$$\begin{cases} x_{k+1} = x_k + u_k \\ y_k = x_k + w_k \end{cases}$$

$A = 1 \quad C = 0$

$B = 1 \quad H = 1 \quad G = [1]$

$D = 0 \,, F = 1$

$$\min E\left(\sum_{k=0}^{T-1} u_k^2 + q x_T^2 \right)$$

$P(0) = \infty$

$\Updownarrow$

Filtro de Kalman: $P^{-1}(0) = 0$ (total falta de informação de x_0)

$$P(k+1) = P(k) + 0 - \frac{P(k)^2}{P(k)+1} = \frac{P(k)\left(P(k)+1\right) - P(k)^2}{P(k)+1}$$

$$= \frac{P(k)}{P(k)+1} \Rightarrow \frac{P(k)+1}{P(k)} = \frac{1}{P(k+1)}$$

$$\Rightarrow \begin{cases} 1 + P^{-1}(k) = P^{-1}(k+1) \\ P^{-1}(0) = 0 \end{cases} \Rightarrow P^{-1}(k) = k$$

$$\Rightarrow P(k) = \frac{1}{k}$$

$$Q(k) = \frac{P(k)}{P(k)+1} = \frac{1/k}{1/k + 1} = \frac{1}{k+1}$$

$$\hat{x}_{k+1|k} = \hat{x}_{k|k-1} + u_k + Q(k)\left(y_k - \hat{x}_{k|k-1}\right) \quad Q(k) = 1/(k+1)$$

$$= \hat{x}_{k|k-1} + u_k + \frac{y_k - \hat{x}_{k|k-1}}{k+1}$$

$$= \frac{k\,\hat{x}_{k|k-1} + (k+1)\,u_k + y_k}{k+1}$$

Controle Ótimo

$$\begin{cases} S(k) = S(k+1) - \dfrac{S(k+1)^2}{S(k+1)+1} = \dfrac{S(k+1)}{S(k+1)+1} \\ S(T) = q \end{cases}$$

$$S(k)^{-1} = S(k+1)^{-1} + 1, \quad S(T)^{-1} = \frac{1}{q}$$

$$S(k)^{-1} = (T-k) + \frac{1}{q} \Rightarrow S(k) = \frac{q}{1+q(T-k)}$$

$$M(k) = \frac{\dfrac{q}{1+q(T-k-1)}}{\dfrac{q}{1+q(T-k-1)} + 1} = \frac{q}{q + 1 + q(T-k-1)}$$

$$= \frac{q}{1+q(T-k)} \Rightarrow u_k = -\frac{q}{1+q(T-k)}\,\hat{x}_{k|k-1}$$

It is important to grasp the remarkable fact that (ii) asserts: *the optimal control u_t is exactly the same as it would be if all unknowns were known and took values equal to their linear least square estimates (equivalently, their conditional means) based upon observations up to time t.* This is the idea known as **certainty equivalence**. As we have seen in the previous section, the distribution of the estimation error $\hat{x}_t - x_t$ does not depend on U_{t-1}. The fact that the problems of optimal estimation and optimal control can be decoupled in this way is known as the **separation principle**.

Proof. The proof is by backward induction. Suppose (11.8) holds at t. Recall that

$$\hat{x}_t = A\hat{x}_{t-1} + Bu_{t-1} + H_t\zeta_t, \qquad \Delta_{t-1} = \hat{x}_{t-1} - x_{t-1}.$$

Then with a quadratic cost of the form $c(x,u) = x^\top Rx + 2u^\top Sx + u^\top Qu$, we have

$$\begin{aligned} F(W_{t-1}) &= \min_{u_{t-1}} E\left[c(x_{t-1}, u_{t-1}) + \hat{x}_t\Pi_t\hat{x}_t + \cdots \mid W_{t-1}, u_{t-1}\right] \\ &= \min_{u_{t-1}} E\big[c(\hat{x}_{t-1} - \Delta_{t-1}, u_{t-1}) \\ &\quad + (A\hat{x}_{t-1} + Bu_{t-1} + H_t\zeta_t)^\top \Pi_t (A\hat{x}_{t-1} + Bu_{t-1} + H_t\zeta_t) \mid W_{t-1}, u_{t-1}\big] \\ &= \min_{u_{t-1}} \left[c(\hat{x}_{t-1}, u_{t-1}) + (A\hat{x}_{t-1} + Bu_{t-1})^\top \Pi_t (A\hat{x}_{t-1} + Bu_{t-1})\right] + \cdots, \end{aligned}$$

where we use the fact that conditional on W_{t-1}, u_{t-1}, both Δ_{t-1} and ζ_t have zero means and are policy independent. This ensures that when we expand the quadratics in powers of Δ_{t-1} and $H_t\zeta_t$ the expected value of the linear terms in these quantities are zero and the expected value of the quadratic terms (represented by $+\cdots$) are policy independent. ■

11.4 Example: inertialess rocket with noisy position sensing

Consider the scalar case of controlling the position of a rocket by inertialess control of its velocity but in the presence of imperfect position sensing.

$$x_t = x_{t-1} + u_{t-1}, \qquad y_t = x_t + \eta_t,$$

where η_t is white noise with variance 1. Suppose it is desired to minimize

$$E\left[\sum_{t=0}^{h-1} u_t^2 + Dx_h^2\right].$$

Notice that the observational relation differs from the usual model of $y_t = Cx_{t-1} + \eta_t$. To derive a Kalman filter formulae for this variation we argue inductively from scratch. Suppose $\hat{x}_{t-1} - x_{t-1} \sim N(0, V_{t-1})$. Consider a linear estimate of x_t,

$$\hat{x}_t = \hat{x}_{t-1} + u_{t-1} + H_t(y_t - \hat{x}_{t-1} - u_{t-1}).$$

(The relevant innovation process is now $\tilde{y}_t = y_t - \hat{x}_{t-1} - u_{t-1}$.) Subtracting the plant equation and substituting for x_t and y_t gives

$$\Delta_t = \Delta_{t-1} + H_t(\eta_t - \Delta_{t-1}).$$

The variance of Δ_t is therefore

$$\operatorname{var}\Delta_t = V_{t-1} - 2H_tV_{t-1} + H_t^2(1 + V_{t-1}).$$

Minimizing this with respect to H_t gives $H_t = V_{t-1}(1 + V_{t-1})^{-1}$, so the variance in the least squares estimate of x_t obeys the recursion,

$$V_t = V_{t-1} - V_{t-1}^2(1 + V_{t-1})^{-1} = V_{t-1}/(1 + V_{t-1}).$$

Hence

$$V_t^{-1} = V_{t-1}^{-1} + 1 = \cdots = V_0^{-1} + t.$$

If there is complete lack of information at the start, then $V_0^{-1} = 0$, $V_t = 1/t$ and

$$\hat{x}_t = \hat{x}_{t-1} + u_{t-1} + \frac{V_{t-1}(y_t - \hat{x}_{t-1} - u_{t-1})}{1 + V_{t-1}} = \frac{(t-1)(\hat{x}_{t-1} + u_{t-1}) + y_t}{t}.$$

As far at the optimal control is concerned, suppose an inductive hypothesis that $F(W_t) = \hat{x}_t^2\Pi_t + \cdots$, where '$\cdots$' denotes policy independent terms. Then

$$\begin{aligned} F(W_{t-1}) &= \inf_u \left\{u^2 + E[\hat{x}_{t-1} + u + H_t(y_t - \hat{x}_{t-1} - u)]^2\Pi_t + \cdots\right\} \\ &= \inf_u \left\{u^2 + (\hat{x}_{t-1} + u)^2\Pi_t + E[H_t(\eta_t - \Delta_{t-1})]^2\Pi_t + \cdots\right\}. \end{aligned}$$

Minimizing over u we obtain the usual Riccati recursion of

$$\Pi_{t-1} = \Pi_t - \Pi_t^2/(1 + \Pi_t) = \Pi_t/(1 + \Pi_t).$$

Hence $\Pi_t = D/(1 + D(h-t))$ and the optimal control is the certainty equivalence control $u_t = -D\hat{x}_t/(1 + D(h-t))$. This is the same control as in the deterministic case, but with x_t replaced by $\hat{x}_t$.