Bifurcation diagrams

PEF5737 - Non-linear dynamics and stability

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1. Objective
2. Algorithm
3. Application to a forced Duffing oscillator

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- To describe a numerical algorithm for obtaining bifurcation diagrams;
- Application of the algorithm to a force Duffing oscillator.

1 Objective

2 Algorithm

3 Application to a forced Duffing oscillator

- A bifurcation diagram offers an idea of the behavior of a dynamical system when the parameters of the mathematical model are varied;
- It also allows identifying bifurcations and chaotic behaviors.
- There is no unique algorithm. This class will present **an** algorithm.

- Firstly, you need a mathematical model (equation of motion) for a particular physical model;
- Your mathematical model is defined by a number of parameters such as mass, damping, stiffness, amplitude/frequency of the external excitation;
- In a number of situations, you are interested in investigating the influence of one parameter of the mathematical model on the response.

Algorithm

- We define a Poincaré's section. For a periodically forced system, the obvious one is that characterized by the stroboscopic frequency of the forcing;
- Create a scheme for numerically integrating the mathematical model when for a given set of parameters of the mathematical model;
- Carry out some tests with the mathematical model. Try to identify the time spanned by the transitory responses. This is important because we will compute the Poincaré's section after the transitory.

Algorithm

- Define a vector containing the values of the parameter you would like to consider in the bifurcation diagram. Finer discretization leads to a better diagram;
- Now, it is time to compute the diagrams. Assume an initial condition for the mathematical model and run the simulation considering the first value of the parameter to be varied (p_1);
- For p_1, plot, in the ordinate axis, the points of the Poincaré's section for one coordinate of the mathematical model (say, x_1). Do not plot the points associated with the transitory response.
- It is important to emphasize that the plotted points do not necessarily means the maximum value of the response. However, provided you have the points in the Poincaré's section for all coordinates of the mathematical model, you can integrate until the maximum displacement is achieved (i.e., integrate until the velocity vanishes). In MATLAB®, use the event location command.

Algorithm

- Some alternatives are found for running the mathematical model for p_2, i.e., the second value of the parameter to be varied. In this algorithm, we use as initial condition for running the simulation with p_2 the last point obtained when the model was run with p_1;
- Repeat the plot procedure, now with p_2 in the abscissa axis and the points in the Poincaré's section in the ordinate axis;
- Run the mathematical model for p_3 using as initial condition the last point obtained when the mathematical model was run with p_2;
- Repeat until the last value p_n is simulated.

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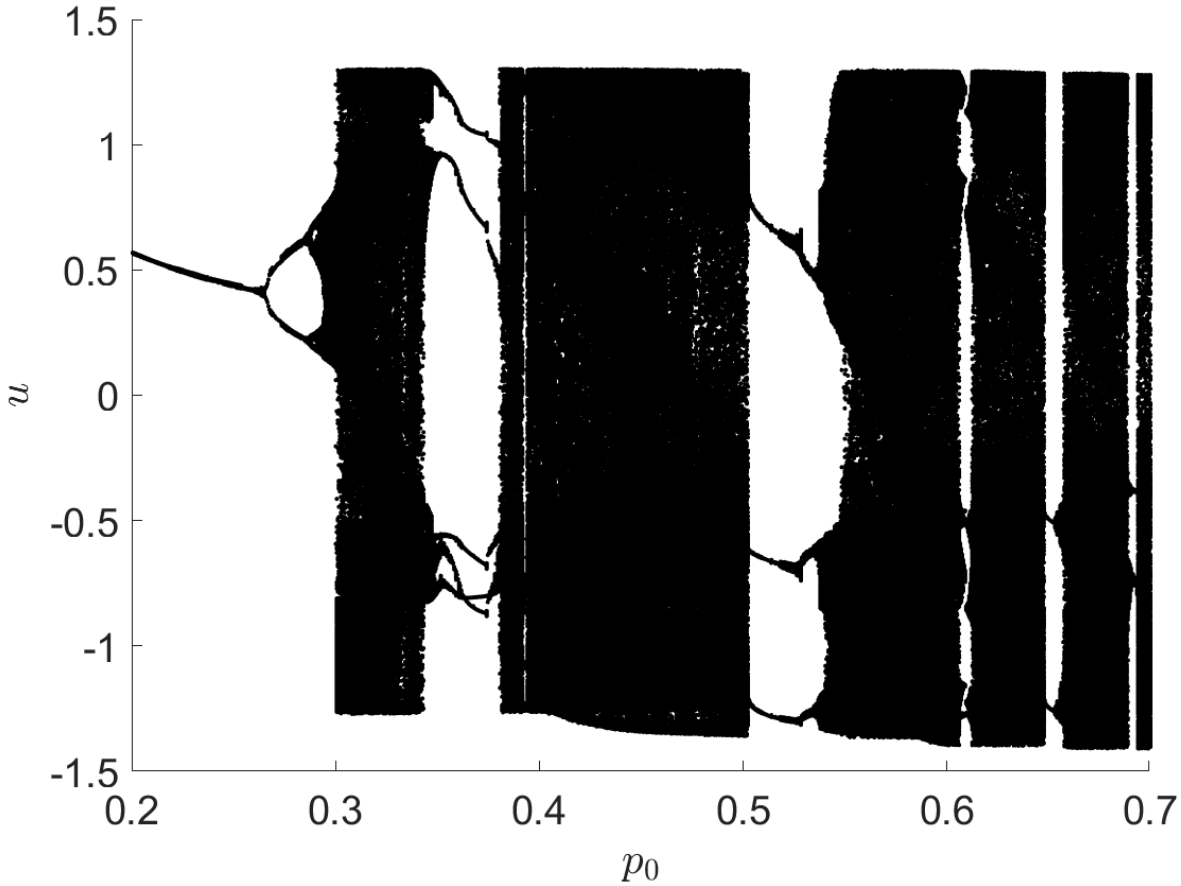
Mathematical model

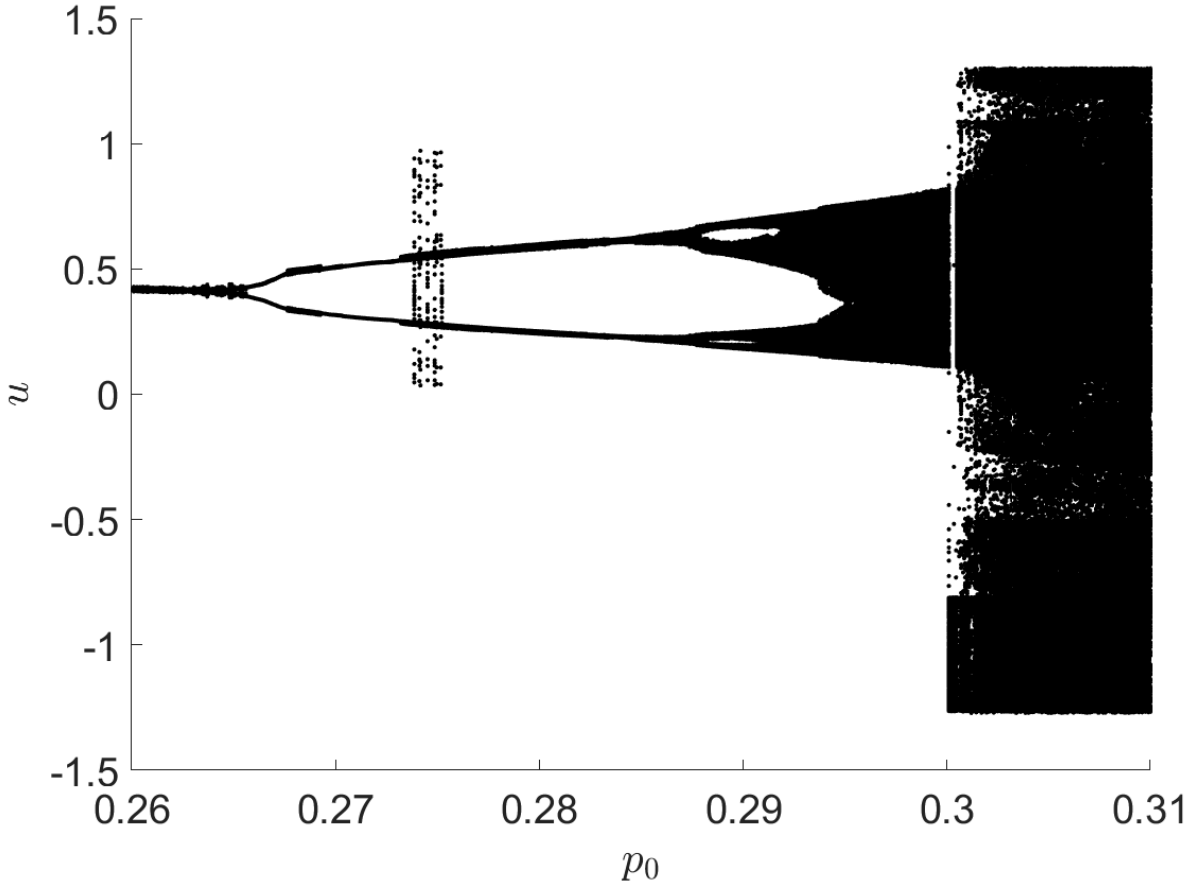
- Force Duffing oscillator: $\ddot{u} + \delta\dot{u} + \alpha u + \beta u^3 = p_0 \cos\omega t$;
- We will investigate the influence of the amplitude of the external excitation in the interval $0.2 \leq p_0 \leq 0.7$;
- The remaining parameters are fixed: $\delta = 0.30, \alpha = -1, \beta = 1, \omega = 1.2$;
- 800 points are considered in the Poincaré's section after disregarding the first 300 forcing cycles;
- The initial condition employed for running the mathematical model with $p_0 = 0.2$ was $u(0) = 1$ and $\dot{u}(0) = 0$.

Results - Bifurcation diagram

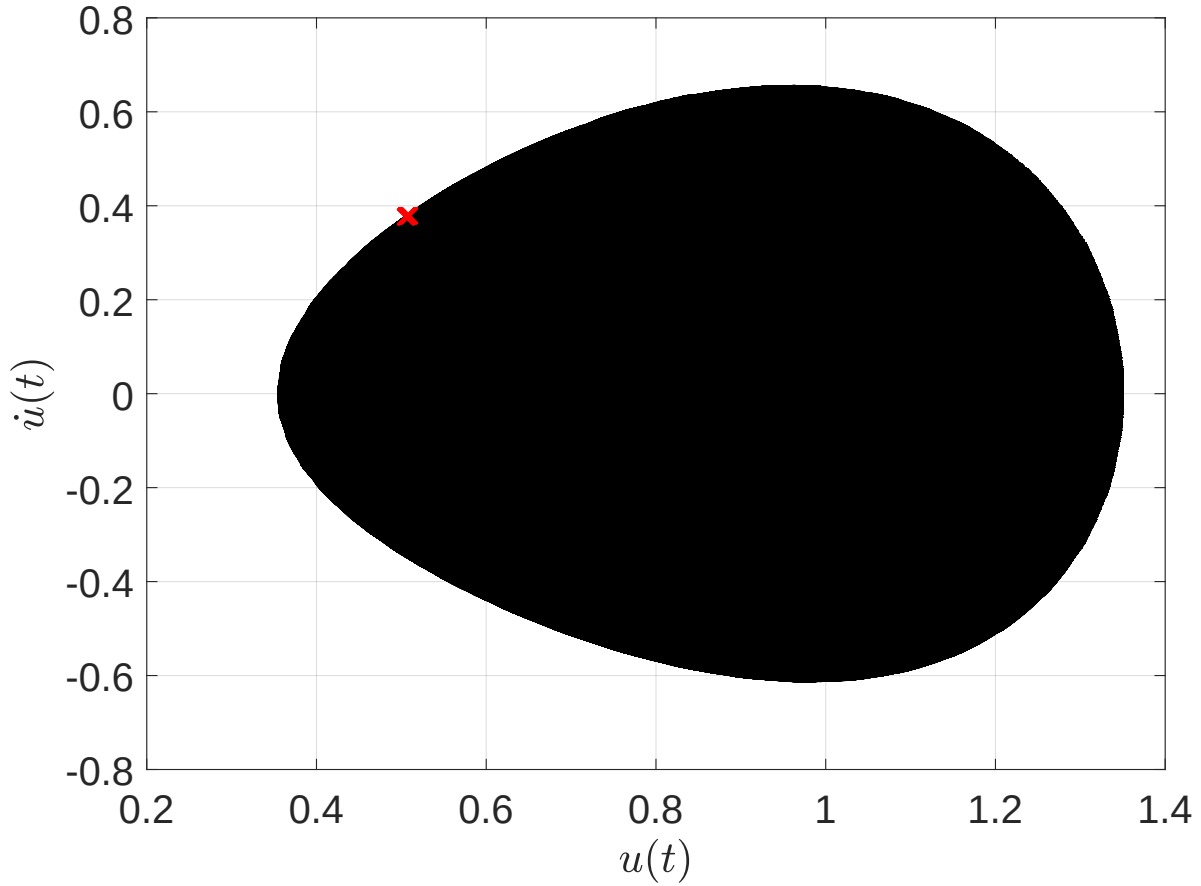


- It is possible to identify values of p_0 in which bifurcations occur. Notice that close to p_0, a period-doubling bifurcation is obtained;
- We emphasize that the indicated points are not necessarily the maximum displacements.

Bifurcation diagram - a closer look

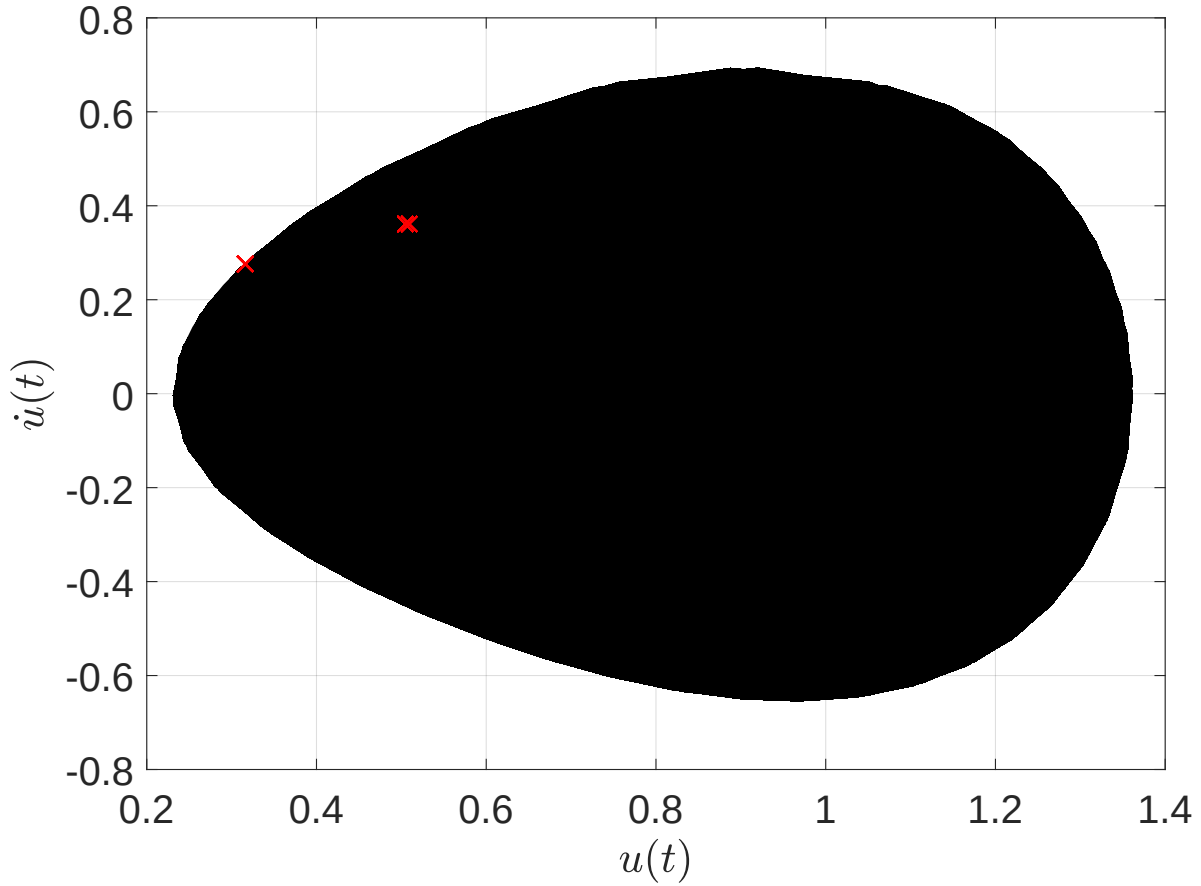


Example of periodic orbit: $p_0 = 0.22$



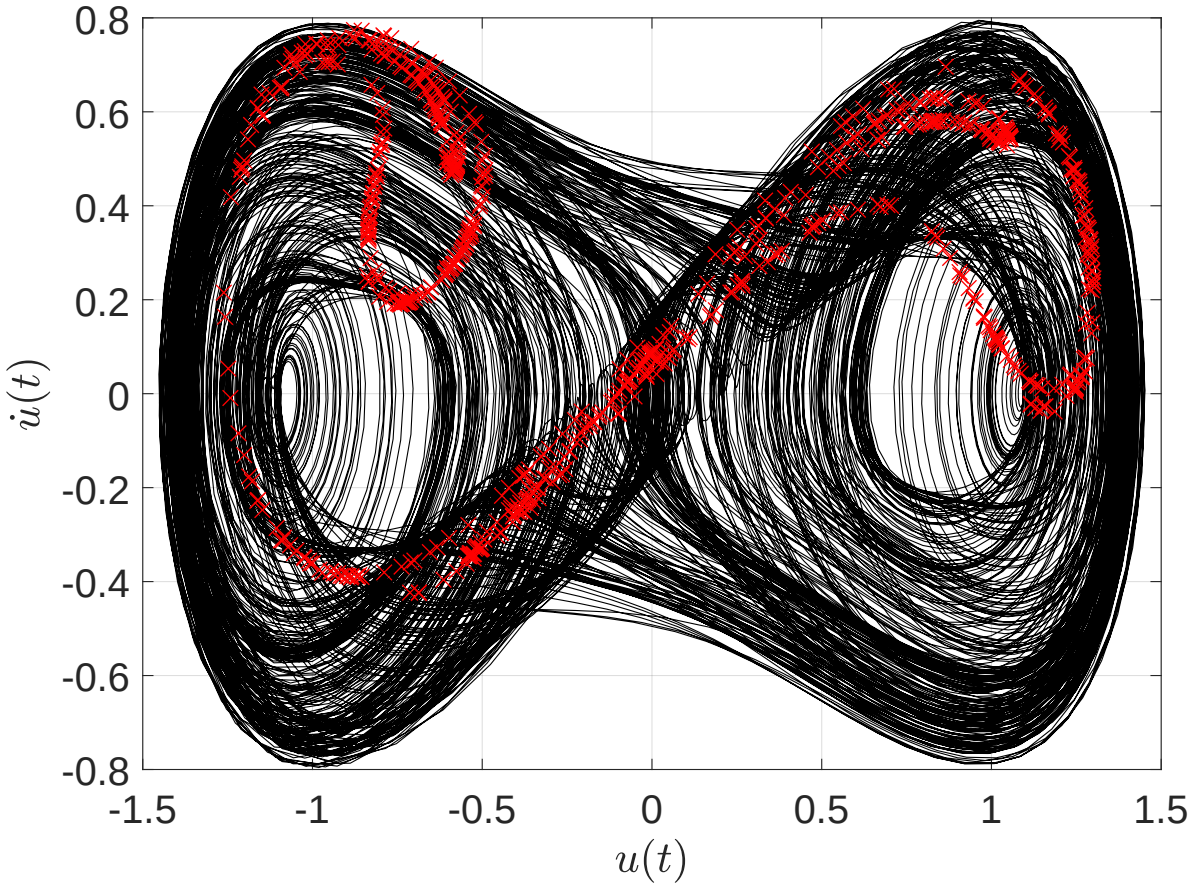
- Period-1 orbit. Red crosses indicate points in the Poincaré's section.

Example of periodic orbit: $p_0 = 0.27$



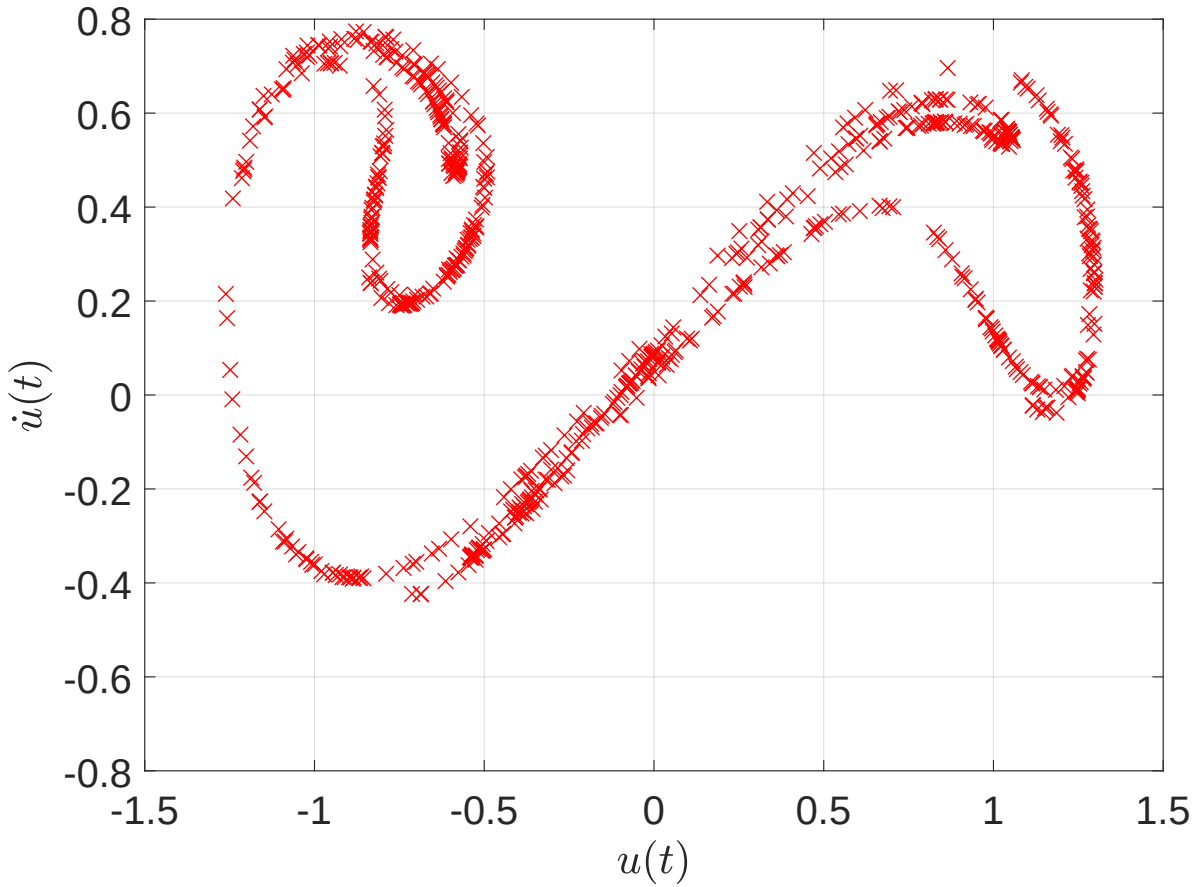
- Period-2 orbit. Red crosses indicate points in the Poincaré's section.

Example of periodic orbit: $p_0 = 0.32$

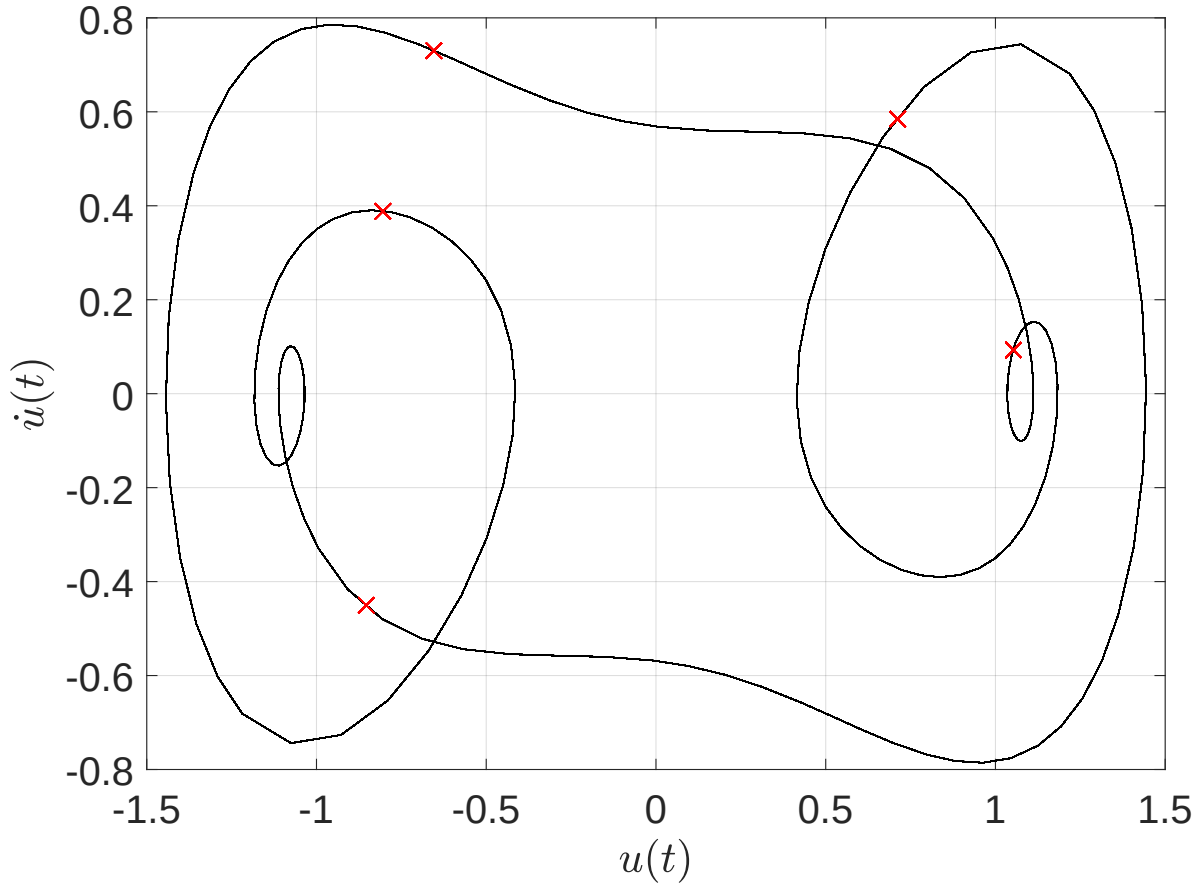


- Chaotic attractor? Red crosses indicate points in the Poincaré's section.

Example of periodic orbit: $p_0 = 0.32$

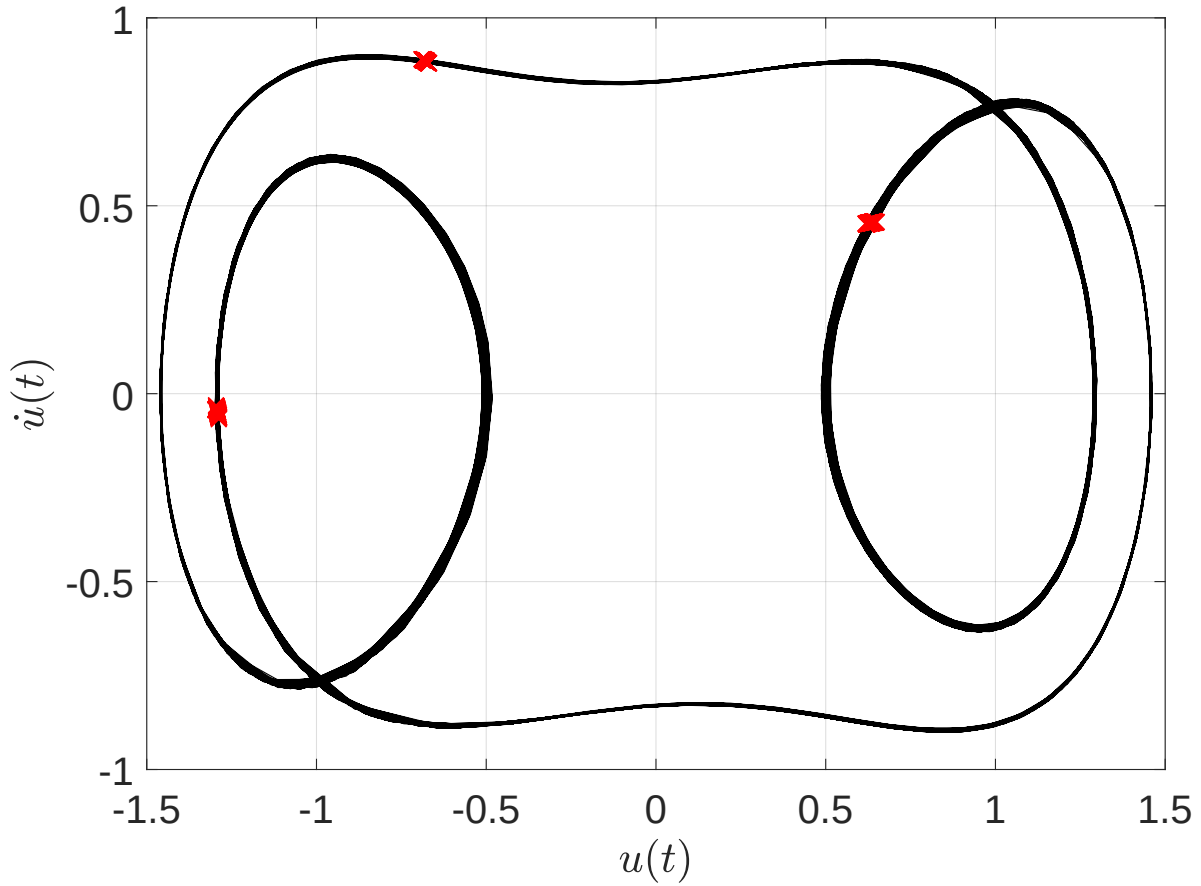


Example of periodic orbit: $p_0 = 0.37$



- Period-5 orbit. Red crosses indicate points in the Poincaré's section.

Example of periodic orbit: $p_0 = 0.52$



- Period-3 orbit. Red crosses indicate points in the Poincaré's section.