

# LOM3203 – Controle e Automação

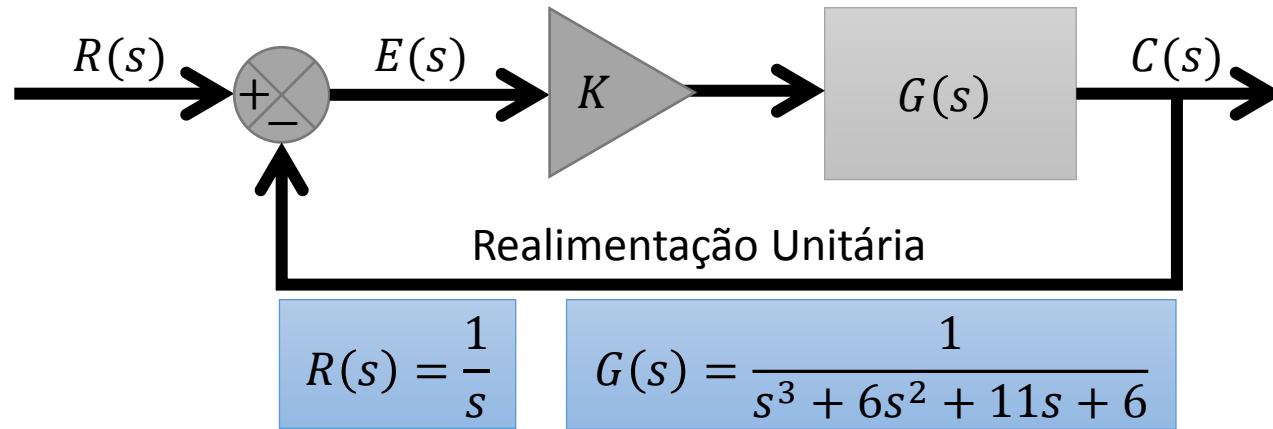
## AULA 5

Prof. Dr. Emerson G. Melo

- ❑ Controle Proporcional (P)
- ❑ Controle Proporcional-Integral (PI)
- ❑ Controle Proporcional-Derivativo (PD)
- ❑ Controle Proporcional-Integral-Derivativo (PID)
- ❑ Sintonia de Controladores PID
- ❑ Sistemas de Segunda Ordem
- ❑ Requisitos de Desempenho
- ❑ Projeto de Controlador PID

# Controle Proporcional (P)

□ A utilização de controle proporcional possibilita diminuir o erro estacionário!!!



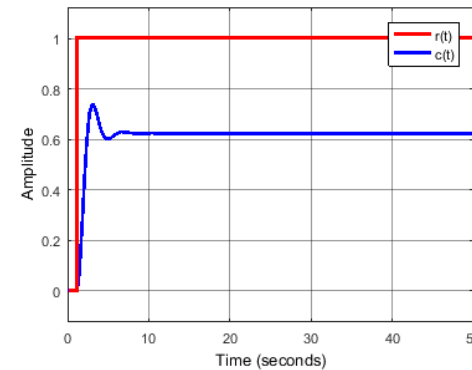
Erro estacionário

$$e_{ss} = \lim_{s \rightarrow 0} \frac{s}{1 + KG(s)} R(s)$$

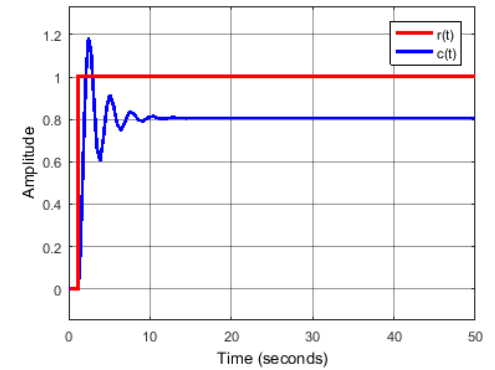
$$e_{ss} = \lim_{s \rightarrow 0} \frac{s}{1 + K(s^3 + 6s^2 + 11s + 6)} \frac{1}{s}$$

$$e_{ss} = \frac{1}{1 + (1/6)K}$$

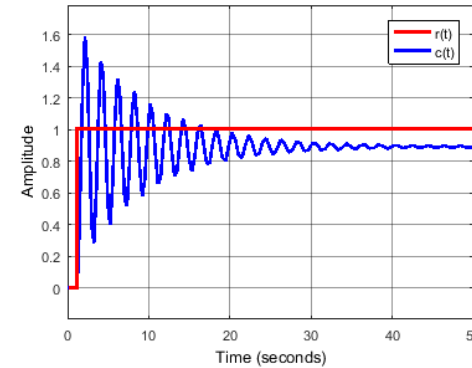
$K = 10; e_{ss} = 0,375$



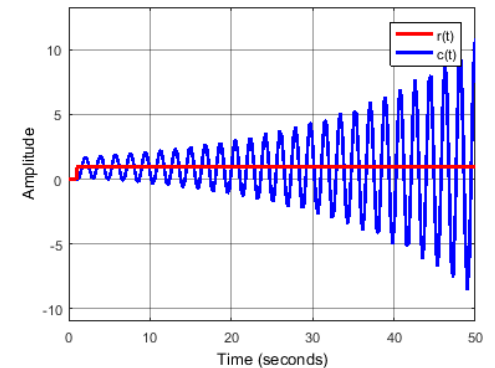
$K = 25; e_{ss} = 0,193$



$K = 50; e_{ss} = 0,107$

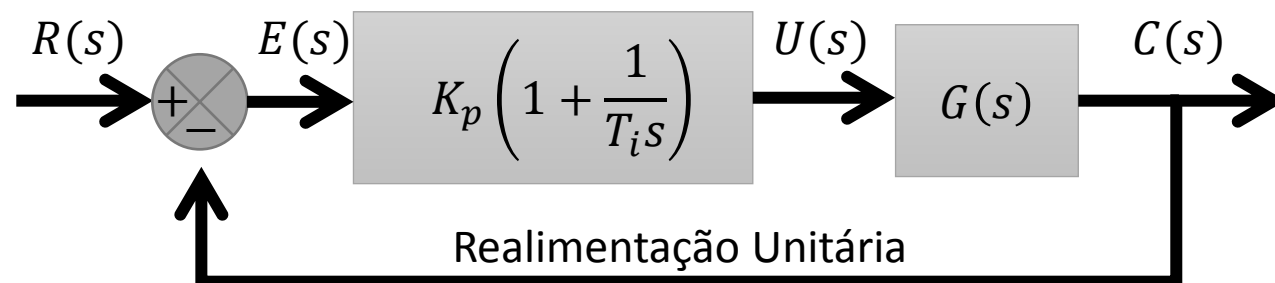


$K = 65; \text{divergiu}$



# Controle Proporcional-Integral (PI)

□ O controle proporcional-integral possibilita eliminar o erro estacionário.



$T_i$  - tempo de integração

$$E(s) = R(s) - C(s)$$

$$E(s) = R(s) - \frac{\left(K_p + \frac{K_p}{T_i s}\right) G(s) R(s)}{1 + \left(K_p + \frac{K_p}{T_i s}\right) G(s)}$$

$$\frac{E(s)}{R(s)} = 1 - \frac{\left(K_p + \frac{K_p}{T_i s}\right) G(s)}{1 + \left(K_p + \frac{K_p}{T_i s}\right) G(s)}$$

$$\frac{E(s)}{R(s)} = \frac{1 + \left(K_p + \frac{K_p}{T_i s}\right) G(s) - \left(K_p + \frac{K_p}{T_i s}\right) G(s)}{1 + \left(K_p + \frac{K_p}{T_i s}\right) G(s)}$$

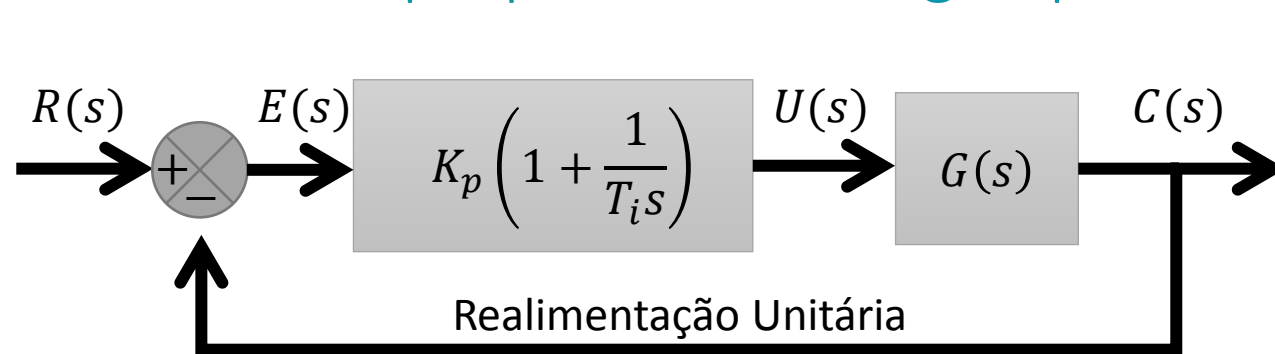
$$\frac{E(s)}{R(s)} = \frac{1}{1 + \left(K_p + \frac{K_p}{T_i s}\right) G(s)} = \frac{1}{1 + K_p G(s) + \frac{K_p G(s)}{T_i s}}$$

$$\frac{E(s)}{R(s)} = \frac{T_i s}{T_i s + T_i s K_p G(s) + K_p G(s)}$$

$$E(s) = \frac{T_i s}{T_i s + T_i s K_p G(s) + K_p G(s)} R(s)$$

# Controle Proporcional-Integral (PI)

□ O controle proporcional-integral possibilita eliminar o erro estacionário.



$$R(s) = \frac{1}{s}$$

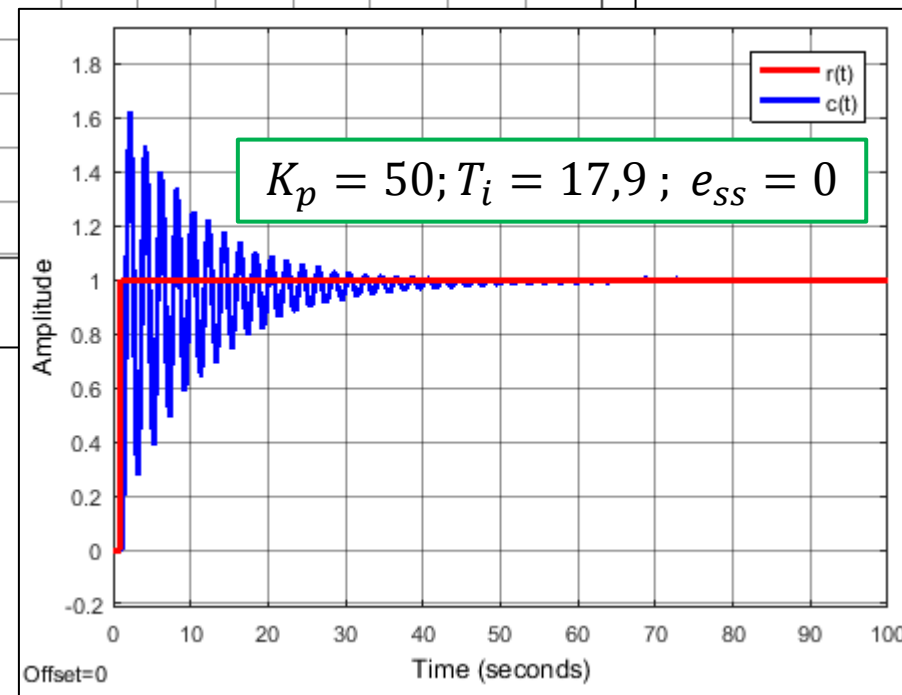
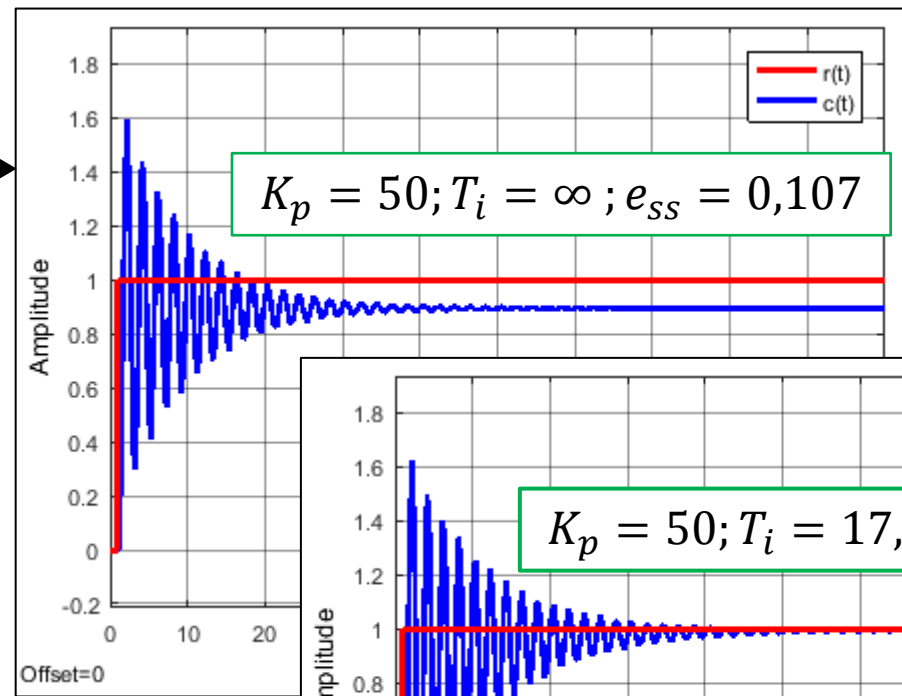
$$G(s) = \frac{1}{s^3 + 6s^2 + 11s + 6}$$

Erro estacionário

$$e_{ss} = \lim_{s \rightarrow 0} s \frac{T_i s}{T_i s + T_i s K_p G(s) + K_p G(s)} R(s)$$

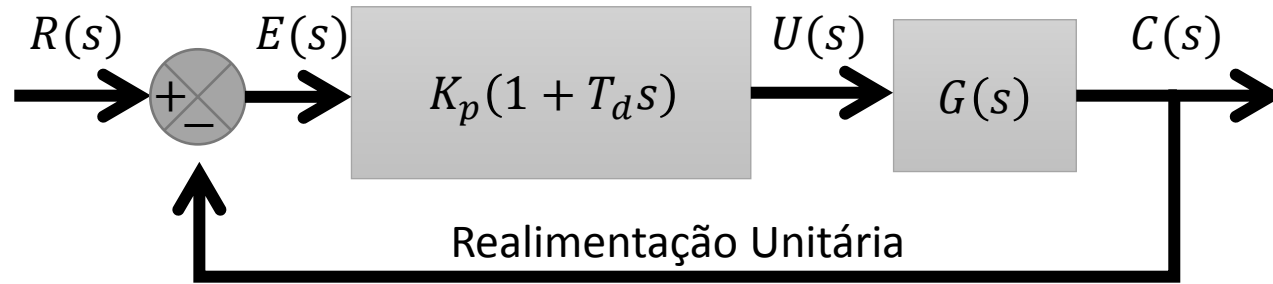
$$e_{ss} = \lim_{s \rightarrow 0} \frac{T_i s^2}{T_i s + T_i s K_p G(s) + K_p G(s)} \frac{1}{s}$$

$$e_{ss} = \lim_{s \rightarrow 0} \frac{T_i s^{\uparrow 0}}{T_i s + T_i s K_p G(s) + K_p G(s)} = 0$$



# Controle Proporcional-Derivativo (PD)

□ O controle proporcional-derivativo melhora a estabilidade do sistema.



$T_d$  - tempo derivativo

$$T(s) = \frac{C(s)}{R(s)} = \frac{K_p(1 + T_d s)G(s)}{1 + K_p(1 + T_d s)G(s)}$$

$$G(s) = \frac{1}{s^3 + 6s^2 + 11s + 6}$$

$$T(s) = \frac{\frac{K_p(1 + T_d s)}{s^3 + 6s^2 + 11s + 6}}{1 + \frac{K_p(1 + T_d s)}{s^3 + 6s^2 + 11s + 6}}$$

$$T(s) = \frac{\frac{K_p(1 + T_d s)}{s^3 + 6s^2 + 11s + 6}}{\frac{s^3 + 6s^2 + 11s + 6 + K_p(1 + T_d s)}{s^3 + 6s^2 + 11s + 6}}$$

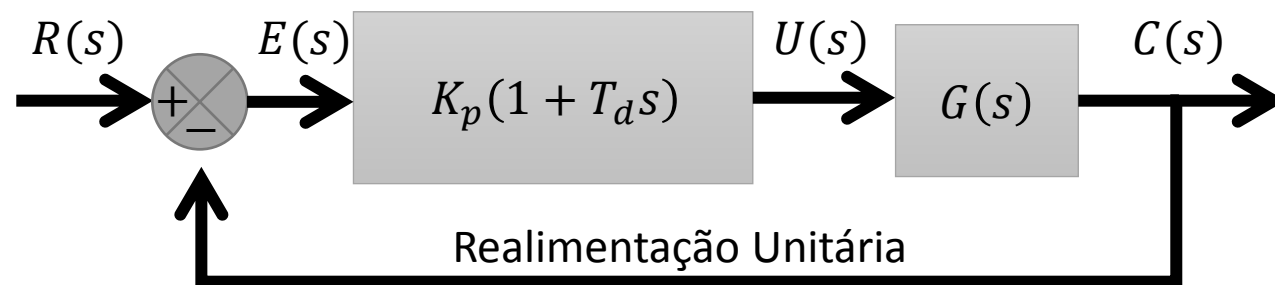
$$T(s) = \frac{K_p(1 + T_d s)}{s^3 + 6s^2 + 11s + 6 + K_p(1 + T_d s)}$$

$$T(s) = \frac{K_p + K_p T_d s}{s^3 + 6s^2 + (11 + K_p T_d)s + 6 + K_p}$$

Aumento do amortecimento do sistema

# Controle Proporcional-Derivativo (PD)

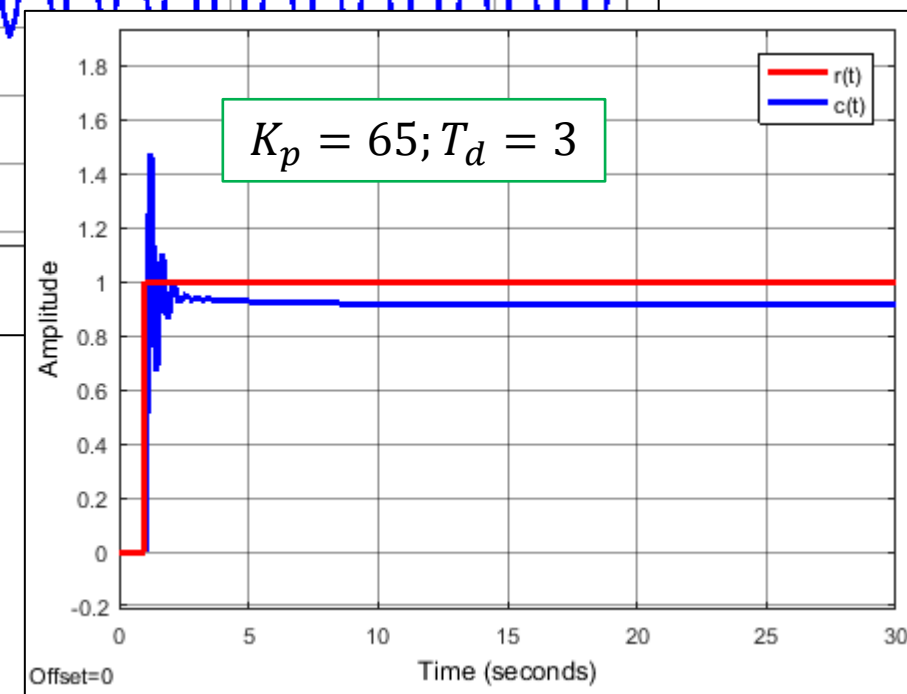
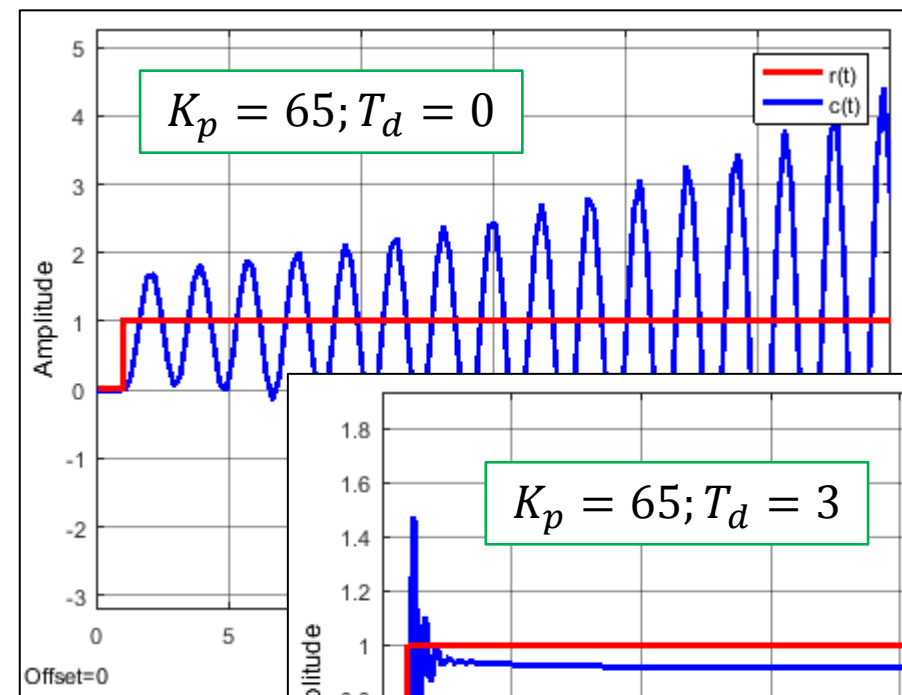
□ O controle proporcional-derivativo melhora a estabilidade do sistema.



$$T(s) = \frac{C(s)}{R(s)} = \frac{K_p(1 + T_d s)G(s)}{1 + K_p(1 + T_d s)G(s)}$$

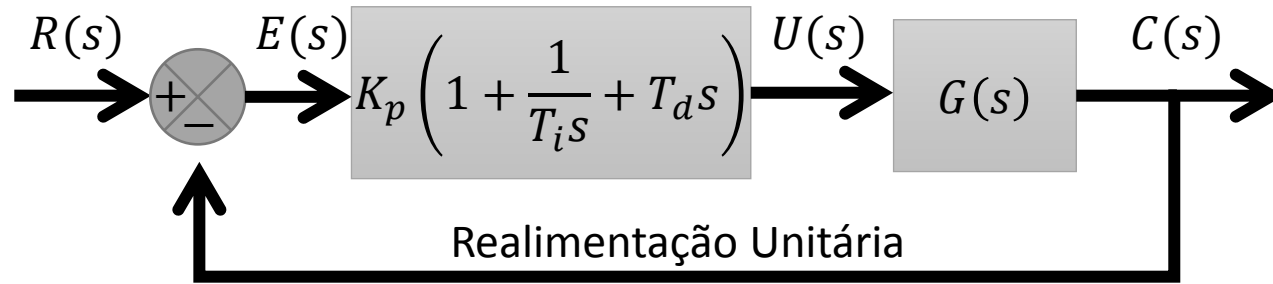
$$G(s) = \frac{1}{s^3 + 6s^2 + 11s + 6}$$

$$T(s) = \frac{K_p + K_p T_d s}{s^3 + 6s^2 + (11 + K_p T_d)s + 6 + K_p}$$



# Controle Proporcional-Integral-Derivativo (PID)

❑ Eliminação do erro estacionário e melhor estabilidade.



$$T(s) = \frac{C(s)}{R(s)} = \frac{K_p \left(1 + \frac{1}{T_i s} + T_d s\right) G(s)}{1 + K_p \left(1 + \frac{1}{T_i s} + T_d s\right) G(s)}$$

$$G(s) = \frac{1}{s^3 + 6s^2 + 11s + 6}$$

$$T(s) = \frac{K_p \left(1 + \frac{1}{T_i s} + T_d s\right)}{s^3 + 6s^2 + 11s + 6} \cdot \frac{K_p \left(1 + \frac{1}{T_i s} + T_d s\right)}{1 + \frac{K_p \left(1 + \frac{1}{T_i s} + T_d s\right)}{s^3 + 6s^2 + 11s + 6}}$$

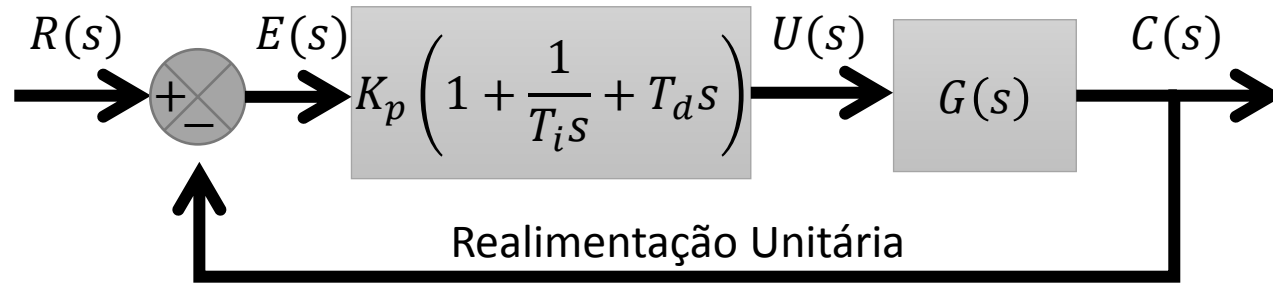
$$T(s) = \frac{\frac{K_p \left(1 + \frac{1}{T_i s} + T_d s\right)}{s^3 + 6s^2 + 11s + 6}}{\frac{s^3 + 6s^2 + 11s + 6 + K_p \left(1 + \frac{1}{T_i s} + T_d s\right)}{s^3 + 6s^2 + 11s + 6}}$$

$$T(s) = \frac{K_p \left(1 + \frac{1}{T_i s} + T_d s\right)}{s^3 + 6s^2 + 11s + 6 + K_p \left(1 + \frac{1}{T_i s} + T_d s\right)}$$



# Controle Proporcional-Integral-Derivativo (PID)

Eliminação do erro estacionário e melhor estabilidade.



$$T(s) = \frac{C(s)}{R(s)} = \frac{K_p \left(1 + \frac{1}{T_i s} + T_d s\right) G(s)}{1 + K_p \left(1 + \frac{1}{T_i s} + T_d s\right) G(s)}$$

$$G(s) = \frac{1}{s^3 + 6s^2 + 11s + 6}$$

$$T(s) = \frac{K_p + \frac{K_p}{T_i s} + K_p T_d s}{s^3 + 6s^2 + 11s + 6 + K_p + \frac{K_p}{T_i s} + K_p T_d s}$$

$$K_i = \frac{K_p}{T_i}$$

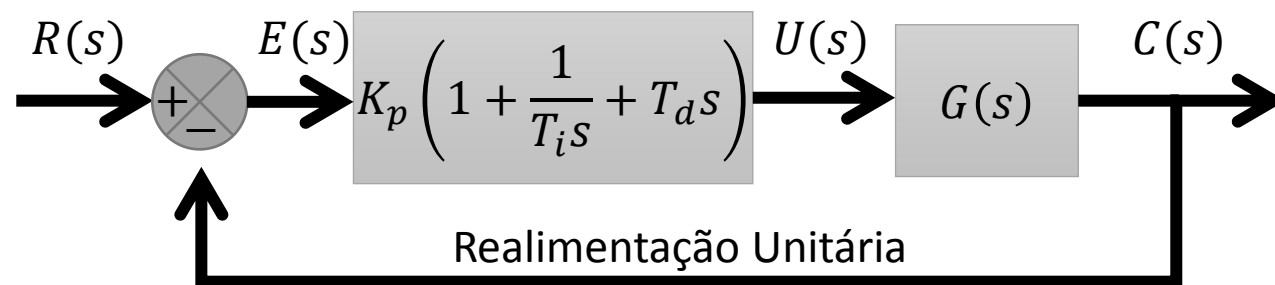
$$K_d = K_p T_d$$

$$T(s) = \frac{K_p + \frac{K_i}{s} + K_d s}{s^3 + 6s^2 + 11s + 6 + K_p + \frac{K_i}{s} + K_d s}$$

$$T(s) = \frac{K_d s^2 + K_p s + K_i}{s^4 + 6s^3 + (11 + K_d)s^2 + (6 + K_p)s + K_i}$$

# Controle Proporcional-Integral-Derivativo (PID)

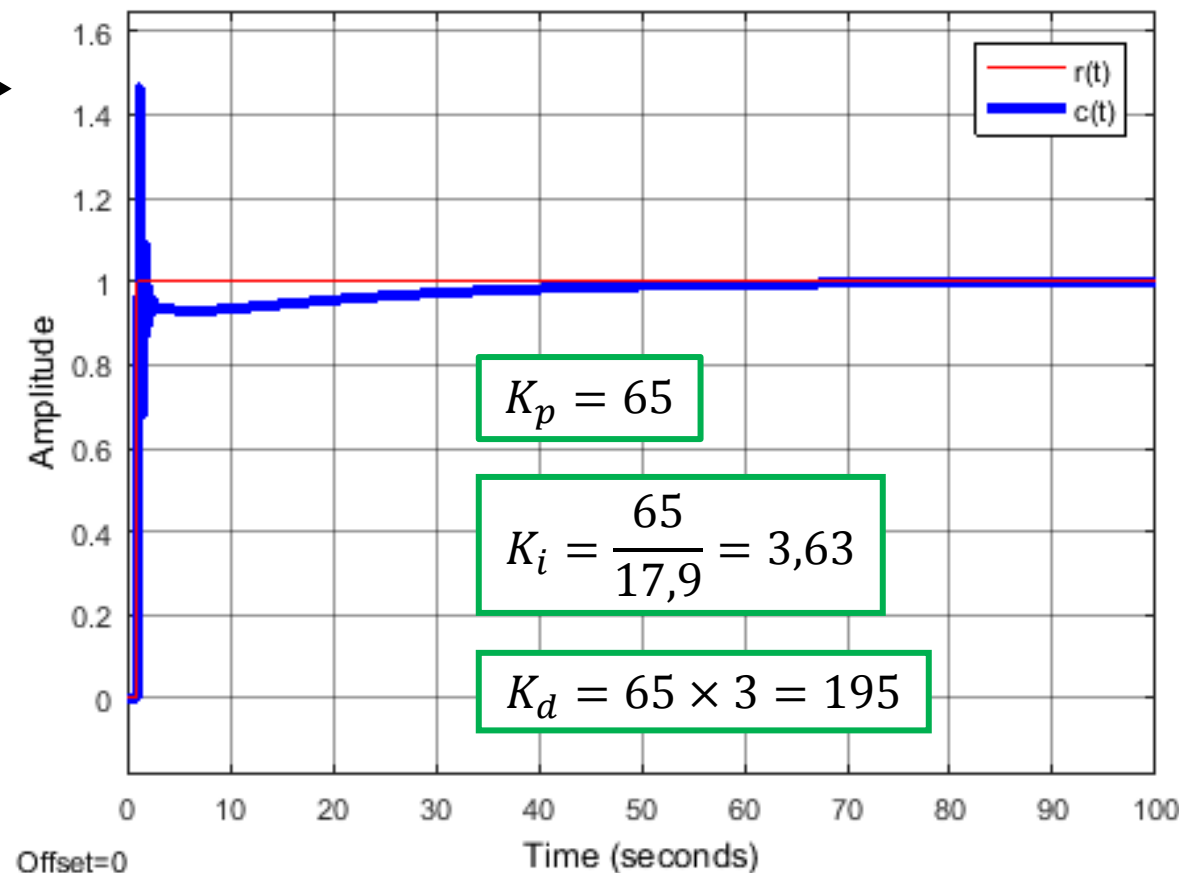
Eliminação do erro estacionário e melhor estabilidade.



$$R(s) = \frac{1}{s}$$

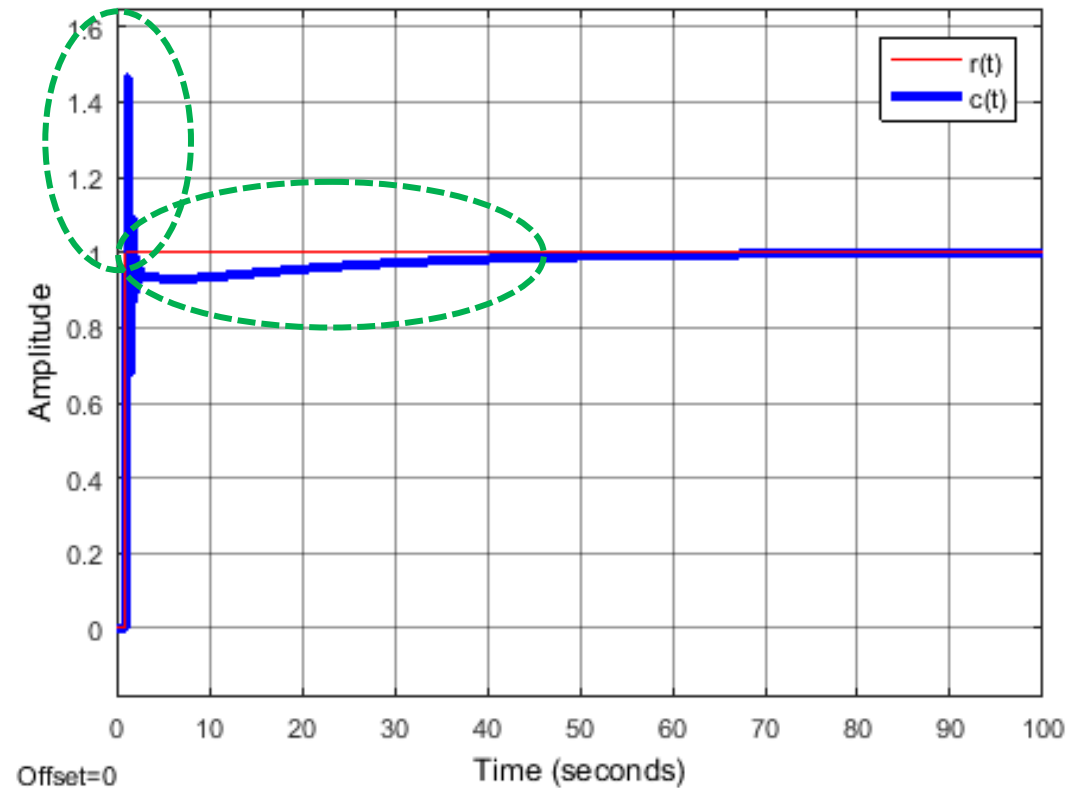
$$G(s) = \frac{1}{s^3 + 6s^2 + 11s + 6}$$

$$T(s) = \frac{K_d s^2 + K_p s + K_i}{s^4 + 6s^3 + (11 + K_d)s^2 + (6 + K_p)s + K_i}$$

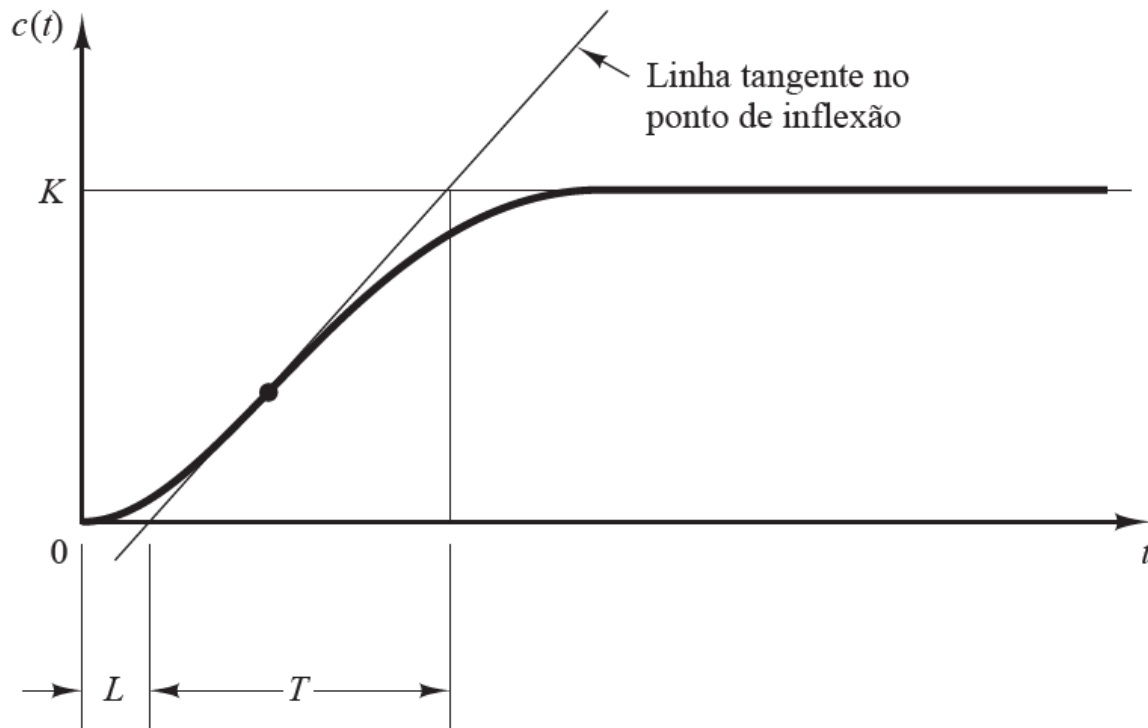


# Sintonia de Controladores PID

- ❑ Cada planta exige um conjunto específico parâmetros de desempenho, como máximo sobressinal, tempo de subida e tempo de acomodação.
- ❑ Regras de Ziegler-Nichols: Sugerem um ponto de partida para o ajuste dos parâmetros  $K_p$ ,  $T_i$  e  $T_d$ .



## Regras de Ziegler-Nichols: Método I

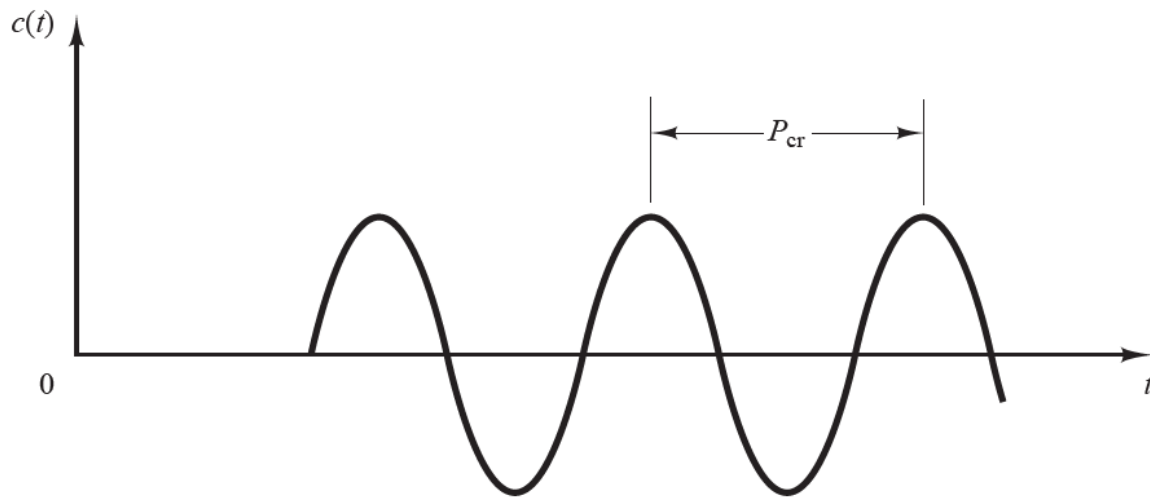


Aplicado nos casos em que planta possui uma curva com rampa em formato de "S"

- 1 – Desenhar uma linha tangente ao ponto de inflexão.
- 2 – Calcular os valores de  $L$  (atraso) e  $T$  (constante de tempo).
- 3 – Calcular os valores de  $K_p$ ,  $T_i$  e  $T_d$ .

| Tipo de controlador | $K_p$            | $T_i$           | $T_d$  |
|---------------------|------------------|-----------------|--------|
| P                   | $\frac{T}{L}$    | $\infty$        | 0      |
| PI                  | $0,9\frac{T}{L}$ | $\frac{L}{0,3}$ | 0      |
| PID                 | $1,2\frac{T}{L}$ | $2L$            | $0,5L$ |

## Regras de Ziegler-Nichols: Método II

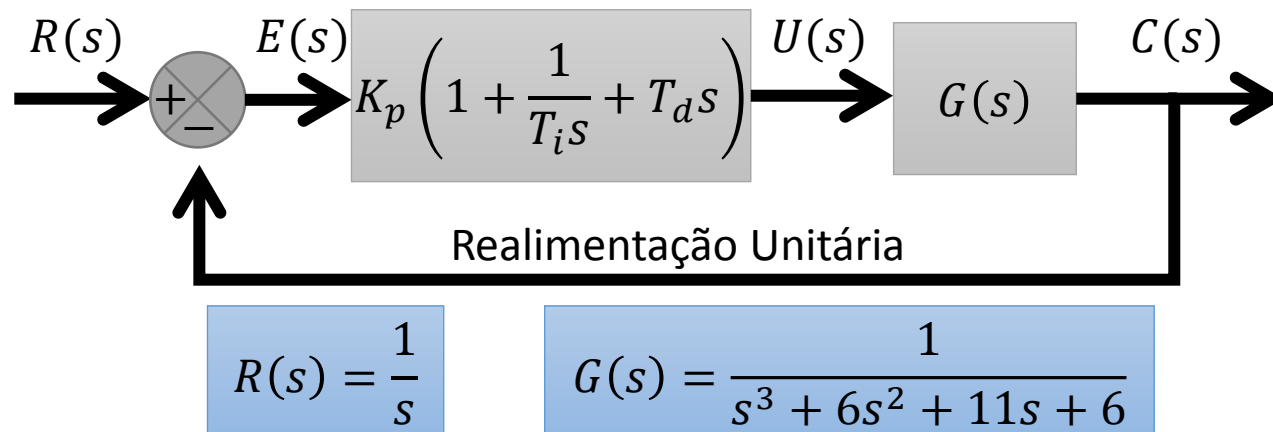


- 1 – Ajustar  $T_i = \infty$  e  $T_d = 0$ .
- 2 – Ajustar  $K_p$  até um valor  $K_{cr}$  no qual a saída produz pela primeira vez uma oscilação sustentada.
- 3 – Obter o período  $P_{cr}$  das oscilações.
- 4 – Calcular os valores de  $K_p$ ,  $T_i$  e  $T_d$ .

| Tipo de controlador | $K_p$        | $T_i$                 | $T_d$         |
|---------------------|--------------|-----------------------|---------------|
| P                   | $0,5K_{cr}$  | $\infty$              | 0             |
| PI                  | $0,45K_{cr}$ | $\frac{1}{1,2}P_{cr}$ | 0             |
| PID                 | $0,6K_{cr}$  | $0,5P_{cr}$           | $0,125P_{cr}$ |

Quando a função de transferência da planta é conhecida os valores de  $K_{cr}$  e  $P_{cr}$  podem ser calculados de forma analítica. Caso contrário, podem ser obtidos de forma experimental.

## Regras de Ziegler-Nichols: Método II



1 – Ajustar  $T_i = \infty$  e  $T_d = 0$ .

$$G_c(s) = K_p \left( 1 + \frac{1}{T_i s} + T_d s \right) = K_p (1 + 0 + 0) = K_{cr}$$

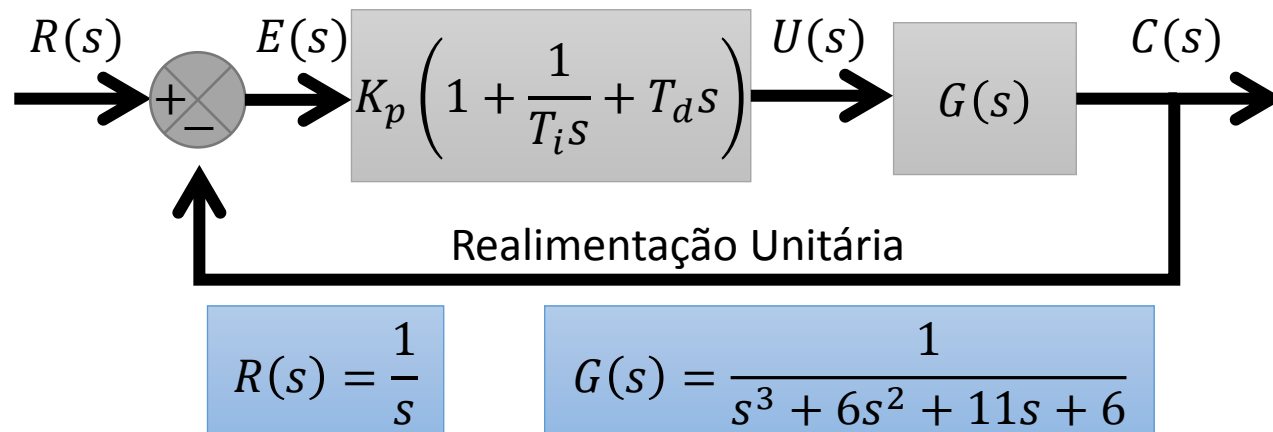
$$G_c(s)G(s) = K_{cr} \frac{1}{s^3 + 6s^2 + 11s + 6}$$

Sistema em Malha Aberta

$$T(s) = \frac{K_{cr}}{s^3 + 6s^2 + 11s + 6 + K_{cr}}$$

Sistema em Malha Fechada

## Regras de Ziegler-Nichols: Método II



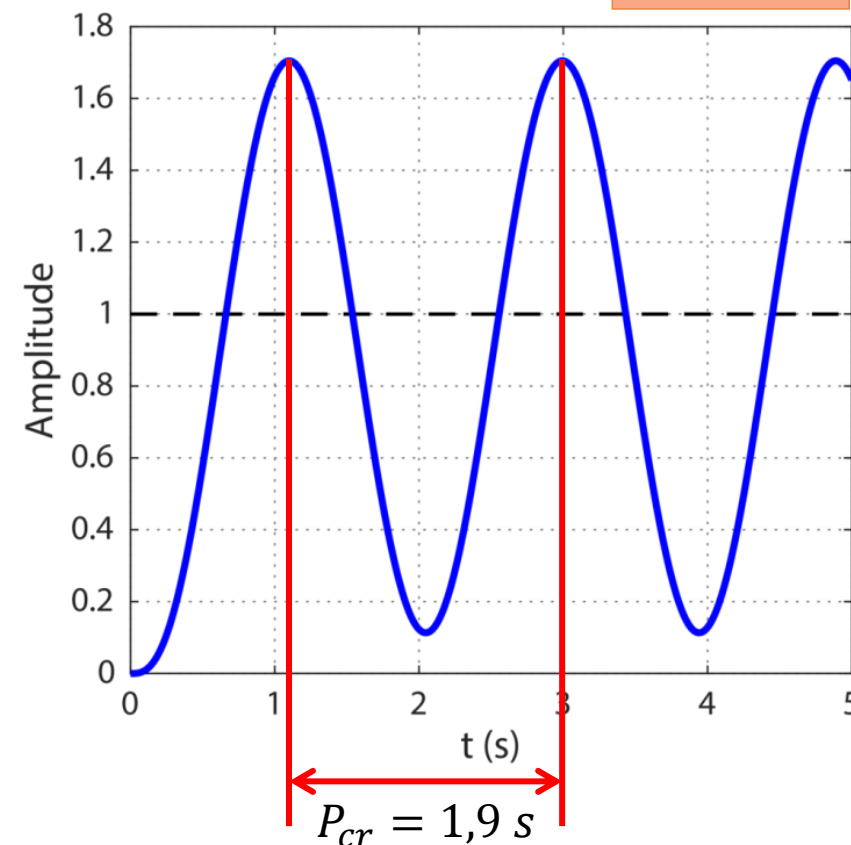
- 1 – Ajustar  $T_i = \infty$  e  $T_d = 0$ .
- 2 – Ajustar  $K_p$  até um valor  $K_{cr}$  no qual a saída produz pela primeira vez uma oscilação sustentada.
- 3 – Obter o período  $P_{cr}$  das oscilações.

$$T(s) = \frac{K_{cr}}{s^3 + 6s^2 + 11s + 6 + K_{cr}}$$

Sistema em Malha Fechada

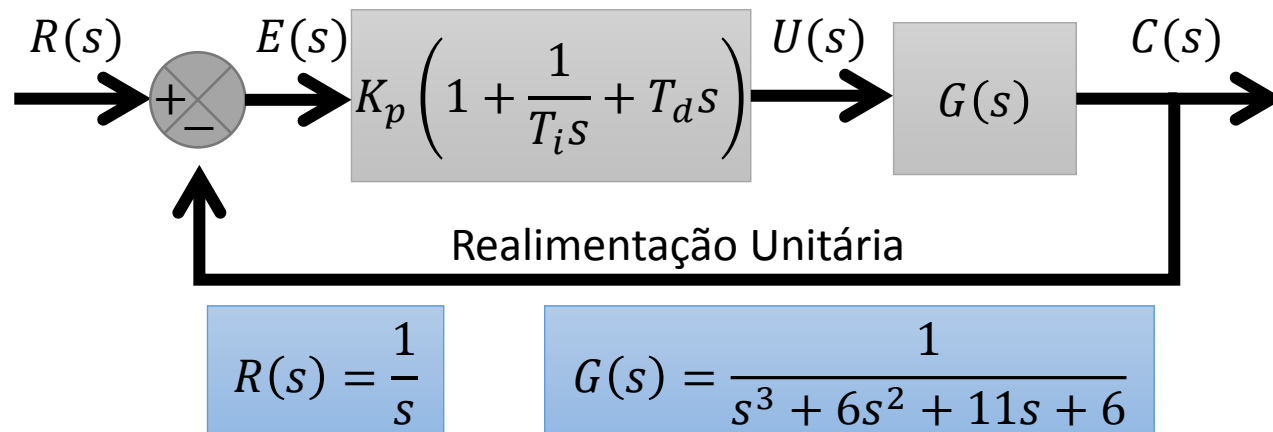
## Método Experimental

$$K_{cr} = 60$$



$$\omega_{cr} = \frac{2\pi}{P_{cr}} = \frac{2\pi}{1,9} = 3,3 \text{ rad/s}$$

## Regras de Ziegler-Nichols: Método II



1 - Ajustar  $T_i = \infty$  e  $T_d = 0$ .

$$G_c(s) = K_p \left( 1 + \frac{1}{T_i s} + T_d s \right) = K_p (1 + 0 + 0) = K_{cr}$$

$$T(s) = \frac{K_{cr}}{s^3 + 6s^2 + 11s + 6 + K_{cr}}$$

Sistema em Malha Fechada

Routh-Hurwitz

|       |                                      |              |
|-------|--------------------------------------|--------------|
| $s^3$ | 1                                    | 11           |
| $s^2$ | 6                                    | $6 + K_{cr}$ |
| $s^1$ | $\frac{60 - K_{cr}}{6}$              | 0            |
| $s^0$ | $-\frac{K_{cr}^2}{6} + 9K_{cr} + 60$ |              |

$$\frac{60 - K_{cr}}{6} \geq 0$$

$$-\frac{K_{cr}^2}{6} + 9K_{cr} + 60 > 0$$

$$K_{cr} = 60$$

Cálculo de  $P_{cr}$ :  $P_{cr} = \frac{2\pi}{\omega_{cr}}$

$$s^3 + 6s^2 + 11s + 6 + K_{cr} = 0$$

$$s^3 + 6s^2 + 11s + 66 = 0$$

$$s = j\omega$$

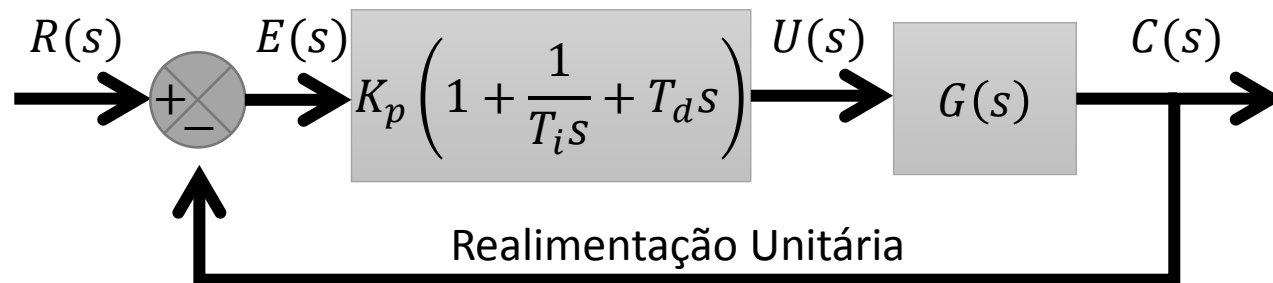
$$(j\omega_{cr})^3 + 6(j\omega_{cr})^2 + 11(j\omega_{cr}) + 66 = 0$$

$$\omega_{cr} = \sqrt{11} = 3,31 \text{ rad/s}$$

$$P_{cr} = 1,898 \text{ s}$$



## Regras de Ziegler-Nichols: Método II



$$R(s) = \frac{1}{s}$$

$$G(s) = \frac{1}{s^3 + 6s^2 + 11s + 6}$$

1 – Ajustar  $T_i = \infty$  e  $T_d = 0$ .

$$G_c(s) = K_p \left( 1 + \frac{1}{T_i s} + T_d s \right) = K_p (1 + 0 + 0) = K_{cr}$$

$$G_c(s)G(s) = K_{cr} \frac{1}{s^3 + 6s^2 + 11s + 6}$$

Sistema em Malha Aberta

## Lugar das Raízes

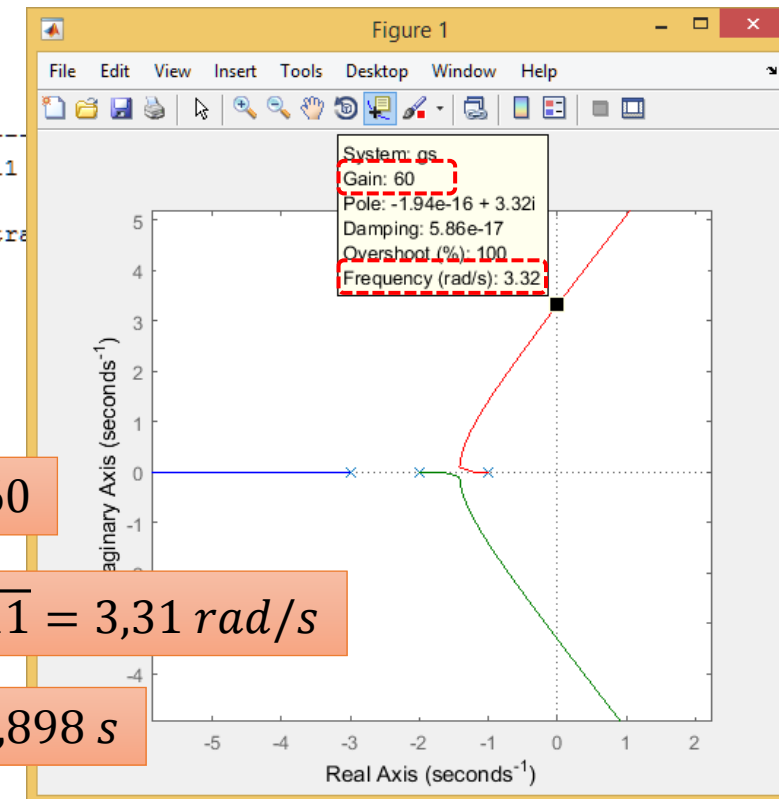
```
Command Window
>> s = tf('s');
>> gs = 1/(s^3 + 6*s^2 + 11*s + 6)

gs =

      1
-----
s^3 + 6 s^2 + 11 s + 6

Continuous-time transfer function.

>> rlocus(gs)
fz >>
```

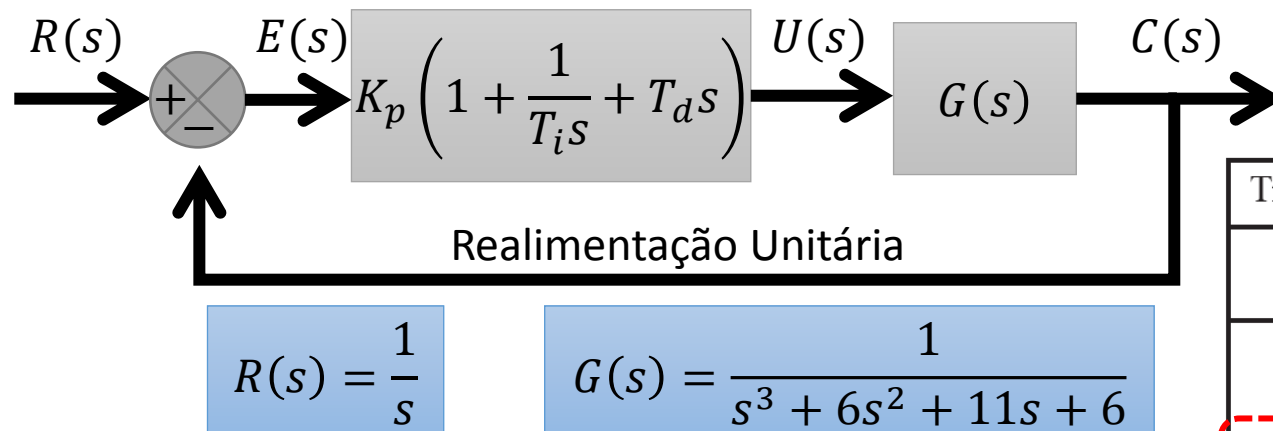


$$K_{cr} = 60$$

$$\omega = \sqrt{11} = 3,31 \text{ rad/s}$$

$$P_{cr} = 1,898 \text{ s}$$

## Regras de Ziegler-Nichols: Método II



- 1 – Ajustar  $T_i = \infty$  e  $T_d = 0$ .
- 2 – Ajustar  $K_p$  até um valor  $K_{cr}$  no qual a saída produz pela primeira vez uma oscilação sustentada.
- 3 – Obter o período  $P_{cr}$  das oscilações.
- 4 – Calcular os valores de  $K_p$ ,  $T_i$  e  $T_d$ .

$K_{cr} = 60$

$P_{cr} = 1,898 \text{ s}$

Calcular os valores de  $K_p$ ,  $T_i$  e  $T_d$ .

| Tipo de controlador | $K_p$        | $T_i$                 | $T_d$         |
|---------------------|--------------|-----------------------|---------------|
| P                   | $0,5K_{cr}$  | $\infty$              | 0             |
| PI                  | $0,45K_{cr}$ | $\frac{1}{1,2}P_{cr}$ | 0             |
| PID                 | $0,6K_{cr}$  | $0,5P_{cr}$           | $0,125P_{cr}$ |

$K_p = 0,6K_{cr} = 36$

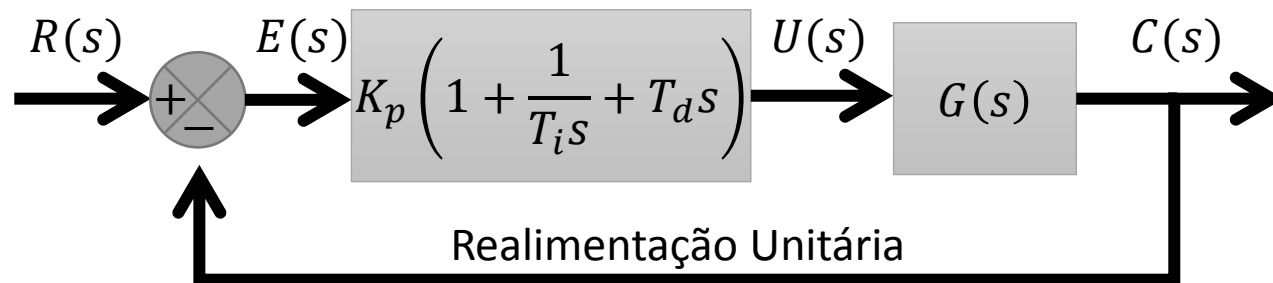
$T_i = 0,5P_{cr} = 0,949$

$T_d = 0,125P_{cr} = 0,237$

$K_i = \frac{K_p}{T_i} = 37,93$

$K_d = K_p T_d = 8,532$

## Regras de Ziegler-Nichols: Método II



$$R(s) = \frac{1}{s}$$

$$G(s) = \frac{1}{s^3 + 6s^2 + 11s + 6}$$

$$K_p = 36$$

$$K_i = 37,93$$

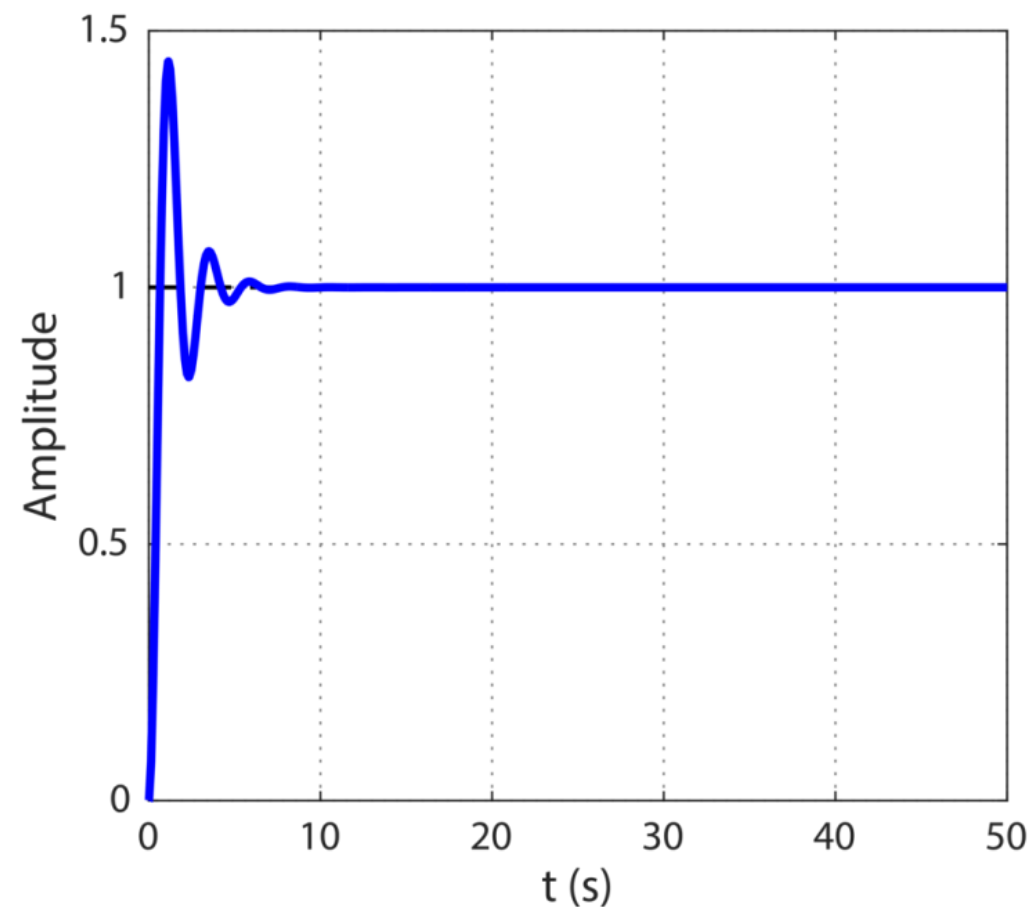
$$K_d = 8,532$$

$$T_i = 0,949$$

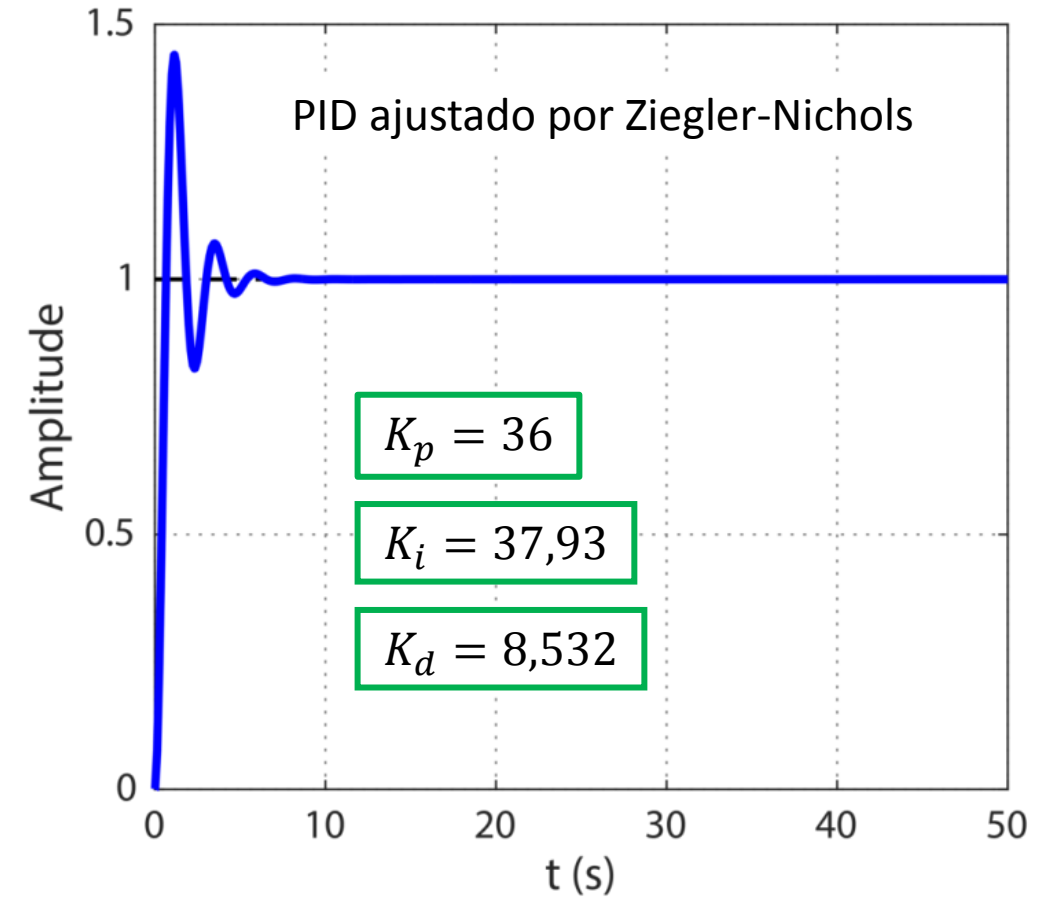
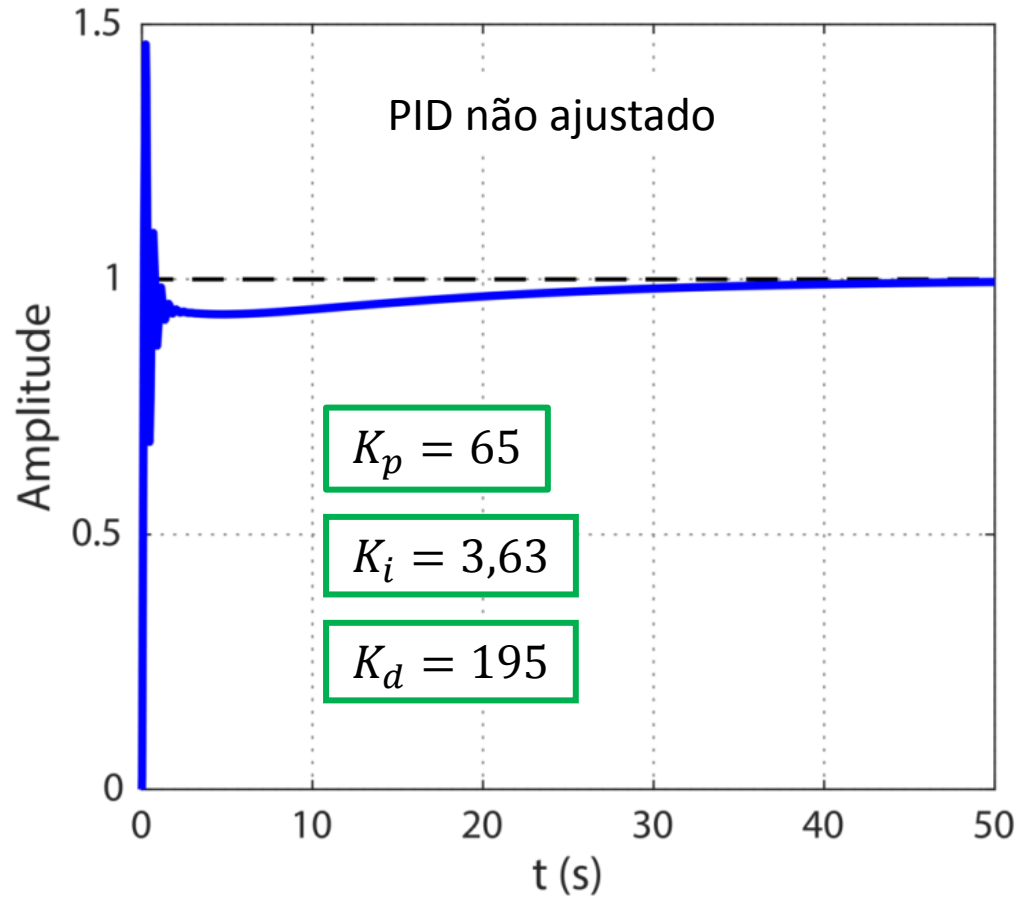
$$T_d = 0,237$$

$$T(s) = \frac{K_d s^2 + K_p s + K_i}{s^4 + 6s^3 + (11 + K_d)s^2 + (6 + K_p)s + K_i}$$

$$T(s) = \frac{8,532s^2 + 36s + 37,93}{s^4 + 6s^3 + 19,532s^2 + 42s + 37,93}$$

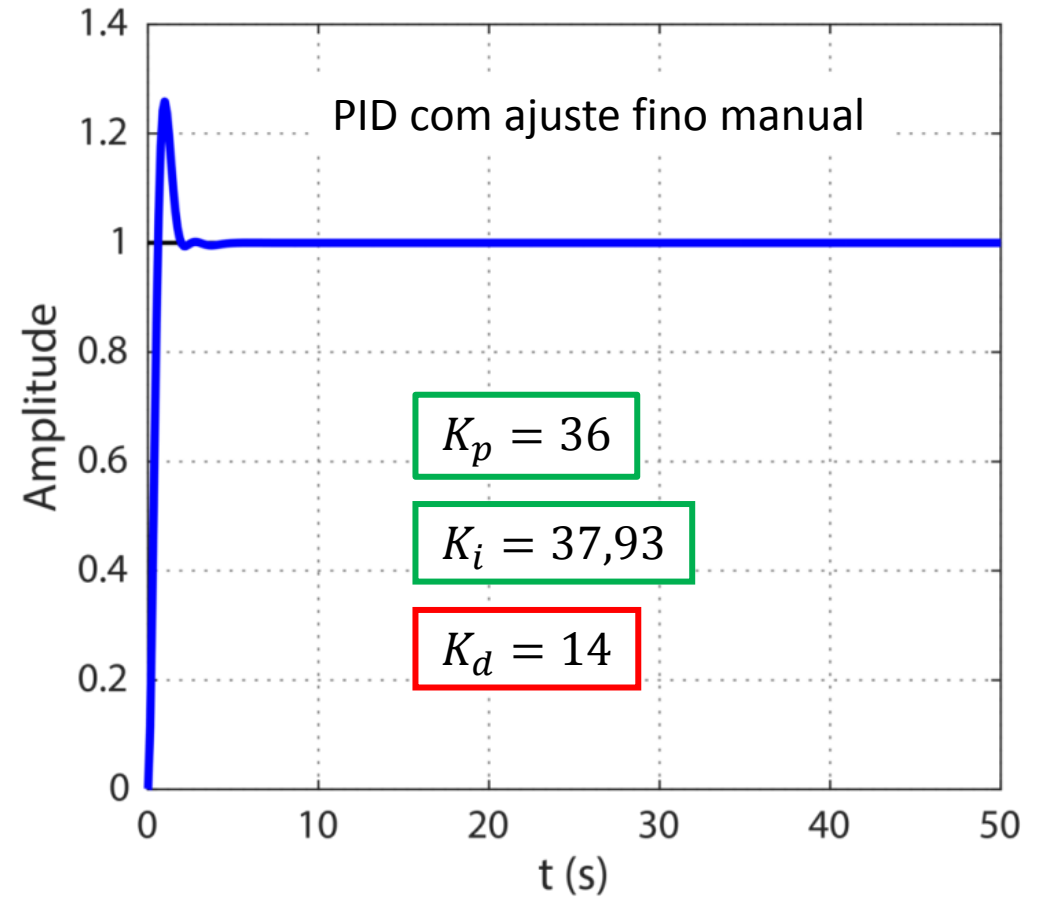
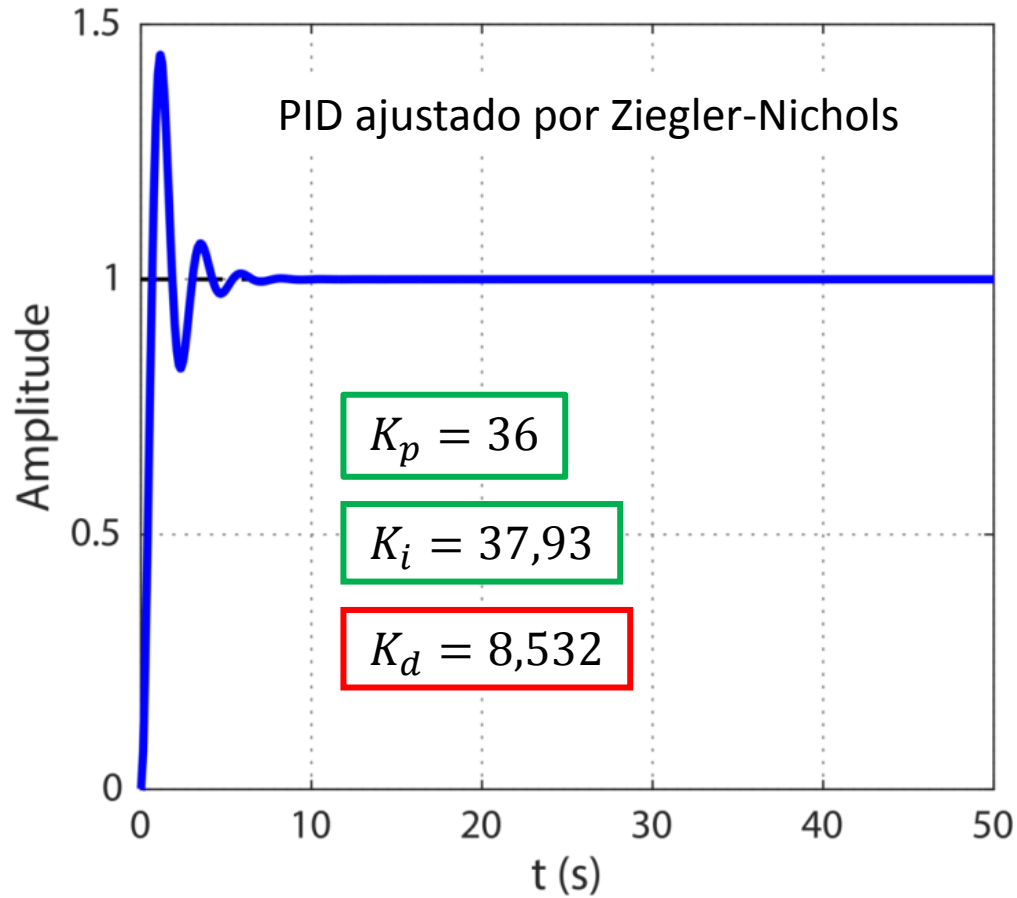


## Regras de Ziegler-Nichols: Método II



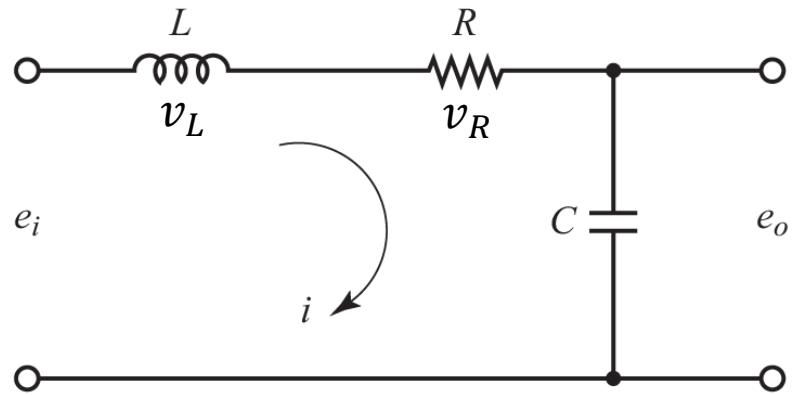
O Overshoot ainda pode ser melhorado por uma ajuste fino manual.

## Regras de Ziegler-Nichols: Método II



# E como aplico isso tudo?





$$v_L(t) = L \frac{di(t)}{dt}$$

$$v_R(t) = Ri(t)$$

$$v_C(t) = \frac{1}{C} \int i(t)dt + v_C(0)$$

## Lei de Kirchhoff das Tensões (LKT)

$$v_L + v_R + v_C - e_i = 0$$

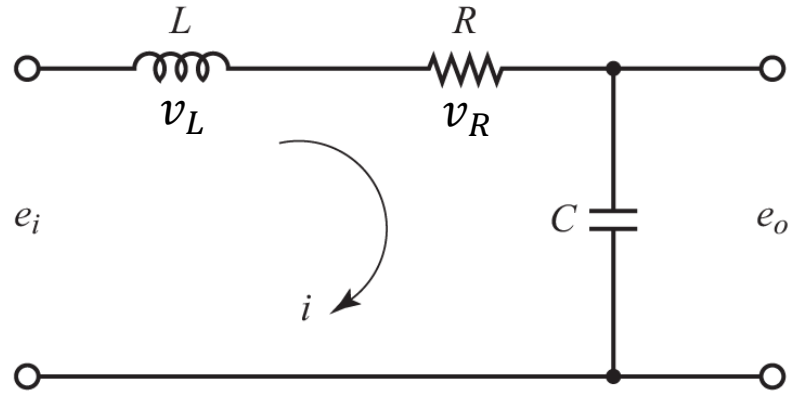
$$L \frac{di(t)}{dt} + Ri(t) + \frac{1}{C} \int i(t)dt + v_C(0) = e_i$$

$$v_C = e_o \quad i(t) = C \frac{de_o(t)}{dt} \quad e_o(0) = 0$$

$$LC \frac{d}{dt} \frac{de_o(t)}{dt} + RC \frac{de_o(t)}{dt} + \frac{C}{C} \int \frac{de_o(t)}{dt} dt = e_i$$

$$LC\ddot{e}_o + RC\dot{e}_o + e_o = e_i$$

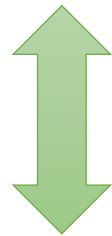
# Sistemas de Segunda Ordem



Domínio do Tempo (t)

$$LC\ddot{e}_o + RC\dot{e}_o + e_o = e_i$$

$$\mathcal{L}\{f(t)\} \Leftrightarrow \mathcal{L}^{-1}\{F(s)\}$$



Domínio das Frequências Complexas (s)

$$LCs^2E_o(s) + RCsE_o(s) + E_o(s) = E_i(s)$$

Função de Transferência de Malha Aberta

$$G(s) = \frac{Y(s)}{U(s)}$$

$$G(s) = \frac{E_o(s)}{E_i(s)} = \frac{1}{LCs^2 + RCs + 1} \quad G(s) = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Parametrização da Função de Transferência de Malha Aberta

Decaimento Exponencial

$$\sigma = \frac{R}{2L}$$

Frequência Natural não Amortecida

$$\omega_n = \frac{1}{\sqrt{LC}}$$

Coefficiente de Amortecimento

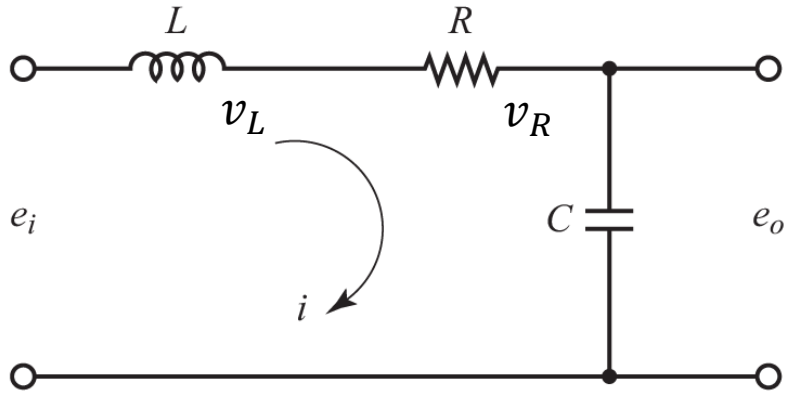
$$\zeta = \frac{\sigma}{\omega_n}$$

$$G(s) = \frac{\omega_n^2}{s^2 + 2\sigma s + \omega_n^2}$$

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



# Sistemas de Segunda Ordem



## Função de Transferência de Malha Aberta

$$G(s) = \frac{1}{LCs^2 + RCs + 1} = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Decaimento Exponencial

$$\sigma = \frac{R}{2L}$$

Frequência Natural não Amortecida

$$\omega_n = \frac{1}{\sqrt{LC}}$$

Coefficiente de Amortecimento

$$\zeta = \frac{\sigma}{\omega_n}$$

## Parametrização da Função de Transferência de Malha Aberta

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Equação Característica

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$\frac{-2\zeta\omega_n \pm \sqrt{(2\zeta\omega_n)^2 - 4\omega_n^2}}{2} = \frac{-2\zeta\omega_n \pm \sqrt{2^2\omega_n^2(\zeta^2 - 1)}}{2} =$$

$$\frac{-2\zeta\omega_n \pm 2\omega_n\sqrt{\zeta^2 - 1}}{2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} =$$

$$\zeta\omega_n \pm \omega_n\sqrt{1 - \zeta^2} = \zeta\omega_n \pm j\omega_d$$

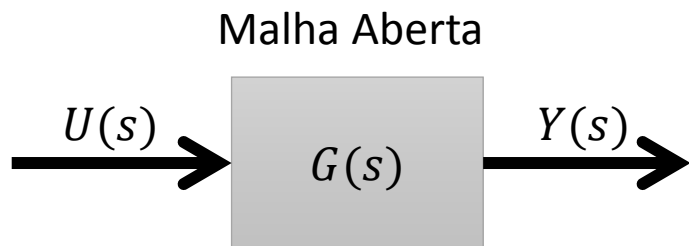
Frequência Natural Amortecida

$$\omega_d = \omega_n\sqrt{1 - \zeta^2}$$

$$G(s) = \frac{\omega_n^2}{(s + \zeta\omega_n + j\omega_d)(s + \zeta\omega_n - j\omega_d)}$$

Demonstre!!!

# Sistemas de Segunda Ordem



Função de Transferência de Malha Aberta

$$G(s) = \frac{\omega_n^2}{(s + \zeta\omega_n + j\omega_d)(s + \zeta\omega_n - j\omega_d)}$$

$$U(s) = \frac{1}{s}$$

Decaimento Exponencial

$$\sigma = \frac{R}{2L}$$

Frequência Natural não Amortecida

$$\omega_n = \frac{1}{\sqrt{LC}}$$

Coefficiente de Amortecimento

$$\zeta = \frac{\sigma}{\omega_n}$$

Frequência Natural Amortecida

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

$$Y(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_d^2} = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

$$\mathcal{L}^{-1}\{Y(s)\} = y(t) = 1 - e^{-\zeta\omega_n t} \left( \cos \omega_d t + \frac{\zeta}{\sqrt{1 - \zeta^2}} \text{sen } \omega_d t \right)$$

$$y(t) = 1 - \frac{e^{-\sigma t}}{\sqrt{1 - \zeta^2}} \text{sen} \left( \omega_d t + \tan^{-1} \frac{\sqrt{1 - \zeta^2}}{\zeta} \right)$$

Sistema não amortecido:

$$\zeta = 0 \quad s = \pm j\omega_n$$

Sistema subamortecido:

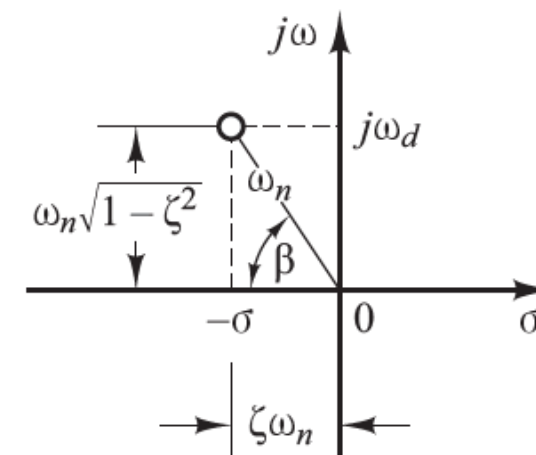
$$0 < \zeta < 1 \quad s = -\sigma \pm j\omega_d$$

Sistema criticamente amortecido:

$$\zeta = 1 \quad s = -\sigma \pm 0$$

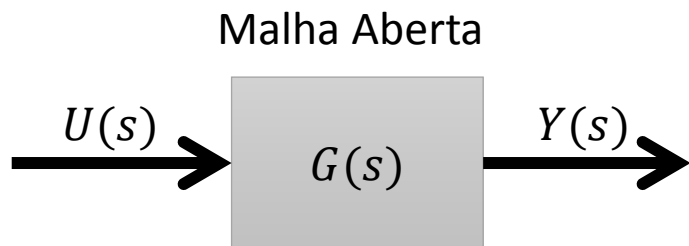
Sistema superamortecido:

$$\zeta > 0 \quad s = -\sigma \pm \omega_d$$



$$\beta = \cos^{-1} \frac{\sigma}{\omega_n} = \cos^{-1} \zeta$$

# Sistemas de Segunda Ordem



Função de Transferência de Malha Aberta

$$G(s) = \frac{\omega_n^2}{(s + \zeta\omega_n + j\omega_d)(s + \zeta\omega_n - j\omega_d)}$$

$$U(s) = \frac{1}{s}$$

Decaimento Exponencial

$$\sigma = \frac{R}{2L}$$

Frequência Natural não Amortecida

$$\omega_n = \frac{1}{\sqrt{LC}}$$

Coefficiente de Amortecimento

$$\zeta = \frac{\sigma}{\omega_n}$$

Frequência Natural Amortecida

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

$$Y(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

$$\mathcal{L}^{-1}\{Y(s)\} = y(t) = 1 - e^{-\zeta\omega_n t} \left( \cos \omega_d t + \frac{\zeta}{\sqrt{1 - \zeta^2}} \text{sen } \omega_d t \right)$$

$$y(t) = 1 - \frac{e^{-\sigma t}}{\sqrt{1 - \zeta^2}} \text{sen} \left( \omega_d t + \tan^{-1} \frac{\sqrt{1 - \zeta^2}}{\zeta} \right)$$

Sistema não amortecido:

$$\zeta = 0 \quad s = \pm j\omega_n$$

Sistema subamortecido:

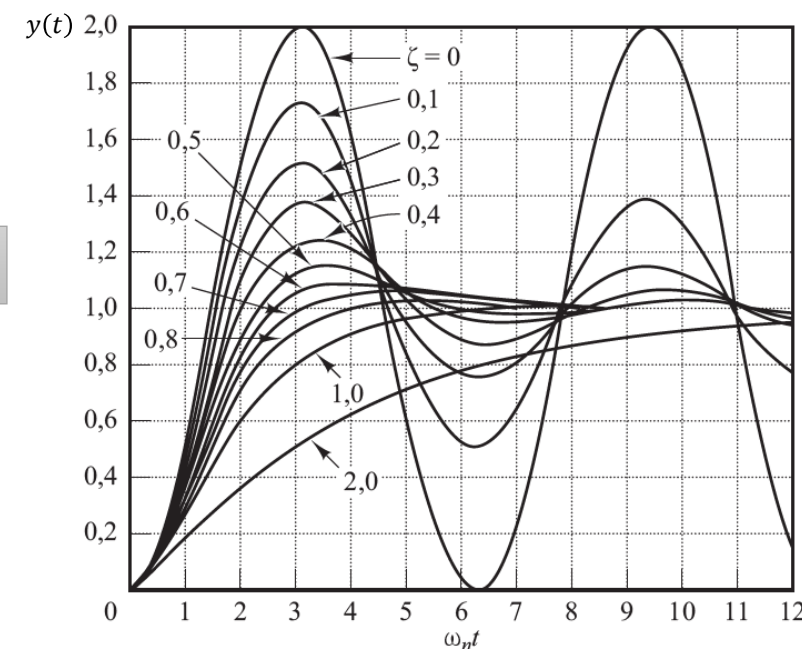
$$0 < \zeta < 1 \quad s = -\sigma \pm j\omega_d$$

Sistema criticamente amortecido:

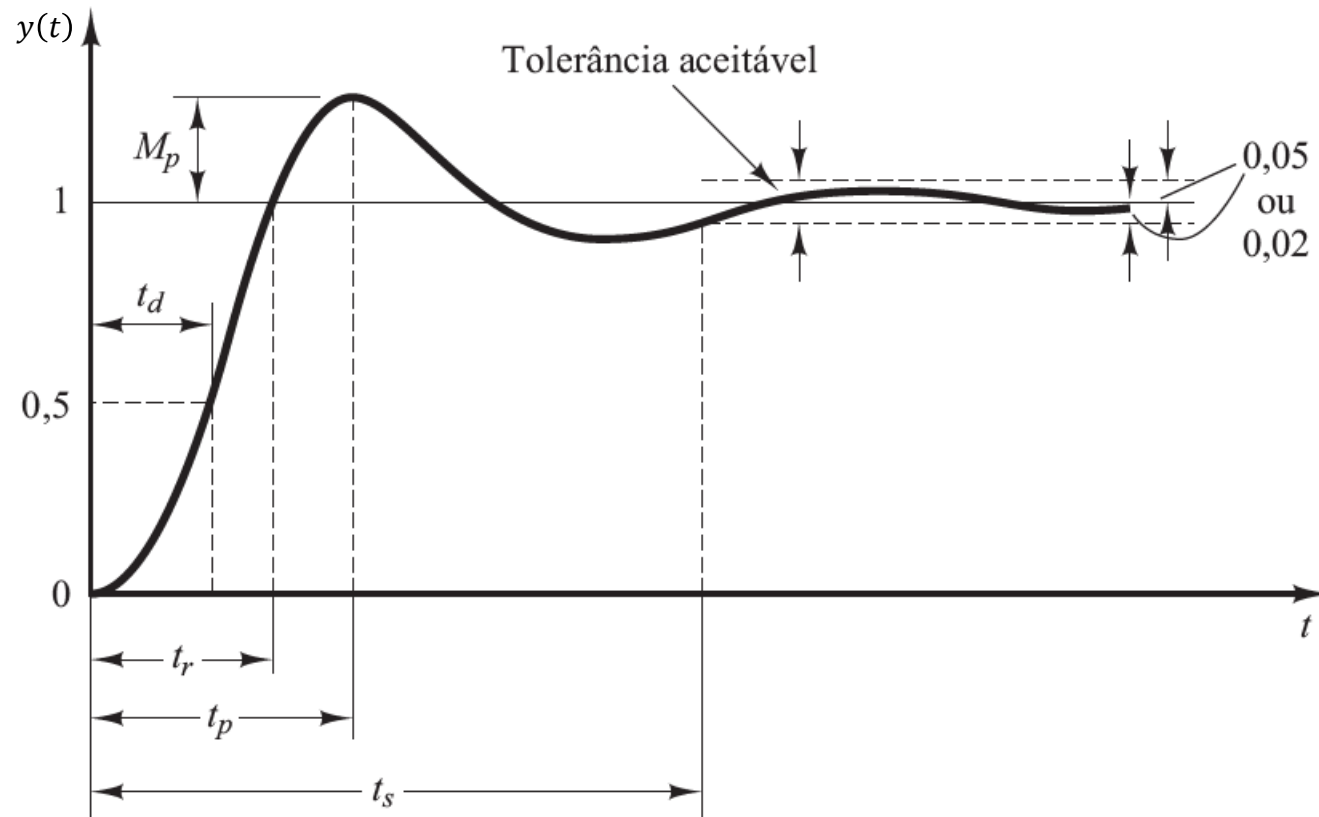
$$\zeta = 1 \quad s = -\sigma \pm 0$$

Sistema superamortecido:

$$\zeta > 1 \quad s = -\sigma \pm \omega_d$$

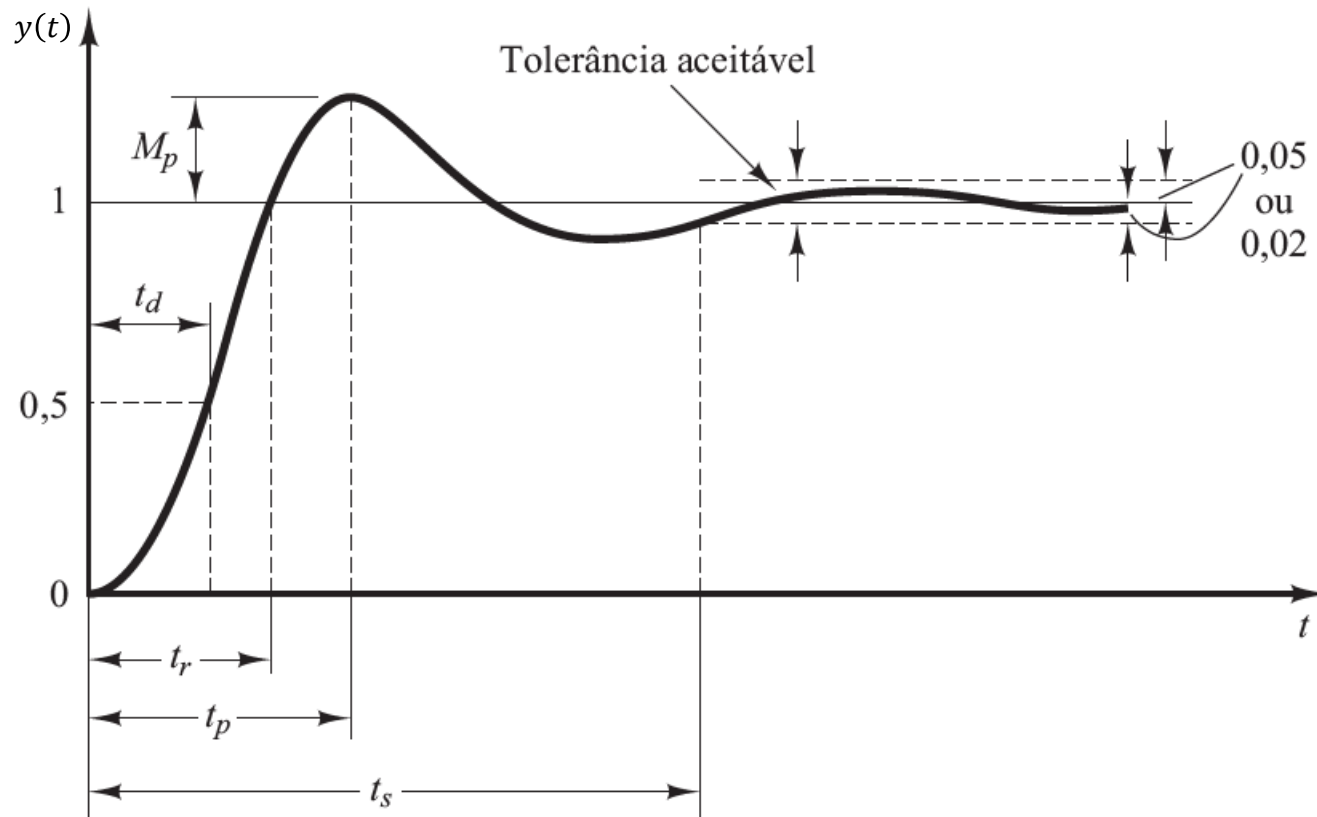


## ☐ Sistemas subamortecidos



- $t_r$  Tempo de subida
- $t_p$  Tempo de pico
- $M_p$  Máximo sobressinal (Overshoot)
- $t_s$  Tempo de acomodação
- $t_d$  Tempo de atraso

## Sistemas subamortecidos



$t_r$  Tempo de subida

$$y(t) = 1 - e^{-\zeta\omega_n t} \left( \cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \text{sen } \omega_d t \right)$$

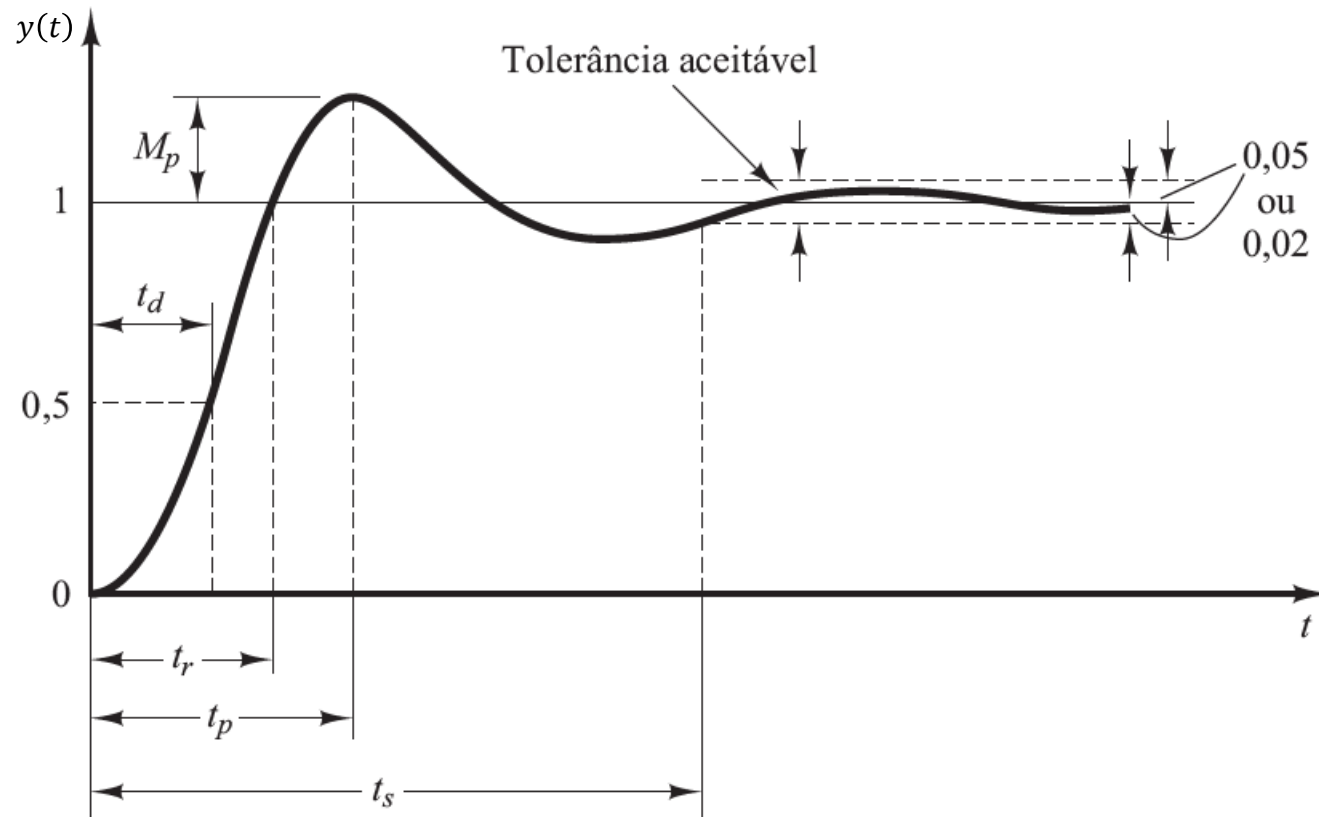
$$1 = 1 - e^{-\zeta\omega_n t} \left( \cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \text{sen } \omega_d t \right)$$

$$\cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \text{sen } \omega_d t = 0$$

$$t_r = \frac{\pi - \beta}{\omega_d} = \frac{\pi - \cos^{-1} \zeta}{\omega_d}$$

$$\beta = \cos^{-1} \zeta$$

## ☐ Sistemas subamortecidos



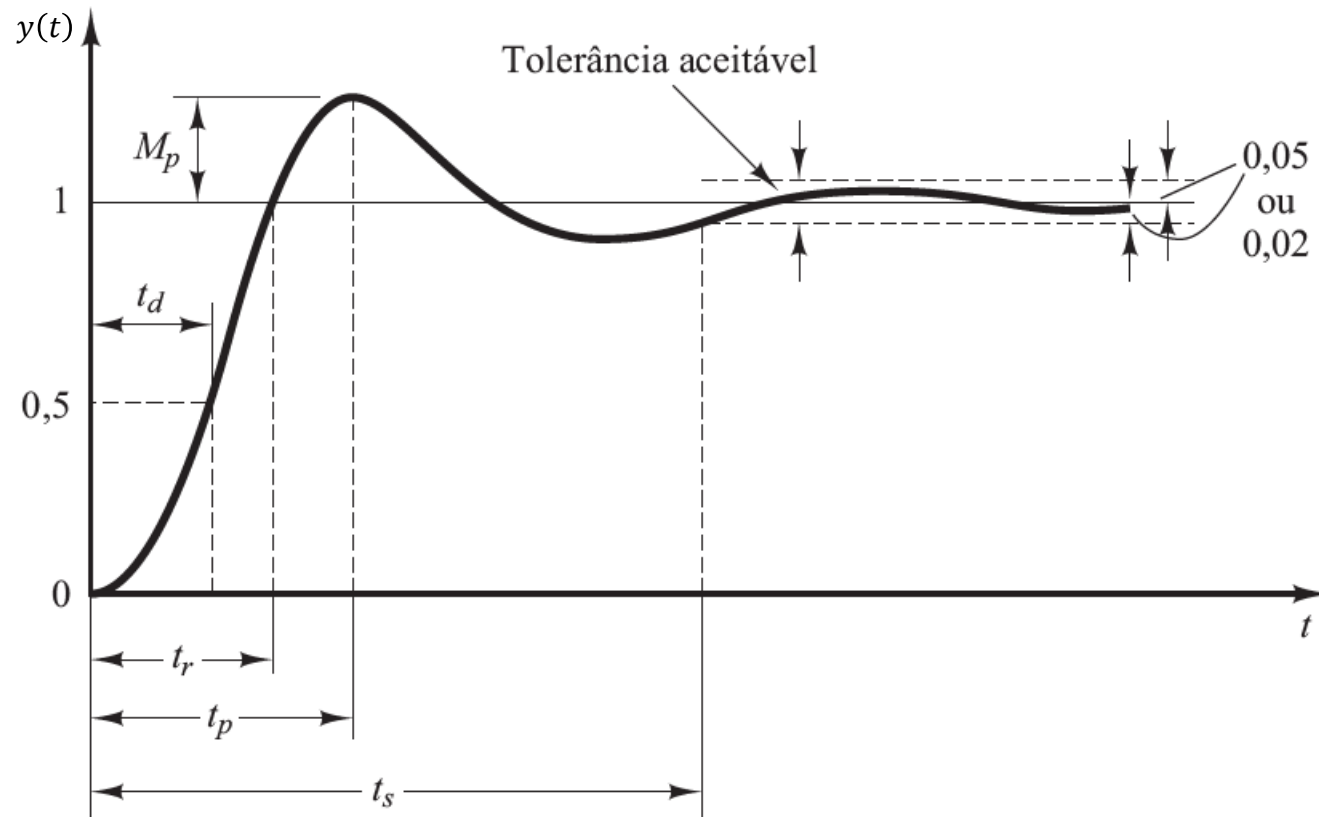
$t_p$  Tempo de pico

$$\frac{dy(t)}{dt} = 0$$

$$t_p = \frac{\pi}{\omega_d}$$

$$\omega_d = \frac{\pi}{t_p}$$

## ☐ Sistemas subamortecidos



$M_p$  Máximo sobressinal (Overshoot)

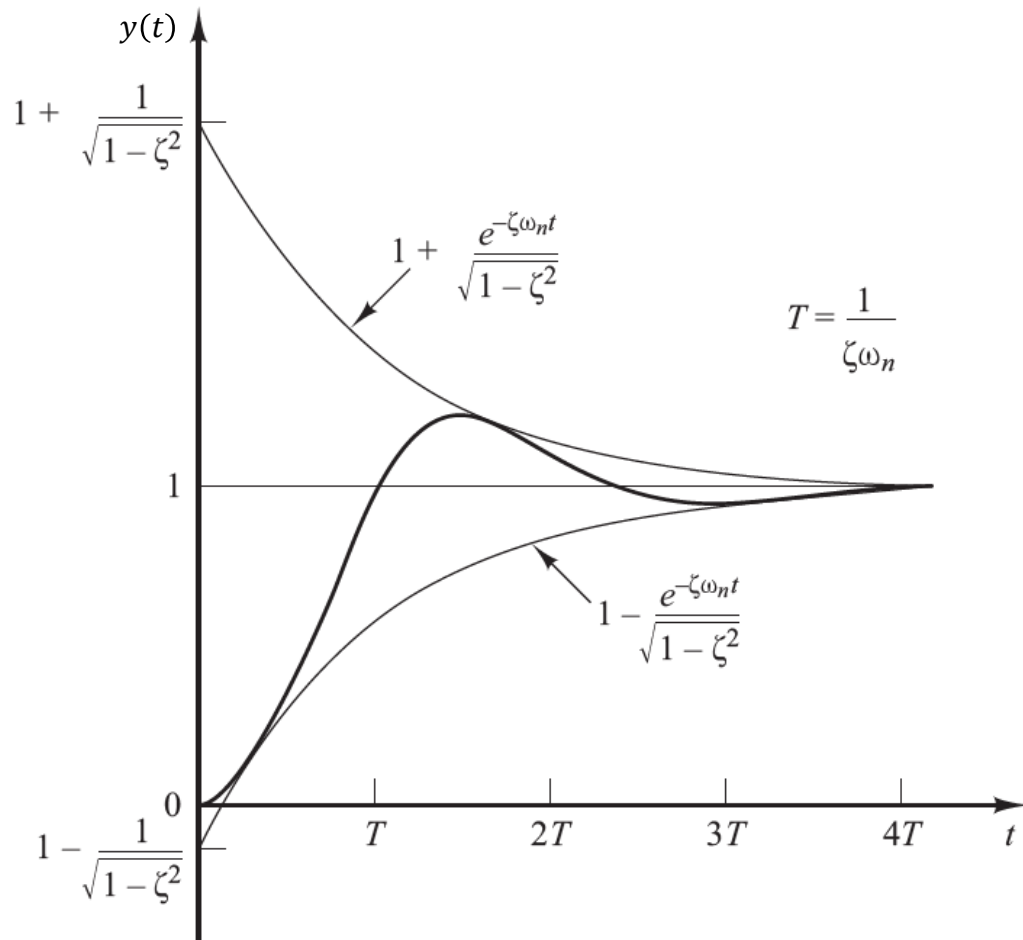
$$t_p = \frac{\pi}{\omega_d}$$

$$M_p = \frac{y(t_p) - y(\infty)}{y(\infty)}$$

$$M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$

$$\zeta = \sqrt{\frac{\ln^2(M_p)}{\pi^2 + \ln^2(M_p)}}$$

## ☐ Sistemas subamortecidos



$t_s$  Tempo de acomodação

$$y(t) = 1 - \frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}} \operatorname{sen} \left( \omega_d t + \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta} \right)$$

Critério de 2%:

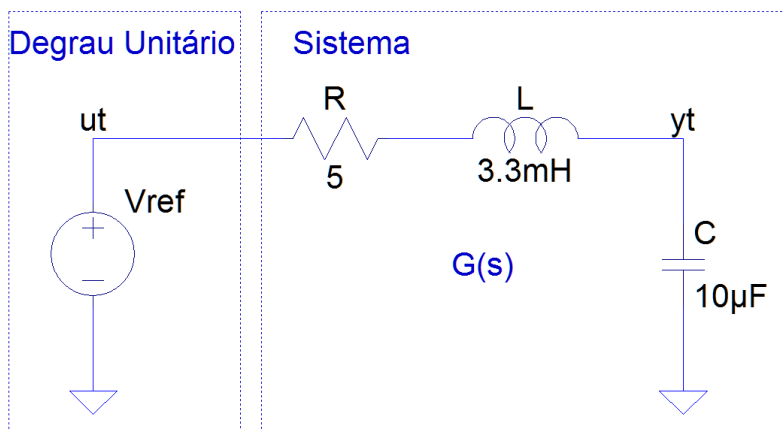
$$t_s = 4T = \frac{4}{\sigma} = \frac{4}{\zeta\omega_n}$$

Critério de 5%:

$$t_s = 3T = \frac{3}{\sigma} = \frac{3}{\zeta\omega_n}$$



## Resposta de Circuito RLC Série



Decaimento Exponencial

$$\sigma = \frac{R}{2L}$$

$$\sigma = \frac{5}{2 \times 3,3 \times 10^{-3}} = 757,6 \text{ rad/s}$$

Frequência Natural não Amortecida

$$\omega_n = \frac{1}{\sqrt{LC}}$$

$$\omega_n = \frac{1}{\sqrt{3,3 \times 10 \times 10^{-9}}} = 5504,8 \text{ rad/s}$$

Coefficiente de Amortecimento

$$\zeta = \frac{\sigma}{\omega_n}$$

$$\zeta = \frac{757,6}{5504,8} = 0,137 \text{ rad/s}$$

Sistema subamortecido:

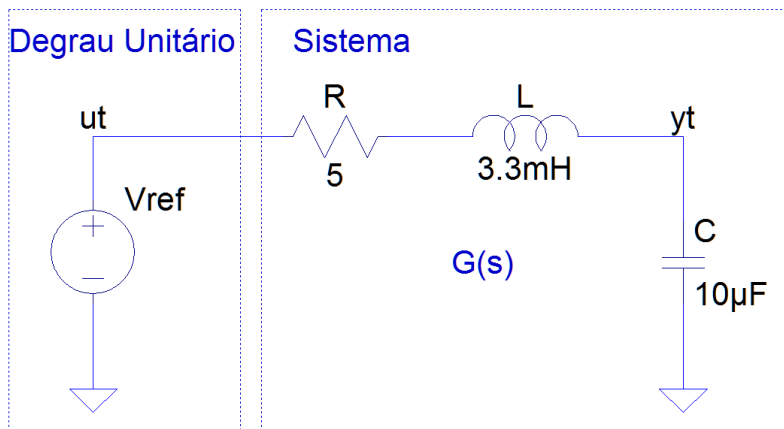
$$0 < \zeta < 1$$

Frequência Natural Amortecida

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

$$\omega_d = 5504,8 \sqrt{1 - 0,137^2} = 5452,9 \text{ rad/s}$$

## Resposta de Circuito RLC Série



$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$G(s) = \frac{5504,8^2}{s^2 + 2 \times 0,137 \times 5504,8s + 5504,8^2} = \frac{30,3 \times 10^6}{s^2 + 1515s + 30,3 \times 10^6}$$

$$\sigma = 757,6 \text{ rad/s}$$

$$\omega_n = 5504,8 \text{ rad/s}$$

$$\zeta = 0,137 \text{ rad/s}$$

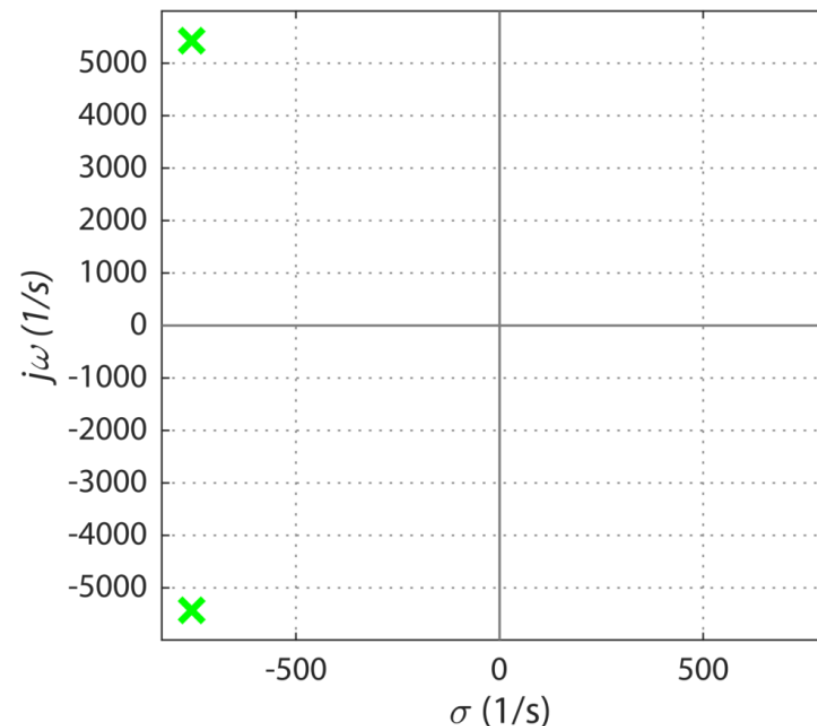
$$\omega_d = 5452,9 \text{ rad/s}$$

$$\beta = \cos^{-1} \zeta = 1,4334 \text{ rad/s}$$

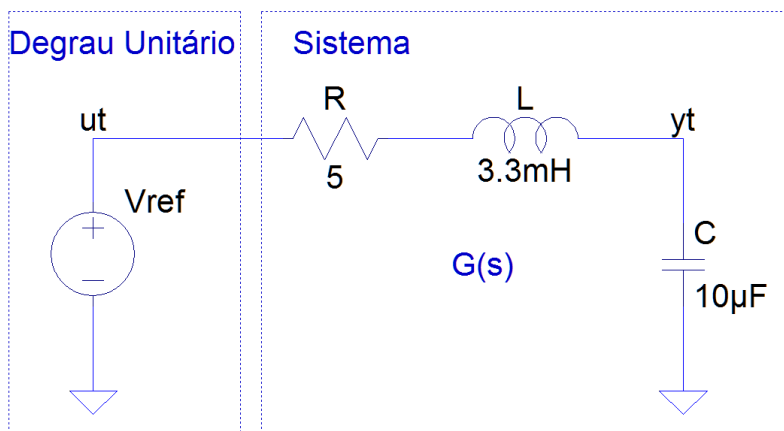
$$s = -\sigma \pm j\omega_d$$

$$s_1 = -757,6 + j5452,9$$

$$s_2 = -757,6 - j5452,9$$



## Resposta de Circuito RLC Série



$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{30,3 \times 10^6}{s^2 + 1508s + 30,3 \times 10^6}$$

### Parâmetros de Desempenho

$$t_r = \frac{\pi - \beta}{\omega_d} = \frac{\pi - 1,4334}{5452,9} = 313 \mu s$$

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{5452,9} = 576 \mu s$$

$$M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100\% = e^{-\frac{0,137\pi}{\sqrt{1-0,137^2}}} \times 100\% = 64,76\%$$

$$t_s = \frac{4}{\zeta\omega_n} = \frac{4}{0,137 \times 5504,8} = 5,3 ms \quad \text{Critério de 2\%:}$$

$$\sigma = 757,6 rad/s$$

$$\omega_n = 5504,8 rad/s$$

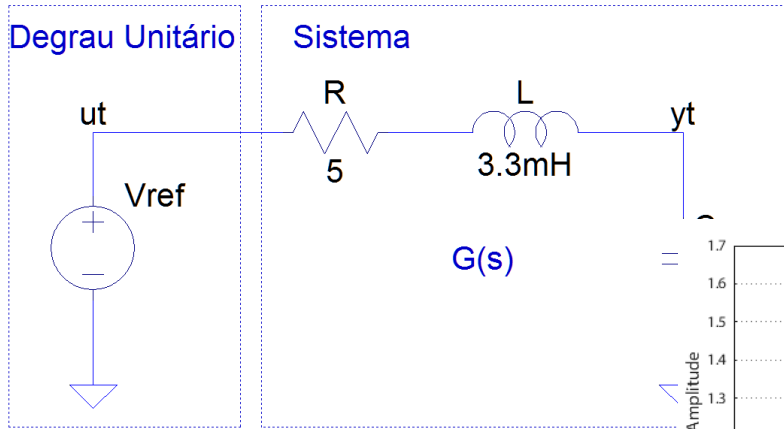
$$\zeta = 0,137 rad/s$$

$$\omega_d = 5452,9 rad/s$$

$$\beta = \cos^{-1} \zeta = 1,4334 rad/s$$

# Requisitos de Desempenho

## Resposta de Circuito RLC Série



$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{30,3 \times 10^6}{s^2 + 1508s + 30,3 \times 10^6}$$

$t_r = 0,313 \text{ ms}$      $t_p = 0,576 \text{ ms}$      $M_p = 64,76\%$      $t_s = 5,3 \text{ ms}$

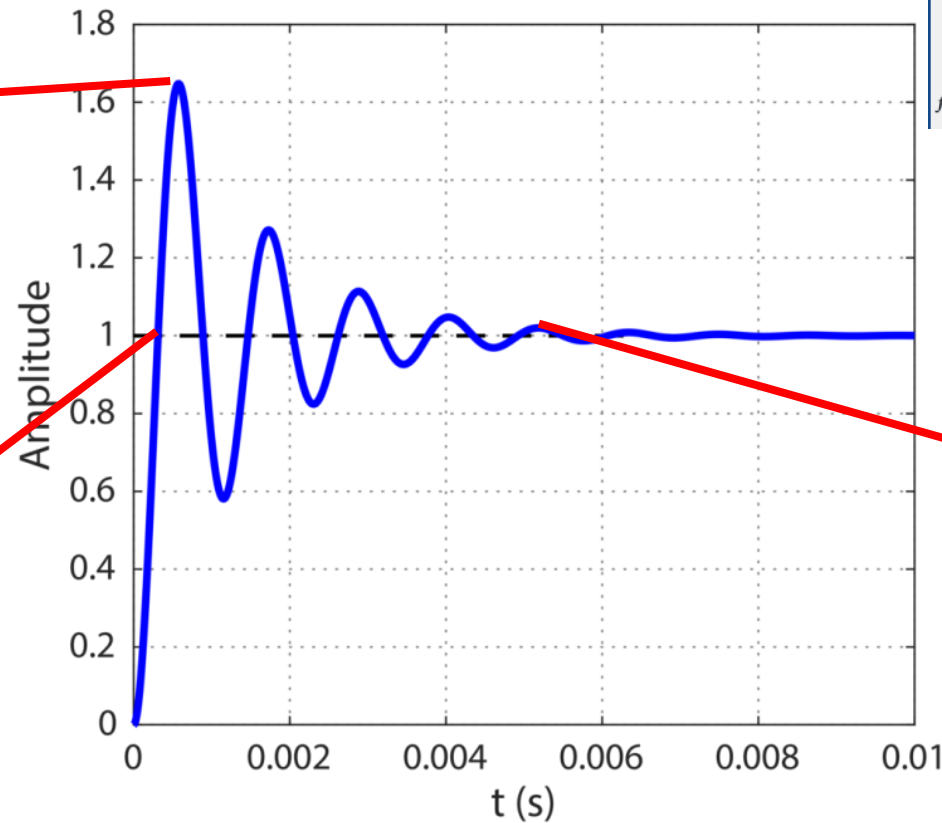
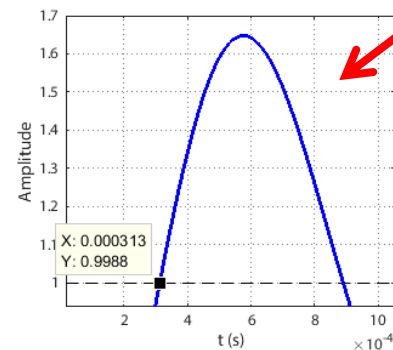
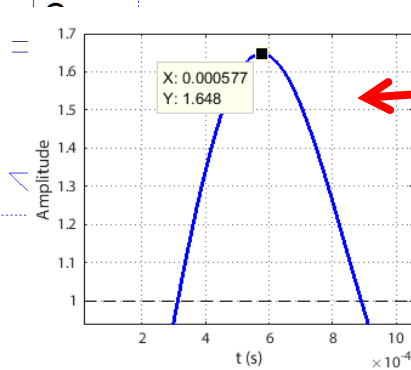
$$\sigma = 757,6 \text{ rad/s}$$

$$\omega_n = 5504,8 \text{ rad/s}$$

$$\zeta = 0,137$$

$$\omega_d = 5452,9 \text{ rad/s}$$

$$\beta = \cos^{-1} \zeta = 1,4334 \text{ rad/s}$$



### MATLAB

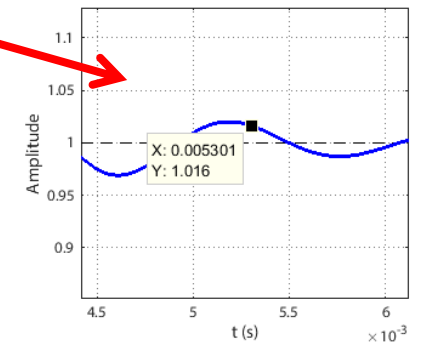
```
Command Window
>> s = tf('s');
>> gs = 30300000/(s^2+1508*s+30300000)

gs =

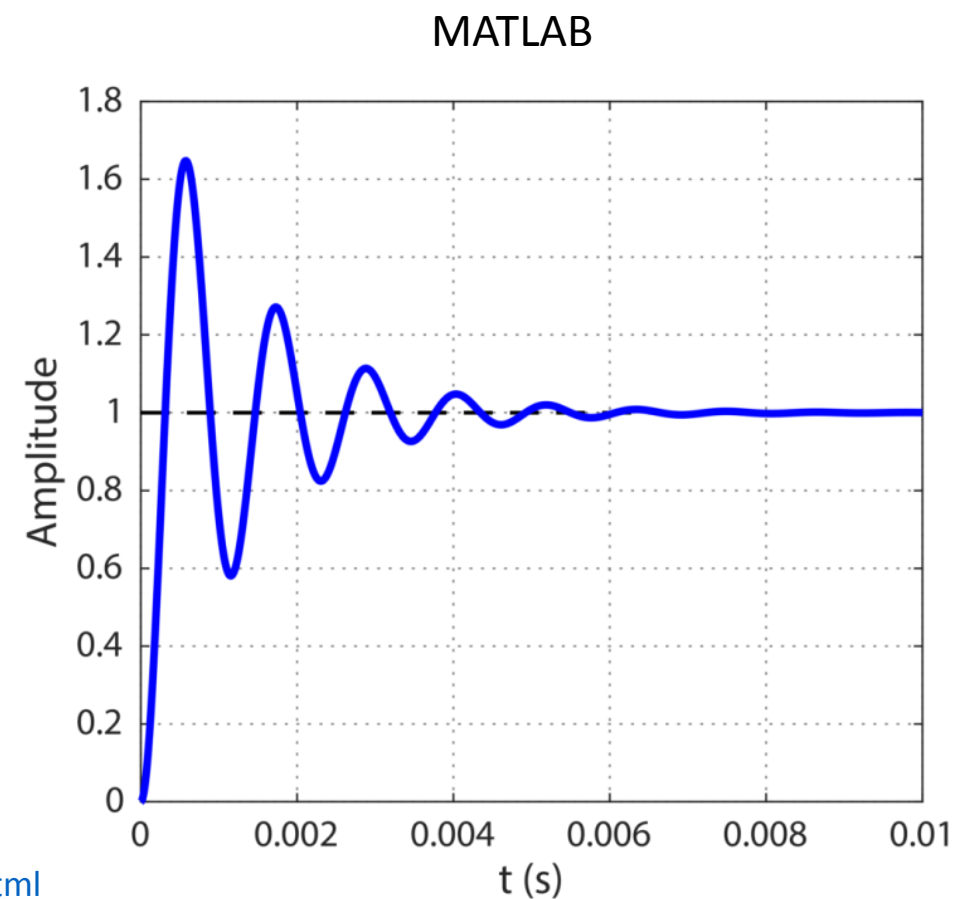
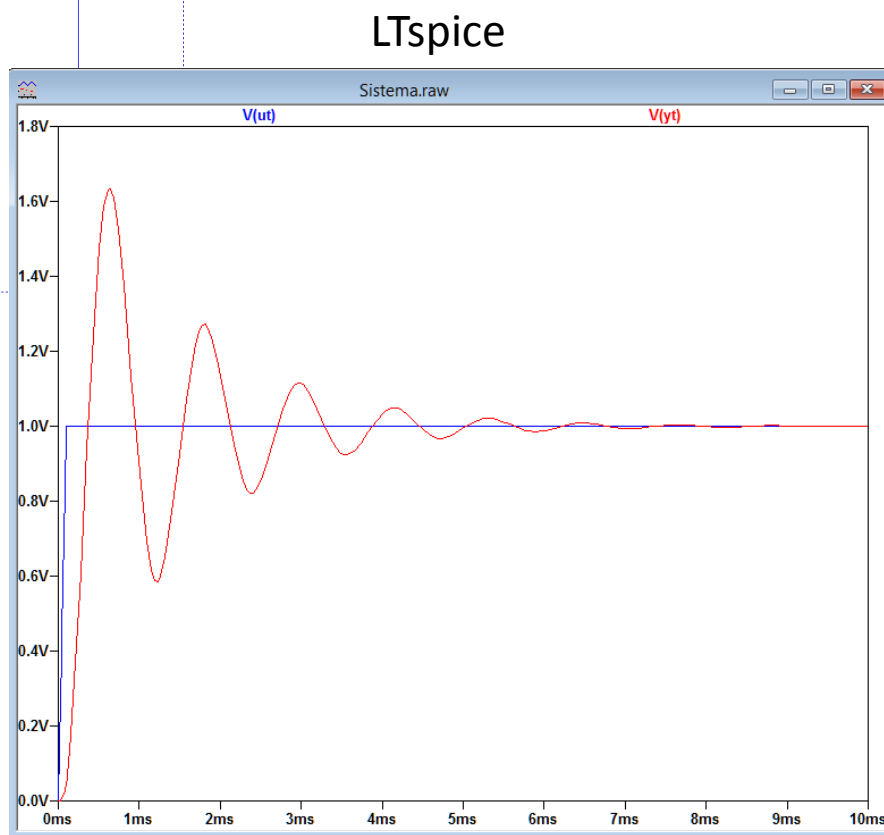
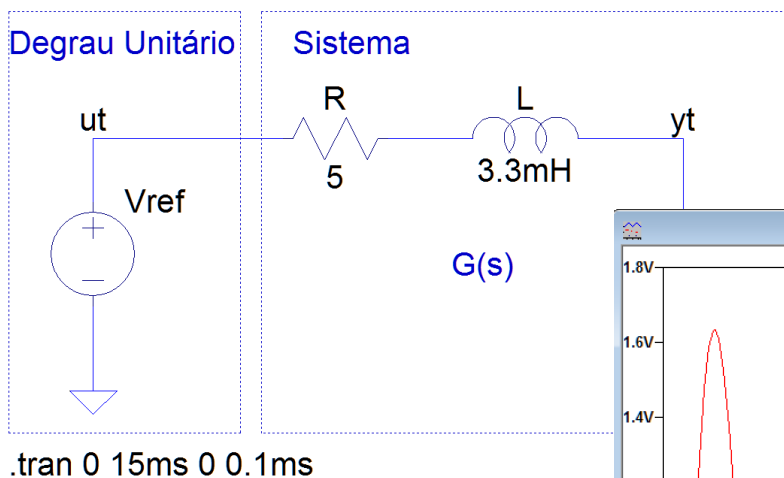
      3.03e07
-----
s^2 + 1508 s + 3.03e07

Continuous-time transfer function.

>> step(gs)
fx >>
```

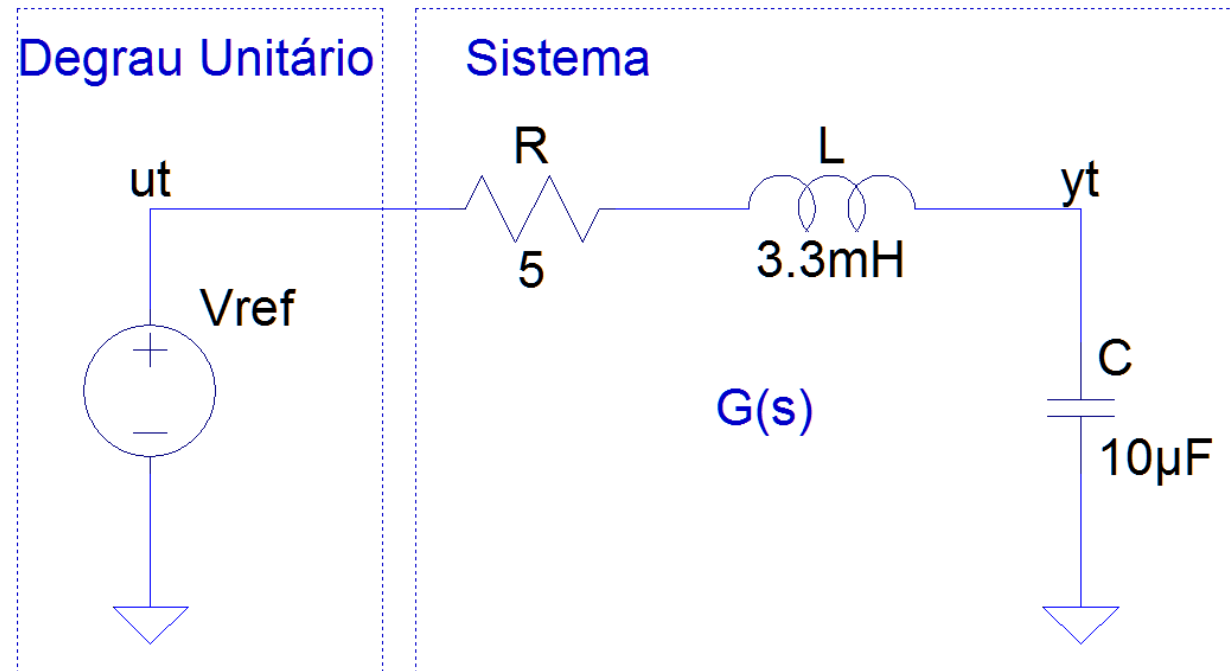


## Resposta de Circuito RLC Série

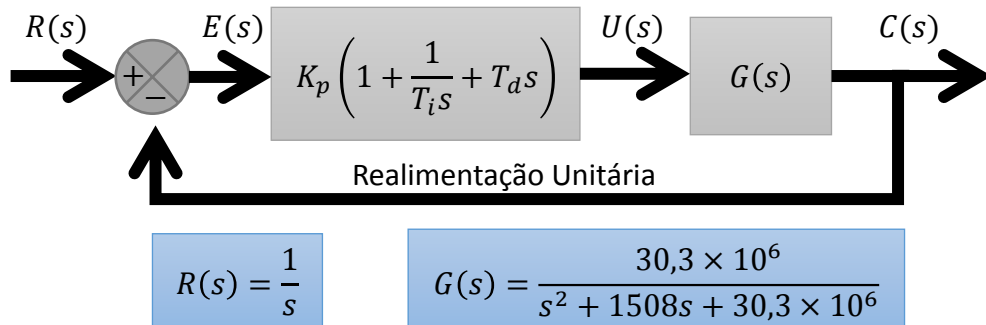


LTspice: <https://www.analog.com/en/design-center/design-tools-and-calculators/ltspice-simulator.html>

- Projeto de um controlador PID para que a resposta do circuito RLC abaixo possua sobressinal menor ou igual a 25%, tempo de acomodação  $< 1$  ms (critério de 2%) e erro estacionário nulo para entrada em degrau.



## Procurando uma solução por Ziegler-Nichols – Método II.

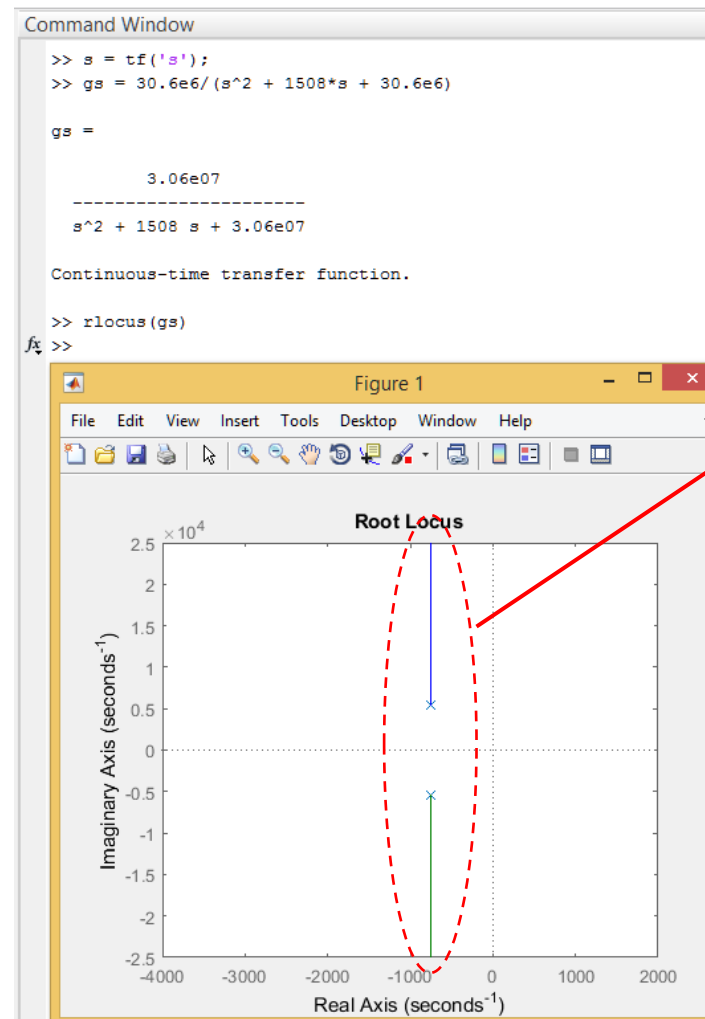


1 – Ajustar  $T_i = \infty$  e  $T_d = 0$ .

$$G_c(s) = K_p \left( 1 + \frac{1}{T_i s} + T_d s \right) = K_p (1 + 0 + 0) = K_{cr}$$

$$G_c(s)G(s) = K_{cr} \frac{30,3 \times 10^6}{s^2 + 1508s + 30,3 \times 10^6}$$

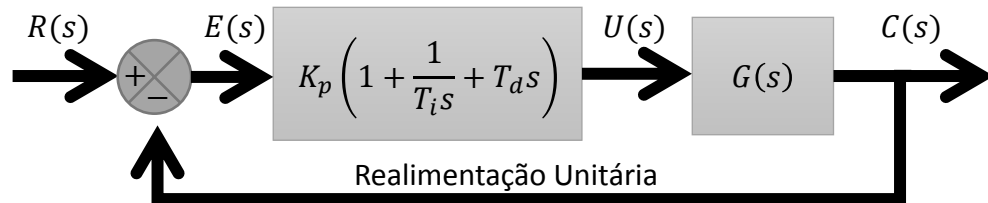
Sistema em Malha Aberta



Os ramos do LGR não cruzam o eixo imaginário, logo não é possível obter uma oscilação sustentada apenas controlando o ganho.

Não é possível obter o Período Crítico para calcular os parâmetros iniciais por Ziegler-Nichols.

## Procurando uma solução pelo Lugar das Raízes.



$$R(s) = \frac{1}{s}$$

$$G(s) = \frac{30,3 \times 10^6}{s^2 + 1508s + 30,3 \times 10^6}$$

$$G_c(s) = K_p \left( 1 + \frac{1}{T_i s} + T_d s \right) = K_p \left( \frac{T_d T_i s^2 + T_i s + 1}{T_i s} \right)$$

$$G_c(s) = K_p \left( T_d s^2 + s + \frac{1}{T_i} \right) = K_p \frac{(s + z_1)(s + z_2)}{s}$$

$$G_c(s) = K_p \frac{(s + z_1)(s + z_2)}{s}$$

$$z_{1,2} = \frac{-1 \pm \sqrt{1 - \frac{4T_d}{T_i}}}{2T_d}$$

Os valores de  $T_d$  e  $T_i$  controlam a posição dos zeros  $z_1$  e  $z_2$ .

Efeito do controlador sobre o sistema:

$$G_c(s)G(s) = K_p \frac{(s + z_1)(s + z_2)}{s} \frac{30,3 \times 10^6}{s^2 + 1508s + 30,3 \times 10^6}$$

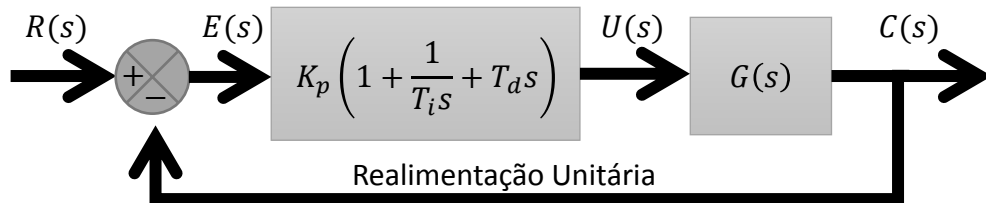
O controlador acrescenta ao sistema **2 zeros** e **1 polo na origem!!**

Assim, alocando valores para  $z_1$  e  $z_2$  é possível observar o LGR e procurar por valores de ganho que satisfaçam aos requisitos de projeto.

Esse processo equivale a escolher valores para  $K_p$ ,  $K_i$  e  $K_d$ !!!



## Procurando uma solução pelo Lugar das Raízes.



$$G_c(s)G(s) = K_p \frac{(s + z_1)(s + z_2)}{s} \frac{30,3 \times 10^6}{s^2 + 1508s + 30,3 \times 10^6}$$

$$G_c(s)G(s) = K_p \frac{30,3 \times 10^6 s^2 + 5,51 \times 10^{11} s + 3,405 \times 10^{15}}{s^3 + 1515s^2 + 30,3 \times 10^6 s}$$

Usando o MATLAB para obter o LGR

Command Window

```
>> s = tf('s');
>> R = 5;
>> L = 0.0033;
>> C = 0.00001;
>> sig = R/(2*L);
>> wn = 1/sqrt(L*C);
>> z = sig/wn;
>> wd = wn*sqrt(1-z^2);
>> gs = wn^2/(s^2+2*z*wn*s+wn^2);
>> z1 = 12*sig + wd*1i;
>> z2 = 12*sig - wd*1i;
>> gcs = ((s+z1)*(s+z2))/s;
>> sys = gcs*gs
```

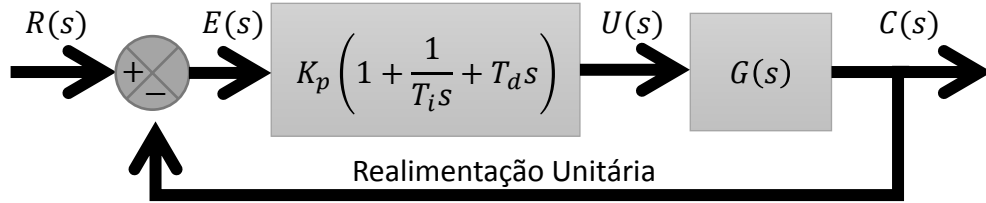
```
sys =

      3.03e07 s^2 + 5.51e11 s + 3.405e15
-----
      s^3 + 1515 s^2 + 3.03e07 s
```

Continuous-time transfer function.

```
>> figure();
>> rlocus(sys);
```

## Procurando uma solução pelo Lugar das Raízes



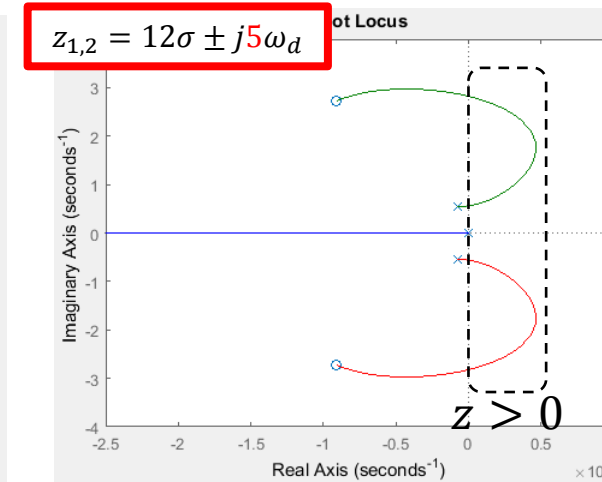
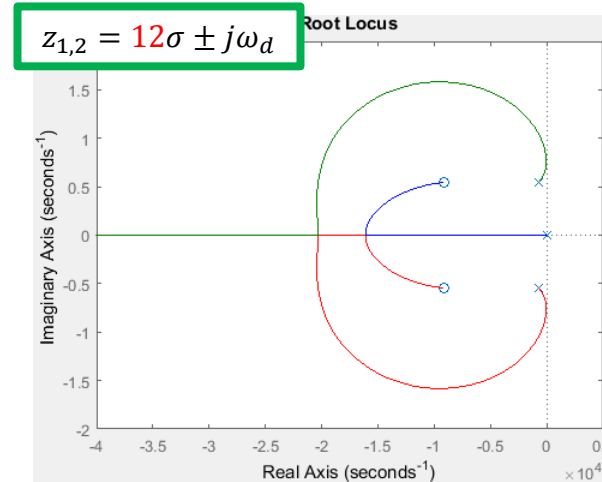
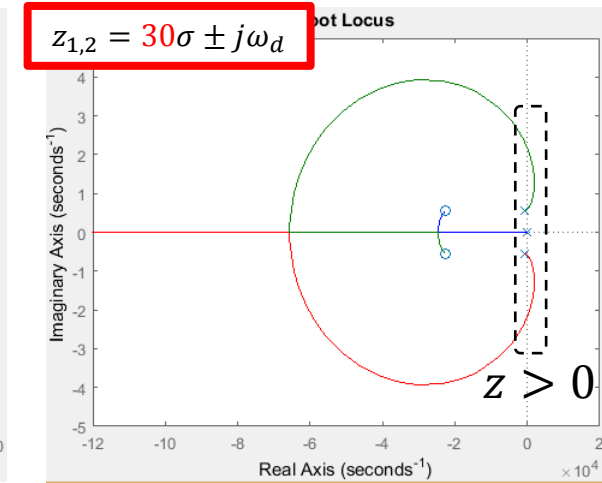
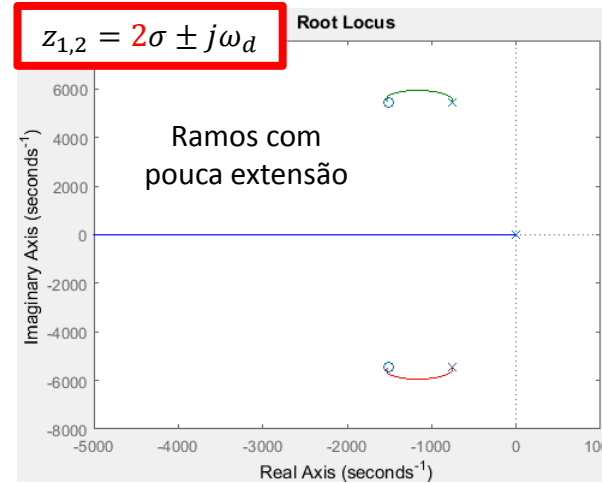
$$G_c(s)G(s) = K_p \frac{30,3 \times 10^6 s^2 + 5,51 \times 10^{11} s + 3,405 \times 10^{15}}{s^3 + 1515 s^2 + 30,3 \times 10^6 s}$$

### Alocação de $z_1$ e $z_2$ :

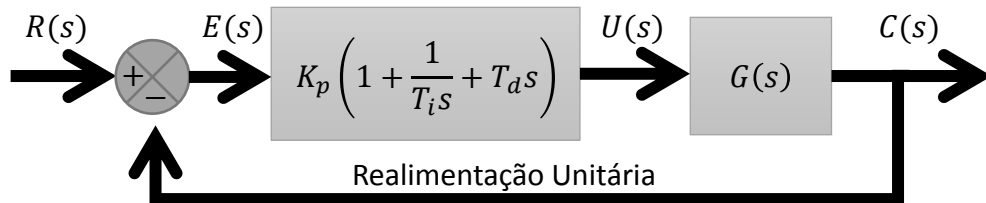
- 1 – Os ramos do LGR precisam estar totalmente no semiplano negativo de  $s$ .
- 2 – Selecionar um par de complexos conjugados:  $z_2 = \hat{z}_1$
- 3 – Dica: Fazer tentativas utilizando múltiplos de  $\sigma$  e  $\omega_d$ .

$$\sigma = 758 \text{ rad/s} \quad \omega_d = 5453 \text{ rad/s}$$

### Ajuste do LGR através da alocação de $z_1$ e $z_2$



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$$G_c(s)G(s) = K_p \frac{30,3 \times 10^6 s^2 + 5,51 \times 10^{11} s + 3,405 \times 10^{15}}{s^3 + 1515 s^2 + 30,3 \times 10^6 s}$$

$$\text{Zeros Alocados: } z_{1,2} = 12\sigma \pm j\omega_d$$

### Seleção do Ganho:

- 1 – Os requisitos de desempenho fornecem indicações para a seleção do ganho.
- 2 – Porém, os valores não são exatos pois o sistema deixa de ser de segunda ordem com a inserção do controlador.

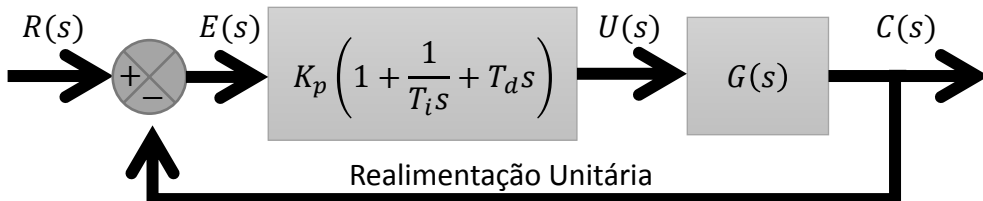
### Critério de 2%:

$$t_s = 4T = \frac{4}{\sigma} = \frac{4}{\zeta\omega_n} \quad \frac{4}{\sigma} < 0,001 \text{ s} \quad \sigma > 4000 \text{ rad/s}$$

### Overshoot < 25%:

$$M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \quad \zeta > \sqrt{\frac{\ln^2(0,25)}{\pi^2 + \ln^2(0,25)}} \quad \zeta > 0,403$$

## Procurando uma solução pelo Lugar das Raízes



$$G_c(s)G(s) = K_p \frac{30,3 \times 10^6 s^2 + 5,51 \times 10^{11} s + 3,405 \times 10^{15}}{s^3 + 1515 s^2 + 30,3 \times 10^6 s}$$

### Command Window

```
>> s = tf('s');
>> R = 5;
>> L = 0.0033;
>> C = 0.00001;
>> sig = R/(2*L);
>> wn = 1/sqrt(L*C);
>> z = sig/wn;
>> wd = wn*sqrt(1-z^2);
>> gs = wn^2/(s^2+2*z*wn*s+wn^2);
>> z1 = 12*sig + wd*1i;
>> z2 = 12*sig - wd*1i;
>> gcs = ((s+z1)*(s+z2))/s;
```

```
>> sys = gcs*gs

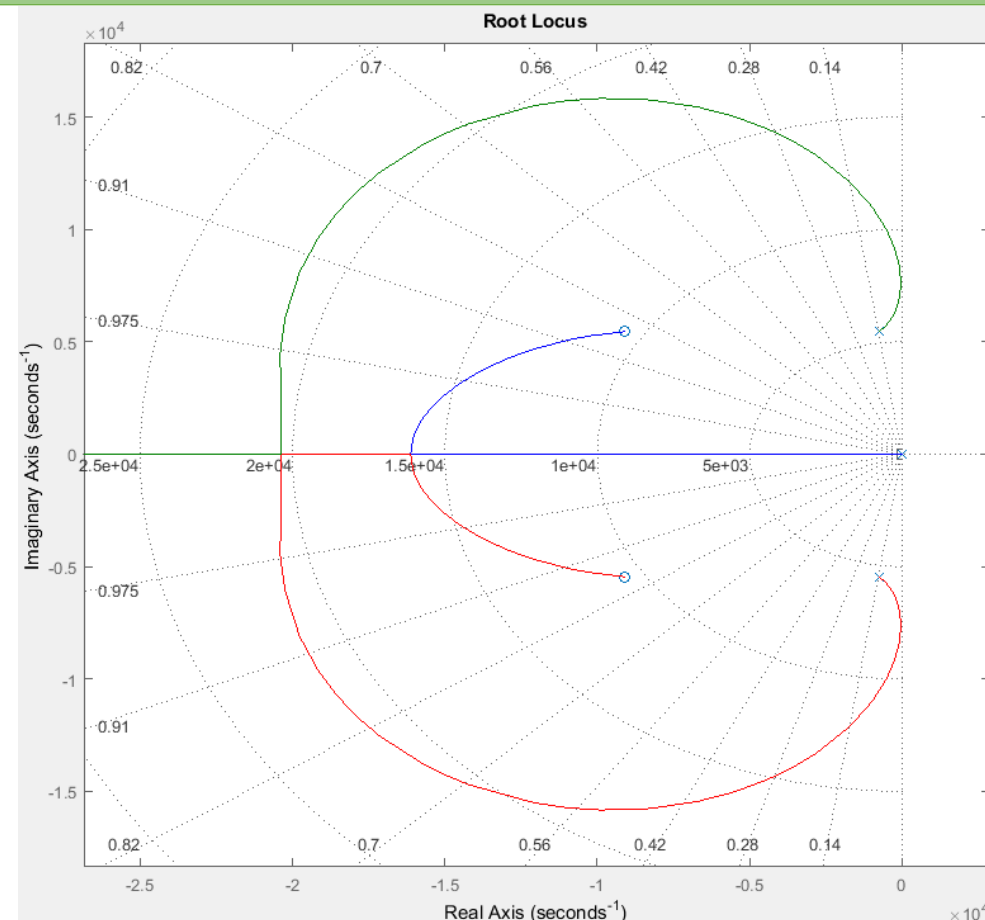
sys =

      3.03e07 s^2 + 5.51e11 s + 3.405e15
-----
      s^3 + 1515 s^2 + 3.03e07 s

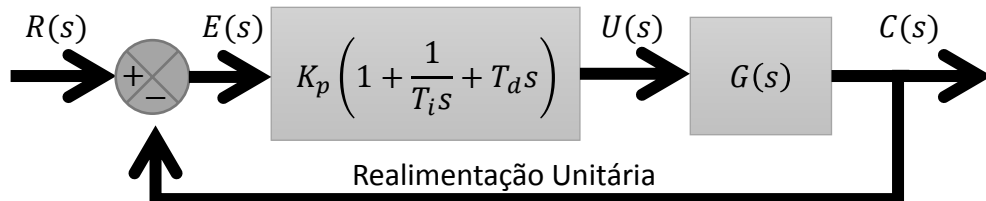
Continuous-time transfer function

>> figure();
>> rlocus(sys);
>> sgrid
fx >>
```

O comando **sgrid** do MATLAB permite visualizar faixas de ganho que resultam em valores constantes de  $\zeta$  e  $\omega_n$ .



## Procurando uma solução pelo Lugar das Raízes



$$G_c(s)G(s) = K_p \frac{30,3 \times 10^6 s^2 + 5,51 \times 10^{11} s + 3,405 \times 10^{15}}{s^3 + 1515 s^2 + 30,3 \times 10^6 s}$$

### Command Window

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>> s = tf('s');
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>> gs = wn^2/(s^2+2*z*wn*s+wn^2);
>> z1 = 12*sig + wd*1i;
>> z2 = 12*sig - wd*1i;
>> gcs = ((s+z1)*(s+z2))/s;
```

```
>> sys = gcs*gs

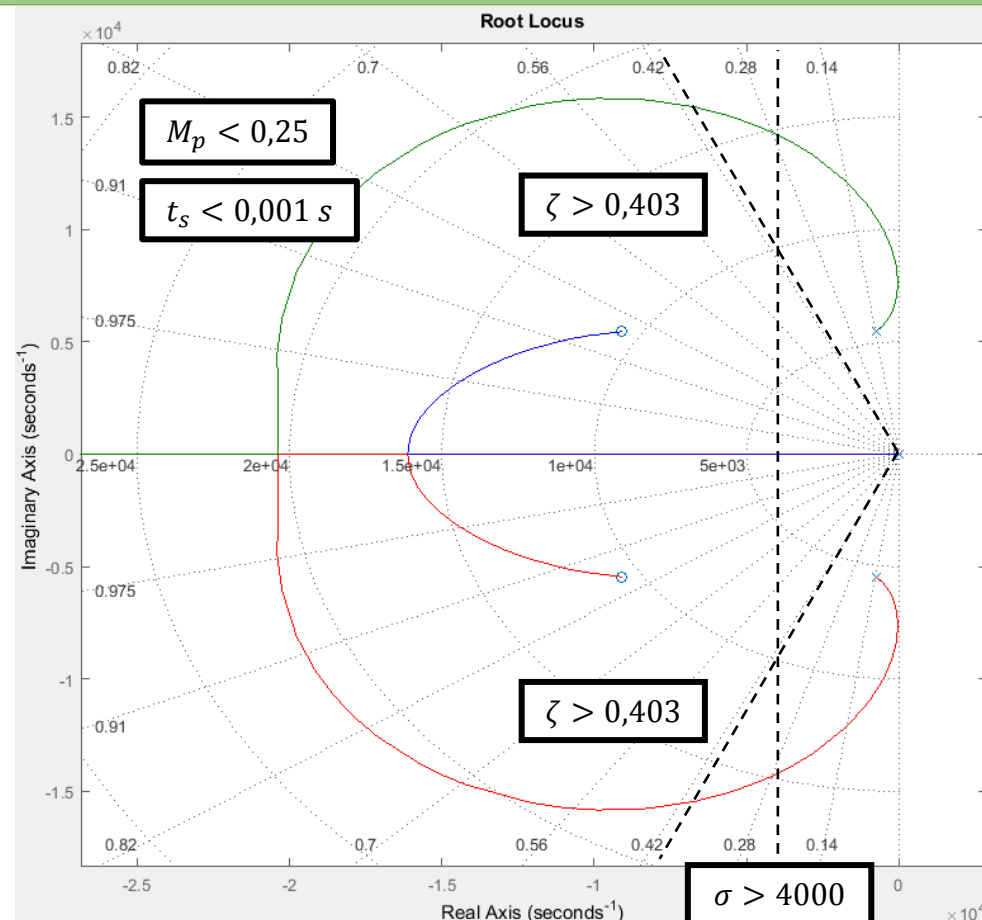
sys =

      3.03e07 s^2 + 5.51e11 s + 3.405e15
-----
      s^3 + 1515 s^2 + 3.03e07 s

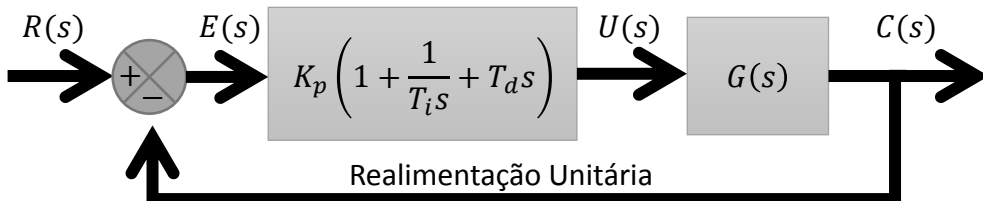
Continuous-time transfer function

>> figure();
>> rlocus(sys);
>> sgrid
fx >>
```

Assim é possível localizar as regiões do LGR que atendem aos requisitos de desempenho.



## Procurando uma solução pelo Lugar das Raízes



$$G_c(s)G(s) = K_p \frac{30,3 \times 10^6 s^2 + 5,51 \times 10^{11} s + 3,405 \times 10^{15}}{s^3 + 1515 s^2 + 30,3 \times 10^6 s}$$

### Command Window

```
>> s = tf('s');
>> R = 5;
>> L = 0.0033;
>> C = 0.00001;
>> sig = R/(2*L);
>> wn = 1/sqrt(L*C);
>> z = sig/wn;
>> wd = wn*sqrt(1-z^2);
>> gs = wn^2/(s^2+2*z*wn*s+wn^2);
>> z1 = 12*sig + wd*1i;
>> z2 = 12*sig - wd*1i;
>> gcs = ((s+z1)*(s+z2))/s;
```

```
>> sys = gcs*gs

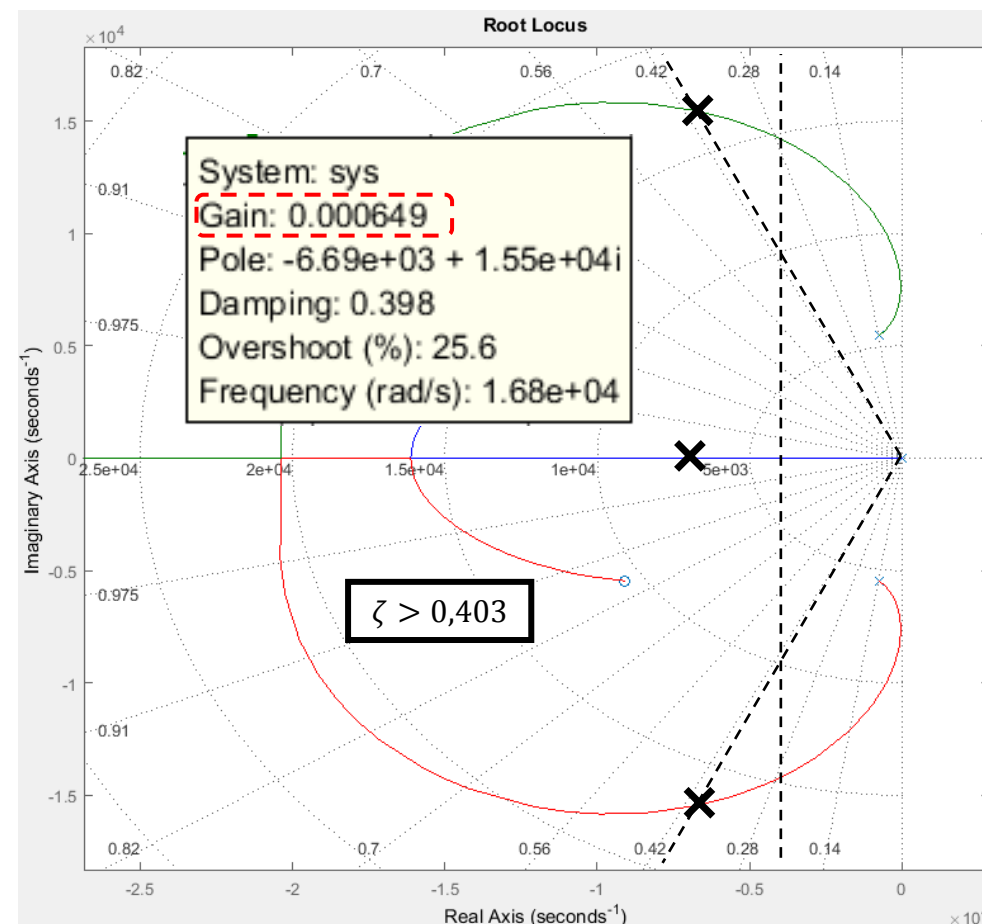
sys =

      3.03e07 s^2 + 5.51e11 s + 3.405e15
-----
      s^3 + 1515 s^2 + 3.03e07 s

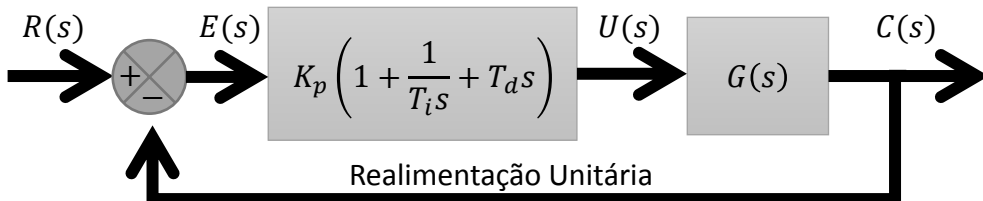
Continuous-time transfer function

>> figure();
>> rlocus(sys);
>> sgrid
fx >>
```

Por fim, seleciona-se o valor do ganho.



## Procurando uma solução pelo Lugar das Raízes



Command Window

```
>> s = tf('s');
>> R = 5;
>> L = 0.0033;
>> C = 0.00001;
>> sig = R/(2*L);
>> wn = 1/sqrt(L*C);
>> z = sig/wn;
>> wd = wn*sqrt(1-z^2);
>> gs = wn^2/(s^2+2*z*wn*s+wn^2);
>> z1 = 12*sig + wd*1i;
>> z2 = 12*sig - wd*1i;
>> gcs = ((s+z1)*(s+z2))/s;
```

```
>> k = 0.000649;
>> sys = feedback(k*gcs*gs,1)

sys =

      1.967e04 s^2 + 3.576e08 s + 2.21e12
-----
      s^3 + 2.118e04 s^2 + 3.879e08 s + 2.21e12

Continuous-time transfer function.

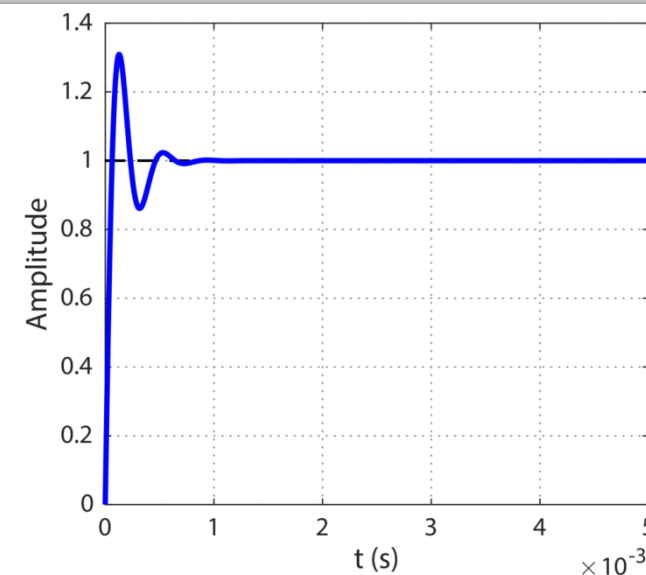
>> t = (0:0.001:5)*1e-3;
>> step(sys,t,opt);
>> stepinfo(sys,'SettlingTimeThreshold',0.02)
```

```
ans =

      RiseTime: 4.9608e-05
      SettlingTime: 5.6274e-04
      SettlingMin: 0.8612
      SettlingMax: 1.3084
      Overshoot: 30.8431
      Undershoot: 0
      Peak: 1.3084
      PeakTime: 1.3072e-04
```

$$T(s) = \frac{K_p G_c(s) G(s)}{1 + K_p G_c(s) G(s)}$$

$$T(s) = \frac{1,967 \times 10^4 s^2 + 3,576 \times 10^8 s + 2,21 \times 10^{12}}{s^3 + 2,118 \times 10^4 s^2 + 3,879 \times 10^8 s + 2,21 \times 10^{12}}$$



Requisito

$$M_p < 0,25$$

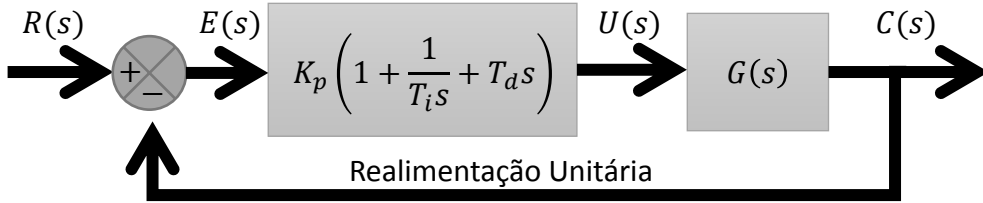
$$t_s < 1 \text{ ms}$$

Resultado

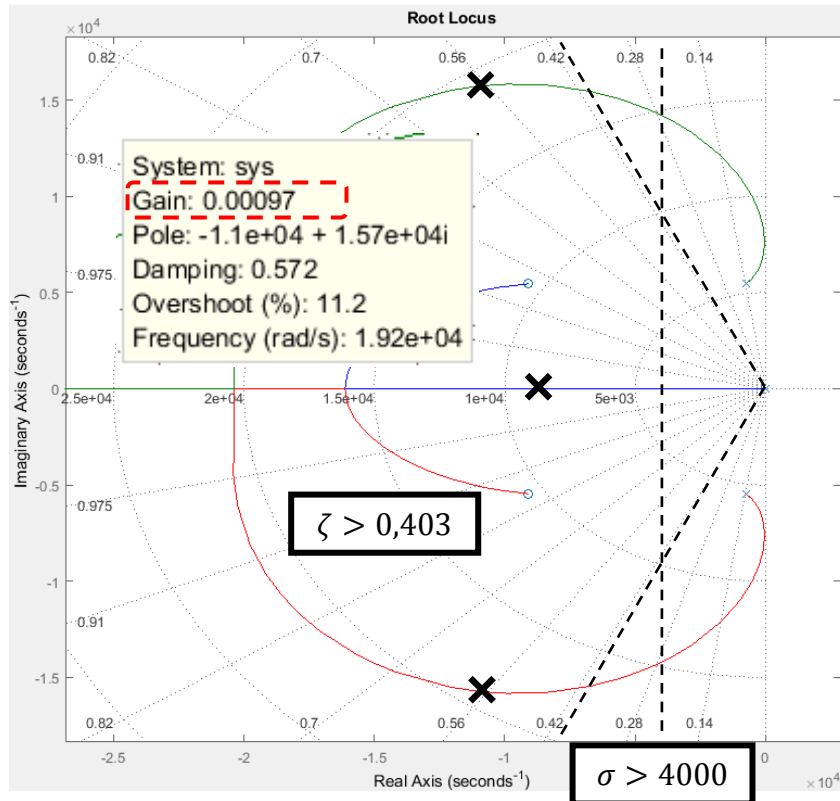
$$M_p = 0,31$$

$$t_s = 562 \mu\text{s}$$

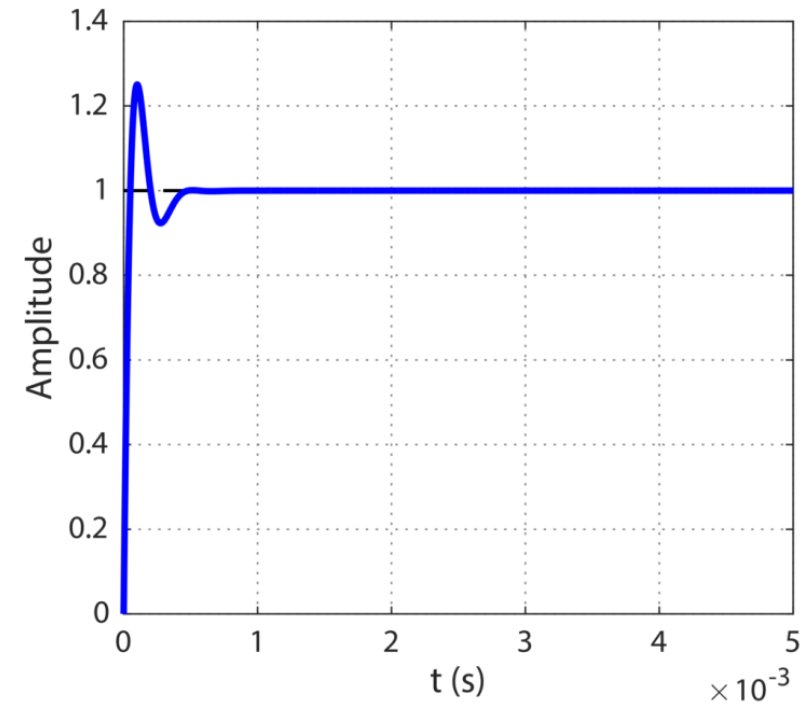
## Procurando uma solução pelo Lugar das Raízes



Selecionando um novo valor de ganho.



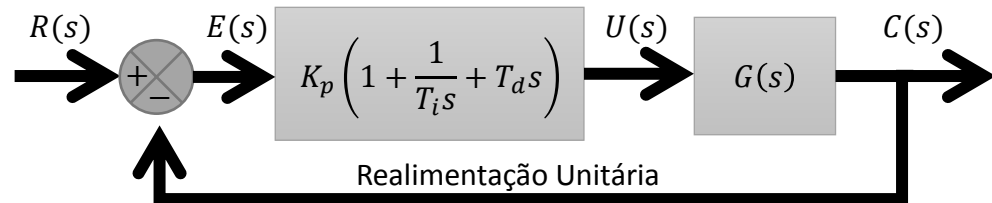
$$T(s) = \frac{2,939 \times 10^4 s^2 + 5,344 \times 10^8 s + 3,303 \times 10^{12}}{s^3 + 3,091 \times 10^4 s^2 + 5,647 \times 10^8 s + 3,303 \times 10^{12}}$$



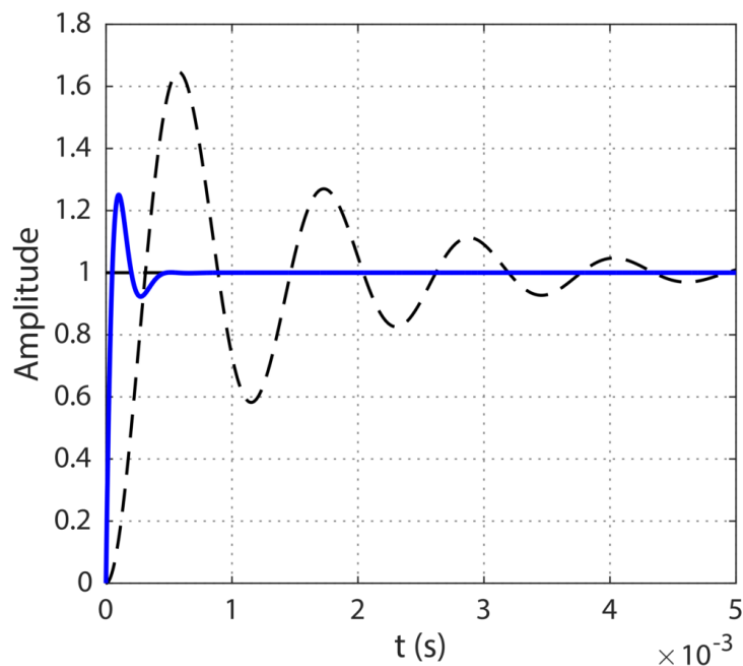
| Requisito            | Resultado               |
|----------------------|-------------------------|
| $M_p < 0,25$         | $M_p = 0,25$            |
| $t_s < 1 \text{ ms}$ | $t_s = 395 \mu\text{s}$ |



## Parâmetros Finais.



$$T(s) = \frac{2,939 \times 10^4 s^2 + 5,344 \times 10^8 s + 3,303 \times 10^{12}}{s^3 + 3,091 \times 10^4 s^2 + 5,647 \times 10^8 s + 3,303 \times 10^{12}}$$



Requisito  
 $M_p < 0,25$   
 $t_s < 1 \text{ ms}$

Resultado  
 $M_p = 0,25$   
 $t_s = 395 \mu\text{s}$

### Parâmetros do Controlador PID

$$G_c(s) = K_p \left( T_d s^2 + s + \frac{1}{T_i} \right) = \frac{0,00097s^2 + 17,63s + 1,09 \times 10^5}{s}$$

$$K_p = 17,63$$

$$K_d = K_p T_d = 0,00097$$

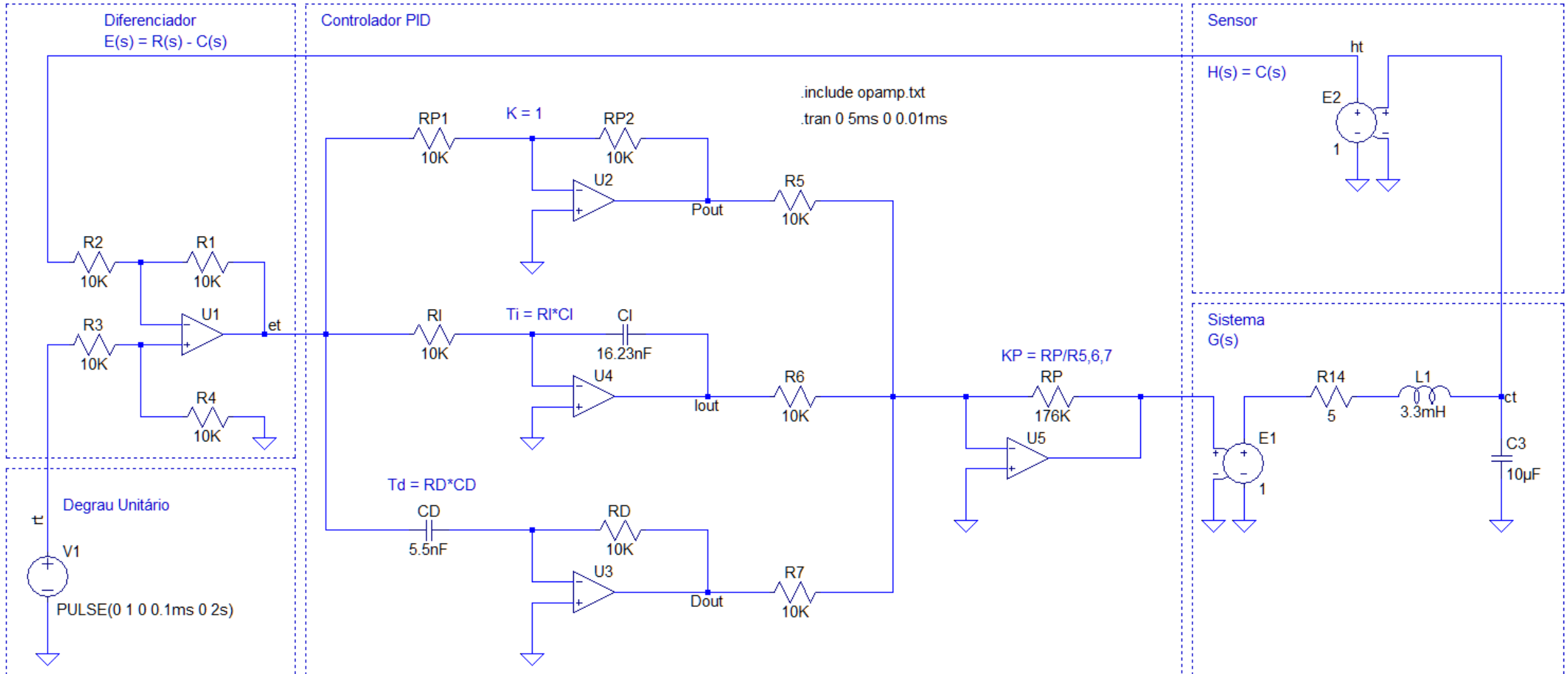
$$T_d = 55 \mu\text{s}$$

$$K_i = \frac{K_p}{T_i} = 1,09 \times 10^5$$

$$T_i = 162,3 \mu\text{s}$$

# Projeto de Controlador PID

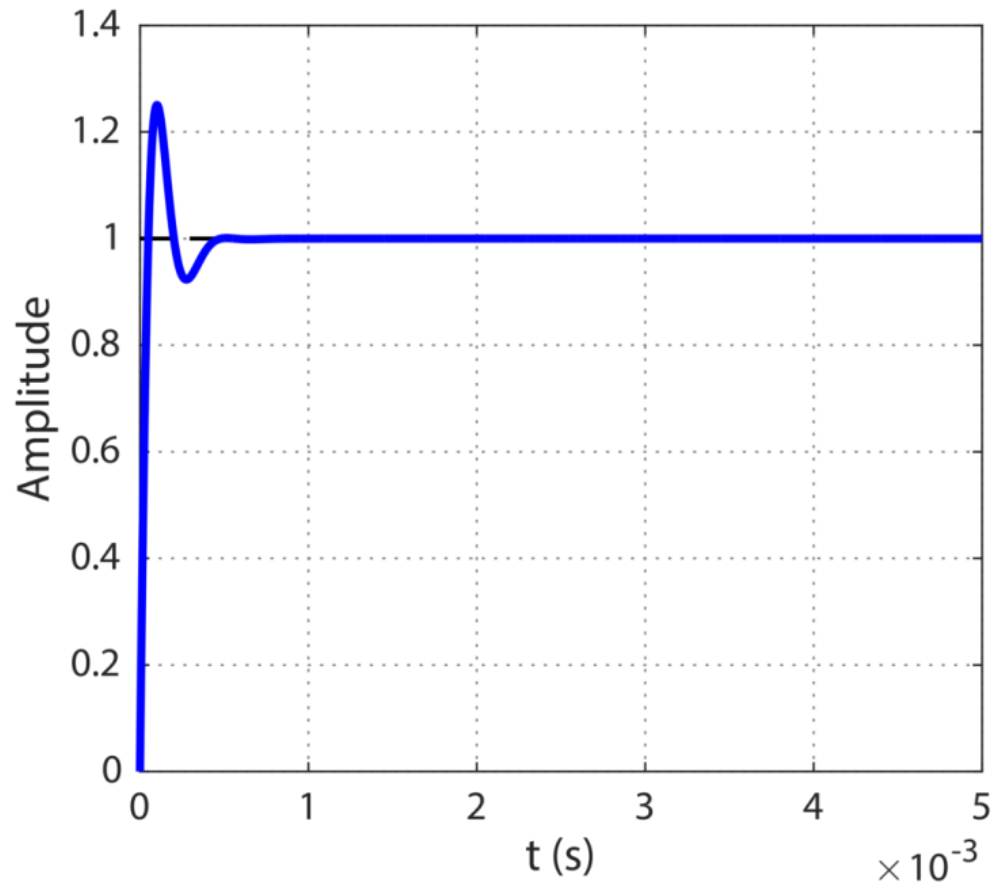
## Verificando.



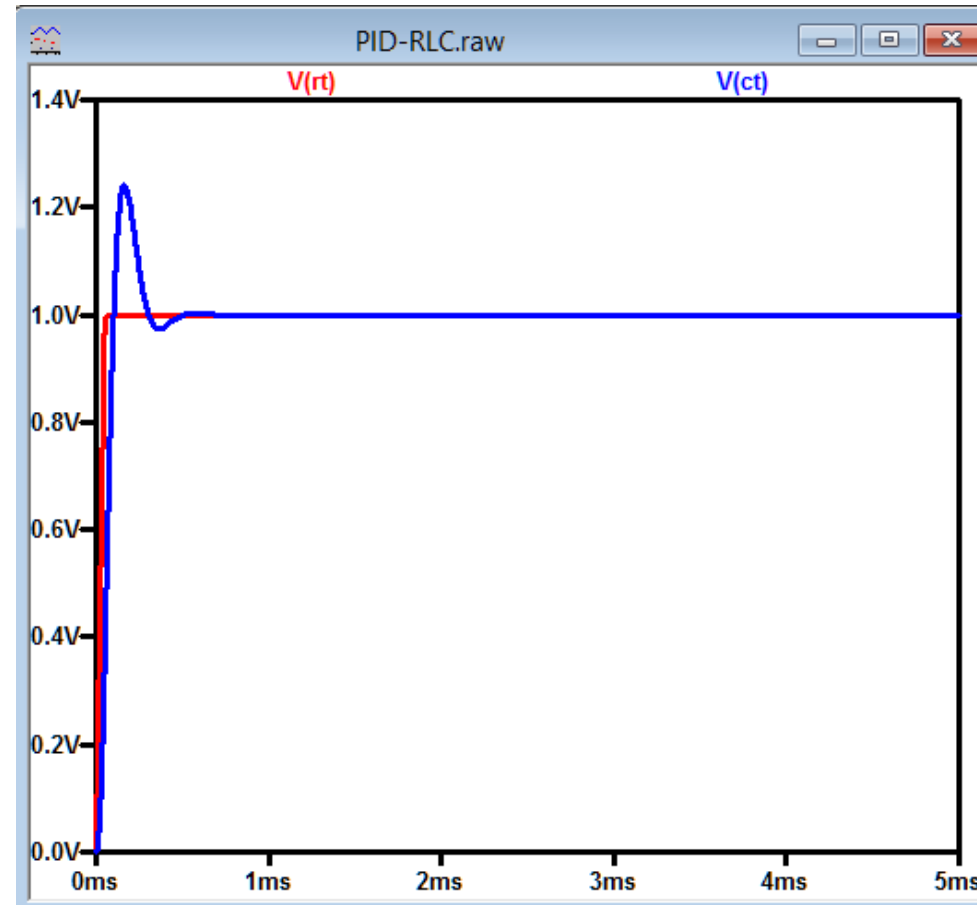
# Projeto de Controlador PID

Verificando.

MATLAB

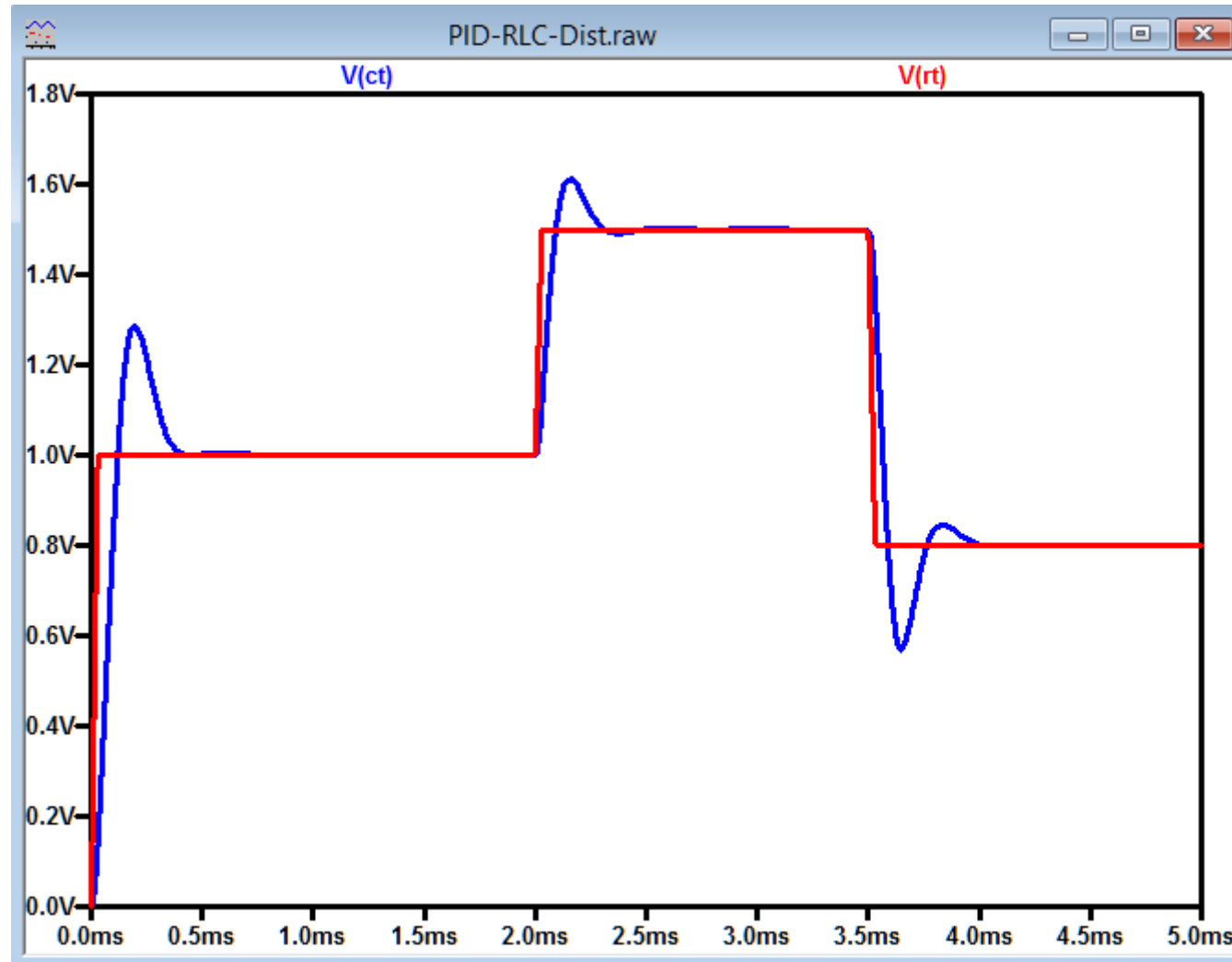


LTspice



# Projeto de Controlador PID

Verificando.



# Referências Bibliográficas

- ❑ K. Ogata, “Engenharia de controle moderno”, 5 ed., São Paulo, Pearson Prentice Hall (2010).
- ❑ C. K. Alexander, e M. N. O. Sadiku, “Fundamentos de Circuitos Elétricos”, 5 ed., Porto Alegre, AMGH (2013).