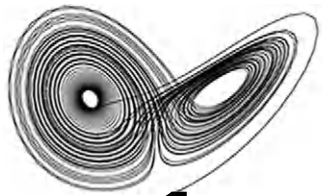

1 First-Order Equations

The purpose of this chapter is to develop some elementary yet important examples of first-order differential equations. The examples here illustrate some of the basic ideas in the theory of ordinary differential equations in the simplest possible setting.

We anticipate that the first few examples will be familiar to readers who have taken an introductory course in differential equations. Later examples, such as the logistic model with harvesting, are included to give the reader a taste of certain topics (e.g., bifurcations, periodic solutions, and Poincaré maps) that we will return to often throughout this book. In later chapters, our treatment of these topics will be much more systematic.

1.1 The Simplest Example

The differential equation familiar to all calculus students,

$$\frac{dx}{dt} = ax,$$

is the simplest. It is also one of the most important. First, what does it mean? Here $x = x(t)$ is an unknown real-valued function of a real variable t and dx/dt is its derivative (we will also use x' or $x'(t)$ for the derivative). In addition, a is a parameter; for each value of a we have a different differential

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equation. The equation tells us that for every value of t the relationship

$$x'(t) = ax(t)$$

is true.

The solutions of this equation are obtained from calculus: if k is any real number, then the function $x(t) = ke^{at}$ is a solution since

$$x'(t) = ake^{at} = ax(t).$$

Moreover, *there are no other solutions.* To see this, let $u(t)$ be any solution and compute the derivative of $u(t)e^{-at}$:

$$\begin{aligned}\frac{d}{dt}\left(u(t)e^{-at}\right) &= u'(t)e^{-at} + u(t)(-ae^{-at})\\ &= au(t)e^{-at} - au(t)e^{-at} = 0.\end{aligned}$$

Therefore, $u(t)e^{-at}$ is a constant k, so $u(t) = ke^{at}$. This proves our assertion. Thus, we have found all possible solutions of this differential equation. We call the collection of all solutions of a differential equation the *general solution* of the equation.

The constant k appearing in this solution is completely determined if the value u_0 of a solution at a single point t_0 is specified. Suppose that a function $x(t)$ satisfying the differential equation is also required to satisfy $x(t_0) = u_0$. Then we must have $ke^{at_0} = u_0$, so that $k = u_0e^{-at_0}$. Thus, we have determined k and this equation therefore has a unique solution satisfying the specified *initial condition* $x(t_0) = u_0$. For simplicity, we often take $t_0 = 0$; then $k = u_0$. There is no loss of generality in taking $t_0 = 0$, for if $u(t)$ is a solution with $u(0) = u_0$, then the function $v(t) = u(t - t_0)$ is a solution with $v(t_0) = u_0$.

It is common to restate this in the form of an *initial value problem*:

$$x' = ax, \quad x(0) = u_0.$$

A solution $x(t)$ of an initial value problem must not only solve the differential equation, but must also take on the prescribed initial value u_0 at $t = 0$.

Note that there is a special solution of this differential equation when $k = 0$. This is the constant solution $x(t) \equiv 0$. A constant solution like this is called an *equilibrium solution* or *equilibrium point* for the equation. Equilibria are often among the most important solutions of differential equations.

The constant a in the equation $x' = ax$ can be considered as a parameter. If a changes, the equation changes and so do the solutions. Can we describe qualitatively the way the solutions change? The sign of a is crucial here:

1. If $a > 0$, $\lim_{t\to\infty} ke^{at}$ equals ∞ when $k > 0$, and equals $-\infty$ when $k < 0$

2. If $a = 0$, $ke^{at} = \text{constant}$
3. If $a < 0$, $\lim_{t\to\infty} ke^{at} = 0$

The qualitative behavior of solutions is vividly illustrated by sketching the graphs of solutions as in Figure 1.1.

Note that the behavior of solutions is quite different when a is positive and negative. When $a > 0$, all nonzero solutions tend away from the equilibrium point at 0 as t increases, whereas when $a < 0$, solutions tend toward the equilibrium point. We say that the equilibrium point is a *source* when nearby solutions tend away from it. The equilibrium point is a *sink* when nearby solutions tend toward it.

We also describe solutions by drawing them on the *phase line*. As the solution $x(t)$ is a function of time, we may view $x(t)$ as a particle moving along the real line. At the equilibrium point, the particle remains at rest (indicated by a solid dot), while any other solution moves up or down the x-axis, as indicated by the arrows in Figure 1.2.

The equation $x' = ax$ is *stable* in a certain sense if $a \neq 0$. More precisely, if a is replaced by another constant b with a sign that is the same as a, then

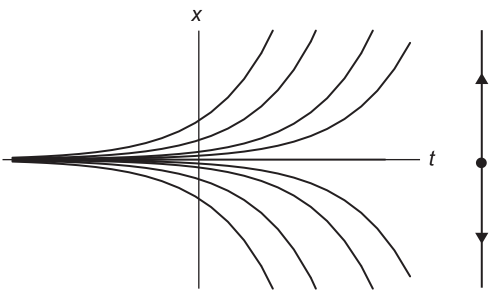

Figure 1.1 The solution graphs and phase line for $x' = ax$ for $a > 0$. Each graph represents a particular solution.

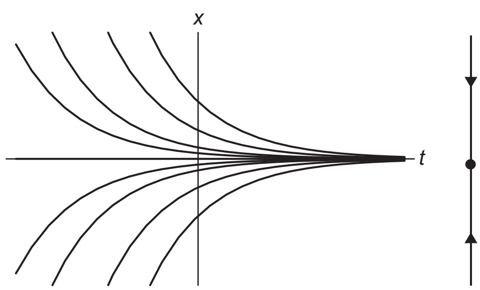

Figure 1.2 The solution graphs and phase line for $x' = ax$ for $a < 0$.

the qualitative behavior of the solutions does not change. But if $a = 0$, the slightest change in a leads to a radical change in the behavior of solutions. We therefore say that we have a *bifurcation* at $a = 0$ in the one-parameter family of equations $x' = ax$. The concept of a bifurcation is one that will arise over and over in subsequent chapters of this book.

1.2 The Logistic Population Model

The differential equation $x' = ax$ can be considered as a simplistic model of population growth when $a > 0$. The quantity $x(t)$ measures the population of some species at time t. The assumption that leads to the differential equation is that the rate of growth of the population (namely, dx/dt) is directly proportional to the size of the population. Of course, this naive assumption omits many circumstances that govern actual population growth, including, for example, the fact that actual populations cannot increase without bound.

To take this restriction into account, we can make the following further assumptions about the population model:

1. If the population is small, the growth rate remains directly proportional to the size of the population.
2. If the population grows too large, however, the growth rate becomes negative.

One differential equation that satisfies these assumptions is the *logistic population growth model.* This differential equation is

$$x' = ax\left(1 - \frac{x}{N}\right).$$

Here a and N are positive parameters: a gives the rate of population growth when x is small, while N represents a sort of "ideal" population or "carrying capacity." Note that if x is small, the differential equation is essentially $x' = ax$ (since the term $1 - (x/N) \approx 1$), but if $x > N$, then $x' < 0$. Thus, this simple equation satisfies the preceding assumptions. We should add here that there are many other differential equations that correspond to these assumptions; our choice is perhaps the simplest.

Without loss of generality, we will assume that $N = 1$. That is, we will choose units so that the carrying capacity is exactly 1 unit of population and $x(t)$ therefore represents the fraction of the ideal population present at time t. Therefore, the logistic equation reduces to

$$x' = f_a(x) = ax(1 - x).$$

This is an example of a first-order, autonomous, nonlinear differential equation. It is *first order* since only the first derivative of x appears in the equation. It is *autonomous* since the right side of the equation depends on x alone, not on time t. Plus, it is *nonlinear* since $f_a(x)$ is a nonlinear function of x. The previous example, $x' = ax$, is a first-order, autonomous, linear differential equation.

The solution of the logistic differential equation is easily found by the tried-and-true calculus method of separation and integration:

$$\int \frac{dx}{x(1-x)} = \int a\,dt.$$

The method of partial fractions allows us to rewrite the left integral as

$$\int \left(\frac{1}{x} + \frac{1}{1-x}\right) dx.$$

Integrating both sides and then solving for x yields

$$x(t) = \frac{Ke^{at}}{1+Ke^{at}},$$

where K is the arbitrary constant that arises from integration. Evaluating this expression at $t = 0$ and solving for K gives

$$K = \frac{x(0)}{1-x(0)}.$$

Using this, we may rewrite this solution as

$$\frac{x(0)e^{at}}{1-x(0)+x(0)e^{at}}.$$

So this solution is valid for any initial population $x(0)$. When $x(0) = 1$, we have an equilibrium solution, since $x(t)$ reduces to $x(t) \equiv 1$. Similarly, $x(t) \equiv 0$ is an equilibrium solution.

Thus, we have "existence" of solutions for the logistic differential equation. We have no guarantee that these are all of the solutions of this equation at this stage; we will return to this issue when we discuss the existence and uniqueness problem for differential equations in Chapter 7.

To get a qualitative feeling for the behavior of solutions, we sketch the *slope field* for this equation. The right side of the differential equation determines the slope of the graph of any solution at each time t. Thus, we may plot little slope lines in the tx–plane as in Figure 1.3, with the slope of the line at (t, x)

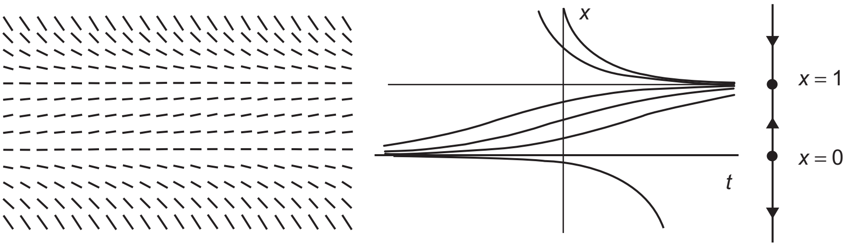

Figure 1.3 Slope field, solution graphs, and phase line for $x' = ax(1-x)$.

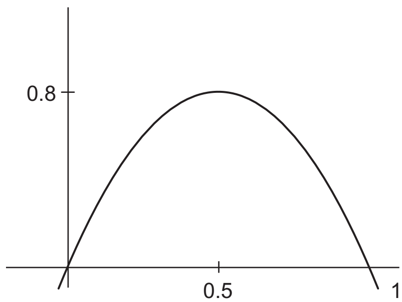

Figure 1.4 The graph of the function $f(x) = ax(1-x)$ with $a = 3.2$.

given by the quantity $ax(1-x)$. Our solutions must therefore have graphs that are tangent to this slope field everywhere. From these graphs, we see immediately that, in agreement with our assumptions, all solutions for which $x(0) > 0$ tend to the ideal population $x(t) \equiv 1$. For $x(0) < 0$, solutions tend to $-\infty$, although these solutions are irrelevant in the context of a population model.

Note that we can also read this behavior from the graph of the function $f_a(x) = ax(1-x)$. This graph, displayed in Figure 1.4, crosses the x-axis at the two points $x = 0$ and $x = 1$, so these represent our equilibrium points. When $0 < x < 1$, we have $f(x) > 0$. Therefore, slopes are positive at any (t, x) with $0 < x < 1$, so solutions must increase in this region. When $x < 0$ or $x > 1$, we have $f(x) < 0$, so solutions must decrease, as we see in both the solution graphs and the phase lines in Figure 1.3.

We may read off the fact that $x = 0$ is a source and $x = 1$ is a sink from the graph of f in similar fashion. Near 0, we have $f(x) > 0$ if $x > 0$, so slopes are positive and solutions increase, but if $x < 0$, then $f(x) < 0$, so slopes are negative and solutions decrease. Thus, nearby solutions move away from 0, so 0 is a source. Similarly, 1 is a sink.

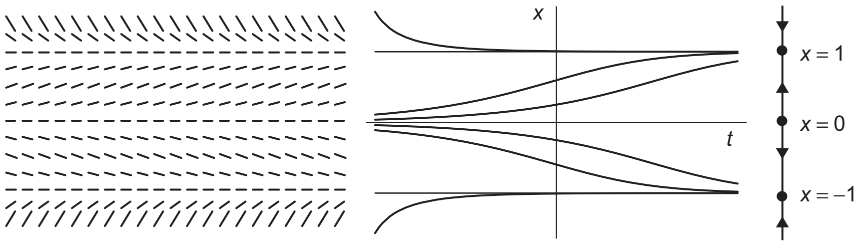

Figure 1.5 Slope field, solution graphs, and phase line for $x' = x - x^3$.

We may also determine this information analytically. We have $f_a'(x) = a - 2ax$ so that $f_a'(0) = a > 0$ and $f_a'(1) = -a < 0$. Since $f_a'(0) > 0$, slopes must increase through the value 0 as x passes through 0. That is, slopes are negative below $x = 0$ and positive above $x = 0$. Thus, solutions must tend away from $x = 0$. In similar fashion, $f_a'(1) < 0$ forces solutions to tend toward $x = 1$, making this equilibrium point a sink. We will encounter many such "derivative tests" like this that predict the qualitative behavior near equilibria in subsequent chapters.

Example. As a further illustration of these qualitative ideas, consider the differential equation

$$x' = g(x) = x - x^3.$$

There are three equilibrium points at $x = 0, \pm 1$. Since $g'(x) = 1 - 3x^2$, we have $g'(0) = 1$, so the equilibrium point 0 is a source. Also, $g'(\pm 1) = -2$, so the equilibrium points at ± 1 are both sinks. Between these equilibria, the sign of the slope field of this equation is nonzero. From this information we can immediately display the phase line, which is shown in Figure 1.5. ■

1.3 Constant Harvesting and Bifurcations

Now let's modify the logistic model to take into account harvesting of the population. Suppose that the population obeys the logistic assumptions with the parameter $a = 1$, but it is also harvested at the constant rate h. The differential equation becomes

$$x' = x(1 - x) - h,$$

where $h \geq 0$ is a new parameter.

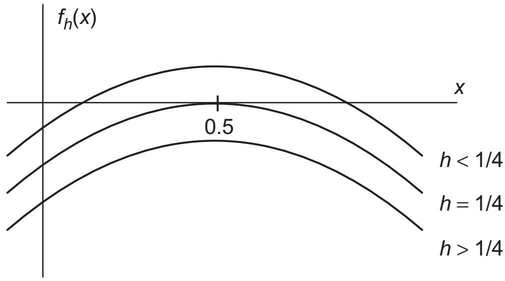

Figure 1.6 The graphs of the function $f_{h(x)} = x(1 - x) - h$.

Rather than solving this equation explicitly (which can be done—see Exercise 6 of this chapter), we use the graph of the function

$$f_h(x) = x(1 - x) - h$$

to "read off" the qualitative behavior of solutions. In Figure 1.6 we display the graph of f_h in three different cases: $0 < h < 1/4$, $h = 1/4$, and $h > 1/4$. It is straightforward to check that f_h has two roots when $0 \le h < 1/4$, one root when $h = 1/4$, and no roots if $h > 1/4$, as illustrated in the graphs. As a consequence, the differential equation has two equilibrium points, x_ℓ and x_r, with $0 \le x_\ell < x_r$ when $0 < h < 1/4$. It is also easy to check that $f_h'(x_\ell) > 0$ so that x_ℓ is a source, and $f_h'(x_r) < 0$ so that x_r is a sink.

As h passes through $h = 1/4$, we encounter another example of a bifurcation. The two equilibria, x_ℓ and x_r, coalesce as h increases through $1/4$ and then disappear when $h > 1/4$. Moreover, when $h > 1/4$, we have $f_h(x) < 0$ for all x. Mathematically, this means that all solutions of the differential equation decrease to $-\infty$ as time goes on.

We record this visually in the *bifurcation diagram.* In Figure 1.7, we plot the parameter h horizontally. Over each h-value we plot the corresponding phase line. The curve in this picture represents the equilibrium points for each value of h. This gives another view of the sink and source merging into a single equilibrium point and then disappearing as h passes through $1/4$.

Ecologically, this bifurcation corresponds to a disaster for the species under study. For rates of harvesting $1/4$ or lower, the population persists, provided the initial population is sufficiently large ($x(0) \ge x_\ell$). But a very small change in the rate of harvesting when $h = 1/4$ leads to a major change in the fate of the population: at any rate of harvesting $h > 1/4$, the species becomes extinct.

This phenomenon highlights the importance of detecting bifurcations in families of differential equations—a procedure that we will encounter many times in later chapters. We should also mention that, despite the simplicity of

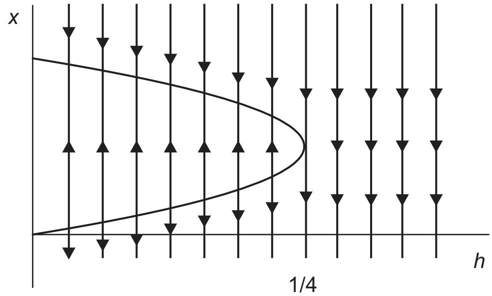

Figure 1.7 The bifurcation diagram for $f_{h(x)} = x(1 - x) - h$.

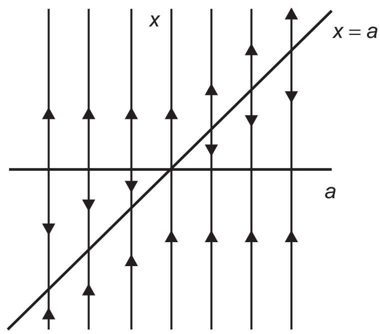

Figure 1.8 The bifurcation diagram for $x' = x^2 - ax$.

this population model, the prediction that small changes in harvesting rates can lead to disastrous changes in population has been observed many times in real situations on earth.

Example. As another example of a bifurcation, consider the family of differential equations

$$x' = g_a(x) = x^2 - ax = x(x - a),$$

which depends on a parameter a. The equilibrium points are given by $x = 0$ and $x = a$. We compute that $g_a'(0) = -a$, so 0 is a sink if $a > 0$ and a source if $a < 0$. Similarly, $g_a'(a) = a$, so $x = a$ is a sink if $a < 0$ and a source if $a > 0$. We have a bifurcation at $a = 0$ since there is only one equilibrium point when $a = 0$. Moreover, the equilibrium point at 0 changes from a source to a sink as a increases through 0. Similarly, the equilibrium at $x = a$ changes from a sink to a source as a passes through 0. The bifurcation diagram for this family is shown in Figure 1.8. ■

1.4 Periodic Harvesting and Periodic Solutions

Now let's change our assumptions on the logistic model to reflect the fact that harvesting does not always occur at a constant rate. For example, populations of many species of fish are harvested at a higher rate in warmer months than in colder months. So, we assume that the population is harvested at a periodic rate. One such model is then

$$x' = f(t,x) = ax(1-x) - h(1+\sin(2\pi t)),$$

where again a and h are positive parameters. Thus, the harvesting reaches a maximum rate $-2h$ at time $t = \frac{1}{4} + n$ where n is an integer (representing the year), and the harvesting reaches its minimum value 0 when $t = \frac{3}{4} + n$, exactly one half year later.

Note that this differential equation now depends explicitly on time; this is an example of a *nonautonomous* differential equation. As in the autonomous case, a solution $x(t)$ of this equation must satisfy $x'(t) = f(t,x(t))$ for all t. Also, this differential equation is no longer separable, so we cannot generate an analytic formula for its solution using the usual methods from calculus. Thus, we are forced to take a more qualitative approach (see Figure 1.9).

To describe the fate of the population in this case, we first note that the right side of the differential equation is periodic with period 1 in the time variable; that is, $f(t+1,x) = f(t,x)$. This fact simplifies the problem of finding solutions somewhat. Suppose that we know the solution of all initial value problems, not for all times but only for $0 \le t \le 1$. Then in fact we know the solutions *for all time.*

For example, suppose $x_1(t)$ is the solution that is defined for $0 \le t \le 1$ and satisfies $x_1(0) = x_0$. Suppose that $x_2(t)$ is the solution that satisfies $x_2(0) =$

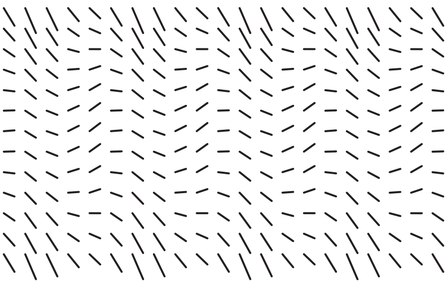

Figure 1.9 The slope field for $f(x) = x(1-x) - h(1+\sin(2\pi t))$.

$x_1(1)$. Then we can extend the solution x_1 by defining $x_1(t+1) = x_2(t)$ for $0 \le t \le 1$. The extended function is a solution since we have

$$\begin{aligned} x_1'(t+1) &= x_2'(t) = f(t, x_2(t)) \\ &= f(t+1, x_1(t+1)). \end{aligned}$$

Thus, if we know the behavior of all solutions in the interval $0 \le t \le 1$, then we can extrapolate in similar fashion to all time intervals and thereby know the behavior of solutions for all time.

Second, suppose that we know the value at time $t = 1$ of the solution satisfying any initial condition $x(0) = x_0$. Then, to each such initial condition x_0, we can associate the value $x(1)$ of the solution $x(t)$ that satisfies $x(0) = x_0$. This gives us a function $p(x_0) = x(1)$. If we compose this function with itself, we derive the value of the solution through x_0 at time 2; that is, $p(p(x_0)) = x(2)$. If we compose this function with itself n times, then we can compute the value of the solution curve at time n and hence we know the fate of the solution curve.

The function p is called a *Poincaré* map for this differential equation. Having such a function allows us to move from the realm of continuous dynamical systems (differential equations) to the often easier-to-understand realm of discrete dynamical systems (iterated functions). For example, suppose that we know that $p(x_0) = x_0$ for some initial condition x_0; that is, x_0 is a *fixed point* for the function p. Then, from our previous observations, we know that $x(n) = x_0$ for each integer n. Moreover, for each time t with $0 < t < 1$, we also have $x(t) = x(t+1)$ and thus $x(t+n) = x(t)$ for each integer n. That is, the solution satisfying the initial condition $x(0) = x_0$ is a periodic function of t with period 1. Such solutions are called *periodic solutions* of the differential equation.

In Figure 1.10, we have displayed several solutions of the logistic equation with periodic harvesting. Note that the solution satisfying the initial condition, $x(0) = x_0$, is a periodic solution, and we have $x_0 = p(x_0) = p(p(x_0))\ldots.$ Similarly, the solution satisfying the initial condition, $x(0) = \hat{x}_0$, also appears to be a periodic solution, so we should have $p(\hat{x}_0) = \hat{x}_0$.

Unfortunately, it is usually the case that computing a Poincaré map for a differential equation is impossible, but for the logistic equation with periodic harvesting we get lucky.

1.5 Computing the Poincaré Map

Before computing the Poincaré map for this equation, we need to introduce some important terminology. To emphasize the dependence of a solution on

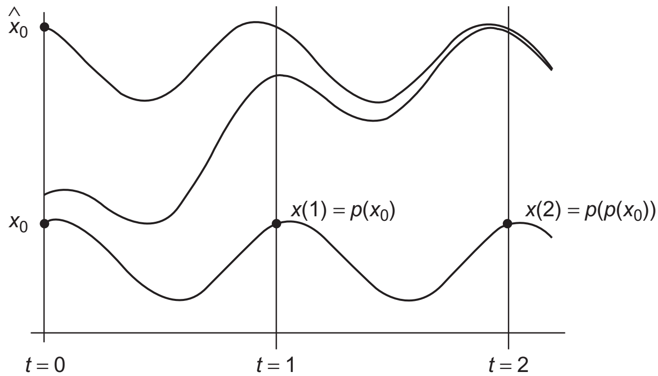

Figure 1.10 The Poincaré map for $x' = 5x(1 - x) - 0.8(1 + \sin(2\pi t))$.

the initial value x_0, we will denote the corresponding solution by $\phi(t, x_0)$. This function, $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, is called the *flow* associated with the differential equation. If we hold the variable x_0 fixed, then the function

$$t \to \phi(t, x_0)$$

is just an alternative expression for the solution of the differential equation satisfying the initial condition x_0. Sometimes we write this function as $\phi_t(x_0)$.

Example. For our first example, $x' = ax$, the flow is given by

$$\phi(t, x_0) = x_0 e^{at}.$$

For the logistic equation (without harvesting), the flow is

$$\phi(t, x_0) = \frac{x(0)e^{at}}{1 - x(0) + x(0)e^{at}}.$$

Now we return to the logistic differential equation with periodic harvesting,

$$x' = f(t, x) = ax(1 - x) - h(1 + \sin(2\pi t)).$$

The solution that satisfies the initial condition, $x(0) = x_0$, is given by $t \to \phi(t, x_0)$. Although we do not have a formula for this expression, we do know that, by the Fundamental Theorem of Calculus, this solution satisfies

$$\phi(t,x_0) = x_0 + \int_0^t f(s,\phi(s,x_0))\, ds$$

since

$$\frac{\partial \phi}{\partial t}(t,x_0) = f(t,\phi(t,x_0))$$

and $\phi(0,x_0) = x_0$.

If we differentiate this solution with respect to x_0, using the Chain Rule, we obtain:

$$\frac{\partial \phi}{\partial x_0}(t,x_0) = 1 + \int_0^t \frac{\partial f}{\partial x_0}(s,\phi(s,x_0)) \cdot \frac{\partial \phi}{\partial x_0}(s,x_0)\, ds.$$

Now let

$$z(t) = \frac{\partial \phi}{\partial x_0}(t,x_0).$$

Note that

$$z(0) = \frac{\partial \phi}{\partial x_0}(0,x_0) = 1.$$

Differentiating z with respect to t, we find

$$\begin{aligned} z'(t) &= \frac{\partial f}{\partial x_0}(t,\phi(t,x_0)) \cdot \frac{\partial \phi}{\partial x_0}(t,x_0) \\ &= \frac{\partial f}{\partial x_0}(t,\phi(t,x_0)) \cdot z(t). \end{aligned}$$

Again, we do not know $\phi(t,x_0)$ explicitly, but this equation does tell us that $z(t)$ solves the differential equation

$$z'(t) = \frac{\partial f}{\partial x_0}(t,\phi(t,x_0))\, z(t)$$

with $z(0) = 1$. Consequently, via separation of variables, we may compute that the solution of this equation is

$$z(t) = \exp \int_0^t \frac{\partial f}{\partial x_0}(s, \phi(s, x_0))\, ds,$$

and so we find

$$\frac{\partial \phi}{\partial x_0}(1, x_0) = \exp \int_0^1 \frac{\partial f}{\partial x_0}(s, \phi(s, x_0))\, ds.$$

Since $p(x_0) = \phi(1, x_0)$, we have determined the derivative $p'(x_0)$ of the Poincaré map; note that $p'(x_0) > 0$. Therefore, p is an increasing function.

Differentiating once more, we find

$$p''(x_0) = p'(x_0) \left(\int_0^1 \frac{\partial^2 f}{\partial x_0 \partial x_0}(s, \phi(s, x_0)) \cdot \exp \left(\int_0^s \frac{\partial f}{\partial x_0}(u, \phi(u, x_0))\, du \right) ds \right),$$

which looks pretty intimidating. However, since

$$f(t, x_0) = a x_0(1 - x_0) - h(1 + \sin(2\pi t)),$$

we have

$$\frac{\partial^2 f}{\partial x_0 \partial x_0} \equiv -2a.$$

Thus, we know in addition that $p''(x_0) < 0$. Consequently, the graph of the Poincaré map is concave down. This implies that the graph of p can cross the diagonal line $y = x$ at most two times; that is, there can be at most two values of x for which $p(x) = x$. Therefore, the Poincaré map has at most two fixed points. These fixed points yield periodic solutions of the original differential equation. These are solutions that satisfy $x(t + 1) = x(t)$ for all t.

Another way to say this is that the flow, $\phi(t, x_0)$, is a periodic function in t with period 1 when the initial condition x_0 is one of the fixed points. We saw these two solutions in the particular case when $h = 0.8$ in Figure 1.10. In Figure 1.11, we again see two solutions that appear to be periodic. Note that one of these appears to attract all nearby solutions, while the other appears to repel them. We'll return to these concepts often and make them more precise later in the book.

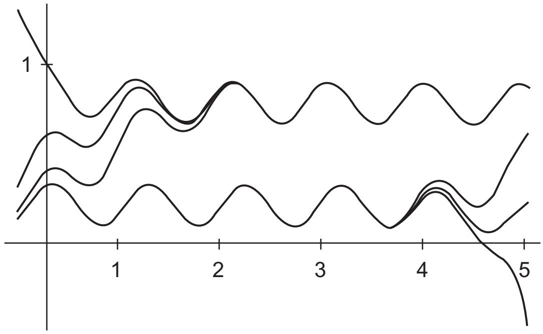

Figure 1.11 Several solutions of $x' = 5x(1-x) - 0.8(1+\sin(2\pi t))$.

Recall that the differential equation also depends on the harvesting parameter h. For small values of h, there will be two fixed points such as shown in Figure 1.11. Differentiating f with respect to h, we find

$$\frac{\partial f}{\partial h}(t, x_0) = -(1 + \sin 2\pi t).$$

Thus, $\partial f/\partial h < 0$ (except when $t = 3/4$). This implies that the slopes of the slope field lines at each point (t, x_0) decrease as h increases. As a consequence, the values of the Poincaré map also decrease as h increases. There is a unique value h_*, therefore, for which the Poincaré map has exactly one fixed point. For $h > h_*$, there are no fixed points for p, so $p(x_0) < x_0$ for all initial values. It then follows that the population again dies out. ■

1.6 Exploration: A Two-Parameter Family

Consider the family of differential equations

$$x' = f_{a,b}(x) = ax - x^3 - b,$$

which depends on two parameters, a and b. The goal of this exploration is to combine all of the ideas in this chapter to put together a complete picture of the two-dimensional parameter plane (the ab–plane) for this differential equation. Feel free to use a computer to experiment with this differential equation at first, but then try the following to verify your observations rigorously:

1. First fix $a = 1$. Use the graph of $f_{1,b}$ to construct the bifurcation diagram for this family of differential equations depending on b.
2. Repeat the previous question for $a = 0$ and then for $a = -1$.
3. What does the bifurcation diagram look like for other values of a?
4. Now fix b and use the graph to construct the bifurcation diagram for this family, which this time depends on a.
5. In the ab–plane, sketch the regions where the corresponding differential equation has different numbers of equilibrium points, including a sketch of the boundary between these regions.
6. Describe, using phase lines and the graph of $f_{a,b}(x)$, the bifurcations that occur as the parameters pass through this boundary.
7. Describe in detail the bifurcations that occur at $a = b = 0$ as a and/or b vary.
8. Consider the differential equation $x' = x - x^3 - b\sin(2\pi t)$, where $|b|$ is small. What can you say about solutions of this equation? Are there any periodic solutions?
9. Experimentally, what happens as $|b|$ increases? Do you observe any bifurcations? Explain what you observe.

EXERCISES

1. Find the general solution of the differential equation $x' = ax + 3$ where a is a parameter. What are the equilibrium points for this equation? For which values of a are the equilibria sinks? For which are they sources?

2. For each of the following differential equations, find all equilibrium solutions and determine whether they are sinks, sources, or neither. Also sketch the phase line.

(a) $x' = x^3 - 3x$
(b) $x' = x^4 - x^2$
(c) $x' = \cos x$
(d) $x' = \sin^2 x$
(e) $x' = |1 - x^2|$

3. Each of the following families of differential equations depends on a parameter a. Sketch the corresponding bifurcation diagrams.

(a) $x' = x^2 - ax$
(b) $x' = x^3 - ax$
(c) $x' = x^3 - x + a$

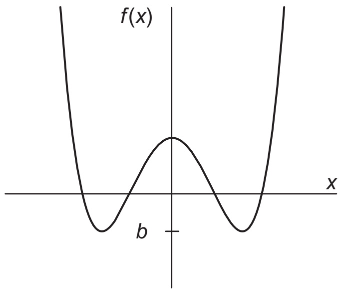

Figure 1.12 Graph of the function *f*.

4. Consider the function $f(x)$ with a graph that is displayed in Figure 1.12.

(a) Sketch the phase line corresponding to the differential equation $x' = f(x)$.
(b) Let $g_a(x) = f(x) + a$. Sketch the bifurcation diagram corresponding to the family of differential equations $x' = g_a(x)$.
(c) Describe the different bifurcations that occur in this family.

5. Consider the family of differential equations

$$x' = ax + \sin x,$$

where a is a parameter.

(a) Sketch the phase line when $a = 0$.
(b) Use the graphs of ax and $\sin x$ to determine the qualitative behavior of all of the bifurcations that occur as a increases from -1 to 1.
(c) Sketch the bifurcation diagram for this family of differential equations.

6. Find the general solution of the logistic differential equation with constant harvesting,

$$x' = x(1 - x) - h,$$

for all values of the parameter $h > 0$.

7. Consider the nonautonomous differential equation

$$x' = \begin{cases} x - 4 & \text{if } t < 5, \\ 2 - x & \text{if } t \geq 5. \end{cases}$$

(a) Find a solution of this equation satisfying $x(0) = 4$. Describe the qualitative behavior of this solution.

(b) Find a solution of this equation satisfying $x(0) = 3$. Describe the qualitative behavior of this solution.

(c) Describe the qualitative behavior of any solution of this system as $t \to \infty$.

8. Consider a first-order linear equation of the form $x' = ax + f(t)$, where $a \in \mathbb{R}$. Let $y(t)$ be any solution of this equation. Prove that the general solution is $y(t) + c\exp(at)$ where $c \in \mathbb{R}$ is arbitrary.

9. Consider a first-order, linear, nonautonomous equation of the form $x'(t) = a(t)x$.

(a) Find a formula involving integrals for the solution of this system.

(b) Prove that your formula gives the general solution of this system.

10. Consider the differential equation $x' = x + \cos t$.

(a) Find the general solution of this equation.

(b) Prove that there is a unique periodic solution for this equation.

(c) Compute the Poincaré map $p\colon \{t = 0\} \to \{t = 2\pi\}$ for this equation and use this to verify again that there is a unique periodic solution.

11. First-order differential equations need not have solutions that are defined for all time.

(a) Find the general solution of the equation $x' = x^2$.

(b) Discuss the domains over which each solution is defined.

(c) Give an example of a differential equation for which the solution satisfying $x(0) = 0$ is defined only for $-1 < t < 1$.

12. First-order differential equations need not have unique solutions satisfying a given initial condition.

(a) Prove that there are infinitely many different solutions of the differential equations $x' = x^{1/3}$ satisfying $x(0) = 0$.

(b) Discuss the corresponding situation that occurs for $x' = x/t$, $x(0) = x_0$.

(c) Discuss the situation that occurs for $x' = x/t^2$, $x(0) = 0$.

13. Let $x' = f(x)$ be an autonomous first-order differential equation with an equilibrium point at x_0.

(a) Suppose $f'(x_0) = 0$. What can you say about the behavior of solutions near x_0? Give examples.

(b) Suppose $f'(x_0) = 0$ and $f''(x_0) \neq 0$. What can you say now?

(c) Suppose $f'(x_0) = f''(x_0) = 0$ but $f'''(x_0) \neq 0$. What can you say now?

14. Consider the first-order nonautonomous equation $x' = p(t)x$, where $p(t)$ is differentiable and periodic with period T. Prove that all solutions of this equation are periodic with period T if and only if

$$\int_0^T p(s)\, ds = 0.$$

15. Consider the differential equation $x' = f(t, x)$, where $f(t, x)$ is continuously differentiable in t and x. Suppose that

$$f(t + T, x) = f(t, x)$$

for all t. Suppose there are constants p, q such that

$$f(t, p) > 0, \quad f(t, q) < 0$$

for all t. Prove that there is a periodic solution $x(t)$ for this equation with $p < x(0) < q$.

16. Consider the differential equation $x' = x^2 - 1 - \cos(t)$. What can be said about the existence of periodic solutions for this equation?