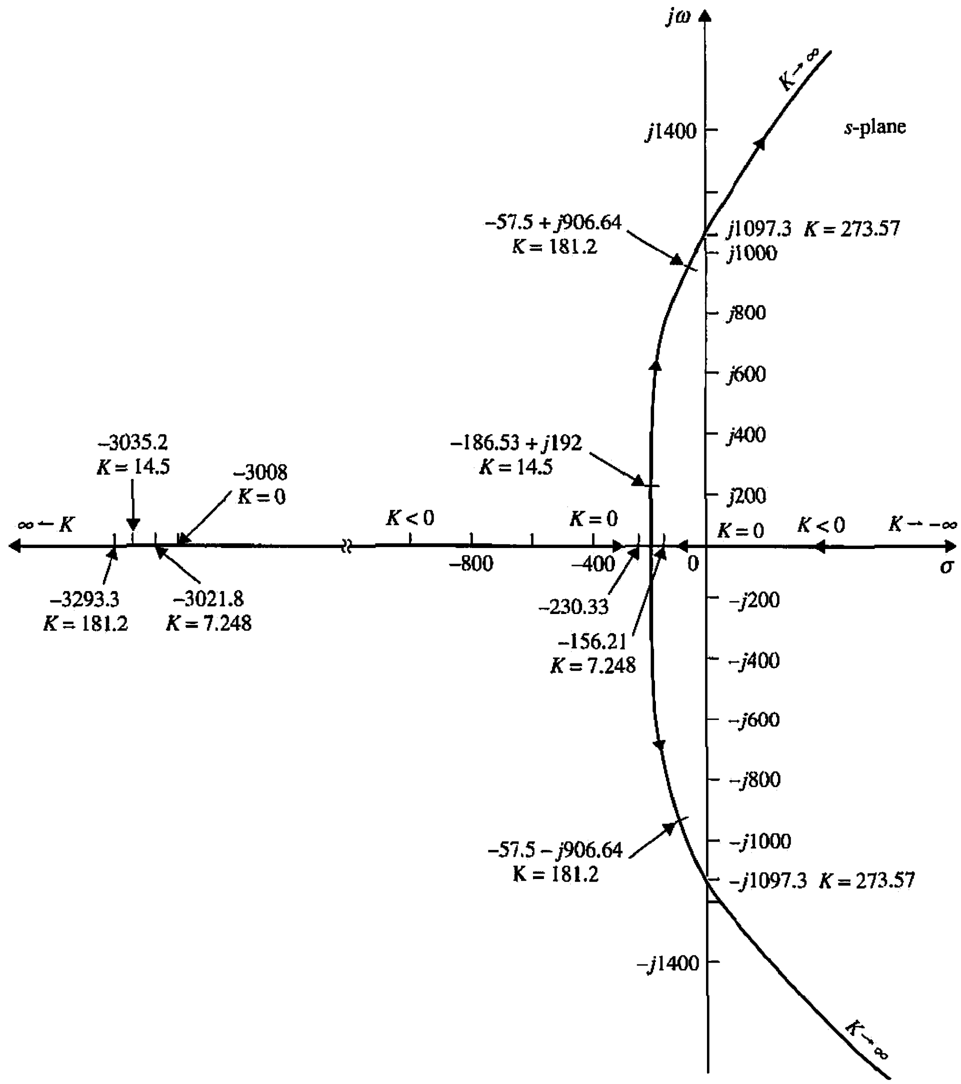


Figure 5-32 Root loci of the third-order attitude-control system.

When $K = 181.2$, the real root at -3293.3 still contributes little to the transient response, but the two complex-conjugate roots at $-57.49 \pm j906.6$ are much closer to the $j\omega$-axis than those of the second-order system for the same K, which are at $-180.6 \pm j884.75$. This explains why the third-order system is a great deal less stable than the second-order system when $K = 181.2$.

By using the Routh-Hurwitz criterion, the marginal value of K for stability is found to be 273.57. With this critical value of K, the closed-loop transfer function becomes

$$\frac{\Theta_y(s)}{\Theta_r(s)} = \frac{1.0872 \times 10^8}{(s + 3408.3)(s^2 + 1.204 \times 10^6)} \tag{5-157}$$

The roots of the characteristic equation are at $s = -3408.3,\ -j1097.3$, and $j1097.3$. These points are shown on the root loci in Fig. 5-32.

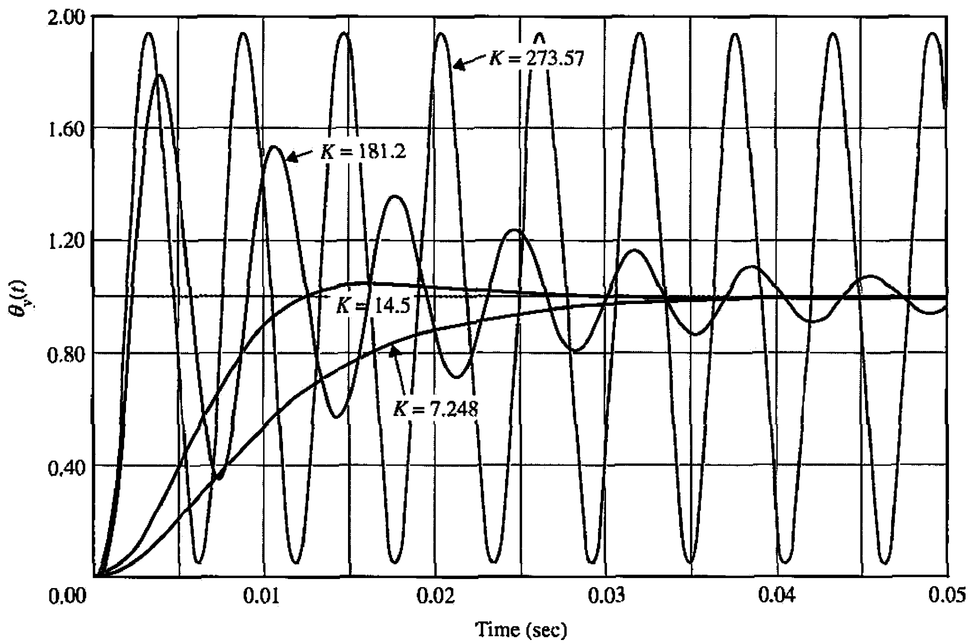


Figure 5-33 Unit-step responses of the third-order attitude-control system.

The unit-step response of the system when $K = 273.57$ is

$$\theta_y(t) = \left[1 - 0.094e^{-3408.3t} - 0.952\sin(1097.3t + 72.16°)\right]u_s(t) \qquad (5\text{-}158)$$

The steady-state response is an undamped sinusoid with a frequency of 1097.3 rad/sec, and the system is said to be marginally stable. When K is greater than 273.57, the two complex-conjugate roots will have positive real parts, the sinusoidal component of the time response will increase with time, and the system is unstable. Thus, we see that the third-order system is capable of being unstable, whereas the second-order system obtained with $L_a = 0$ is stable for all finite positive values of K.

Fig. 5-33 shows the unit-step responses of the third-order system for the three values of K used. The responses for $K = 7.248$ and $K = 14.5$ are very close to those of the second-order system with the same values of K that are shown in Fig. 5-29. However, the two responses for $K = 181.2$ are quite different.

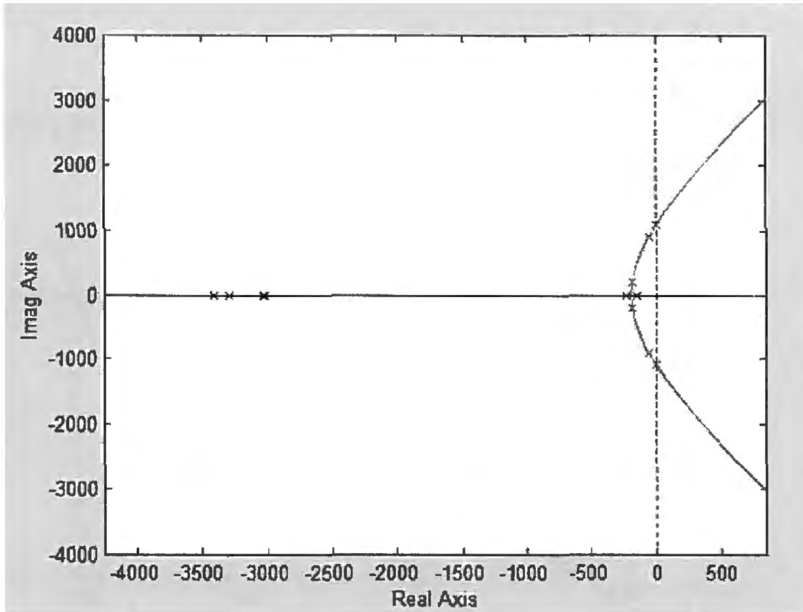
Toolbox 5-8-3

The root locus plot in Fig. 5-32 is obtained by the following MATLAB commands

```
for k=[7.248 14.5 181.2 273.57]

t=0:0.001:0.05;
num = [1.5*(10^7)*k];
den = [1 3408.3 1204000 1.5*(10^7)*k];
rlocus(num,den)

hold on;
end
```



Steady-State Response

From Eq. (5-152), we see that, when the inductance is restored, the third-order system is still of type 1. The value of K_v is still the same as that given in Eq. (5-150). Thus, the inductance of the motor does not affect the steady-state performance of the system, provided that the system is stable. This is expected, since L_a affects only the rate of change and not the final value of the motor current. A good engineer should always try to interpret the analytical results with the physical system.

5-9 BASIC CONTROL SYSTEMS AND EFFECTS OF ADDING POLES AND ZEROS TO TRANSFER FUNCTIONS

The position-control system discussed in the preceding section reveals important properties of the time responses of typical second- and third-order closed-loop systems. Specifically, the effects on the transient response relative to the location of the roots of the characteristic equation are demonstrated.

In all previous examples of control systems we have discussed thus far, the controller has been typically a simple amplifier with a constant gain K. This type of control action is formally known as **proportional control**, because the control signal at the output of the controller is simply related to the input of the controller by a proportional constant.

Intuitively, one should also be able to use the derivative or integral of the input signal, in addition to the proportional operation. Therefore, we can consider a more general continuous-data controller to be one that contains such components as adders or summers (addition or subtraction), amplifiers, attenuators, differentiators, and integrators — see Section 4-3-3 and Chapter 9 for more details. For example, one of the best-known controllers used in practice is the PID controller, which stands for **proportional**, **integral**, and **derivative**. The integral and derivative components of the PID controller have individual performance implications, and their applications require an understanding of the basics of these elements.

All in all, what these controllers do is *add additional poles and zeros* to the open- or closed-loop transfer function of the overall system. As a result, it is important to appreciate the effects of adding poles and zeros to a transfer function first. We show that—although the roots of the characteristic equation of the system, which are the poles of the closed-loop transfer function, affect the transient response of linear time-invariant control systems, particularly the stability—the zeros of the transfer function are also important. Thus, the addition of poles and zeros and/or cancellation of undesirable poles and zeros of the transfer function often are necessary in achieving satisfactory time-domain performance of control systems.

In this section, we show that the addition of poles and zeros to forward-path and closed-loop transfer functions has varying effects on the transient response of the closed-loop system.

5-9-1 Addition of a Pole to the Forward-Path Transfer Function: Unity-Feedback Systems

For the position-control system described in Section 5-8, when the motor inductance is neglected, the system is of the second order, and the forward-path transfer function is of the prototype given in Eq. (5-131). When the motor inductance is restored, the system is of the third order, and the forward-path transfer function is given in Eq. (5-149). Comparing the two transfer functions of Eqs. (5-131) and (5-149), we see that the effect of the motor inductance is equivalent to adding a pole at $s = -3008$ to the forward-path transfer function of Eq. (5-131) while shifting the pole at -361.2 to -400.26, and the proportional constant is also increased. The apparent effect of adding a pole to the forward-path transfer function is that the third-order system can now become unstable if the value of the amplifier gain K exceeds 273.57. As shown by the root-loci diagrams of Fig. 5-32 and Fig. 5-34, the new pole of $G(s)$ at $s = -3008$ essentially "pushes" and "bends" the complex-conjugate portion of the root loci of the second-order system toward the right-half s-plane. Actually, because of the specific value of the inductance chosen, the additional pole of the third-order system is far to the left of the pole at -400.26, so its effect is small except when the value of K is relatively large.

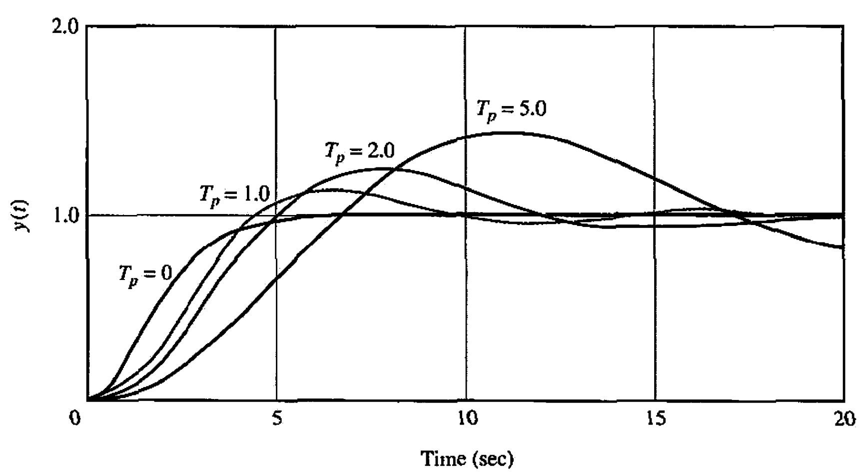


Figure 5-34 Unit-step responses of the system with the closed-loop transfer function in Eq. (5-160): $\zeta = 1$; $\omega_n = 1$; and $T_p = 0, 1, 2,$ and 5.

To study the general effect of the addition of a pole, and its relative location, to a forward-path transfer function of a unity-feedback system, consider the transfer function

$$G(s) = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)(1 + T_p s)} \tag{5-159}$$

The pole at $s = -1/T_p$ is considered to be added to the prototype second-order transfer function. The closed-loop transfer function is written

$$M(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)} = \frac{\omega_n^2}{T_p s^3 + (1 + 2\zeta\omega_n T_p)s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{5-160}$$

Fig. 5-34 illustrates the unit-step responses of the closed-loop system when $\omega_n = 1$; $\zeta = 1$; and $T_p = 0$, 1, 2, and 5. These responses again show *that the addition of a pole to the forward-path transfer function generally has the effect of increasing the maximum overshoot of the closed-loop system.*

As the value of T_p increases, the pole at $-1/T_p$ moves closer to the origin in the s-plane, and the maximum overshoot increases. These responses also show that the added pole increases the rise time of the step response. This is not surprising, because the additional pole has the effect of reducing the bandwidth (see Chapter 8) of the system, thus cutting out the high-frequency components of the signal transmitted through the system.

Toolbox 5-9-1

The corresponding responses for Fig. 5-34 are obtained by the following sequence of MATLAB functions

```
clear all
w=1; l=1;
for Tp=[0 1 2 5];

t=0:0.001:20;
num = [w];
den = [Tp 1+2*l*w*Tp 2*l*w w^2];

step(num,den,t);
hold on;
end
xlabel('Time(secs)')
ylabel('apos;y(t)')
title('Unit-step responses of the system')
```

The corresponding responses for Fig. 5-37 are obtained by the following sequence of MATLAB functions

```
clear all
w=1;l=0.25;
for Tp=[0 0.2 0.667 1];

t=0:0.001:20;
num = [w];
den = [Tp 1+2*l*w*Tp 2*l*w w^2];
```

```
step(num,den,t);
hold on;
end
xlabel('Time(secs)')
ylabel('y(t)')
title('Unit-step responses of the system')
```

The same conclusion can be drawn from the unit-step responses of Fig. 5-35, which are obtained with $\omega_n = 1$; $\zeta = 0.25$; and $T_p = 0, 0.2, 0.667$, and 1.0. In this case, when T_p is greater than 0.667, the amplitude of the unit-step response increases with time, and the system is unstable.

5-9-2 Addition of a Pole to the Closed-Loop Transfer Function

Because the poles of the closed-loop transfer function are roots of the characteristic equation, they control the transient response of the system directly. Consider the closed-loop transfer function

$$M(s) = \frac{Y(s)}{R(s)} = \frac{\omega_n^2}{\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)\left(1 + T_p s\right)} \tag{5-161}$$

where the term $\left(1 + T_p s\right)$ is added to a prototype second-order transfer function. Fig. 5-36 illustrates the unit-step response of the system with $\omega_n = 1.0$; $\zeta = 0.5$; and $T_p = 0$, 0.5, 1.0, 2.0, and 4.0. As the pole at $s = -1/T_p$ is moved toward the origin

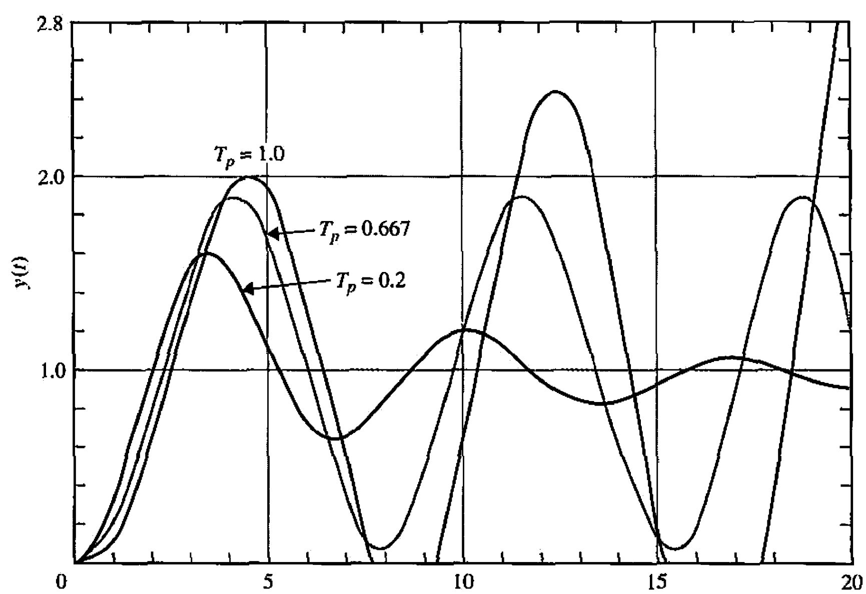


Figure 5-35 Unit-step responses of the system with the closed-loop transfer function in Eq. (5-160): $\zeta = 0.25$; $\omega_n = 1$; and $T_p = 0, 0.2, 0.667$, and 1.0.

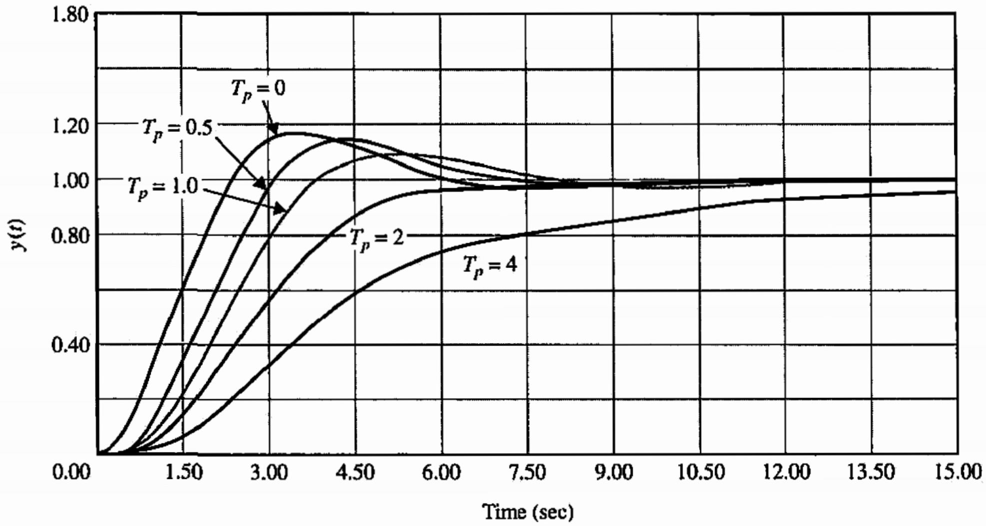


Figure 5-36 Unit-step responses of the system with the closed-loop transfer function in *Eq. (5-161): $\zeta = 0.5$; $\omega_n = 1$; and $T_p = 0, 0.5, 1.0, 2.0$, and 4.0.*

in the s-plane, the rise time increases and the maximum overshoot decreases. Thus, as far as the overshoot is concerned, adding a pole to the closed-loop transfer function has just the opposite effect to that of adding a pole to the forward-path transfer function.

Toolbox 5-9-2

The corresponding responses for Fig. 5-36 are obtained by the following sequence of MATLAB functions

```
clear all
w=1;l=0.5;
for Tp=[0 0.5 1 2];

t=0:0.001:15;
num = [w^2];
den = conv([1 2*l*w w^2],[Tp 1]);

step(num,den,t);
hold on;
end
xlabel('Time(secs)')
ylabel('y(t)')
title('Unit-step responses of the system')
```

5-9-3 Addition of a Zero to the Closed-Loop Transfer Function

Fig. 5-37 shows the unit-step responses of the closed-loop system with the transfer function

$$M(s) = \frac{Y(s)}{R(s)} = \frac{\omega_n^2(1 + T_z s)}{(s^2 + 2\zeta\omega_n s + \omega_n^2)} \tag{5-162}$$

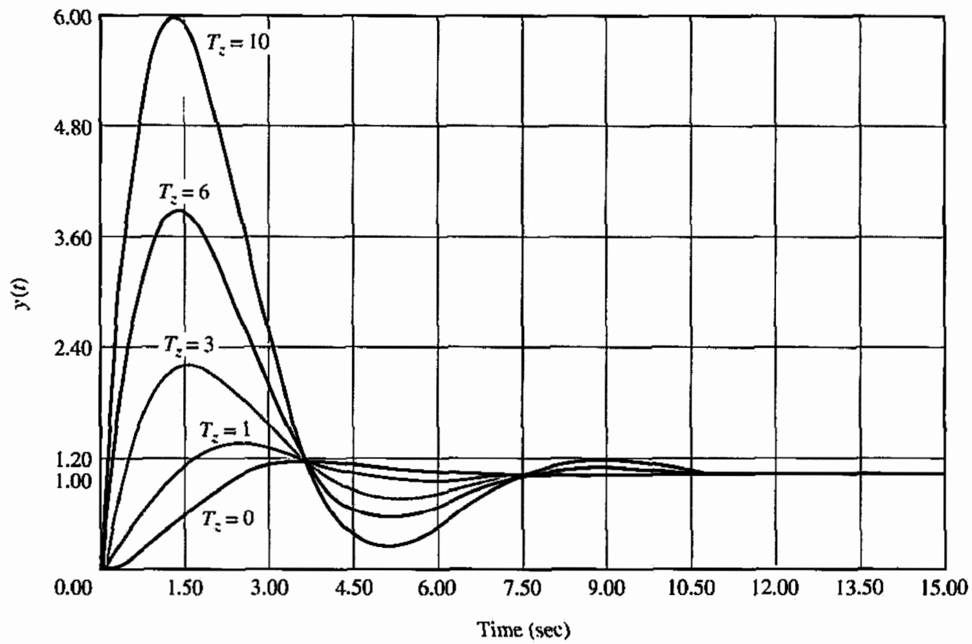


Figure 5-37 Unit-step responses of the system with the closed-loop transfer function in Eq. (5-162): $T_z = 0, 1, 2, 3, 6,$ and 10.

where $\omega_n = 1$; $\zeta = 0.5$; and $T_z = 0, 1, 2, 3, 6,$ and 10. In this case, we see that adding a zero to the closed-loop transfer function decreases the rise time and increases the maximum overshoot of the step response.

We can analyze the general case by writing Eq. (5-162) as

$$M(s) = \frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} + \frac{T_z \omega_n^2 s}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{5-163}$$

For a unit-step input, let the output response that corresponds to the first term of the right side of Eq. (5-163) be $y_1(t)$. Then, the total unit-step response is

$$y(t) = y_1(t) + T_z \frac{dy_1(t)}{dt} \tag{5-164}$$

Fig. 5-38 shows why the addition of the zero at $s = -1/T_z$ reduces the rise time and increases the maximum overshoot, according to Eq. (5-164). In fact, as T_z approaches infinity, the maximum overshoot also approaches infinity, and yet the system is still stable as long as the overshoot is finite and ζ is positive.

5-9-4 Addition of a Zero to the Forward-Path Transfer Function: Unity-Feedback Systems

Let us consider that a zero at $-1/T_z$ is added to the forward-path transfer function of a third-order system, so

$$G(s) = \frac{6(1 + T_z s)}{s(s+1)(s+2)} \tag{5-165}$$

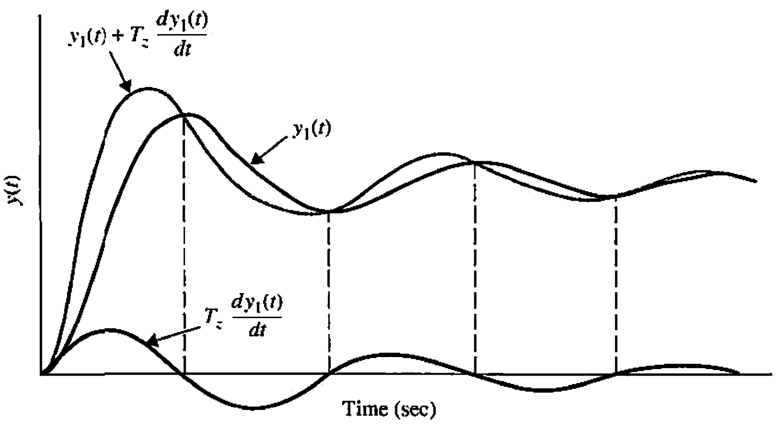


Figure 5-38 Unit-step responses showing the effect of adding a zero to the closed-loop transfer function.

The closed-loop transfer function is

$$M(s) = \frac{Y(s)}{R(s)} = \frac{6(1 + T_z s)}{s^3 + 3s^2 + (2 + 6T_z)s + 6} \tag{5-166}$$

The difference between this case and that of adding a zero to the closed-loop transfer function is that, in the present case, not only the term $(1 + T_z s)$ appears in the numerator of $M(s)$, but the denominator of $M(s)$ also contains T_z. The term $(1 + T_z s)$ in the numerator of $M(s)$ increases the maximum overshoot, but T_z appears in the coefficient of the s term in the denominator, which has the effect of improving damping, or reducing the maximum overshoot. Fig. 5-39 illustrates the unit-step responses when $T_z = 0$, 0.2, 0.5, 2.0, 5.0, and 10.0. Notice that, when $T_z = 0$, the closed-loop system is on the verge of becoming unstable. When $T_z = 0.2$ and 0.5, the maximum overshoots are reduced, mainly

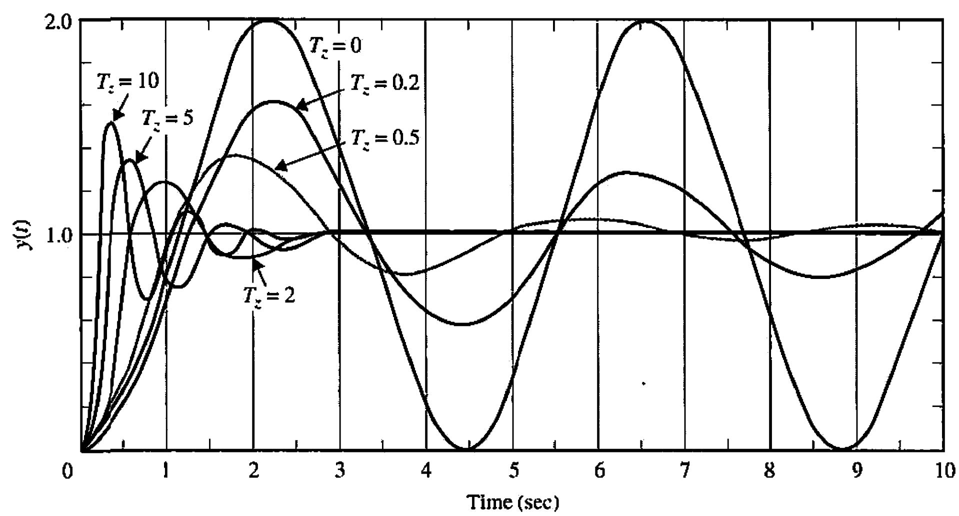


Figure 5-39 Unit-step responses of the system with the closed-loop transfer function in Eq. (5-166): $T_z = 0$, 0.2, 0.5, 2.0, 5.0, and 10.0.

because of the improved damping. As T_z increases beyond 2.0, although the damping is still further improved, the $(1 + T_z s)$ term in the numerator becomes more dominant, so the maximum overshoot actually becomes greater as T_z is increased further.

An important finding from these discussions is that, although the characteristic equation roots are generally used to study the relative damping and relative stability of linear control systems, the zeros of the transfer function should not be overlooked in their effects on the transient performance of the system.

Toolbox 5-9-3

The corresponding responses for Fig. 5-39 are obtained by the following sequence of MATLAB functions

```
clear all
w=1;l=0.5;
for Tz=[0 0.2 0.5 3 5];
t=0:0.001:15;
num = [6*Tz 6];
den = [1 3 2+6*Tz 6];

step(num,den,t);
hold on;
end
xlabel('Time(secs)')
ylabel('y(t)')
title('Unit-step responses of the system')
```

5-10 DOMINANT POLES AND ZEROS OF TRANSFER FUNCTIONS

From the discussions given in the preceding sections, it becomes apparent that the location of the poles and zeros of a transfer function in the s-plane greatly affects the transient response of the system. For analysis and design purposes, it is important to sort out the poles that have a dominant effect on the transient response and call these the dominant poles.

Because most control systems in practice are of orders higher than two, it would be useful to establish guidelines on the approximation of high-order systems by lower-order ones insofar as the transient response is concerned. In design, we can use the dominant poles to control the dynamic performance of the system, whereas the insignificant poles are used for the purpose of ensuring that the controller transfer function can be realized by physical components.

For all practical purposes, we can divide the s-plane into regions in which the dominant and insignificant poles can lie, as shown in Fig. 5-40. We intentionally do not assign specific values to the coordinates, since these are all relative to a given system.

The poles that are *close* to the imaginary axis in the left-half s-plane give rise to transient responses that will decay relatively slowly, whereas the poles that are *far away* from the axis (relative to the dominant poles) correspond to fast-decaying time responses. The distance D between the dominant region and the least significant region shown in Fig. 5-40 will be subject to discussion. The question is: How large a pole is considered to be really large? It has been recognized in practice and in the literature that if the magnitude of the real part of a pole is at least 5 to 10 times that of a dominant pole or a pair of complex

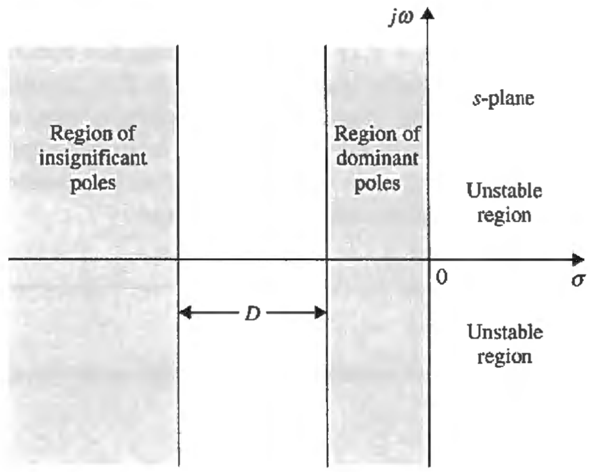


Figure 5-40 Regions of dominant and insignificant poles in the s-plane.

dominant poles, then the pole may be regarded as insignificant insofar as the transient response is concerned. The zeros that are *close* to the imaginary axis in the left-half s-plane affect the transient responses more significantly, whereas the zeros that are *far away* from the axis (relative to the dominant poles) have a smaller effect on the time response.

We must point out that the regions shown in Fig. 5-40 are selected merely for the definitions of dominant and insignificant poles. For design purposes, such as in pole-placement design, the dominant poles and the insignificant poles should most likely be located in the tinted regions in Fig. 5-41. Again, we do not show any absolute coordinates, except that the desired region of the dominant poles is centered around the line that corresponds to $\zeta = 0.707$. It should also be noted that, while designing, we cannot place the insignificant poles arbitrarily far to the left in the s-plane or these may require unrealistic system parameter values when the pencil-and-paper design is implemented by physical components.

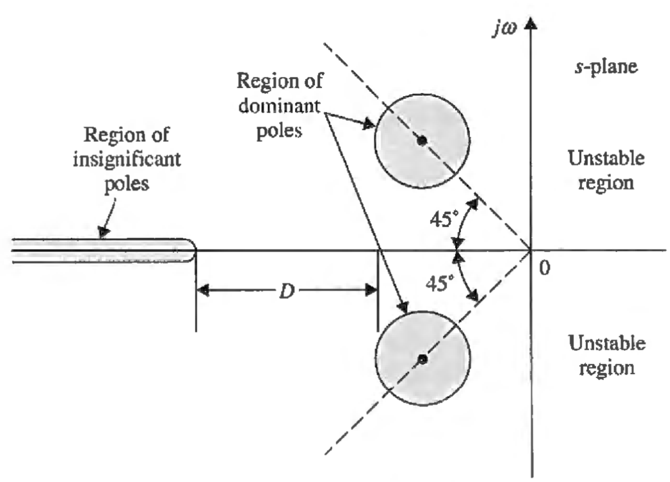


Figure 5-41 Regions of dominant and insignificant poles in the s-plane for design purposes.

5-10-1 Summary of Effects of Poles and Zeros

Based on previous observations, we can summarize the following:

1. Complex-conjugate poles of the closed-loop transfer function lead to a step response that is underdamped. If all system poles are real, the step response is overdamped. However, zeros of the closed-loop transfer function may cause overshoot even if the system is overdamped.
2. The response of a system is dominated by those poles closest to the origin in the s-plane. Transients due to those poles, which are farther to the left, decay faster.
3. The farther to the left in the s-plane the system's dominant poles are, the faster the system will respond and the greater its bandwidth will be.
4. The farther to the left in the s-plane the system's dominant poles are, the more expensive it will be and the larger its internal signals will be. While this can be justified analytically, it is obvious that striking a nail harder with a hammer drives the nail in faster but requires more energy per strike. Similarly, a sports car can accelerate faster, but it uses more fuel than an average car.
5. When a pole and zero of a system transfer function nearly cancel each other, the portion of the system response associated with the pole will have a small magnitude.

5-10-2 The Relative Damping Ratio

When a system is higher than the second order, we can no longer strictly use the damping ratio ζ and the natural undamped frequency ω_n, which are defined for the prototype second-order systems. However, if the system dynamics can be accurately represented by a pair of complex-conjugate dominant poles, then we can still use ζ and ω_n to indicate the dynamics of the transient response, and the damping ratio in this case is referred to as the relative damping ratio of the system. For example, consider the closed-loop transfer function

$$M(s) = \frac{Y(s)}{R(s)} = \frac{20}{(s+10)(s^2+2s+2)} \qquad (5\text{-}167)$$

The pole at $s = -10$ is 10 times the real part of the complex conjugate poles, which are at $-1 \pm j1$. We can refer to the relative damping ratio of the system as 0.707.

5-10-3 The Proper Way of Neglecting the Insignificant Poles with Consideration of the Steady-State Response

Thus far, we have provided guidelines for neglecting insignificant poles of a transfer function from the standpoint of the transient response. However, going through with the mechanics, the steady-state performance must also be considered. Let us consider the transfer function in Eq. (5-167); the pole at -10 can be neglected from the transient standpoint. To do this, we should first express Eq. (5-167) as

$$M(s) = \frac{20}{10(s/10+1)(s^2+2s+2)} \qquad (5\text{-}168)$$

Then we reason that $|s/10| \ll 1$ when the absolute value of s is much smaller than 10, because of the dominant nature of the complex poles. The term $s/10$ can be neglected when compared with 1. Then, Eq. (5-168) is approximated by

$$M(s) \cong \frac{20}{10(s^2 + 2s + 2)} \tag{5-169}$$

This way, the steady-state performance of the third-order system will not be affected by the approximation. In other words, the third-order system described by Eq. (5-167) and the second-order system approximated by Eq. (5-169) all have a final value of unity when a unit-step input is applied. On the other hand, if we simply throw away the term $(s + 10)$ in Eq. (5-167), the approximating second-order system will have a steady-state value of 5 when a unit-step input is applied.

▸ 5-11 BASIC CONTROL SYSTEMS UTILIZING ADDITION OF POLES AND ZEROS

In practice we can control the response of a system by adding poles and zeros or a simple amplifier with a constant gain K to its transfer function. So far in this chapter, we have discussed the effect of adding a simple gain in the time response—i.e., proportional control. In this section, we look at controllers that include derivative or integral of the input signal in addition to the proportional operation.

▸ EXAMPLE 5-11-1 Fig. 5-42 shows the block diagram of a feedback control system that arbitrarily has a second-order prototype process with the transfer function

$$G_p(s) = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)} \tag{5-170}$$

The series controller in this case is a proportional-derivative (PD) type with the transfer function

$$G_c(s) = K_P + K_D s \tag{5-171}$$

In this case, the forward-path transfer function of the compensated system is

$$G(s) = \frac{Y(s)}{E(s)} = G_c(s)G_p(s) = \frac{\omega_n^2(K_P + K_D s)}{s(s + 2\zeta\omega_n)} \tag{5-172}$$

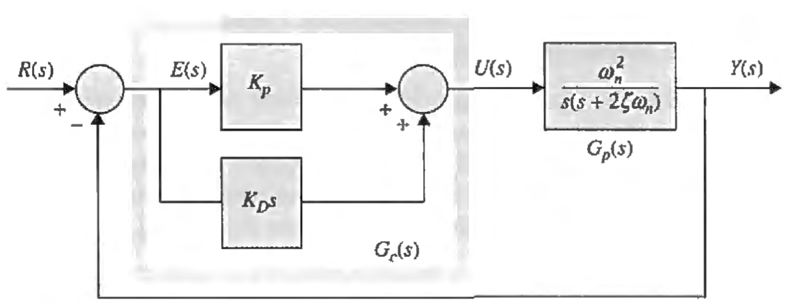


Figure 5-42 Control system with PD controller.

which shows that the PD control is equivalent to adding a simple zero at $s = -K_P/K_D$ to the forward-path transfer function. Consider the second-order model

$$G(s) = \frac{2}{s(s+2)} \tag{5-173}$$

Rewriting the transfer function of the PD controller as

$$G_c(s) = (K_P + K_D s) \tag{5-174}$$

the forward-path transfer function of the system becomes

$$G(s) = \frac{Y(s)}{E(s)} = \frac{2(K_P + K_D s)}{s(s+2)} \tag{5-175}$$

The closed-loop transfer function is

$$\frac{Y(s)}{R(s)} = \frac{2(K_P + K_D s)}{s^2 + (2 + 2K_D)s + 2K_P} \tag{5-176}$$

Eq. (5-176) shows that the effects of the PD controller are the following:

1. Adding a zero at $s = -K_P/K_D$ to the closed-loop transfer function.
2. Increasing the *damping term*, which is the coefficient of the s term in the denominator, from 2 to $2 + 2K_D$.

We should quickly point out that Eq. (5-175) no longer represents a prototype second-order system, since the transient response is also affected by the zero of the transfer function at $s = -K_P/K_D$. It turns out that for this second-order system, as the value of K_D increases, the zero will move very close to the origin and effectively cancel the pole of $G(s)$ at $s = 0$. Thus, as K_D increases, the transfer function in Eq. (5-175) approaches that of a first-order system with the pole at $s = -2$, and the closed-loop system will not have any overshoot. In general, for higher-order systems, however, the zero at $s = -K_P/K_D$ may increase the overshoot when K_D becomes very large.

The characteristic equation is written as

$$s^2 + (2 + 2K_D)s + 2K_P = 0 \tag{5-177}$$

Ignoring the zero of the transfer function in equation (5-177) and comparing (5-177) to prototype second-order system characteristic equation

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \tag{5-178}$$

we get the damping ratio and natural frequency values of

$$\zeta = \frac{1 + K_D}{\sqrt{2K_P}} \qquad \omega_n = \sqrt{2K_P} \tag{5-179}$$

which clearly show the positive effect of K_D on damping. For $K_P = 8$, if we wish to have critical damping, $\zeta = 1$, Eq. (5-179) gives $K_D = 3$. Fig. 5-43 shows the unit-step responses of the closed-loop system with $K_P = 8$ and $K_D = 3$. With the PD control, the maximum overshoot is 2%. In the present case, although K_D is chosen for critical damping, the overshoot is due to the zero at $s = -K_P/K_D$ of the closed-loop transfer function. Upon selecting a smaller $K_P = 1$, for $\zeta = 1$, Eq. (5-179) gives $K_D = 0.414$. Fig. 5-43 shows a critically damped unit-step response in this case, which implies the zero at $s = -K_P/K_D$ of the closed-loop transfer function has a smaller impact on the response of the system, and the overall response is similar to that of a prototype second-order system. However, in either case, upon *increasing* K_D, *the general conclusion is that the PD controller decreases the maximum overshoot, the rise time, and the settling time.*

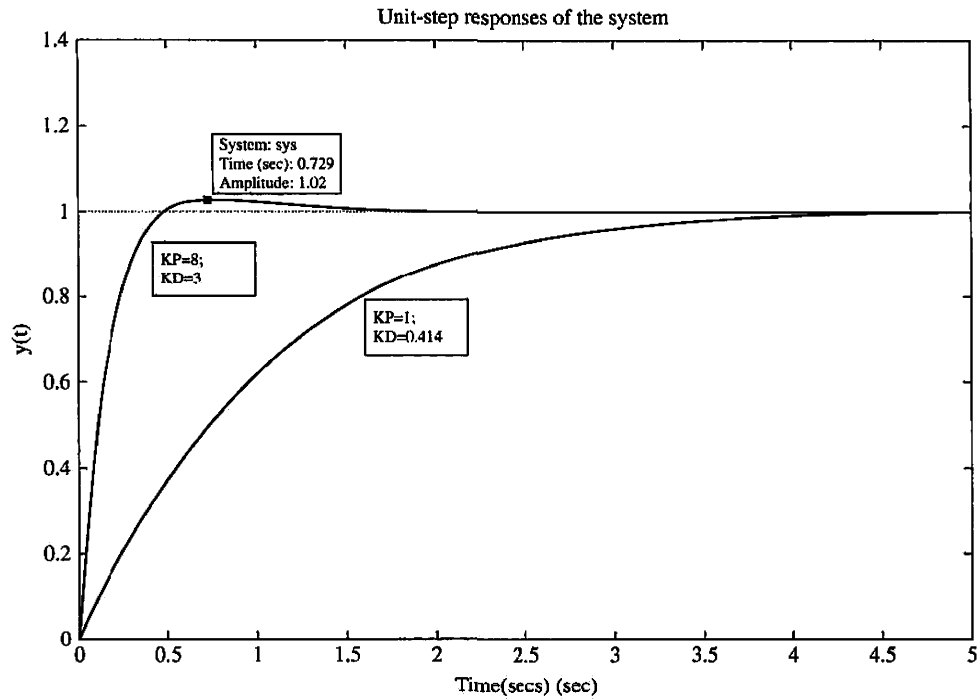


Figure 5-43 Unit-step response of Eq. (5-176) for two sets of K_D and K_P values.

Toolbox 5-11-1

The corresponding responses for Fig. 5-43 are obtained by the following sequence of MATLAB functions

```
clear all
t=0:0.001:5;

num = [2*3 16]; % KP=4 and KD=3
den = [1 2+2*3 16];
step(num,den,t);

hold on;

num = [2*.414 2]; % KP=1 and KD=0.414
den = [1 2+2*.414 2];
step(num,den,t);

xlabel('Time(secs)')
ylabel('y(t)')
title('Unit-step responses of the system')
```

EXAMPLE 5-11-2 We saw in the previous example that the PD controller can improve the damping and rise time of a control system. Because the PD controller does not change the system type, it may not fulfill the compensation objectives in many situations involving steady-state error. For this purpose, an integral controller may be used. The integral part of the PID controller produces a signal that is proportional to

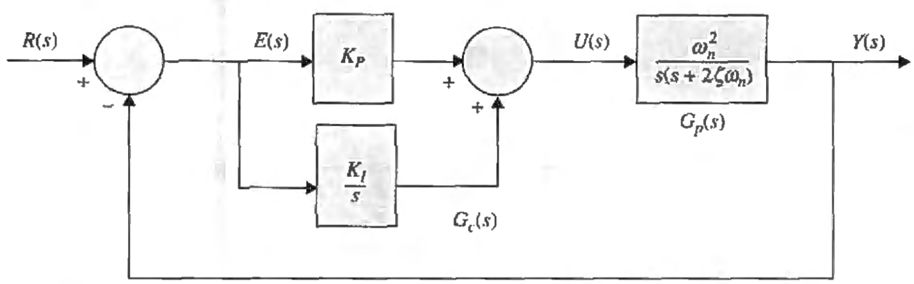


Figure 5-44 Control system with PI controller.

the time integral of the input of the controller. Fig. 5-44 illustrates the block diagram of the prototype second-order system with a series PI controller. The transfer function of the PI controller is

$$G_c(s) = K_P + \frac{K_I}{s} \tag{5-180}$$

Using the circuit elements given in Table 4-4 in Chapter 4, the forward-path transfer function of the compensated system is

$$G(s) = G_c(s)G_p(s) = \frac{\omega_n^2(K_P s + K_I)}{s^2(s + 2\zeta\omega_n)} \tag{5-181}$$

Clearly, the immediate effects of the PI controller are the following:

1. Adding a zero at $s = -K_I/K_P$ to the forward-path transfer function.
2. Adding a pole at $s = 0$ to the forward-path transfer function. This means that the system type is increased by one. Thus, the steady-state error of the original system is improved by one order; that is, if the steady-state error to a given input is constant, the PI control reduces it to zero (provided that the compensated system remains stable).

Consider the second-order model

$$G_p(s) = \frac{2}{(s+1)(s+2)} \tag{5-182}$$

The system in Fig. 5-44, with the forward-path transfer function in Eq. (5-182), will now have a zero steady-state error when the reference input is a step function. However, because the system is now of the third order, *it may be less stable* than the original second-order system or even become *unstable* if the parameters K_P and K_I are not properly chosen. In the case of a type 0 system with a PD control, the magnitude of the steady-state error is inversely proportional to K_P. When a type 0 system is converted to type 1 using a PI controller, the steady-state error due to a step input is always zero if the system is stable. The problem is then to choose the proper combination of K_P and K_I so that the transient response is satisfactory.

The pole-zero configuration of the PI controller in Eq. (5-180) is shown in Fig. 5-45. At first glance, it may seem that PI control will improve the steady-state error at the expense of stability. However, we shall show that, if the location of the zero of $G_c(s)$ is selected properly, both the damping and the steady-state error can be improved. Because the PI controller is essentially a low-pass filter, the compensated system usually will have a slower rise time and longer settling time. *A viable method of designing the PI control is to select the zero at $s = -K_I/K_P$ so that it is relatively close to the origin and away from the most significant poles of the process; the values of K_P and K_I should be relatively small.*

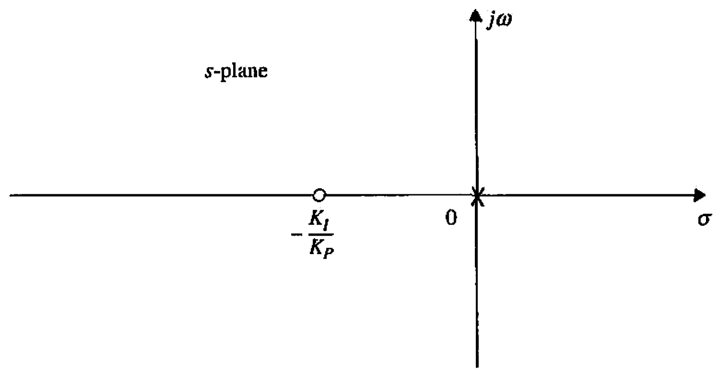


Figure 5-45 Pole-zero configuration of a PI controller.

Applying the PI controller of Eq. (5-180), the forward-path transfer function of the system becomes

$$G(s) = G_c(s)G_P(s) = \frac{2K_P(s + K_I/K_P)}{s(s+1)(s+2)} = \frac{2K_P(s + K_I/K_P)}{s^3 + 3s^2 + 2s} \tag{5-183}$$

The steady-state error due to a step input $u_s(t)$ is zero. The closed-loop transfer function is

$$\frac{Y(s)}{R(s)} = \frac{2K_P(s + K_I/K_P)}{s^3 + 3s^2 + 2(1 + K_P)s + 2K_I} \tag{5-184}$$

The characteristic equation of the closed-loop system is

$$s^3 + 3s^2 + 2(1 + K_P)s + 2K_I = 0 \tag{5-185}$$

Applying Routh's test to Eq. (5-185) yields the result that the system is stable for $0 < K_I/K_P < 13.5$. This means that the zero of $G(s)$ at $s = -K_I/K_P$ cannot be placed too far to the left in the left-half s-plane, or the system will be unstable. Let us place the zero at $-K_I/K_P$ relatively close to the origin. For the present case, the most significant pole of $G_p(s)$ is at -1. Thus, K_I/K_P should be chosen so that the following condition is satisfied:

$$\frac{K_I}{K_P} \ll 1 \tag{5-186}$$

With the condition in Eq. (5-186) satisfied, Eq. (5-184) can be approximated by

$$G(s) \cong \frac{2K_P}{s^2 + 3s + 2 + 2K_P} \tag{5-187}$$

where the term K_I/K_P in the numerator and K_I in the denominator are neglected. As a design criterion, we assume a desired percent maximum overshoot value of about 4.3 for a unit-step input, which utilizing expression (5-104) results in a relative damping ratio of 0.707. From the denominator of Eq. (5-187) compared with a prototype second-order system, we get natural frequency value of $\omega_n = 2.1213$ rad/s and the required proportional gain of $K_P = 1.25$. This should also be true for the third-order system with the PI controller if the value of K_I/K_P satisfies Eq. (5-186). Thus, to achieve this, we pick a small K_I. If K_I is too small, however, the system time response is slow and the desired steady-state error requirement is not met fast enough. Upon increasing K_I to 1.125, the desired response is met, as shown in Fig. 5-46.

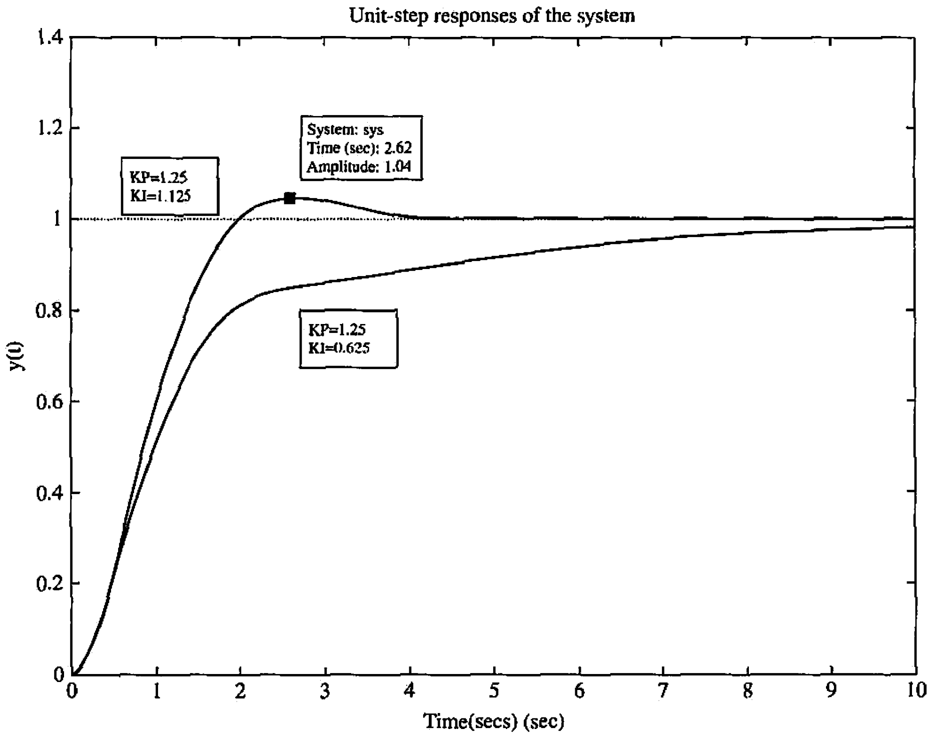


Figure 5-46 Unit-step response of Eq. (5-185) for two sets of K_I and K_P values.

Toolbox 5-11-2

The corresponding responses for Fig. 5-46 are obtained by the following sequence of MATLAB functions

```
clear all
t=0:0.001:10;

num= [2*1.25 1.125]; %KP=1.25 and KI=0.625
den= [1 3 2+2*1.25 1.125];
step(num,den,t);

hold on;

num= [2*1.25 2*1.125]; %KP=1.25 and KI=1.125
den= [1 3 2+2*1.25 2*1.125];
step(num,den,t);

xlabel('Time(secs)')
ylabel('y(t)')
title('Unit-step responses of the system')
```

5-12 MATLAB TOOLS

In this chapter we provided MATLAB toolboxes for finding the time response of simple control systems. We also introduced the concepts of root contours and root locus and included MATLAB codes to draw them for simple control examples. In Chapters 6 and 9, where we address more complex control-system modeling and analysis, we will introduce the Automatic Control Systems software (ACSYS) that utilizes MATLAB and SIMULINK m-files and GUIs (graphical user interface) for the analysis of more complex control engineering problems.

The reader is especially encouraged to explore the Control Lab software tools presented in Chapter 6 that simulate dc motor speed and position control topics discussed earlier in this chapter. These simulation tools provide the user with virtual experiments and design projects using systems involving dc motors, sensors, electronic components, and mechanical components.

▶ 5-13 SUMMARY

This chapter was devoted to the time-domain analysis of linear continuous-data control systems. The time response of control systems is divided into the transient and the steady-state responses. The steady-state error is a measure of the accuracy of the system as time approaches infinity. When the system has unity feedback for the step, ramp, and parabolic inputs, the steady-state error is characterized by the error constants K_p, K_v, and K_a, respectively, as well as the system **type**. When applying the steady-state error analysis, the final-value theorem of the Laplace transform is the basis; it should be ascertained that the closed-loop system is stable or the error analysis will be invalid. The error constants are not defined for systems with nonunity feedback. For nonunity-feedback systems, a method of determining the steady-state error was introduced by using the closed-loop transfer function.

The transient response is characterized by such criteria as the **maximum overshoot, rise time, delay time**, and **settling time**, and such parameters as **damping ratio, natural undamped frequency**, and **time constant**. The analytical expressions of these parameters can all be related to the system parameters simply if the transfer function is of the second-order prototype. For second-order systems that are not of the prototype and for higher-order systems, the analytical relationships between the transient parameters and the system constants are difficult to determine. Computer simulations are recommended for these systems.

Time-domain analysis of a position-control system was conducted. The transient and steady-state analyses were carried out first by approximating the system as a second-order system. The effect of varying the amplifier gain K on the transient and steady-state performance was demonstrated. The concept of the root-locus technique was introduced, and the system was then analyzed as a third-order system. It was shown that the second-order approximation was accurate only for low values of K.

The effects of adding poles and zeros to the forward-path and closed-loop transfer functions were demonstrated. The dominant poles of transfer functions were also discussed. This established the significance of the location of the poles of the transfer function in the s-plane and under what conditions the insignificant poles (and zeros) could be neglected with regard to the transient response.

Later in the chapter, simple controllers—namely the PD, PI, and PID—were introduced. Designs were carried out in the time-domain (and s-domain). The time-domain design may be characterized by specifications such as the relative damping ratio, maximum overshoot, rise time, delay time, settling time, or simply the location of the characteristic-equation roots, keeping in mind that the zeros of the system transfer function also affect the transient response. The performance is generally measured by the step response and the steady-state error.

MATLAB toolboxes and the Automatic Control System software tool are good tools to study the time response of control systems. Through the GUI approach provided by **ACSYS**, these programs are intended to create a user-friendly environment to reduce the complexity of control systems design. See Chapters 6 and 9 for more detail.

▶ REVIEW QUESTIONS

1. Give the definitions of the error constants K_p, K_v, and K_a.
2. Specify the type of input to which the error constant K_p is dedicated.

3. Specify the type of input to which the error constant K_v is dedicated.

4. Specify the type of input to which the error constant K_a is dedicated.

5. Define an error constant if the input to a unity-feedback control system is described by $r(t) = t^3 u_s(t)/6$.

6. Give the definition of the system type of a linear time-invariant system.

7. If a unity-feedback control system type is 2, then it is certain that the steady-state error of the system to a step input or a ramp input will be zero. (T) (F)

8. Linear and nonlinear frictions will generally degrade the steady-state error of a control system. (T) (F)

9. The maximum overshoot of a unit-step response of the second-order prototype system will never exceed 100% when the damping ratio ζ and the natural undamped frequency ω_n are all positive. (T) (F)

10. For the second-order prototype system, when the undamped natural frequency ω_n increases, the maximum overshoot of the output stays the same. (T) (F)

11. The maximum overshoot of the following system will never exceed 100% when ζ, ω_n, and T are all positive.

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2(1 + Ts)}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

(T) (F)

12. Increasing the undamped natural frequency will generally reduce the rise time of the step response. (T) (F)

13. Increasing the undamped natural frequency will generally reduce the settling time of the step response. (T) (F)

14. Adding a zero to the forward-path transfer function will generally improve the system damping and thus will always reduce the maximum overshoot of the system. (T) (F)

15. Given the following characteristic equation of a linear control system, increasing the value of K will increase the frequency of oscillation of the system.

$$s^3 + 3s^2 + 5s + K = 0$$

(T) (F)

16. For the characteristic equation given in question 15, increasing the coefficient of the s^2 term will generally improve the damping of the system. (T) (F)

17. The location of the roots of the characteristic equation in the s-plane will give a definite indication on the maximum overshoot of the transient response of the system. (T) (F)

18. The following transfer function $G(s)$ can be approximated by $G_L(s)$ because the pole at -20 is much larger than the dominant pole at $s = -1$.

$$G(s) = \frac{10}{s(s+1)(s+20)} \qquad G_L(s) = \frac{10}{s(s+1)}$$

(T) (F)

19. What is a PD controller? Write its input–output transfer function.

20. A PD controller has the constants K_D and K_P. Give the effects of these constants on the steady-state error of the system. Does the PD control change the type of a system?

21. Give the effects of the PD control on rise time and settling time of a control system.

22. How does the PD controller affect the bandwidth of a control system?

23. Once the value of K_D of a PD controller is fixed, increasing the value of K_P will increase the phase margin monotonically. (T) (F)

24. If a PD controller is designed so that the characteristic-equation roots have better damping than the original system, then the maximum overshoot of the system is always reduced. **(T) (F)**

25. What does it mean when a control system is described as being robust?

26. A system compensated with a PD controller is usually more robust than the system compensated with a PI controller. **(T) (F)**

27. What is a PI controller? Write its input–output transfer function.

28. A PI controller has the constants K_P and K_I. Give the effects of the PI controller on the steady-state error of the system. Does the PI control change the system type?

29. Give the effects of the PI control on the rise time and settling time of a control system.

Answers to these review questions can be found on this book's companion Web site: www.wiley.com/college/golnaraghi.

➢ REFERENCES

1. J. C. Willems and S. K. Mitter, "Controllability, Observability, Pole Allocation, and State Reconstruction," *IEEE Trans. Automatic Control*, Vol. AC-16, pp. 582–595, Dec. 1971.
2. H. W. Smith and E. J. Davison, "Design of Industrial Regulators," *Proc. IEE (London)*, Vol. 119, pp. 1210–1216, Aug. 1972.
3. F. N. Bailey and S. Meshkat, "Root Locus Design of a Robust Speed Control," *Proc. Incremental Motion Control Symposium*, pp. 49–54, June 1983.
4. M. Vidyasagar, "On Undershoot and Nonminimum Phase Zeros," *IEEE Trans. Automatic Control*, Vol. AC-31, p. 440, May 1986.
5. T. Norimatsu and M. Ito, "On the Zero Non-Regular Control System," *J. Inst. Elec. Eng. Japan*, Vol. 81, pp. 567–575, 1961.
6. K. Ogata, *Modern Control Engineering*, 4th Ed., Prentice Hall, NJ, 2002.
7. G. F. Franklin and J. D. Powell, *Feedback Control of Dynamic Systems*, 5th Ed., Prentice-Hall, NJ, 2006.
8. J. J. Distefano, III, A. R. Stubberud, and I. J. Williams, *Schaum's Outline of Theory and Problems of Feedback and Control Systems*, 2nd Ed. McGraw-Hill, 1990.

➢ PROBLEMS

In addition to using the conventional approaches, use MATLAB to solve the problems in this chapter.

5-1. A pair of complex-conjugate poles in the s-plane is required to meet the various specifications that follow. For each specification, sketch the region in the s-plane in which the poles should be located.

(a) $\zeta \geq 0.707$	$\omega_n \geq 2$ rad/sec	(positive damping)
(b) $0 \leq \zeta \leq 0.707$	$\omega_n \leq 2$ rad/sec	(positive damping)
(c) $\zeta \leq 0.5$	$1 \leq \omega_n \leq 5$ rad/sec	(positive damping)
(d) $0.5 \leq \zeta \leq 0.707$	$\omega_n \leq 5$ rad/sec	(positive and negative damping)

5-2. Determine the type of the following unity-feedback systems for which the forward-path transfer functions are given.

(a) $G(s) = \dfrac{K}{(1+s)(1+10s)(1+20s)}$ **(b)** $G(s) = \dfrac{10e^{-0.2s}}{(1+s)(1+10s)(1+20s)}$

(c) $G(s) = \dfrac{10(s+1)}{s(s+5)(s+6)}$ **(d)** $G(s) = \dfrac{100(s-1)}{s^2(s+5)(s+6)^2}$

(e) $G(s) = \dfrac{10(s+1)}{s^3(s^2+5s+5)}$ (f) $G(s) = \dfrac{100}{s^3(s+2)^2}$

(g) $G(s) = \dfrac{5(s+2)}{s^2(s+4)}$ (h) $G(s) = \dfrac{8(s+1)}{(s^2+2s+3)(s+1)}$

5-3. Determine the step, ramp, and parabolic error constants of the following unity-feedback control systems. The forward-path transfer functions are given.

(a) $G(s) = \dfrac{1000}{(1+0.1s)(1+10s)}$ (b) $G(s) = \dfrac{100}{s(s^2+10s+100)}$

(c) $G(s) = \dfrac{K}{s(1+0.1s)(1+0.5s)}$ (d) $G(s) = \dfrac{100}{s^2(s^2+10s+100)}$

(e) $G(s) = \dfrac{1000}{s(s+10)(s+100)}$ (f) $G(s) = \dfrac{K(1+2s)(1+4s)}{s^2(s^2+s+1)}$

5-4. For the unity-feedback control systems described in Problem 5-2, determine the steady-state error for a unit-step input, a unit-ramp input, and a parabolic input, $(t^2/2)u_s(t)$. Check the stability of the system before applying the final-value theorem.

5-5. The following transfer functions are given for a single-loop nonunity-feedback control system. Find the steady-state errors due to a unit-step input, a unit-ramp input, and a parabolic input, $(t^2/2)u_s(t)$.

(a) $G(s) = \dfrac{1}{(s^2+s+2)}$ $H(s) = \dfrac{1}{(s+1)}$

(b) $G(s) = \dfrac{1}{s(s+5)}$ $H(s) = 5$

(c) $G(s) = \dfrac{1}{s^2(s+10)}$ $H(s) = \dfrac{s+1}{s+5}$

(d) $G(s) = \dfrac{1}{s^2(s+12)}$ $H(s) = 5(s+2)$

5-6. Find the steady-state errors of the following single-loop control systems for a unit-step input, a unit-ramp input, and a parabolic input, $(t^2/2)u_s(t)$. For systems that include a parameter K, find its value so that the answers are valid.

(a) $M(s) = \dfrac{s+4}{s^4+16s^3+48s^2+4s+4}, \ K_H = 1$

(b) $M(s) = \dfrac{K(s+3)}{s^3+3s^2+(K+2)s+3K}, \ K_H = 1$

(c) $M(s) = \dfrac{s+5}{s^4+15s^3+50s^2+10s}, \ H(s) = \dfrac{10s}{s+5}$

(d) $M(s) = \dfrac{K(s+5)}{s^4+17s^3+60s^2+5Ks+5K}, \ K_H = 1$

5-7. The output of the system shown in Fig. 5P-8 has a transfer function Y/X. Find the poles and zeros of the closed loop system and the system type.

5-8. Find the position, velocity, and acceleration error constants for the system given in Fig. 5P-8.

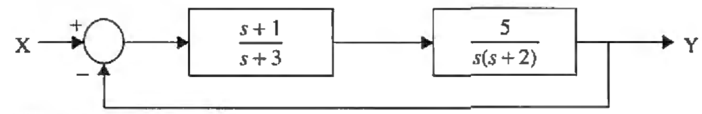


Figure 5P-8

5-9. Find the steady-state error for Problem 5-8 for (a) a unit-step input, (b) a unit-ramp input, and (c) a unit-parabolic input.

5-10. Repeat Problem 5-8 for the system given in Fig. 5P-10.

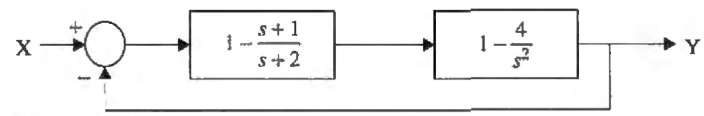


Figure 5P-10

5-11. Find the steady-state error of the system given in Problem 5-10 when the input is

$$X = \frac{5}{2s} - \frac{3}{s^2} + \frac{4}{s^3}$$

5-12. Find the rise time of the following first-order system:

$$G(s) = \frac{1-k}{s-k} \quad with\ |k| < 1$$

5-13. The block diagram of a control system is shown in Fig. 5P-13. Find the step-, ramp-, and parabolic-error constants. The error signal is defined to be $e(t)$. Find the steady-state errors in terms of K and K_t when the following inputs are applied. Assume that the system is stable.

(a) $r(t) = u_s(t)$

(b) $r(t) = tu_s(t)$

(c) $r(t) = (t^2/2)u_s(t)$

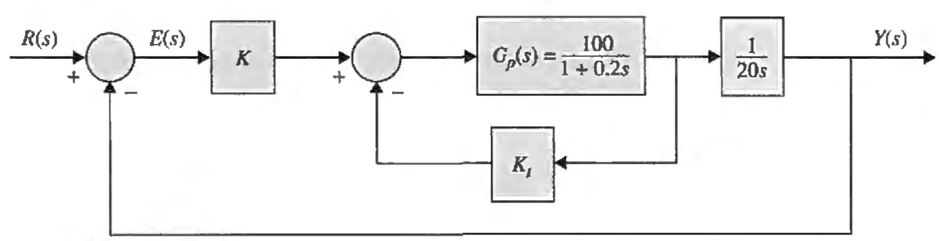


Figure 5P-13

5-14. Repeat Problem 5-13 when the transfer function of the process is, instead,

$$G_p(s) = \frac{100}{(1+0.1s)(1+0.5s)}$$

What constraints must be made, if any, on the values of K and K_t so that the answers are valid? Determine the minimum steady-state error that can be achieved with a unit-ramp input by varying the values of K and K_t.

5-15. For the position-control system shown in Fig. 3P-7, determine the following.

(a) Find the steady-state value of the error signal $\theta_e(t)$ in terms of the system parameters when the input is a unit-step function.

(b) Repeat part (a) when the input is a unit-ramp function. Assume that the system is stable.

5-16. The block diagram of a feedback control system is shown in Fig. 5P-16. The error signal is defined to be $e(t)$.

(a) Find the steady-state error of the system in terms of K and K_t when the input is a unit-ramp function. Give the constraints on the values of K and K_t so that the answer is valid. Let $n(t) = 0$ for this part.

(b) Find the steady-state value of $y(t)$ when $n(t)$ is a unit-step function. Let $r(t) = 0$. Assume that the system is stable.

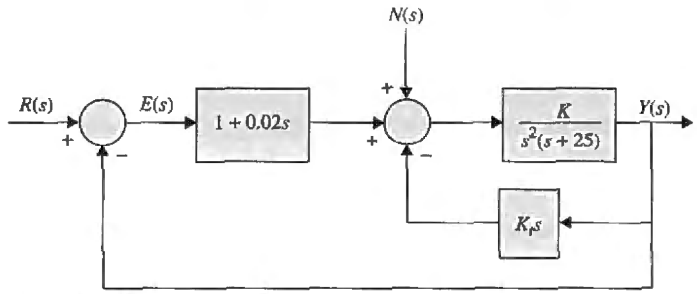


Figure 5P-16

5-17. The block diagram of a linear control system is shown in Fig. 5P-17, where $r(t)$ is the reference input and $n(t)$ is the disturbance.

(a) Find the steady-state value of $e(t)$ when $n(t) = 0$ and $r(t) = tu_s(t)$. Find the conditions on the values of α and K so that the solution is valid.

(b) Find the steady-state value of $y(t)$ when $r(t) = 0$ and $n(t) = u_s(t)$.

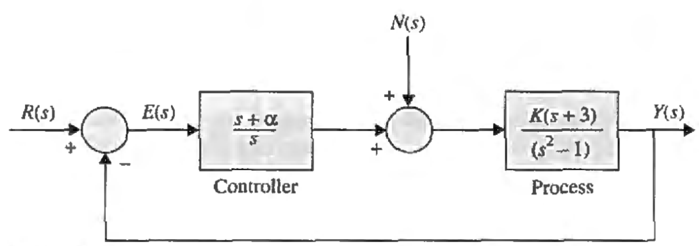


Figure 5P-17

5-18. The unit-step response of a linear control system is shown in Fig. 5P-18. Find the transfer function of a second-order prototype system to model the system.

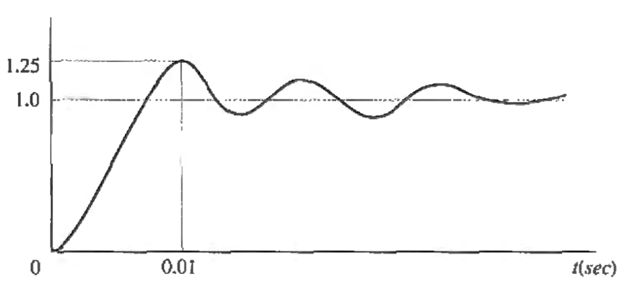


Figure 5P-18

5-19. For the control system shown in Fig. 5P-13, find the values of K and K_t so that the maximum overshoot of the output is approximately 4.3% and the rise time t_r is approximately 0.2 sec. Use Eq. (5-98) for the rise-time relationship. Simulate the system with any time-response simulation program to check the accuracy of your solutions.

5-20. Repeat Problem 5-19 with a maximum overshoot of 10% and a rise time of 0.1 sec.

5-21. Repeat Problem 5-19 with a maximum overshoot of 20% and a rise time of 0.05 sec.

5-22. For the control system shown in Fig. 5P-13, find the values of K and K_t so that the maximum overshoot of the output is approximately 4.3% and the delay time t_d is approximately 0.1 sec. Use Eq. (5-96) for the delay-time relationship. Simulate the system with a computer program to check the accuracy of your solutions.

5-23. Repeat Problem 5-22 with a maximum overshoot of 10% and a delay time of 0.05 sec.

5-24. Repeat Problem 5-22 with a maximum overshoot of 20% and a delay time of 0.01 sec.

5-25. For the control system shown in Fig. 5P-13, find the values of K and K_t so that the damping ratio of the system is 0.6 and the settling time of the unit-step response is 0.1 sec. Use Eq. (5-102) for the settling time relationship. Simulate the system with a computer program to check the accuracy of your results.

5-26. **(a)** Repeat Problem 5-25 with a maximum overshoot of 10% and a settling time of 0.05 sec.

(b) Repeat Problem 5-25 with a maximum overshoot of 20% and a settling time of 0.01 sec.

5-27. Repeat Problem 5-25 with a damping ratio of 0.707 and a settling time of 0.1 sec. Use Eq. (5-103) for the settling time relationship.

5-28. The forward-path transfer function of a control system with unity feedback is

$$G(s) = \frac{K}{s(s+a)(s+30)}$$

where a and K are real constants.

(a) Find the values of a and K so that the relative damping ratio of the complex roots of the characteristic equation is 0.5 and the rise time of the unit-step response is approximately 1 sec. Use Eq. (5-98) as an approximation of the rise time. With the values of a and K found, determine the actual rise time using computer simulation.

(b) With the values of a and K found in part (a), find the steady-state errors of the system when the reference input is (i) a unit-step function and (ii) a unit-ramp function.

5-29. The block diagram of a linear control system is shown in Fig. 5P-29.

(a) By means of trial and error, find the value of K so that the characteristic equation has two equal real roots and the system is stable. You may use any root-finding computer program to solve this problem.

(b) Find the unit-step response of the system when K has the value found in part (a). Use any computer simulation program for this. Set all the initial conditions to zero.

(c) Repeat part (b) when $K = -1$. What is peculiar about the step response for small t, and what may have caused it?

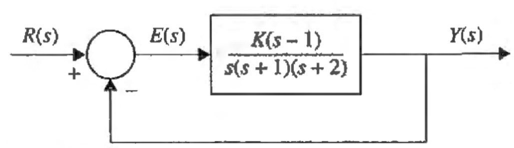


Figure 5P-29

5-30. A controlled process is represented by the following dynamic equations:

$$\frac{dx_1(t)}{dt} = -x_1(t) + 5x_2(t)$$
$$\frac{dx_2(t)}{dt} = -6x_1(t) + u(t)$$
$$y(t) = x_1(t)$$

The control is obtained through state feedback with

$$u(t) = -k_1x_1(t) - k_2x_2(t) + r(t)$$

where k_1 and k_2 are real constants, and $r(t)$ is the reference input.

(a) Find the locus in the k_1-versus-k_2 plane (k_1 = vertical axis) on which the overall system has a natural undamped frequency of 10 rad/sec.

(b) Find the locus in the k_1-versus-k_2 plane on which the overall system has a damping ratio of 0.707.

(c) Find the values of k_1 and k_2 such that $\zeta = 0.707$ and $\omega_n = 10$ rad/sec.

(d) Let the error signal be defined as $e(t) = r(t) - y(t)$. Find the steady-state error when $r(t) = u_s(t)$ and k_1 and k_2 are at the values found in part (c).

(e) Find the locus in the k_1-versus-k_2 plane on which the steady-state error due to a unit-step input is zero.

5-31. The block diagram of a linear control system is shown in Fig. 5P-31. Construct a parameter plane of K_p versus K_d (K_p is the vertical axis), and show the following trajectories or regions in the plane.

(a) Unstable and stable regions

(b) Trajectories on which the damping is critical ($\zeta = 1$)

(c) Region in which the system is overdamped ($\zeta > 1$)

(d) Region in which the system is underdamped ($\zeta < 1$)

(e) Trajectory on which the parabolic-error constant K_a is 1000 sec^{-2}

(f) Trajectory on which the natural undamped frequency ω_n is 50 rad/sec

(g) Trajectory on which the system is either uncontrollable or unobservable (hint: look for pole-zero cancellation)

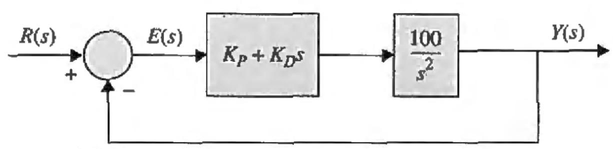


Figure 5P-31

5-32. The block diagram of a linear control system is shown in Fig. 5P-32. The fixed parameters of the system are given as $T = 0.1$, $J = 0.01$, and $K_i = 10$.

(a) When $r(t) = tu_s(t)$ and $T_d(t) = 0$, determine how the values of K and K_t affect the steady-state value of $e(t)$. Find the restrictions on K and K_t so that the system is stable.

(b) Let $r(t) = 0$. Determine how the values of K and K_t affect the steady-state value of $y(t)$ when the disturbance input $T_d(t) = u_s(t)$.

(c) Let $K_t = 0.01$ and $r(t) = 0$. Find the minimum steady-state value of $y(t)$ that can be obtained by varying K, when $T_d(t)$ is a unit-step function. Find the value of this K. From the transient standpoint, would you operate the system at this value of K? Explain.

(d) Assume that it is desired to operate the system with the value of K as selected in part (c). Find the value of K_t so that the complex roots of the characteristic equation will have a real part of -2.5. Find all three roots of the characteristic equation.

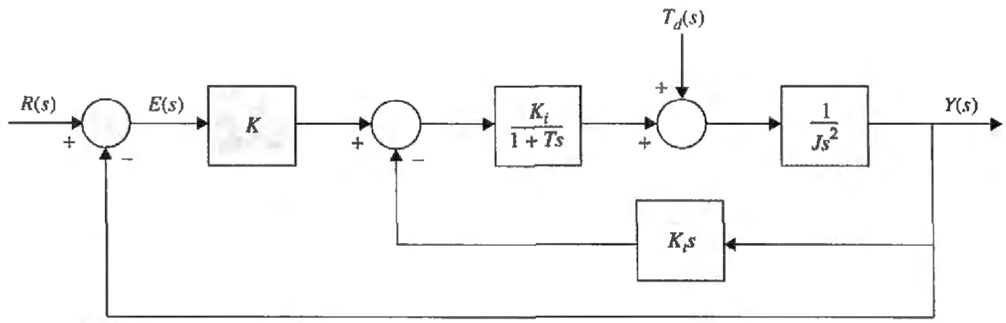


Figure 5P-32

5-33. Consider a second-order unity feedback system with $\zeta = 0.6$ and $\omega_n = 5$ rad/sec. Calculate the rise time, peak time, maximum overshoot, and settling time when a unit-step input is applied to the system.

5-34. Fig. 5P-34 shows the block diagram of a servomotor. Assume $J = 1$ kg-m^2 and $B = 1$ N-m/rad/sec. If the maximum overshoot of the unit-step input and the peak time are 0.2 and 0.1 sec., respectively,

(a) Find its damping ratio and natural frequency.

(b) Find the gain K and velocity feedback K_f. Also, calculate the rise time and settling time.

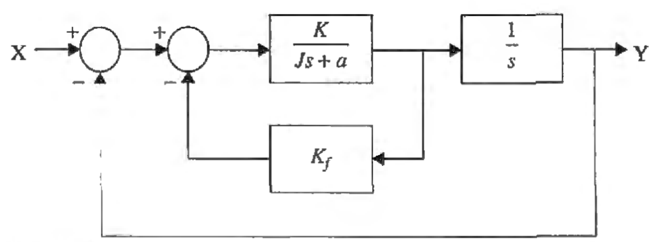


Figure 5P-34

5-35. Find the unit-step response of the following systems assuming zero initial conditions:

(a)
$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-1 & -1\\ 6.5 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$
$$\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

(b)
$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$$
$$y_1 = [1 \quad 0]\begin{bmatrix}x_1\\ x_2\end{bmatrix} + [0]u$$

(c)
$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ -1 & -1 & 0\\ 1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 1\\ 0\end{bmatrix}u$$
$$y = [0 \quad 0 \quad 1]\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

5-36. Use MATLAB to solve Problem 5-35.

5-37. Find the impulse response of the given systems in Problem 5-35.

5-38. Use MATLAB to solve Problem 5-37.

5-39. Fig. 5P-39 shows a mechanical system.

(a) Find the differential equation of the system.

(b) Use MATLAB to find the unit-step input response of the system.

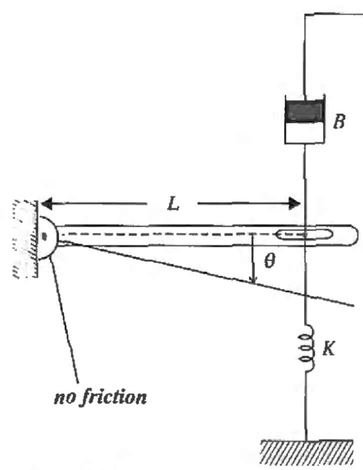


Figure 5P-39

5-40. The dc-motor control system for controlling a printwheel described in Problem 4-49 has the forward-path transfer function

$$G(s) = \frac{\Theta_o(s)}{\Theta_e(s)} = \frac{nK_sK_iK_LK}{\Delta(s)}$$

where $\Delta(s) = s\big[L_aJ_mJ_Ls^4 + J_L(R_aJ_m + B_mL_a)s^3$

$+ (n^2K_LL_aJ_L + K_LL_aJ_m + K_iK_bJ_L + R_aB_mJ_L)s^2$

$+ (n^2R_aK_LJ_L + R_aK_LJ_m + B_mK_LL_a)s + R_aB_mK_L + K_iK_bK_L\big]$

where $K_i = 9$ oz-in./A, $K_b = 0.636$ V/rad/sec, $R_a = 5\Omega$, $L_a = 1$ mH, $K_s = 1$ V/rad, $n = 1/10$, $J_m = J_L =$ 0.001 oz-in.-sec^2, and $B_m \cong 0$. The characteristic equation of the closed-loop system is

$$\Delta(s) + nK_sK_iK_LK = 0$$

(a) Let $K_L = 10{,}000$ oz-in./rad. Write the forward-path transfer function $G(s)$ and find the poles of $G(s)$. Find the critical value of K for the closed-loop system to be stable. Find the roots of the characteristic equation of the closed-loop system when K is at marginal stability.

(b) Repeat part (a) when $K_L = 1000$ oz-in./rad.

(c) Repeat part (a) when $K_L = \infty$; that is, the motor shaft is rigid.

(d) Compare the results of parts (a), (b), and (c), and comment on the effects of the values of K_L on the poles of $G(s)$ and the roots of the characteristic equation.

5-41. The block diagram of the guided-missile attitude-control system described in Problem 4-20 is shown in Fig. 5P-41. The command input is $r(t)$, and $d(t)$ represents disturbance input. The objective of this problem is to study the effect of the controller $G_c(s)$ on the steady-state and transient responses of the system.

(a) Let $G_c(s) = 1$. Find the steady-state error of the system when $r(t)$ is a unit-step function. Set $d(t) = 0$.

(b) Let $G_c(s) = (s + \alpha)/s$. Find the steady-state error when $r(t)$ is a unit-step function.

(c) Obtain the unit-step response of the system for $0 \le t \le 0.5$ sec with $G_c(s)$ as given in part (b) and $\alpha = 5$, 50, and 500. Assume zero initial conditions. Record the maximum overshoot of $y(t)$ for each case. Use any available computer simulation program. Comment on the effect of varying the value of α of the controller on the transient response.

(d) Set $r(t) = 0$ and $G_c(s) = 1$. Find the steady-state value of $y(t)$ when $d(t) = u_s(t)$.

(e) Let $G_c(s) = (s + \alpha)/s$. Find the steady-state value of $y(t)$ when $d(t) = u_s(t)$.

(f) Obtain the output response for $0 \le t \le 0.5$ sec, with $G_c(s)$ as given in part (e) when $r(t) = 0$ and $d(t) = u_s(t)$; $\alpha = 5$, 50, and 500. Use zero initial conditions.

(g) Comment on the effect of varying the value of α of the controller on the transient response of $y(t)$ and $d(t)$.

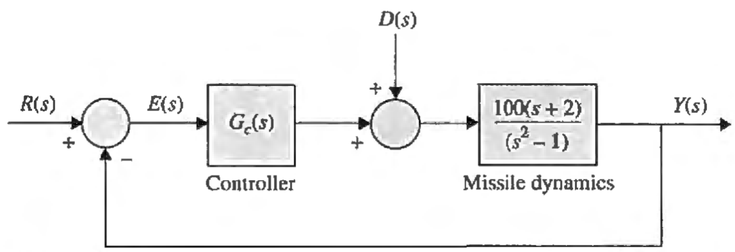


Figure 5P-41

5-42. The block diagram shown in Fig. 5P-42 represents a liquid-level control system. The liquid level is represented by $h(t)$, and N denotes the number of inlets.

(a) Because one of the poles of the open-loop transfer function is relatively far to the left on the real axis of the s-plane at $s = -10$, it is suggested that this pole can be neglected. Approximate the system by a second-order system by neglecting the pole of $G(s)$ at $s = -10$. The approximation should be valid for both the transient and the steady-state responses. Apply the formulas for the maximum overshoot and the peak time t_{max} to the second-order model for $N = 1$ and $N = 10$.

(b) Obtain the unit-step response (with zero initial conditions) of the original third-order system with $N = 1$ and then with $N = 10$. Compare the responses of the original system with those of the second-order approximating system. Comment on the accuracy of the approximation as a function of N.

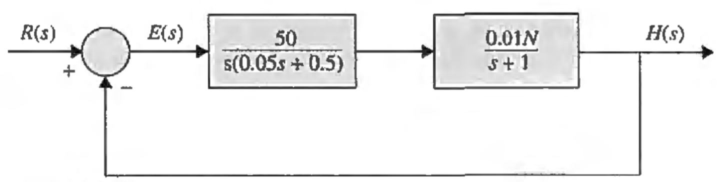


Figure 5P-42

5-43. The forward-path transfer function of a unity-feedback control system is

$$G(s) = \frac{1 + T_z s}{s(s+1)^2}$$

Compute and plot the unit-step responses of the closed-loop system for $T_z = 0$, 0.5, 1.0, 10.0, and 50.0. Assume zero initial conditions. Use any computer simulation program that is available. Comment on the effects of the various values of T_z on the step response.

5-44. The forward-path transfer function of a unity-feedback control system is

$$G(s) = \frac{1}{s(s+1)^2(1+T_p s)}$$

Compute and plot the unit-step responses of the closed-loop system for $T_p = 0, 0.5$, and 0.707. Assume zero initial conditions. Use any computer simulation program. Find the critical value of T_p so that the closed-loop system is marginally stable. Comment on the effects of the pole at $s = -1/T_p$ in $G(s)$.

5-45. Compare and plot the unit-step responses of the unity-feedback closed-loop systems with the forward-path transfer functions given. Assume zero initial conditions.

(a) $G(s) = \dfrac{1+T_z s}{s(s+0.55)(s+1.5)}$ For $T_z = 0,\ 1,\ 5,\ 20$

(b) $G(s) = \dfrac{1+T_z s}{(s^2+2s+2)}$ For $T_z = 0,\ 1,\ 5,\ 20$

(c) $G(s) = \dfrac{2}{(s^2+2s+2)(1+T_p s)}$ For $T_p = 0,\ 0.5,\ 1.0$

(d) $G(s) = \dfrac{10}{s(s+5)(1+T_p s)}$ For $T_p = 0,\ 0.5,\ 1.0$

(e) $G(s) = \dfrac{K}{s(s+1.25)(s^2+2.5s+10)}$

- **(i)** For $K = 5$
- **(ii)** For $K = 10$
- **(iii)** For $K = 30$

(f) $G(s) = \dfrac{K(s+2.5)}{s(s+1.25)(s^2+2.5s+10)}$

- **(i)** For $K = 5$
- **(ii)** For $K = 10$
- **(iii)** For $K = 30$

5-46. Fig. 5P-46 shows the block diagram of a servomotor with tachometer feedback.

(a) Find the error signal $E(s)$ in the presence of the reference input $X(s)$ and disturbance input $D(s)$.

(b) Calculate the steady-state error of the system when $X(s)$ is a unit ramp and D(s) is a unit step.

(c) Use MATLAB to plot the response of the system for part (b).

(d) Use MATLAB to plot the response of the system when $X(s)$ is a unit-step input and $D(s)$ is a unit impulse input.

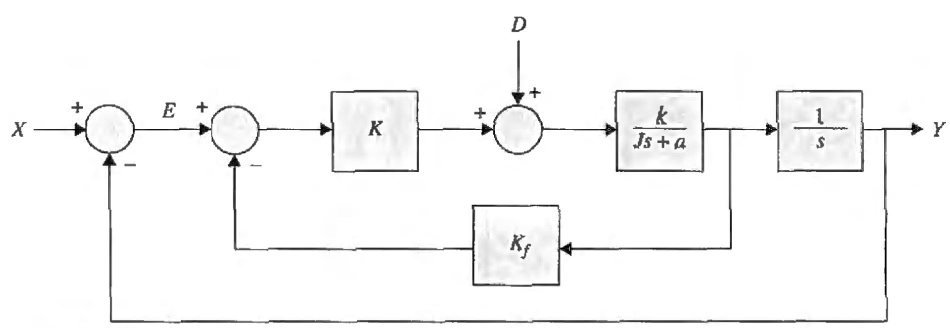


Figure 5P-46

5-47. The feedforward transfer function of a stable unity feedback system is $G(s)$. If the closed-loop transfer function can be rewritten as

$$\frac{Y(s)}{X(s)} = \frac{G(s)}{1+G(s)} = \frac{(A_1 s+1)(A_2 s+1)\ldots(A_n s+1)}{(B_1 s+1)(B_2 s+1)\ldots(B_m s+1)}$$

(a) Find the steady-state error to a unit-step input.

(b) Calculate $\frac{1}{K} = \frac{1}{\lim_{s\to 0} sG(s)}$.

5-48. If the maximum overshoot and 1% settling time of the unit-step response of the closed-loop system shown in Fig. 5P-48 are no more than 25% and 0.1 sec, find the gain K and pole location P of the compensator. Also, use MATLAB to plot the unit-step input response of the system and verify your controller design.

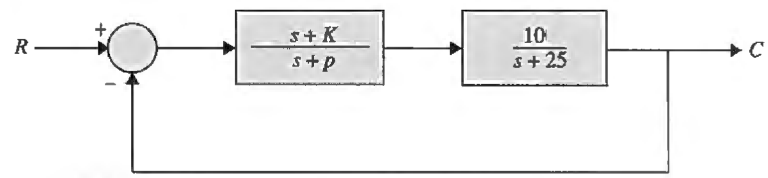


Figure 5P-48

5-49. If a given second-order system is required to have a peak time less than t, find the region in the s-plane corresponding to the poles that meet this specification.

5-50. A unity feedback control system shown in Fig. 5P-50(a) is designed so that its closed-loop poles lie within the region shown in Fig. 5P-50(b).

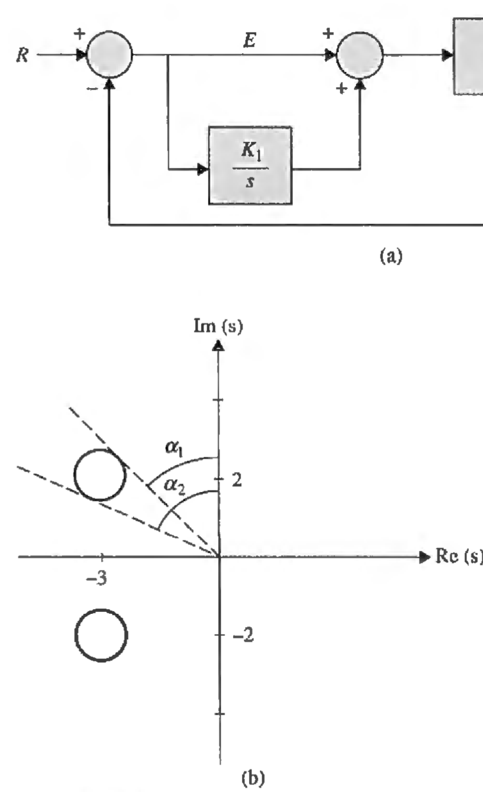


Figure 5P-50

(a) Find the values for ω_n and ζ.

(b) If $K_p = 2$ and $P = 2$, then find the values for K and K_I.

(c) Show that, regardless of values K_p and p, the controller can be designed to place the poles anywhere in the left side of the s-plane.

5-51. The equation of a dc motor is given by

$$J_m \ddot{\theta}_m + \left(B + \frac{K_1 K_2}{R}\right)\dot{\theta}_m = \frac{K_1}{R} v$$

Assuming $J_m = 0.02$ kg-m^2, B = 0.002 N-m-sec, $K_1 = 0.04$ N-m/A, $K_2 = 0.04$ V-sec, and $R = 20\,\Omega$.

(a) Find the transfer function between the applied voltage and the motor speed.

(b) Calculate the steady-state speed of the motor after applying a voltage of 10 V.

(c) Determine the transfer function between the applied voltage and the shaft angle θ_m.

(d) Including a closed-loop feedback to part (c) such that $v = K(\theta_p - \theta_m)$, where K is the feedback gain, obtain the transfer function between θ_p and θ_m.

(e) If the maximum overshoot is less than 25%, determine K.

(f) If the rise time is less than 3 sec, determine K.

(g) Use MATLAB to plot the step response of the position servo system for $K = 0.5, 1.0$, and 2.0. Find the rise time and overshoot.

5-52. In the unity feedback closed-loop system in a configuration similar to that in Fig. 5P-48, the plant transfer function is

$$G(s) = \frac{1}{s(s+3)}$$

and the controller transfer function is

$$H(s) = \frac{k(s+a)}{(s+b)}$$

Design the controller parameters so that the closed-loop system has a 10% overshoot for a unit step input and a 1% settling time of 1.5.sec.

5-53. An autopilot is designed to maintain the pitch attitude α of an airplane. The transfer function between pitch angle α and elevator angle β are given by

$$\frac{\alpha(s)}{\beta(s)} = \frac{60(s+1)(s+2)}{(s^2+6s+40)(s^2+0.04s+0.07)}$$

The autopilot pitch controller uses the pitch error e to adjust the elevator as

$$\frac{\beta(s)}{E(s)} = \frac{K(s+3)}{s+10}$$

Use a unity feedback configuration and utilize MATLAB to find K with an overshoot of less than 10% and a rise time faster than 0.5 sec for a unit-step input. Explain controller design difficulties for complex systems.

5-54. The block diagram of a control system with a series controller is shown in Fig. 5P-54. Find the transfer function of the controller $G_c(s)$ so that the following specifications are satisfied:

(a) The ramp-error constant K_v is 5.

(b) The closed-loop transfer function is of the form

$$M(s) = \frac{Y(s)}{R(s)} = \frac{K}{(s^2+20s+200)(s+a)}$$

where K and a are real constants. Use MATLAB to find the values of K and a, and confirm the results.

The design strategy is to place the closed-loop poles at $-10 + j10$ and $-10 - j10$, and then adjust the values of K and a to satisfy the steady-state requirement. The value of a is large so that it will not affect the transient response appreciably. Find the maximum overshoot of the designed system.

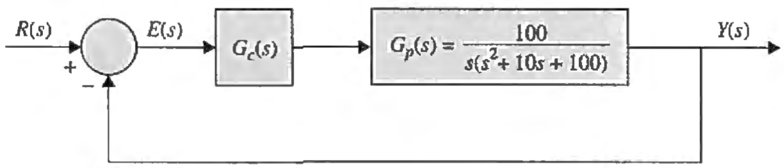


Figure 5P-54

5-55. Repeat Problem 5-54 if the ramp-error constant is to be 9. What is the maximum value of K_v that can be realized? Comment on the difficulties that may arise in attempting to realize a very large K_v.

5-56. A control system with a PD controller is shown in Fig. 5P-56. Use MATLAB to

(a) Find the values of K_P and K_D so that the ramp-error constant K_v is 1000 and the damping ratio is 0.5.

(b) Find the values of K_P and K_D so that the ramp-error constant K_v is 1000 and the damping ratio is 0.707.

(c) Find the values of K_P and K_D so that the ramp-error constant K_v is 1000 and the damping ratio is 1.0.

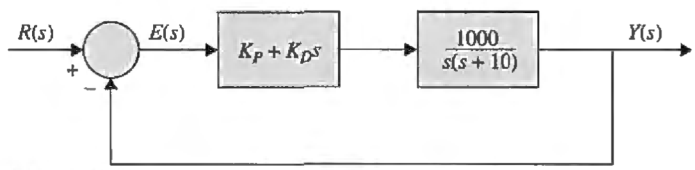


Figure 5P-56

5-57. For the control system shown in Figure 5P-56, set the value of K_P so that the ramp-error constant is 1000. Use MATLAB to

(a) Vary the value of K_D from 0.2 to 1.0 in increments of 0.2 and determine the values of rise time and maximum overshoot of the system.

(b) Vary the value of K_D from 0.2 to 1.0 in increments of 0.2 and find the value of K_D so that the maximum overshoot is minimum.

5-58. Consider the second-order model of the aircraft attitude control system shown in Fig. 5-29. The transfer function of the process is $G_P(s) = \frac{4500K}{s(s+361.2)}$. Use MATLAB to design a series PD controller with the transfer function $G_c(s) = K_P + K_D s$ so that the following performance specifications are satisfied:

Steady-state error due to a unit-ramp input ≤ 0.001

Maximum overshoot $\leq 5\%$

Rise time $t_r \leq 0.005$ sec

Settling time $t_s \leq 0.005$ sec

5-59. Fig. 5P-59 shows the block diagram of the liquid-level control system described in Problem 5-42. The number of inlets is denoted by N. Set $N = 20$. Use MATLAB to design the PD controller so that with a unit-step input the tank is filled to within 5% of the reference level in less than 3 sec without overshoot.

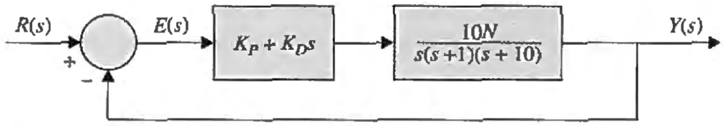


Figure 5P-59

5-60. For the liquid-level control system described in Problem 5-59, set K_P so that the ramp-error constant is 1. Use MATLAB to vary K_D from 0 to 0.5 and determine the values of rise time and maximum overshoot of the system.

5-61. A control system with a type 0 process $G_p(s)$ and a PI controller is shown in Fig. 5P-61. Use MATLAB to

(a) Find the value of K_I so that the ramp-error constant K_v is 10.

(b) Find the value of K_P so that the magnitude of the imaginary parts of the complex roots of the characteristic equation of the system is 15 rad/sec. Find the roots of the characteristic equation.

(c) Sketch the root contours of the characteristic equation with the value of K_I as determined in part (a) and for $0 \leq K_P < \infty$.

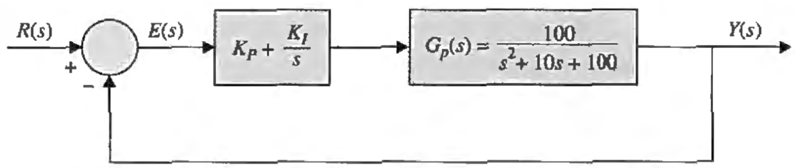


Figure 5P-61

5-62. For the control system described in Problem 5-61, set K_I so that the ramp-error constant is 10. Use MATLAB to vary K_P and determine the values of rise time and maximum overshoot of the system.

5-63. For the control system shown in Fig. 5P-61, use MATLAB to perform the following:

(a) Find the value of K_I so that the ramp-error constant K_v is 100.

(b) With the value of K_I found in part (a), find the critical value of K_P so that the system is stable. Sketch the root contours of the characteristic equation for $0 \leq K_P < \infty$.

(c) Show that the maximum overshoot is high for both large and small values of K_P. Use the value of K_I found in part (a). Find the value of K_P when the maximum overshoot is a minimum. What is the value of this maximum overshoot?

5-64. Repeat Problem 5-63 for $K_v = 10$.

5-65. A control system with a type 0 process and a PID controller is shown in Fig. 5P-65. Use MATLAB to design the controller parameters so that the following specifications are satisfied:

Ramp-error constant $K_v = 100$

Rise time $t_r < 0.01$ sec.

Maximum overshoot $< 2\%$

Plot the unit-step response of the designed system.

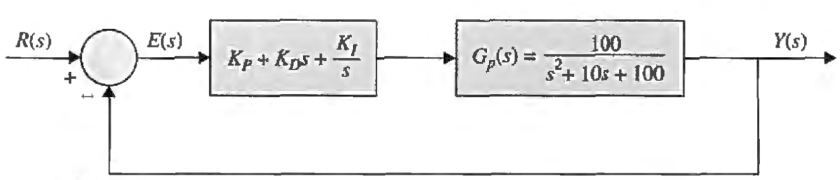


Figure 5P-65

5-66. Consider the quarter-car model of vehicle suspension systems in Example 4-11-3. The Laplace transform between the base acceleration and displacement is given by

$$\frac{Z(s)}{\ddot{Y}(s)} = \frac{-1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

(a) It is desired to design a proportional controller. Use MATLAB to design the controller parameters where the rise time is no more than 0.05 sec and the overshoot is no more than 3%. Plot the unit-step response of the designed system.

(b) It is desired to design a PD controller. Use MATLAB to design the controller parameters where the rise time is no more than 0.05 sec and the overshoot is no more than 3%. Plot the unit-step response of the designed system.

(c) It is desired to design a PI controller. Use MATLAB to design the controller parameters where the rise time is no more than 0.05 sec and the overshoot is no more than 3%. Plot the unit-step response of the designed system.

(d) It is desired to design a PID controller. Use MATLAB to design the controller parameters where the rise time is no more than 0.05 sec and the overshoot is no more than 3%. Plot the unit-step response of the designed system.

5-67. Consider the spring-mass system shown in Fig. 5P-67.

Its transfer function is given by $\frac{Y(s)}{F(s)} = \frac{1}{Ms^2+Bs+K}$.

Repeat Problem 5-66 where $M = 1$ kg, B = 10 N.s/m, $K = 20$ N/m.

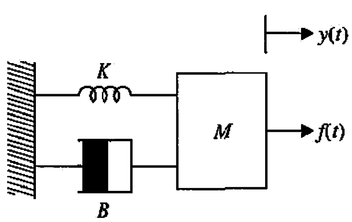


Figure 5P-67

5-68. Consider the vehicle suspension system hitting a bump described in Problem 4-3. Use MATLAB to design a proportional controller where the 1% settling time is less than 0.1 sec and the overshoot is no more than 2%. Assume $m = 25$ kg, $J = 5$ kg-m^2, $K = 100$ N/m, and $r = 0.35$ m. Plot the impulse response of the system.

5-69. Consider the train system described in Problem 4-6. Use MATLAB to design a proportional controller where the peak time is less than 0.05 sec and the overshoot is no more than 4%. Assume $M = 1$ kg, $m = 0.5$ kg, $k = 1$ N/m, $\mu = 0.002$ sec/m, and $g = 9.8$ m/s^2.

5-70. Consider the inverted pendulum described in Problem 4-9, where $M = 0.5$ kg, $m = 0.2$ kg, $\mu = 0.1$ N/m/sec (friction of the cart), $I = 0.006$ kg-m^2, $g = 9.8$ m/s^2, and 1 = 0.3 m.
Use MATLAB to design a PD controller where the rise time is less than 0.2 sec and the overshoot is no more than 10%.