DECISION MAKING – THE ANALYTIC HIERARCHY AND NETWORK PROCESSES (AHP/ANP)

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Abstract

This is the first part of an introduction to multicriteria decision making using the Analytic Hierarchy Process (AHP) and its generalization, the Analytic Network Process (ANP). The discussion involves individual and group decisions both with the independence of the criteria from the alternatives as in the AHP and also with dependence and feedback in the entire decision structure as in the ANP. This part explains the Analytic Hierarchy Process, with examples, and presents in some detail the mathematical foundations. An exposition of the Analytic Network Process and its applications will appear in later issues of this journal.

Keywords: Decision making, Analytic Hierarchy Process (AHP), Analytic Network Process (ANP)

1. Introduction (Saaty 1977, 1994, 2000a, 2000b and 2001)

Decision making involves criteria and alternatives to choose from. The criteria usually have different importance and the alternatives in turn differ in our preference for them on each criterion. To make such tradeoffs and choices we need a way to measure. Measuring needs a good understanding of methods of measurement and different scales of measurement.

Many people think that measurement needs a physical scale with a zero and a unit to apply to objects or phenomena. That is not true. Surprisingly enough, we can also derive accurate and reliable relative scales that do not have a zero or a unit by using our understanding and judgments that are, after all, the most fundamental determinants of why we want to measure something. In reality we do that all the time and we do it subconsciously without thinking about it. Physical scales help our understanding and use of the things that we know how to measure. After we obtain readings from a physical scale, they still need to be interpreted according to what they mean and how adequate or inadequate they are to satisfy some need we have. But the number of things we don't know how to measure is infinitely larger than the things we know how to measure, and it is highly unlikely that we will ever find ways to measure everything on a physical scale with a unit. Scales of measurement are inventions of a technological mind. Our minds and ways of understanding we have had with us and will always have. The brain is an electrical device of neurons whose

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firings and synthesis must perform measurement with great accuracy to give us all the meaning and understanding that we have to enable us to survive and reach out to control a complex world. Can we rely on our minds to be accurate guides with their judgments? The answer depends on how well we know the phenomena to which we apply measurement and how good our judgments are to represent our understanding. In our own personal affairs we are the best judges of what may be good for us. In situations involving many people, we need the judgments from all the participants. In general we think that there are people who are more expert than others in some areas and their judgments should have precedence over the judgments of those who know less as in fact is often the case in practice.

Judgments expressed in the form of comparisons are fundamental in our biological makeup. They are intrinsic in the operations of our brains and that of animals and one might even say of plants since, for example, they control how much sunlight to admit. We all make decisions every moment, consciously or unconsciously, today and tomorrow, now and forever, it seems. Decision-making is a fundamental process that is integral in everything we do. How do we do it? The Harvard psychologist Arthur Blumenthal tells us in his book The Process of Cognition, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1977, that there are two types of judgment: "Comparative judgment which is the identification of some relation between two stimuli both present to the observer, and absolute judgment which involves the relation between a single stimulus and some information held in short term memory about some former comparison stimuli or about some previously experienced measurement scale using which the observer rates the single stimulus."

When we think about it, both these processes involve making comparisons. Comparisons imply that all things we know are understood in relative terms to other things. It does not seem possible to know an absolute in itself independently of something else that influences it or that it influences. The question then is how do we make comparisons in a scientific way and derive from these comparisons scales of relative measurement? When we have many scales with respect to a diversity of criteria and subcriteria, how do we synthesize these scales to obtain an overall relative scale? Can we validate this process so that we can trust its reliability? What can we say about other ways people have proposed to deal with judgment and measurement, how do they relate to this fundamental idea of comparisons, and can they be relied on for validity? These are all questions we need to consider in making a decision. It is useful to remember that there are many people in the world who only know their feelings and may know nothing about numbers and never heard of them but can still make good decisions, how do they do it? It is unlikely that by guessing at numbers and assigning them directly to the alternatives to indicate order under a criterion will yield meaningful priorities because the numbers are arbitrary. Even if they are taken from a scale for a particular criterion, how would we combine them across the criteria as they would likely be from different scales? Our answer to this conundrum is to derive a relative scale for the criteria with respect to the goal and to derive relative scales for the alternatives with respect to each of the criteria and use a weighting and adding process that will make these scales alike. The scale we derive under each criterion is the same priority scale that measures the preference we have for the alternatives with respect to each criterion, and the importance we attribute to the criteria in terms of the goal. As we shall see below, the judgments made use absolute numbers and the priorities derived from them are also absolute numbers that represent relative dominance. Among the many applications made by companies and governments, now perhaps numbering in the thousands, the Analytic Hierarchy Process was used by IBM as part of its quality improvement strategy to design its AS/400 computer and win the prestigious Malcolm Baldrige National Quality Award (Bauer et al. 1992).

2. Deriving a Scale of Priorities from Pairwise Comparisons

Suppose we wish to derive a scale of relative importance according to size (volume) of three apples A, B, C shown in Figure 1. Assume that their volumes are known respectively as S_1, S_2 and S_3. For each position in the matrix the volume of the apple at the left is compared with that of the apple at the top and the ratio is entered. A matrix of judgments $A=(a_{ij})$ is constructed with respect to a particular property the elements have in common. It is reciprocal, that is, $a_{ji}=1/a_{ij}$, and $a_{ii}=1$. For the matrix in Figure 1, it is necessary to make only three judgments with the remainder being automatically determined. There are $n(n-1)/2$ judgments required for a matrix of order n. Sometimes one (particularly an expert who knows well what the judgments should be) may wish to make a minimum set of judgments and construct a consistent matrix defined as one whose entries satisfy $a_{ij}a_{jk}=a_{ik}$, $i,j,k=1,\cdots,n$. To do this one can enter $n-1$ judgments in a row or in a column, or in a spanning set with at least one judgment in every row and column, and construct the rest of the entries in the matrix using the consistency condition. Redundancy in the number of judgments generally improves the validity of the final answer because the judgments of the few elements one chooses to compare may be more biased.

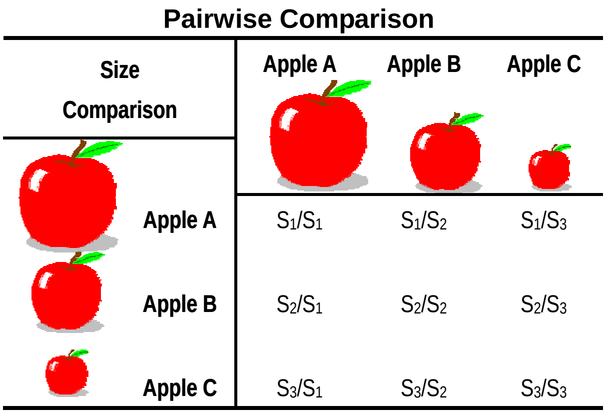
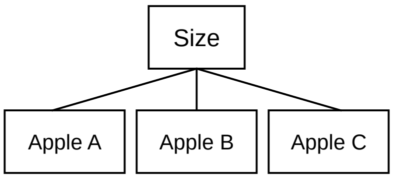


Figure 1 Reciprocal structure of pairwise comparison matrix for apples

Assume that we know the volumes of the apples so that the values we enter in Figure 2 are consistent. Apple A is twice as big in volume as apple B, and apple B is three times as big as apple C, so we enter a 2 in the (1,2) position, and so on. Ones are entered on the diagonal by default as every entity equals itself on any criterion. Note that in the (2, 3) position we can enter the value 3 because we know the judgments are consistent as they are based on actual measurements. We can deduce the value this way: from the first row A = 2B and A = 6C, and thus B = 3C.

Pairwise Comparison



Size Comparison	Apple A	Apple B	Apple C	Relative Size of Apples from Any Column Normalized	Priorities
Apple A	1	2	6	6/10	A
Apple B	1/2	1	3	3/10	B
Apple C	1/6	1/3	1	1/10	C

Figure 2 Pairwise comparison matrix for apples using judgments

If we did not have actual measurements, we could not be certain that the judgments in the first row are accurate, and we would not mind estimating the value in the (2, 3) position directly by comparing apple B with apple C. We are then very likely to be inconsistent. How inconsistent can we be before we think it is intolerable? Later we give an actual measure of inconsistency and argue that a consistency of about 10% is considered acceptable.

We obtain from the consistent pairwise comparison matrix above a vector of priorities showing the relative sizes of the apples. Note that we do not have to go to all this trouble to derive the relative volumes of the apples. We could simply have normalized the actual measurements. The reason we did so is to lay the foundation for what to do when we have no measures for the property in question. When judgments are consistent as they are here, this vector of priorities can be obtained in two ways: dividing the elements in any column by the sum of its entries (normalizing it), or by summing the entries in each row to obtain the overall dominance in size of that alternative relative to the others and normalizing the resulting column of values. Incidentally, calculating dominance plays an important role in computing the priorities when judgments are inconsistent for then an alternative may dominate another by different magnitudes by transiting to it through intermediate alternatives. Thus the story is very different if the judgments are inconsistent, and we need to allow inconsistent judgments for good reasons. In sports, team A beats team B, team B beats team C, but team C beats team A. How would we admit such an occurrence in our attempt to explain the real world if we do not allow inconsistency? Most theories have taken a stand against such an occurrence with an axiom that assumes transitivity and prohibits intransitivity, although one does not have to be intransitive to be inconsistent in the values obtained. Others have wished it away by saying that it should not happen in human thinking. But it does, and we offer a theory below that copes with intransitivity.

3. The Fundamental Scale of the AHP for Making Comparisons with Judgments

If we were to use judgments instead of ratios, we would estimate the ratios as numbers using the Fundamental Scale of the AHP, shown in Table 1 and derived analytically later in the paper, and enter these judgments in the matrix. A judgment is made on a pair of elements with respect to a property they have in common. The smaller element is considered to be the unit and one estimates how many times more important, preferable or likely, more generally “dominant”, the other is by using a number from the Fundamental Scale. Dominance is often interpreted as importance when comparing the criteria and as preference when comparing the alternatives with respect to the criteria. It can also be interpreted as likelihood as in the likelihood of a person getting elected as president, or other terms that fit the situation.

The set of objects being pairwise compared must be homogeneous. That is, the dominance of the largest object must be no more than 9 times the smallest one (this is the widest span we use for many good reasons discussed elsewhere in the AHP literature). Things that differ by more than this range can be clustered into homogeneous groups and dealt with by using this scale. If measurements from an existing scale are used, they can simply be normalized without regard to homogeneity. When the elements being compared are very close, they should be compared with other more contrasting elements, and the larger of the two should be favored a little in the judgments over the smaller. We have found this approach to be effective to bring out the actual priorities of the two close elements. Otherwise we have proposed the use of a scale between 1 and 2 using decimals and similar judgments to the Fundamental Scale below. We note that human judgment is relatively insensitive to such small decimal changes.

Table 2 shows how an audience of about 30 people, using consensus to arrive at each judgment, provided judgments to estimate the *dominance* of the consumption of drinks in the United States (which drink is consumed more in the US and how much more than another drink?). The derived vector of relative consumption and the actual vector, obtained by normalizing the consumption given in official statistical data sources, are at the bottom of the table.

Table 1 The Fundamental scale of absolute numbers

Intensity of Importance	*Definition*	*Explanation*
1	Equal importance	Two activities contribute equally to the objective
2	Weak or slight	
3	Moderate importance	Experience and judgment slightly favor one activity over another
4	Moderate plus	
5	Strong importance	Experience and judgment strongly favor one activity over another
6	Strong plus	
7	Very strong or demonstrated importance	An activity is favored very strongly over another; its dominance demonstrated in practice
8	Very, very strong	
9	Extreme importance	The evidence favoring one activity over another is of the highest possible order of affirmation
Reciprocals of above	If activity *i* has one of the above nonzero numbers assigned to it when compared with activity *j*, then *j* has the reciprocal value when compared with *i*	A reasonable assumption
Rationals	Ratios arising from the scale	If consistency were to be forced by obtaining *n* numerical values to span the matrix

Table 2 Relative consumption of drinks

Which Drink Is Consumed More in the U.S.?

An Example of Estimation Using Judgments

Drink Consumption in the U.S.	**Coffee**	**Wine**	**Tea**	**Beer**	**Sodas**	**Milk**	**Water**
Coffee	1	9	5	2	1	1	1/2
Wine	1/9	1	1/3	1/9	1/9	1/9	1/9
Tea	1/5	2	1	1/3	1/4	1/3	1/9
Beer	1/2	9	3	1	1/2	1	1/3
Sodas	1	9	4	2	1	2	1/2
Milk	1	9	3	1	1/2	1	1/3
Water	2	9	9	3	2	3	1

The derived scale based on the judgments in the matrix is:

Coffee	Wine	Tea	Beer	Sodas	Milk	Water
.177	.019	.042	.116	.190	.129	.327

with a consistency ratio of .022.

The actual consumption (from statistical sources) is:

.180	.010	.040	.120	.180	.140	.330

If the objects are not homogeneous, they may be divided into groups that are homogeneous. If necessary additional objects can be added merely to fill out the intervening clusters to move from the smallest object to the largest one. Figure 3 shows how this process works in comparing a cherry tomato with a water melon, which appears to be two orders of magnitude bigger in size, by introducing intermediate objects in stages.

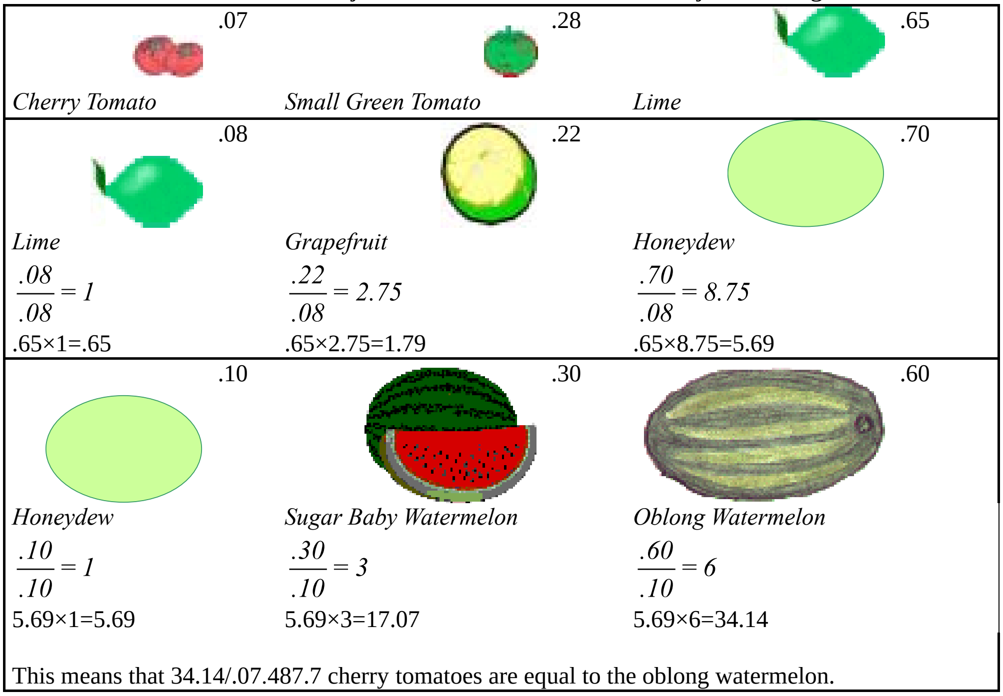


Figure 3 Clustering to compare non-homogeneous objects

4. Scales of Measurement

Mathematically a scale is a triple, a set of numbers, a set of objects and a mapping of the objects to the numbers. There are two ways to perform measurement, one is by using an instrument and making the correspondence directly, and the other is by using judgment. When using judgments one can either assign numbers to the objects by guessing their value on some scale of measurement when there is one, or derive a scale by considering a subset of objects in some fashion such as comparing them in pairs, thus making the correspondence indirect. In addition there are two kinds of origin; one is an absolute origin as in absolute temperature where nothing falls below that reading; and the other where the origin is a dividing point of positive and negative values with no bound on either side such as with a thermometer. Underlying both these ways are the following kinds (there can be more) of general scales:

Nominal Scale invariant under one to one correspondence where a number is assigned to each object; for example, handing out numbers for order of service to people in a queue.

Ordinal Scale invariant under monotone transformations, where things are ordered by number but the magnitudes of the numbers only serve to designate order, increasing or decreasing; for example, assigning two numbers 1 and 2, to two people to indicate that one is taller than the other, without including any information about their actual heights. The smaller number may be assigned to the taller person and vice versa.

Interval Scale invariant under a positive linear transformation; for example, the linear transformation F = (9/5) C + 32 for converting a Celsius to a Fahrenheit temperature reading. Note that one cannot add two readings x_1 and x_2 on an interval scale because then $y_1 + y_2 = (a\, x_1 + b) + (a\, x_2 + b) = a\,(x_1 + x_2) + 2b$ which is of the form $ax + 2b$ and not of the form $ax + b$. However, one can take an average of such readings because dividing by 2 yields the correct form.

Ratio Scale invariant under a similarity transformation, $y = ax$, $a > 0$. An example is converting weight measured in pounds to kilograms by using the similarity transformation $K = 2.2\ P$. The ratio of the weights of the two objects is the same regardless of whether the measurements are done in pounds or in kilograms. Zero is not the measurement of anything; it applies to objects that do not have the property and in addition one cannot divide by zero to preserve ratios in a meaningful way. Note that one can add two readings from a ratio scale, but not multiply them because $a^2 x_1 x_2$ does not have the form ax. The ratio of two readings from a ratio scale such as 6 kg/ 3 kg = 2 is a number that belongs to an absolute scale that says that the 6 kg object is twice heavier than the 3 kg object. The ratio 2 cannot be changed by some formula to another number. Thus we introduce the next scale.

Absolute Scale invariant under the identity transformation $x = x$; for example, numbers used in counting the people in a room.

There are also other less well-known scales like a logarithmic and a log-normal scale.

The fundamental scale of the AHP is a scale of absolute numbers used to answer the basic question in all pairwise comparisons: **how many times more dominant is one element than the other with respect to a certain criterion or attribute?** The derived scale, obtained by solving a system of homogeneous linear equations whose coefficients are absolute numbers, is also an absolute scale of relative numbers. Such a relative scale does not have a unit nor does it have an absolute zero. The derived scale is like probabilities in not having a unit or an absolute zero.

In a judgment matrix A, instead of assigning two numbers w_i and w_j (that generally we do not know), as one does with tangibles, and forming the ratio w_i / w_j we assign a single number drawn from the fundamental scale of absolute numbers shown in Table 1 above to represent the ratio $(w_i / w_j)/1$. It is a nearest integer approximation to the ratio w_i / w_j. The ratio of two numbers from a ratio scale (invariant under multiplication by a positive constant) is an absolute number (invariant under the identity transformation) and is dimensionless. In other words it is not measured on a scale with a unit starting from zero. The numbers of an absolute scale are defined in terms of similarity or equivalence. The (absolute) number of a class is the class of all those classes that are similar to it; that is they can be put into one-to-one correspondence with it. But that is not our complete story about absolute numbers transformed to relative form – relative absolute numbers. We now continue our account.

The derived scale will reveal what w_i and w_j are. This is a central fact about the relative measurement approach. It needs a fundamental scale to express numerically the relative dominance relationship by using the smaller or lesser element as the unit of each comparison. Some people who do not understand this and regard the AHP as controversial, forget that most people in the world don't think in terms of numbers but of how they feel about intensities of dominance. They think that the AHP would have a greater theoretical strength if the judgments were made in terms of "ratios of preference differences". I think that the layman would find this proposal laughable as I do for its paucity of understanding, taking the difference of non-existing numbers which one is trying to find in the first place. He needs first to see a utility doctor who would help him create an interval scale utility function so he can take values from it to form differences and then form their ratios to get one judgment!

5. From Consistency to Inconsistency

Consistency is essential in human thinking because it enables us to order the world according to dominance. It is a necessary condition for thinking about the world in a scientific way, but it is not sufficient because a mentally disturbed person can think in a perfectly consistent way about a world that does not exist. We need actual knowledge about the world to validate our thinking. But if we were always consistent we would not be able to change our minds. New knowledge often requires that we see things in a new light that can contradict what we thought was correct before. Thus we live with the contradiction that we must be consistent to capture valid knowledge about the world but at the same time be ready to change our minds and be inconsistent if new information requires that we think differently than we thought before. It is clear that large inconsistency unsettles our thinking and thus we need to change our minds in small steps to integrate new information in the old total scheme. This means that inconsistency must be large enough to allow for change in our consistent understanding but small enough to make it possible to adapt our old beliefs to new information. *This means that inconsistency must be precisely one order of magnitude less important than consistency, or simply 10% of the total concern with consistent measurement. If it were larger it would disrupt consistent measurement and if it were smaller it would make insignificant contribution to change in measurement*.

The paired comparisons process using actual measurements for the elements being compared leads to the following consistent reciprocal matrix:

$$\begin{array}{c} \\ \\ A_1 \\ A_2 \\ \vdots \\ A_n \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & \cdots & A_n \\ w_1 & w_2 & \cdots & w_n \end{array} \\ \begin{bmatrix} w_1/w_1 & w_1/w_2 & \cdots & w_1/w_n \\ w_2/w_1 & w_2/w_2 & \cdots & w_2/w_n \\ & & \vdots & \\ w_n/w_1 & w_n/w_2 & \cdots & w_n/w_n \end{bmatrix} \end{array}$$

We note that we can recover the vector $w = (w_1, ..., w_n)$ by solving the system of equations defined by:

$$Aw = \begin{bmatrix} w_1/w_1 & w_1/w_2 & \cdots & w_1/w_n \\ w_2/w_1 & w_2/w_2 & \cdots & w_2/w_n \\ & & \vdots & \\ w_n/w_1 & w_n/w_2 & \cdots & w_n/w_n \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = n \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = nw$$

Solving this homogeneous system of linear equations $Aw = nw$ to find w is a trivial eigenvalue problem, because the existence of a solution depends on whether or not n is an eigenvalue of the characteristic equation of A. But A has rank one and thus all its eigenvalues but one are equal to zero. The sum of the eigenvalues of a matrix is equal to its trace, the sum of its diagonal elements, which in this case is equal to n. Thus n is the largest or the principal eigenvalue of A and w is its corresponding principal eigenvector that is positive and unique to within multiplication by a constant, and thus belongs to a ratio scale. We now know what must be done to recover the weights w_i, whether they are known in advance or not.

We said earlier that an n by n matrix $A = (a_{ij})$ is consistent if $a_{ij}a_{jk} = a_{ik}$, $i, j, k = 1,...,n$ holds among its entries. We have for a consistent matrix $A^k = n^{k-1}A$, a constant times the original matrix. In normalized form both A and A^k have the same principal eigenvector. That is not so for an inconsistent matrix. A consistent matrix always has the form

$$A = \left(\frac{w_i}{w_j} \right).$$

Of course, real-world pairwise comparison matrices are very unlikely to be consistent.

In the inconsistent case, the normalized sum of the rows of each power of the matrix contributes to the final priority vector. Using Cesaro summability and the well-known theorem of Perron, we are led to derive the priorities in the form of the principal right eigenvector. Now we give an elegant mathematical discussion, based on the concept of invariance, to show why we still need for an inconsistent matrix the principal right eigenvector for our priority vector. It is clear that no matter what method we use to derive the weights w_i, we need to get them back as proportional to the expression

$$\sum_{j=1}^{n} a_{ij} w_j \quad i = 1,...,n,$$

that is, we must solve

$$\sum_{j=1}^{n} a_{ij} w_j = cw_i \quad i = 1,...,n.$$

Otherwise $\sum_{j=1}^{n} a_{ij} w_j \quad i = 1,...,n$

would yield another set of different weights and they in turn can be used to form new expressions

$$\sum_{j=1}^{n} a_{ij} w_j \quad i = 1, ..., n ,$$

and so on ad infinitum. Unless we solve the principal eigenvalue problem, our quest for priorities becomes meaningless.

We learn from the consistent case that what we get on the right is proportional to the sum on the left that involves the same ratio scale used to weight the judgments that we are looking for. Thus we have the proportionality constant c. A better way to see this is to use the derived vector of priorities to weight each row of the matrix and take the sum. This yields a new vector of priorities (relative dominance of each element) represented in the comparisons. This vector can again be used to weight the rows and obtain still another vector of priorities. In the limit (if one exists), the limit vector itself can be used to weight the rows and get the limit vector back perhaps proportionately. Our general problem possibly with inconsistent judgments takes the form:

$$Aw = \begin{bmatrix} 1 & a_{12} & \cdots & a_{1n} \\ 1/a_{12} & 1 & \cdots & a_{2n} \\ & & \vdots & \\ 1/a_{1n} & 1/a_{2n} & \cdots & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = cw$$

This homogeneous system of linear equations $Aw = cw$ has a solution w if c is the principal eigenvalue of A. That this is the case can be shown using an argument that involves both left and right eigenvectors of A. Two vectors $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$ are orthogonal if their scalar product $x_1 y_1 + ... + x_n y_n$ is equal to zero. It is known that any left eigenvector of a matrix corresponding to an eigenvalue is orthogonal to any right eigenvector corresponding to a different eigenvalue. This property is known as biorthogonality (Horn and Johnson 1985).

Theorem *For a given positive matrix A, the only positive vector w and only positive constant c that satisfy* $Aw = cw$, *is a vector w that is a positive multiple of the principal eigenvector of A, and the only such c is the principal eigenvalue of A.*

Proof. We know that the right principal eigenvector and the principal eigenvalue satisfy our requirements. We also know that the algebraic multiplicity of the principal eigenvalue is one, and that there is a positive left eigenvector of A (call it z) corresponding to the principal eigenvalue. Suppose there is a positive vector y and a (necessarily positive) scalar d such that $Ay = dy$. If d and c are not equal, then by biorthogonality y is orthogonal to z, which is impossible since both vectors are positive. If c and d are equal, then y and w are dependent since c has algebraic multiplicity one, and y is a positive multiple of w. This completes the proof.

6. An Example of an AHP Decision

The simple decision is to choose the best city in which to live. We shall show how to make this decision using both methods of the AHP which conform with what Blumenthal said. We do it first with relative (comparative) measurement and second with absolute measurement. With the relative measurement method the criteria are pairwise compared with respect to the goal, the alternatives are pairwise compared with respect to each criterion and the results are synthesized or combined using a weighting and adding process to give an overall ranking of the alternatives. With the absolute measurement method standards are established for each criterion and the cities are rated one-by-one against the standards rather than being compared with each other.

6.1 Making the Decision with a Relative Measurement Model

The relative measurement model for picking the best city in which to live is shown below in Figure 4 (example by Mary Reiter).

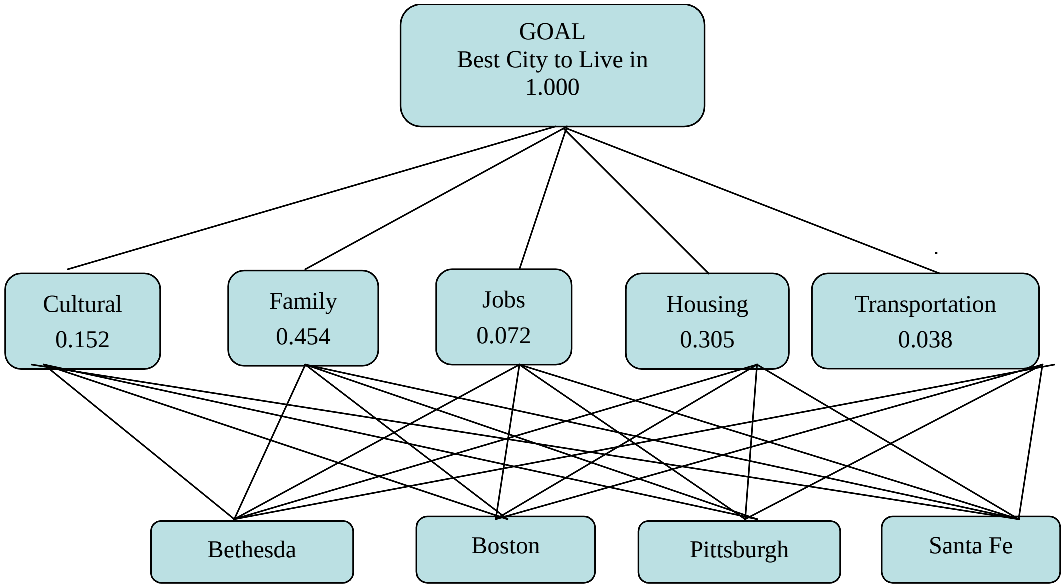


Figure 4 Relative model for choosing best city to live in

Entering Judgments

For each cell in the comparison matrix there is associated a row criterion (listed on the left), call it X, and a column criterion (on the top), call it Y. One answers this question for the cell: How much more important is X than Y in choosing a best city in which to live? The judgments, shown in Table 3, are entered using the Fundamental Scale of the AHP. Fractional values between the integers such as 4.32 can also be used when they are known from measurement.

The Number of Judgments and Consistency

In this decision there are 10 judgments to be entered. As we shall see later, inconsistency for a judgment matrix can be computed as a function of its maximum eigenvalue λ_{max} and the order n of the matrix. The time gained, from making fewer judgments than 10 along a spanning tree for example can be offset by not having sufficient redundancy in the judgments to fine tune and improve the overall outcome. There can be no inconsistency when the minimum number of judgments is used.

Next the alternatives are pairwise compared with respect to each of the criteria. The judgments and the derived priorities for the alternatives are shown in Table 4. The priority vectors are the principal eigenvectors of the pairwise comparison matrices. They are in the distributive form, that is, they have been normalized by dividing each element of the principal eigenvector by the sum of its elements so that they sum to 1. The priority vectors can be transformed to their idealized form by selecting the largest element in the vector and dividing all the elements by it so that it takes on the value 1, with the others proportionately less. The element (or elements) with a priority of 1 become the ideal(s). Later we explain why we use these two forms of synthesis.

Synthesis

The outcome of the distributive form is shown in Table 5 and that for the ideal form is shown in Table 6. The columns in Table 5 are the priority vectors for the cities from Table 4 and the columns in Table 6 are these same vectors in idealized form with respect to each criterion. Using either form the totals vector is obtained by multiplying the priority of each criterion times the priority of each alternative with respect to it and summing. The overall priority vector is obtained from the totals vector by normalizing: dividing each element in the totals vector by the sum of its elements. The final outcome with either form of synthesis is that Pittsburgh is the highest ranked city for this individual. Though the final priorities are somewhat different the order is the same: Pittsburgh, Boston, Bethesda and Santa Fe. The ratios of the final priorities are meaningful. Pittsburgh is almost twice as preferred as Bethesda.

When synthesizing in the distributive form the totals vector and the overall priorities vector are the same. When synthesizing in the ideal form as shown in Table 6 they are not. Ideal synthesis gives slightly different results from distributive synthesis in this case.

Table 3 Criteria weights with respect to the goal

GOAL	Culture	Family	Housing	Jobs	Transportation	Priorities
Culture	1	1/5	3	1/2	5	0.152
Family	5	1	7	1	7	0.433
Housing	1/3	1/7	1	1/4	3	0.072
Job	2	1	4	1	7	0.305
Transportation	1/5	1/7	1/3	1/7	1	0.038

Inconsistency 0.05

Table 4 Alternatives' weights with respect to criteria

Culture	Bethesda	Boston	Pittsburgh	Santa Fe	Priorities
Bethesda	1	1/2	1	1/2	0.163
Boston	2	1	2.5	1	0.345
Pittsburgh	1	1/2.5	1	1/2.5	0.146
Santa Fe	2	1	2.5	1	0.345

Inconsistency .002

Family	Bethesda	Boston	Pittsburgh	Santa Fe	Priorities
Bethesda	1	2	1/3	4	0.210
Boston	1	1	1/8	2	0.098
Pittsburgh	3	8	1	9	0.635
Santa Fe	1/4	1/2	1/9	1	0.057

Inconsistency .012

Housing	Bethesda	Boston	Pittsburgh	Santa Fe	Priorities
Bethesda	1	5	1/2	2.5	0.262
Boston	1/5	1	1/9	1/4	0.047
Pittsburgh	2	9	1	7	0.571
Santa Fe	1/2.5	4	1/7	1	0.120

Inconsistency .012

Jobs	Bethesda	Boston	Pittsburgh	Santa Fe	Priorities
Bethesda	1	1/2	3	4	0.279
Boston	2	1	6	8	0.559
Pittsburgh	1/3	1/6	1	1	0.087
Santa Fe	1/4	1/8	1	1	0.075

Inconsistency .004

Transportation	Bethesda	Boston	Pittsburgh	Santa Fe	Priorities
Bethesda	1	1.5	1/2	4	0.249
Boston	1/1.5	1	1/3.5	2.5	0.157
Pittsburgh	2	3.5	1	9	0.533
Santa Fe	1/4	1/2.5	1/9	1	0.061

Inconsistency .001

Table 5 Synthesis using the distributive mode to obtain the overall priorities for the alternatives

Synthesis	Cultural	Family	Housing	Jobs	Transport	Totals	Overall Priorities
	0.152	***0.433***	***0.072***	***0.305***	***0.038***	**(Weight and add)**	**(Normalize Totals)**
Bethesda	0.163	0.210	0.262	0.279	0.249	0.229	0.229
Boston	0.345	0.098	0.047	0.559	0.157	0.275	0.275
Pittsburgh	0.146	0.635	0.571	0.087	0.533	0.385	0.385
Santa Fe	0.345	0.057	0.120	0.075	0.061	0.111	0.111

Table 6 Synthesis using the ideal mode to obtain the overall priorities for the alternatives

Alternatives	Cultural *0.152*	Family *0.433*	Housing *0.072*	Jobs *0.305*	Transport *0.038*	Totals (Weight and add)	Overall Priorities (Normalize Totals)
Bethesda	0.474	0.330	0.459	0.500	0.467	0.418	0.224
Boston	1.000	0.155	0.082	1.000	0.295	0.541	0.290
Pittsburgh	0.424	1.000	1.000	0.155	1.000	0.655	0.351
Santa Fe	1.000	0.089	0.209	0.135	0.115	0.251	0.135

Ideal Synthesis Prevents Rank Reversal (Saaty 2001, Saaty and Vargas 1984a)

An important distinction to make between measurement in physics and measurement in decision making is that in the first we usually seek measurements that approximate to the weight and length of things, whereas in human action we seek to order actions according to priorities. In mathematics a distinction is made between *metric topology* that deals with the measurement of length, mass and time and *order topology* that deals with the ordering of priorities through the concept of *dominance* rather than closeness used in metric methods. We have seen that the principal eigenvector of a matrix is necessary to capture dominance priorities. When we have a matrix of judgments we derive its priorities in the form of its principal eigenvector. When we deal with a hierarchy the principle of hierarchic composition involves weighting and adding as a special case of the more general principle of network composition in which priorities are also derived as the principal eigenvector of a stochastic matrix which involves weighting and adding in the process of raising a matrix to powers. Some scholars whose specialization is in the physical sciences are perhaps unaware of the methods of order topology and have used various arguments to justify why they would use a metric approach to derive priorities and also to obtain the overall synthesis. It may be worthwhile to discuss this at some length in the following paragraph.

Ideal synthesis should be used when one wishes to prevent reversals in rank of the original set of alternatives from occurring when a new dominated alternative is added. With the distributive form rank reversal can occur to account for the presence of many other alternatives in cases where adding many things of the same kind or of nearly the same kind can depreciate the value of any of them. It has been established that 92% of the time, there is no rank reversal in the distributive mode when a new dominated alternative is added (Saaty and Vargas 1993). We note that uniqueness or manyness are not criteria that can be included when the alternatives are assumed to be independent of one another, for then to rank an alternative one would have to see how many other alternatives there are thus creating dependence among them.

Both the distributive and ideal modes are necessary for use in the AHP. We have shown that idealization is essential and is independent of what method one may use. There are people who have made it an obsession to find ways to avoid rank reversal in every decision and wish to alter the synthesis of the AHP away from normalization or idealization. They are likely to obtain outcomes that are not compatible with what the real outcome of a decision should be, because in decision-making we also want uniqueness of the answer we get.

Here is a failed attempt by some people to do things their metric way to preserve rank other than by the ideal form. The multiplicative approach to the AHP uses the familiar methods of taking the geometric mean to obtain the priorities of the alternatives for each criterion without normalization, and then raising them to the powers of the criteria and again taking the geometric mean to perform synthesis in a distorted way to always preserve rank. It is essentially a consequence of attempting to minimize the logarithmic least squares expression (Saaty 2000a, Saaty and Vargas 1984b)

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\log a_{ij} - \log\frac{w_i}{w_j}\right)^2 .$$

It does not work when the same measurement is used for the alternatives with respect to several criteria as one can easily verify and that should be sufficient to throw it out. Second and more seriously, the multiplicative method has an untenable mathematical problem. Assume that an alternative has a priority 0.2 with respect to each of two criteria whose respective priorities are 0.3 and 0.5. It is logical to assume that this alternative should have a higher priority with respect to the more important criterion, the one with the value of 0.5, after the weighting is performed. But $0.2^{0.5} < 0.2^{0.3}$ and alas it does not, it has a smaller priority. One would think that the procedure of ranking in this way would have been abandoned at first knowledge of this observation.

We conclude that in order to preserve rank indiscriminately from any other alternative, one can use the rating approach of the AHP described below in which alternatives are evaluated one at a time using the ideal mode. In addition, by deriving priorities from paired comparisons, rank is always preserved if one idealizes only the first time, and then compares each alternative with the ideal, allowing the value to exceed one. On the other hand, idealizing repeatedly, only preserves rank from irrelevant alternatives.

Remark On occasion someone has suggested the use of Pareto optimality instead of weighting the priorities of the alternatives by the priorities of the criteria and adding to find the best alternative. It is known that a concave function for the synthesis, if one could be found, would serve the purpose of finding the best alternative when it is known what it should be. But if the best alternative is already known for some property that it has which makes it the best, then one has a single not a multiple criteria decision. Naturally a multiple criteria problem may not yield the expected outcome. This is a special case of when the weights of the criteria depend on those of the alternatives. We will see in Part 2 that the final overall choice is automatically made in the process of finding the priorities of the criteria as they depend on the alternatives. Pareto optimality plays no role to determine the best outcome in that general case.

6.2 Making the Decision with an Absolute or Ratings Model

Using the absolute or ratings method of the AHP, categories (intensities) or standards are established for the criteria and cities are rated one at a time by selecting the appropriate category under each criterion rather than compared against other cities. The standards are prioritized for each criterion by making pairwise comparisons. For example, the standards for the criterion Job Opportunities are: Excellent, Above Average, Average, Below Average and Poor. Judgments are entered for such questions as: "How much more preferable is Excellent than Above Average for this criterion? Each city is then rated by selecting the appropriate category for it for each criterion. The city's score is then computed by weighting the priority of the selected category by the priority of the criterion and summing for all the criteria. The prioritized categories are essentially absolute scales, abstract yardsticks, which have been derived and are unique to each criterion. Judgment is still required to select the appropriate category under a criterion for a city, but the cities are no longer compared against each other. In absolute measurement, the cities are scored independently of each other. In relative measurement, there is dependence, as a city's performance depends on what other cities there are in the comparison group. Figure 5 and Tables 7, 8 and 9 represent what one does in the ratings or absolute measurement approach of the AHP. Table 7 illustrates the pairwise comparisons of the intensities under one criterion. The process must be repeated to compare the intensities for each of the other criteria. We caution that such intensities and their priorities are only appropriate for our given problem and should not be used with the same priorities for all criteria nor carelessly in other problems.

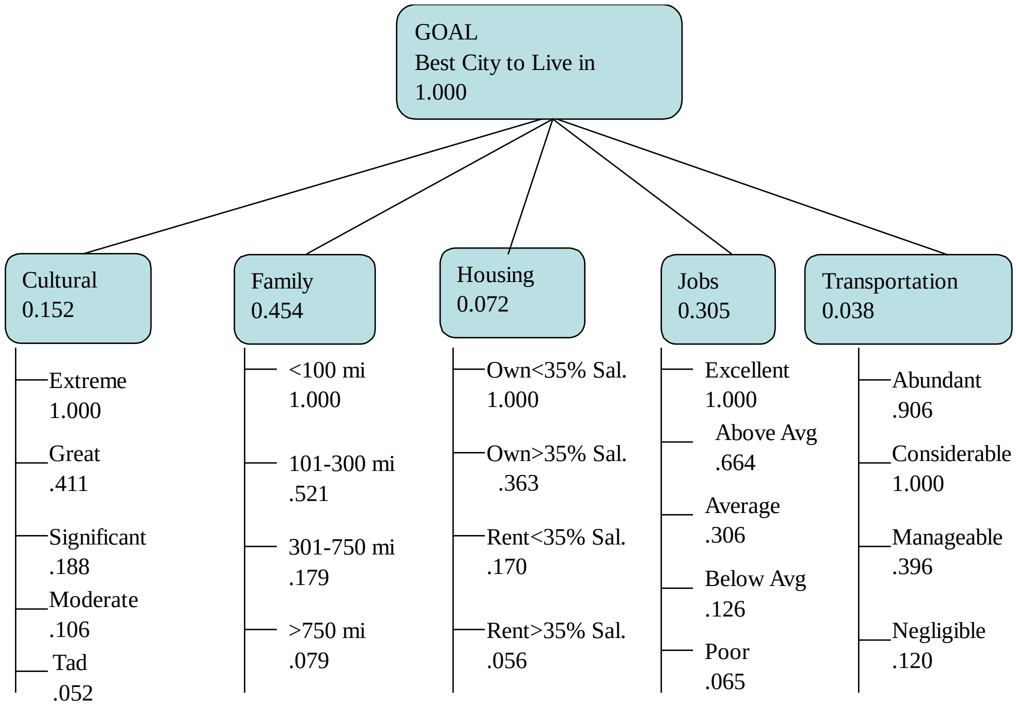


Figure 5 Absolute or ratings mode for choosing best city to live in

Table 7 Deriving priorities for the cultural criterion categories

	Extreme	Great	Significant	Moderate	Tad	Derived Priorities	Idealized Priorities
Extreme	1	5	6	8	9	.569	1.000
Great	1/5	1	4	5	7	.234	.411
Significant	1/6	1/4	1	3	5	.107	.188
Moderate	1/8	1/5	1/3	1	4	.060	.106
Tad	1/9	1/7	1/5	1/4	1	.030	.052

Inconsistency = .112

Table 8 Verbal ratings of cities under each criterion

Alternatives	Cultural *.195*	Family *.394*	Housing *.056*	Jobs *.325*	Transport *.030*	Total Score	Priorities (Normal.)
Pittsburgh	Signific.	<100 mi	Own>35%	Average	Manageable	**.562**	**.294**
Boston	Extreme	301-750 mi	Rent>35%	Above Avg	Abundant	**.512**	**.267**
Bethesda	Great	101-300 mi	Rent<35%	Excellent	Considerable	**.650**	**.339**
Santa Fe	Signific.	>750 mi	Own>35%	Average	Negligible	**.191**	**.100**

Table 9 Priorities of ratings of cities under each criterion

Alternatives	Cultural *.195*	Family *.394*	Housing *.056*	Jobs *.325*	Transport *.030*	Total Score	Priorities (Normalized)
Pittsburgh	0.188	1.000	0.363	0.306	0.396	**.562**	**.294**
Boston	1.000	0.179	0.056	0.664	0.906	**.512**	**.267**
Bethesda	0.411	0.521	0.170	1.000	1.000	**.650**	**.339**
Santa Fe	0.188	0.079	0.363	0.306	0.120	**.191**	**.100**

When the intensities are intangible, like excellent, very good and so on down to poor, there may be alternatives that fall above or below that range because what is excellent for one group of alternatives may not be applicable to alternatives that are much better or much worse than the given alternatives. In that case we need to expand the intensities by putting them into categories. We may use the same names for them but we may have order of magnitude categories in which we compare the elements in each category or even use the same scale but then combine that category with an adjacent category using the top or bottom rated intensity as a pivot as in the cherry and watermelon example. To determine which category an alternative should be rated on, we first start with any alternative and rate it. From then on before rating a new alternative we need to compare it with the previous alternative if it is better or worse and in doing that we need to reason through and insert hypothetical alternatives to place it correctly just as we did in the cherry-watermelon example. In real life, alternatives that naturally occur in a certain activity tend to be alike or homogeneous. Even when they are not alike they generally differ by one or two categories of intensity on each criterion. When they differ by more, they are unlikely to be considered as serious contenders and are assigned a zero value. The concern is usually from the top rated alternatives downwards for the intensities of each criterion.

7. Sanctioning China? An Application with Benefits, Costs and Risks

This example was developed in February 1995 when media were voicing strong opinion about whether the US government should sanction China about intellectual property rights. I and my coauthor Professor Jen Shang sent our analysis to Mr. Mickey Kantor, the then chief US negotiator before his trip to Beijing. Mr. Kantor acknowledged reading our article in a very positive way by calling me. We are not taking credit that the US did not sanction China. But we are quite happy that the outcome of the decision was along the lines of our recommendation. The model we used, shown in Figure 6, is a three part Benefits, Costs and Risks model. Note that in this example we formed the (marginal) ratio of the benefits which are positive, to the costs and risks both of which are opposite because we must respond to the question, which alternative is more costly (risky) for a give criterion.

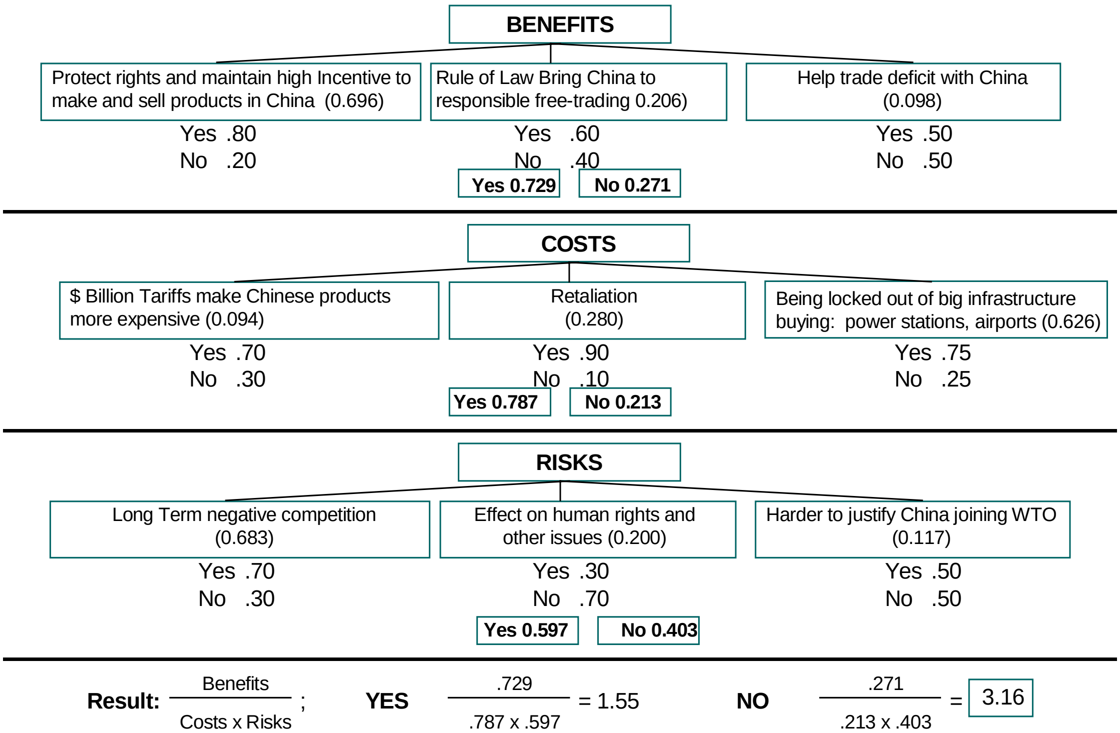


Figure 6 China trade sanction model

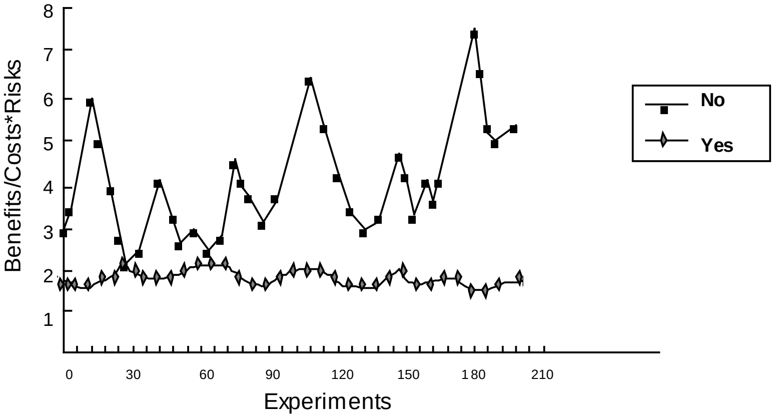


Figure 7 Sensitivity analysis of the outcome of the China decision

In later examples, instead of forming ratios of absolute numbers, we use subtraction and negative priorities (Saaty and Ozdemir 2003b) after carefully rating one at a time the highest ranked alternative under each of the benefits (B), opportunities (O), the costs (C) and the risks (R) or as a collective we refer to them as (BOCR). The highest ranked alternative is often different under each. In this manner instead of obtaining marginal (per unit) results by forming the ratio we obtain the totals. It is clear from this analysis that the US should not have taken any action that would be averse to cultivating a successful working relation between the two great countries for the foreseeable future.

8. Stimulus-Response and the Fundamental Scale

We shall see in Part 2 that a hierarchy is a special case of a network whose priorities and interactions are represented in a supermatrix *W*. As a result, hierarchic synthesis of priorities is a special case of network synthesis of priorities (Saaty 2001). The limit priorities of *W*, limit as $k \to \infty$ of W^k, yield its network synthesis. Equivalently, because *W* is column stochastic, its network synthesis can be obtained by solving the principal eigenvalue problem $Ww = w$ with $\lambda_{\max} = 1$. Invariance of the eigenvector makes additive hierarchic synthesis necessary to obtain priorities for a hierarchy.

To be able to perceive and sense objects in the environment our brains miniaturize them within our system of neurons so that we have a proportional relationship between what we perceive and what is out there. Without proportionality we cannot coordinate our thinking with our actions with the accuracy needed to control the environment. Proportionality with respect to a single stimulus requires that our response to a proportionately amplified or attenuated stimulus we receive from a source should be proportional to what our response would be to the original value of that stimulus. If $w(s)$ is our response to a stimulus of magnitude s, then the foregoing gives rise to the functional equation $w(as) = b\, w(s)$. This equation can also be obtained as the necessary condition for solving the Fredholm equation of the second kind:

$$\int_a^b K(s,t)\, w(t)\, dt \;=\; \lambda_{\max}\, w(s)$$

obtained as the continuous generalization of the discrete formulation $Aw = \lambda_{\max} w$ for deriving priorities where instead of the positive reciprocal matrix A in the principal eigenvalue problem, we have a positive kernel, $K(s,t) > 0,$ with $K(s,t)\ K(t,s) = 1$ that is also consistent i.e. $K(s,t)\ K(t,u) = K(s,u)$, for all $s,\ t,$ and u . The solution of this functional equation in the real domain is given by

$$w(s) = Ce^{\log b \frac{\log s}{\log a}} P\left(\frac{\log s}{\log a}\right)$$

where P is a periodic function of period 1 and $P(0) = 1$. One of the simplest such examples with $u = \log s / \log a$ is $P(u)$ = cos (u/2B) for which $P(0) = 1$.

The logarithmic law of response to stimuli can be obtained as a first order approximation to this solution through series expansions of the exponential and of the cosine functions as:

$$v(u) = C_1 e^{-\beta u} P(u) \approx C_2 \log s + C_3$$

$\log ab \equiv -\beta, \beta > 0$. The expression on the right is known as the Weber-Fechner law of logarithmic response $M = a \log s + b, a \neq 0$ to a stimulus of magnitude s. This law was empirically established and tested in 1860 by Gustav Theodor Fechner who used a law formulated by Ernest Heinrich Weber regarding discrimination between two nearby values of a stimulus. We have now shown that that Fechner's version can be derived by starting with a functional equation for stimulus response.

The integer-valued scale of response used in making paired comparison judgments can be derived from the logarithmic response function as follows. The larger the stimulus, the larger a change in it is needed for that change to be detectable. The ratio of successive just noticeable differences (the well-known "jnd" in psychology) is equal to the ratio of their corresponding successive stimuli values. Proportionality is maintained. Thus, starting with a stimulus s_0 successive magnitudes of the new stimuli take the form:

$$s_1 = s_0 + \Delta s_0 = s_0 + \frac{\Delta s_0}{s_0} s_0 = s_0(1+r)$$

$$s_2 = s_1 + \Delta s_1 = s_1(1+r) = s_0(1+r)^2 \equiv s_0 \alpha^2$$

$$\vdots$$

$$s_n = s_{n-1}\alpha = s_0 \alpha^n\ (n = 0, 1, 2, ...)$$

We consider the responses to these stimuli to be measured on a ratio scale (b=0). A typical response has the form $M_i = a \log \alpha^i$, i =1, $\cdots$,n, or one after another they have the form:

$$M_1 = a \log \alpha, M_2 = 2a \log \alpha, \cdots, M_n = na \log \alpha$$

We take the ratios $M_i / M_{1,}$, i = 1,$\cdots$, n, of these responses in which the first is the smallest and serves as the unit of comparison, thus obtaining the *integer* values 1, 2, $\cdots$, n of the fundamental scale of the AHP. It appears that numbers are intrinsic to our ability to make comparisons, and that they were not an invention by our primitive ancestors. We must be grateful to them for the discovery of the symbolism. In a less mathematical vein, we note that we are able to distinguish ordinally between high, medium and low at one level and for each of them in a second level below that also distinguish between high, medium and low giving us nine different categories. We assign the value one to (low, low) which is the smallest and the value nine to (high, high) which is the highest, thus covering the spectrum of possibilities between two levels, and giving the value nine for the top of the paired comparisons scale as compared with the lowest value on the scale. Because of increase in inconsistency when we compare more than about 7 elements, we don't need to keep in mind more than 7 ± 2 elements. This was first conjectured by the psychologist George Miller in the 1950's and explained in the AHP in the 1970's (Saaty and Ozdemir 2003a). Finally, we note that the scale just derived is attached to the importance we assign to judgments. If we have an exact measurement such as 2.375 and want to use it as it is for our judgment without attaching significance to it, we can use its entire value without approximation.

A person may not be schooled in the use of numbers and there are many in our world who do not, but still have feelings, judgments and understanding that enable him or her to make accurate comparisons (equal, moderate, strong, very strong and extreme and compromises between these intensities). Such judgments can be applied successfully to compare stimuli that are not too disparate but homogeneous in magnitude. By homogeneous we mean that they fall within specified bounds. Table 1, the Fundamental Scale for paired comparisons, summarizes the foregoing discussion.

The idea of using time dependent judgments has been examined in detail and will not be discussed in this paper (Saaty 2003).

9. When Is a Positive Reciprocal Matrix Consistent? (Saaty 2000a)

Let $A = [a_{ij}]$ be an n-by-n positive reciprocal matrix, so all $a_{ii} = 1$ and $a_{ij} = 1/a_{ji}$ for all $i,j = 1, \cdots, n$. Let $w = [w_i]$ be the Perron vector of A, let $D = \text{diag}(w_1, \cdots, w_n)$ be the n-by-n diagonal matrix whose main diagonal entries are the entries of w, and set $E \,/\, D^{-1}AD = [a_{ij} w_j / w_i] = [\varepsilon_{ij}]$. Then E is similar to A and is a positive reciprocal matrix since $\varepsilon_{ji} = a_{ji}w_i/w_j = (a_{ij}\, w_j / w_i)^{-1} = 1/\varepsilon_{ij}$. Moreover, all the row sums of E are equal to the principal eigenvalue of A:

$$\begin{aligned}\sum_{j=1}^{n} \varepsilon_{ij} &= \sum_j a_{ij} w_j / w_i = [Aw]_i / w_i \\ &= \lambda_{\max} w_i / w_i = \lambda_{\max}\end{aligned}.$$

The computation

$$\begin{aligned} n\lambda_{\max} &= \sum_{i=1}^{n} \Big(\sum_{j=1}^{n} \varepsilon_{ij} \Big) = \sum_{i=1}^{n} \varepsilon_{ii} + \sum_{\substack{i,j=1 \\ i \neq j}}^{n} (\varepsilon_{ij} + \varepsilon_{ji}) \\ &= n + \sum_{\substack{i,j=1 \\ i \neq j}}^{n} (\varepsilon_{ij} + \varepsilon_{ij}^{-1}) \geq n + (n^2 - n)/2 = n^2 \end{aligned} \tag{1}$$

reveals that $\lambda_{\max} \geq n$. Moreover, since $x + 1/x \geq 2$ for all x > 0, with equality if and only if $x = 1$, we see that $\lambda_{\max} = n$ if and only if all $\varepsilon_{ij} = 1$, which is equivalent to having all $a_{ij} = w_i / w_j$.

The foregoing arguments show that a positive reciprocal matrix A has $\lambda_{\max} \geq n$, with equality if and only if A is consistent. As our measure of deviation of A from consistency, we choose the *consistency index*

$$\mu \equiv \frac{\lambda_{\max} - n}{n-1}.$$

We have seen that $\mu \geq 0$ and $\mu = 0$ if and only if A is consistent. These two desirable properties explain the term "n" in the numerator of μ; what about the term "n–1" in the denominator? Since trace $A = n$ is the sum of all the eigenvalues of A, if we denote the eigenvalues of A that are different from $\lambda_{\max}$ by $\lambda_2, ..., \lambda_{n-1}$, we see that

$$n = \lambda_{\max} + \sum_{i=2}^{n} \lambda_i ,$$

so

$$n - \lambda_{\max} = \sum_{i=2}^{n} \lambda_i \text{ and } \mu = -\frac{1}{n-1} \sum_{i=2}^{n} \lambda_i$$

is the negative average of the non-principal eigenvalues of A.

It is an easy, but instructive, computation to show that $\lambda_{\max} = 2$ for every 2-by-2 positive reciprocal matrix:

$$\begin{bmatrix} 1 & \alpha \\ \alpha^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1+\alpha \\ (1+\alpha)\alpha^{-1} \end{bmatrix} = 2 \begin{bmatrix} 1+\alpha \\ (1+\alpha)\alpha^{-1} \end{bmatrix}$$

Thus, every 2-by-2 positive reciprocal matrix is consistent.

Not every 3-by-3 positive reciprocal matrix is consistent, but in this case we are fortunate to have again explicit formulas for the principal eigenvalue and eigenvector. For

$$A = \begin{bmatrix} 1 & a & b \\ 1/a & 1 & c \\ 1/b & 1/c & 1 \end{bmatrix},$$

we have $\lambda_{\max} = 1 + d + d^{-1}$, $d = (ac/b)^{1/3}$ and

$$\begin{aligned} w_1 &= bd/(1+bd+\frac{c}{d}) \\ w_2 &= c/d(1+bd+\frac{c}{d}), \\ w_3 &= 1/(1+bd+\frac{c}{d}) \end{aligned} \qquad (2)$$

Note that $\lambda_{\max} = 3$ when d = 1 or $c = b/a$, which is true if and only if A is consistent.

In order to get some feel for what the consistency index might be telling us about a positive n-by-n reciprocal matrix A, consider the following simulation: choose the entries of A above the main diagonal at random from the 17 values {1/9, 1/8, $\cdots$, 1/2, 1, 2, $\cdots$, 8, 9}. Then fill in the entries of A below the diagonal by taking reciprocals. Put ones down the main diagonal and compute the consistency index. Do this 50,000 times and take the average, which we call the *random index*. Table 4 shows the values obtained from one set of such simulations, for matrices of size 1, 2, $\cdots$, 10.

Since it would be pointless to try to discern any priority ranking from a set of random comparison judgments, we should probably be uncomfortable about proceeding unless the consistency index of a pairwise comparison matrix is very much smaller than the corresponding random index value in Table 10. The *consistency ratio* (C.R.) of a pairwise comparison matrix is the ratio of its consistency index μ to the corresponding random index value in Table 10.

Table 10 Random index

n	1	2	3	4	5	6	7	8	9	10
Random Index	0	0	.52	.89	1.11	1.25	1.35	1.40	1.45	1.49

If the C.R. is larger than desired, we do three things: 1) Find the most inconsistent judgment in the matrix, 2) Determine the range of values to which that judgment can be changed corresponding to which the inconsistency would be improved, 3) Ask the decision maker to consider, if he can, changing his judgment to a plausible value in that range. If he is unwilling, we try with the second most inconsistent judgment and so on. If no judgment is changed the decision is postponed until better understanding of the criteria is obtained. Three methods are plausible for changing the judgments to improve inconsistency. All require theoretical investigation of convergence and efficiency. The first uses an explicit formula for the partial derivatives of the principal eigenvalue with respect to the matrix entries.

For a given positive reciprocal matrix $A=[a_{ij}]$ and a given pair of distinct indices $k > l$, define $A(t)=[a_{ij}(t)]$ by $a_{kl}(t) \equiv a_{kl} + t$, $a_{lk}(t) \equiv (a_{lk} + t)^{-1}$, and $a_{ij}(t) \equiv a_{ij}$ for all $i \neq k, j \neq l$, so $A(0) = A$. Let $\lambda_{\max}(t)$ denote the Perron eigenvalue of $A(t)$ for all t in a neighborhood of $t = 0$ that is small enough to ensure that all entries of the reciprocal matrix $A(t)$ are positive there. Finally, let $v = [v_i]$ be the unique positive eigenvector of the positive matrix A^T that is normalized so that $v^T w = 1$. Then a classical perturbation formula (Horn and Johnson 1985, theorem 6.3.12) tells us that

$$\left.\frac{\mathrm{d}\lambda_{\max}(t)}{\mathrm{d}t}\right|_{t=0} = \frac{v^T A'(0) w}{v^T w} = v^T A'(0) w$$

$$= v_k w_l - \frac{1}{a_{kl}^2} v_l w_k .$$

We conclude that

$$\frac{\partial \lambda_{\max}}{\partial a_{ij}} = v_i w_j - a_{ji}{}^2 v_j w_i \quad \text{for all } i,j=1,\cdots,n.$$

Because we are operating within the set of positive reciprocal matrices we have:

$$\frac{\partial \lambda_{\max}}{\partial a_{ji}} = -\frac{\partial \lambda_{\max}}{\partial a_{ij}} \quad \text{for all } i \text{ and } j.$$

Thus, to identify an entry of A whose adjustment within the class of reciprocal matrices would result in the largest rate of change in $\lambda_{\max}$ we should examine the $n(n-1)/2$ values $\{v_i w_j - a_{ji}{}^2 v_j w_i\}, i > j$ and select (any) one of largest absolute value (Harker 1987). It is significant to note here that if one compares more than about seven elements in a homogeneous group, the rise in inconsistency is generally so small that it is then difficult to determine which judgment should be changed (Saaty and Ozdemir 2003a).

10. Nonlinearity and Multilinear Forms in the AHP

Hierarchic composition produces sums of products of priorities. These define a special kind of mathematical function known as a multilinear form. It is useful for us to examine briefly how these forms that arise here naturally, may tell us something useful about the real world. This subject is wide open for investigation.

A **monomial** is a single term that is a product of one coefficient and several variables each with an exponent indicating a power (often restricted to be a non-negative integer) of that variable. Examples are $-3x^5, a^2x^3y^2$,

$-a$. A **polynomial** is the sum or difference of monomials as $-5x^3y^7z^2+2xy^4+7$ from which we can define a polynomial in one variable as $2x^4-x^2+x+1$. A polynomial is a rational integral algebraic expression with nonnegative powers of the variables. The coefficients of a polynomial can be real or complex. A **multinomial** is another term for polynomial, although one would prefer the former to apply to several variables and the latter to a single variable. A **form** is a polynomial in several variables in which the sum of the powers of the variables in each term is equal to that in any other term. A form is binary, ternary etc depending on whether it has two, three etc. variables. It is linear, quadratic etc the sum of the degrees of the variables that is the same in each term. For example $7xz-3y^2+2yz$ is a ternary quadratic form. A **multilinear** form is a form in which the variables are divided into sets so that in each term a variable from every set appears to the first power. It has the general form

$$\sum_{i,j,\cdots,l=1}^{n} a_{ij\cdots l}x_i y_j \ldots z_l$$

with m sets of variables with n variables in each set, $x_1,x_2,\cdots,x_n;y_1,y_2,\cdots,y_n;z_1,z_2,\cdots,z_n$ and because it is linear in the variables of each set, it is called a multilinear form. When m=1, the form

$$\sum_{i=1}^{n} a_i x_i$$

is known as a linear form. When m=2, the form

$$\sum_{i,j=1}^{n} a_{ij}x_i y_i$$

is known as a bilinear form. Any form can be obtained from a multilinear form by identifying certain of the variables. Conversely, transforming any form to a multilinear form is carried out by polarization. For example $x_1^2+2x_1x_2+x_2^2$ can be written as a multilinear form $x_1y_1+x_1y_2+x_2y_1+x_2y_2$ with y_1 identified with x_1 and y_2 identified with x_2. A multilinear form is a particular case of a multilinear mapping or operator (not defined here) as a result of which one can think of it as symmetric, skew symmetric, alternating, symmetrized and skew symmetrized forms. Let us now turn to hierarchies using more uniform notation.

Hierarchic composition yields multilinear forms which are of course nonlinear and have the form

$$\sum_{i_1,\cdots,i_p} x_1^{i_1}x_2^{i_2}\cdots x_p^{i_p}$$

The richer the structure of a hierarchy in breadth and depth the more complex are the derived multilinear forms from it. There seems to be a good opportunity to investigate the relationship obtained by composition to covariant tensors and their algebraic properties. More concretely we have the following covariant tensor for the priority of the ith element in the hth level of the hierarchy.

$$w_i^h=\sum_{i_2,\cdots,i_{h-1}=1}^{N_{h-1},\cdots,N_1} w_{i_1i_2}^{h-1}\cdots w_{i_{h-2}i_{h-1}}^{2}w_{i_{h-1}}^{1} \quad i_1\equiv i$$

The composite vector for the entire hth level is represented by the vector with covariant tensorial components. Similarly, the left eigenvector approach to a hierarchy gives rise to a vector with contravariant tensor components. Tensors, are generalizations of scalars (which have no indices), vectors (which have a single index), and matrices or arrays (which have two indices) to an arbitrary number of indices. They are widely known and used in physics and engineering.

Another interpretation follows the lines of polynomial approximation. We see above that polynomials in one and in several variables are intimately linked to multilinear forms. The **Weierstrass approximation theorem** states that every continuous function defined on an interval [a,b] can be uniformly approximated as closely as desired by a polynomial function. It assures us that one can get arbitrarily close to any continuous function as the polynomial order is increased. Because polynomials are the simplest functions, and computers can directly evaluate polynomials, this theorem has both practical and theoretical relevance. The **Stone-Weierstrass theorem** generalizes the Weierstrass approximation theorem in two directions: instead of the compact interval [a,b], an arbitrary compact Hausdorff space X is considered, and instead of the algebra of polynomial functions, approximation with elements from other subalgebras is investigated. Thus we see that the multilinear forms generated in the AHP represent or converge closely to a continuous function in many variables, differentiable or non-differentiable, assumed to underlie our understanding of a complex decision. In the ANP, raising the matrix to infinite powers generates a multilinear form that is an infinite series of numerical terms that converges to some limit. Performing sensitivity analysis generates a large number of limit points presumed to lie on a function to which the multilinear form as a function of its variables converges. As a result, the ANP, discussed in Part 2, is more likely to provide accurate answers about real world decisions than the AHP with its truncated relations.

11. The Analytic Hierarchy Process and Resource Allocation (Saaty et al. 2003)

Intangible resources such as quality, care, attention, and intelligence are often needed to develop a plan, design a system or solve a problem. Thus far, resource allocation models have not dealt with intangibles directly, but rather by assigning them worth in terms of such phenomena as time and money. Although there is no direct scale of measurement for an intangible, it can be measured in relative terms together with tangibles. A ratio scale of priorities can thus be derived for both. These priorities serve as coefficients in an optimization framework to derive relative amounts of resources to be allocated. For intangible resources, because there is no unit of measurement, no absolute amount of a resource can be specified. However, in the presence of tangibles, it becomes possible to compute their absolute equivalents because of the proportionality inherent in their priorities. The coefficients of a mathematical linear programming (LP) model can be represented with priorities obtained with relative (i.e., pairwise comparisons) measurement as shown in the previous section. The result is that when measurement scales exist, the solution to the relative linear programming (RLP) model (with coefficients normalized to unity to make them correspond to priorities obtained with relative measurement) and the solution to the absolute linear programming (LP) model (the "usual" model with measurements on physical scales are the same to within a multiplicative constant. It is then possible to construct LP models using solely relative measurement to optimize the allocation of intangible resources, as follows:

Traditional LP ⇔ Relative LP

Decision Variables: $\bar{x} = (\bar{x}_1, \cdots, \bar{x}_n)^T$

$$\bar{w} = (w_1, \cdots, w_n)^T$$

Objective Function: $\sum_j c_j x_j \;\rightarrow\; {}_R c_j = \frac{c_j}{\sum_k |c_k|}$

$$\rightarrow \quad \sum_j {}_R c_j w_j$$

Constraints: $\sum_j a_{ij} x_j \le b_i$

$$\rightarrow \left\{ \begin{array}{l} {}_R a_{ij} = \dfrac{a_{ij}}{\sum_k |a_{ik}|} \\ {}_R b_i = \dfrac{\frac{b_i}{\sum_k |a_{ik}|}}{\sum_h \frac{|b_h|}{\sum_k |a_{hk}|}} \\ w_j = \dfrac{x_j}{\sum_h \frac{|b_h|}{\sum_k |a_{hk}|}} \end{array} \right\} \rightarrow$$

$$\sum_j {}_R a_{ij} w_j \le {}_R b_i$$

Primal:
$$\begin{array}{l} \text{Max} \sum_j c_j x_j \\ \text{s.t.}: \sum_j a_{ij} x_j \le b_i \\ x_j \ge 0 \end{array} \quad \Leftrightarrow \quad \begin{array}{l} \text{Max} \sum_j {}_R c_j w_j \\ \text{s.t.}: \sum_j {}_R a_{ij} w_j \le {}_R b_i \\ w_j \ge 0 \end{array}$$

Dual:
$$\begin{array}{l} \text{Min} \sum_i b_i y_i \\ \text{s.t.}: \sum_i a_{ij} y_i \ge c_j \\ y_i \ge 0 \end{array} \quad \Leftrightarrow \quad \begin{array}{l} \text{Min} \sum_i {}_R b_i v_i \\ \text{s.t.}: \sum_i {}_R a_{ij} v_i \ge {}_R c_j \\ v_j \ge 0 \end{array}$$

It is significant to note that all coefficients in the relative formulation are unit free, although their relative magnitudes are preserved. Thus, the underlying magnitudes they represent can be compared in pairs.

There are three places where intangibles can arise in an LP model, the objective function and in estimating the left side and the right side of the coefficients of the constraints. The most common is in the objective function wherein the coefficients can be estimated as priorities, with the rest of the model formulated in the usual way. This presents no practical complications since the solution is the same if the objective function coefficients are given in relative terms, which is tantamount to dividing by a constant. For general treatment and examples, see (Saaty et al. 2003).

12. Group Decision Making

Here we consider two issues in group decision making. The first is how to aggregate individual judgments, and the second is how to construct a group choice from individual choices. The reciprocal property plays an important role in combining the judgments of several individuals to obtain a judgment for a group. Judgments must be combined so that the reciprocal of the synthesized judgments must be equal to the syntheses of the reciprocals of these judgments. It has been proved that the geometric mean is the unique way to do that. If the individuals are experts, they my not wish to combine their judgments but only their final outcome from a hierarchy. In that case one takes the geometric mean of the final outcomes. If the individuals have different priorities of importance their judgments (final outcomes) are raised to the power of their priorities and then the geometric mean is formed.

How to Aggregate Individual Judgments

Let the function $f(x_1,...,x_n)$ for synthesizing the judgments given by n judges, satisfy the

(i) *Separability condition* (S): $f(x_1,...,x_n) = g(x_1)...g(x_n)$, for all $x_1,...,x_n$ in an interval P of positive numbers, where *g* is a function mapping P onto a proper interval J and is a continuous, associative and cancellative operation. [(S) means that the influences of the individual judgments can be separated as above.]

(ii) *Unanimity condition* (U): $f(x,...,x) = x$ for all x in P. [(U) means that if all individuals give the same judgment *x*, that judgment should also be the synthesized judgment.]

(iii) *Homogeneity condition* (H): $f(ux_1,...,ux_n) = uf(x_1,...,x_n)$ where $u > 0$ and x_k, ux_k $(k=1,2,\cdots,n)$ are all in P. [For ratio judgments (H) means that if all individuals judge a ratio *u* times as large as another ratio, then the synthesized judgment should also be *u* times as large.]

(iv) *Power conditions* (P_p): $f(x_1^p,...,x_n^p) = f^p(x_1,...,x_n)$. [($P_2$) for example means that if the *k*th individual judges the length of a side of a square to be x_k, the synthesized judgment on the area of that square will be given by the square of the synthesized judgment on the length of its side.]

Special case ($R=P_{-1}$):

$$f\left(\frac{1}{x_1},...,\frac{1}{x_n}\right) = 1/f(x_1,...,x_n).$$

[(R) is of particular importance in ratio judgments. It means that the synthesized value of the reciprocal of the individual judgments should be the reciprocal of the synthesized value of the original judgments.]

Aczel and Saaty (Saaty 2000b) proved the following theorem:

Theorem *The general separable (S) synthesizing functions satisfying the unanimity (U) and homogeneity (H) conditions are the geometric mean and the root-mean-power. If moreover the reciprocal property (R) is assumed even for a single n-tuple* $(x_1,...,x_n)$ *of the judgments of n individuals, where not all* x_k *are equal, then only the geometric mean satisfies all the above conditions.*

In any rational consensus, those who know more should, accordingly, influence the consensus more strongly than those who are less knowledgeable. Some people are clearly wiser and more sensible in such matters than others, others may be more powerful and their opinions should be given appropriately greater weight. For such unequal importance of voters not all *g*'s in (S) are the same function. In place of (S), the weighted separability property (WS) is now: $f(x_1,...,x_n) = g_1(x_1)...g_n(x_n)$ [(WS) implies that not all judging individuals have the same weight when the judgments are synthesized and the different influences are reflected in the different functions $(g_1,...,g_n)$.]

In this situation, Aczel and Alsina (Saaty 2000b) proved the following theorem:

Theorem *The general weighted-separable (WS) synthesizing functions with the unanimity (U) and homogeneity (H) properties are the weighted geometric mean* $f(x_1,x_2,\dots,x_n)= = x_1^{q_1} x_2^{q_2} \cdots x_n^{q_n}$ *and the weighted root-mean-powers* $f(x_1,x_2,\dots,x_n)=\sqrt[\gamma]{q_1 x_1^{\gamma}+q_2 x_2^{\gamma}\dots+q_n x_n^{\gamma}}$, *where* $q_1+...+q_n=1$, $q_k>0, k=1,...,n$, $\gamma>0$ *, but otherwise* $q_1,...,q_n,\gamma$ *are arbitrary constants.*

If *f* also has the reciprocal property (R) and for a single set of entries $(x_1,...,x_n)$ of judgments of *n* individuals, where not all x_k are equal, then *only the weighted geometric mean* applies. We give the following theorem which is an explicit statement of the synthesis problem that follows from the previous results, and applies to the second and third cases of the deterministic approach:

Theorem *If* $x_1^{(i)},...,x_n^{(i)}$ $i=1, \cdots, m$ *are rankings of n alternatives by m independent judges and if* a_i *is the importance of judge i developed from a hierarchy for evaluating the judges, and hence*

$$\sum_{i=1}^{m} a_i = 1 \text{, then } \left(\prod_{i=1}^{m} x_1^{a_i} \right), ..., \left(\prod_{i=1}^{m} x_n^{a_i} \right)$$

are the combined ranks of the alternatives for the m judges.

The power or priority of judge *i* is simply a replication of the judgment of that judge (as if there are as many other judges as indicated by his/her power a_i), which implies multiplying his/her ratio by itself a_i times, and the result follows.

The first requires knowledge of the functions which the particular alternative performs and how well it compares with a standard or benchmark. The second requires comparison with the other alternatives to determine its importance.

On the Construction of Group Choice from Individual Choices

Given a group of individuals, a set of alternatives (with cardinality greater than 2), and individual ordinal preferences for the alternatives, Arrow proved with his Impossibility Theorem that it is impossible to derive a rational group choice (construct a social choice function that aggregates individual preferences) from ordinal preferences of the individuals that satisfy the following four conditions, i.e., at least one of them is violated:

Decisiveness: the aggregation procedure must generally produce a group order.

Unanimity: if all individuals prefer alternative A to alternative B, then the aggregation procedure must produce a group order indicating that the group prefers A to B.

Independence of irrelevant alternatives: given two sets of alternatives which both include A and B, if all individuals prefer A to B in both sets, then the aggregation procedure must produce a group order indicating that the group, given any of the two sets of alternatives, prefers A to B.

No dictator: no single individual preferences determine the group order.

Using the ratio scale approach of the AHP, it can be shown that because now the individual preferences are cardinal rather than ordinal, it is *possible* to derive a rational group choice satisfying the above four conditions. It is possible because: a) Individual priority scales can always be derived from a set of pairwise cardinal preference judgments as long as they form at least a minimal spanning tree in the completely connected graph of the elements being compared; and b) The cardinal preference judgments associated with group choice belong to an absolute scale that represents the relative intensity of the group preferences (Saaty and Vargas 2003).

13. Axioms of the AHP (Saaty 2000b)

The AHP includes four axioms. Informally, they are concerned with the reciprocal relation, comparison of homogeneous elements, hierarchic and systems dependence, and expectations about the validity of the rank and value of the outcome and their dependence on the structure used and its extension. The formalism for introducing the axioms would take us far afield in this presentation although we recommend examining them very highly to the reader by reference to my book on Fundamentals of the AHP.

14. How to Structure a Hierarchy - Relationship to Automatic Control

What kinds of hierarchies are there and how should they be structured to meet certain needs? What is the main purpose of arranging goals, attributes, issues, and stakeholders in a hierarchy? Most problems arise because we do not know the internal dynamics of a system in sufficient detail to identify cause-effect relationships. If we were able to do so, the problem could be reduced to one of social engineering, as we would know at what points in the system intervention is necessary to bring about the desired objective. The crucial contribution of the AHP is that it enables us to make practical decisions based on a "pre-causal" understanding – namely, on our feelings and judgments about the relative impact of one variable on another (Saaty 2000b).

Briefly, when constructing hierarchies one must include enough relevant detail to represent the problem as thoroughly as possible, but not so much as to include the whole universe in a small decision. One needs to: Consider the environment surrounding the problem. Identify the issues or attributes that one feels influence and contribute to the solution. Identify the participants associated with the problem. Arranging the goals, attributes, issues, and stakeholders in a hierarchy serves three purposes: It provides an overall view of the complex relationships inherent in the situation; it captures the spread of influence from the more important and general criteria to the less important ones; and it permits the decision maker to assess whether he or she is comparing issues of the same order of magnitude in weight or impact on the solution.

Two General Structures of Hierarchies

1) Generic Hierarchy for Forward Planning

The levels of the hierarchy successively descend from the goal down to:

– Time Horizons
– Uncontrollable Environmental Constraints
– Risk Scenarios
– Controllable Systemic Constraints
– Overall Objectives of the Systems
– Stakeholders
– Stakeholder Objectives (Separate for each)
– Stakeholder Policies (Separate for each)
– Exploratory Scenarios (Outcomes)
– Composite or Logical Scenario (Out- come)

Most prediction problems are of this kind. Contingency Planning policies must be devised to deal with unexpected occurrences and exploratory scenarios are included to allow for such a possibility. The exploratory scenarios are what each stakeholder would pursue if alone with no other stakeholders around.

2) The Backward Planning Hierarchy

The levels of this hierarchy successively descend from the goal of choosing a best outcome to:

– Anticipatory Scenarios
– Problems and Opportunities
– Actors and Coalitions
– Actor Objectives
– Actor Policies
– Particular Control Policies of a particular actor to Influence the Outcome

Most decision problems are of this kind. Planning involves testing the impact of the high priority policies in the bottom level. These policies are added to the policies of that particular actor in the forward process which results in a second forward process hierarchy. The iterations are repeated to close the gap between the dominant contrast scenarios (or composite scenario) of the forward process and the anticipatory scenarios of the backward process. See my book on planning (Saaty and Kearns 1991).

In a hierarchy or network alternatives can be evaluated not simply in terms of the usual criteria but also separately in terms of control criteria that would expedite and ensure their implementation. That way an alternative that looks best under “state” criteria may not look as good under control criteria and may not come out best even if it is the most desired.

15. Judgments Feelings and Measurement

Because decision making involves judgments, preferences, feelings, and risk taking, it appears that it belongs in part to meta rational thinking. In rational thinking one uses logic based on explicit assumptions to derive one’s conclusions. In decision making one elicits information about comparisons and preferences that belong to the domain of feelings and emotions.

A question that puzzles all of us brought up in the use of models is that usually a model is based on data from measurement that anyone can validate on their own. In the AHP we rely on the judgment of people. Where does this judgment originate, and how can we trust the subjective understanding of people to tell us something “objective” about the real world? We must assume that any understanding registers somewhere in our nervous system and we carry it with us. In the end we are the ones who provide the criteria and ways of understanding. At bottom all knowledge is subjectively derived. In this regard psychologists make the distinction between our cognitive and our affective (feeling) abilities. The changes in state of an organism due to the dynamic stresses in the psychological situation experienced are directly apprehended as sensations or perceptions belonging to our cognitive ability. The state itself is apprehended as feeling (affect), a global effect arising from a pattern of visceral impulses that is not easily localizable.

While "thinking" is generally thought to be carried out in the neo-cortex of the brain, feelings and partly emotions are associated with the autonomic (sympathetic and parasympathetic) nervous system that in part is known to operate independently of the thought processes of the brain. There is very little conscious control over many activities of the autonomic nervous system. It is as if there are two persons in each of us. One that looks out at the environment to give us information for survival of hazards, and another that looks inside to keep our system running. The sympathetic division, located in the spinal cord from its first thoracic to its third lumbar segments prepares the body in times of stress by dilating the blood vessels in the heart, muscles, and other vital organs, speeding the heart and blood flow (by stimulating production of adrenaline that liberates sugar from the liver) and constricting it in the skin. The parasympathetic division has two parts one originating in the midbrain, pons and medulla and consists of four cranial nerves mostly opposing sympathetic action as needed, and the other division comes from cells in the second, third and fourth segments of the sacral part of the spinal cord both stimulating parts of the body and inhibiting others like constricting the bronchi in the lungs.

Most animals have small brains but have effective autonomic systems to run their bodies, perhaps better in some ways than we have. Our brain looks out to the environment to provide data for adjustment and survival. Philosophically, decision making must be subject to the laws of science but its assumptions cannot be stated explicitly because of the use of feelings and intuition to express preference. Science has not yet learned enough about where emotions and feelings fit rationally into our system of logical thinking.

It has been pointed out to this author that there is a classification of types or levels of consciousness that originated in India which shows that truth belongs to different domains of existence of which logical thinking is only a part and not necessarily the ultimate means of discovering ideas and meaning. They are: 1) physical (matter and energy in the form of solids, liquids and gasses), 2) etheric (electromagnetic, subatomic particles), 3) emotional (feeling, emotion, desire, imagination, personal power), 4) mental (intellectual, understanding, beliefs, thoughts, knowledge, and cognitive processes), 5) causal (personal individuality, the enlivening source of life and consciousness), 6) physical to causal (the personality as a unit is made of several bodies: the mental/intellectual, the emotional, the etheric and the physical), 7) the different bodies combined, 8) manasic (consciousness of a bigger reality beyond the physical world), 9) social or religious buddhic/christic, (wider consciousness beyond individuality and integration with others with love and harmony), 10) atmic (identification not with individuals, not with groups, but with all pervading life-equanimity and peacefulness towards all.) Atmic consciousness is characterized by omnipotence and an extreme power of will that makes nearly all possible is that of pure equanimity with undifferentiated awareness - identification, not with individuality, not with groups of beings, but with all pervading life itself. It is the transcendence of both pain and bliss, extremely intense peace, 11) monadic (the generator of consciousness for all the previous levels, the power station from which will, love and intelligence are derived), and 12) logoic (the universal God consciousness encompassing all the beings living on the multiple levels mentioned above of which we are the atoms.) Decision making, even as we try to explain it with logic, belongs to the tenth or atmic level of consciousness.

16. Conclusions

A reliable decision theory, as any scientific theory, should be able to describe and account for how people make decisions and how to generalize on that as a foundation for organizing human thinking in a workable and harmonious way with what our instincts and feelings tell us. Thus we need to be aware of how to present our theories and validate them so they can provide a basis for further developments in the future. How do we know that we have valid answers about the real world when we make a decision? After all that decision depends on our feelings and preferences. Do they survive well and capture what happens in the real world? How long should it take to find that out?

The Analytic Hierarchy Process (AHP) and its generalization to dependence and feedback the Analytic Network Process (ANP) are our conscious analytical digitalization or discretization of thoughts in the brain of continuous natural processes that go on in our intuitive learning systems that are both mental thinking processes as well as long standing feelings, reflexes, preferences and judgments whose origins are tied to our autonomic system consisting of sympathetic and parasympathetic nervous systems. It is as if we are a form of intelligent life that uses the brain to obtain information about the global environment, but is otherwise self-sufficient to exist in the local environment. The AHP/ANP helps us in unfolding the complexity that is within us. The ANP will show greater depth and more widely usable applications of these ideas.

I know of at least four books in Chinese on the AHP. They are:

Saaty,T.L., *The Analytic Hierarchy Process-Applications to Resource Allocation, Management and Conflict*, translated by Shubo Xu, Press of Coal Industry, China, 334 pages, 1989.

Xu, Shubo, *Applied Decision Making Methods – The Analytic Hierarchy Process*, Press of Tianjin University, Tianjin, 1988.

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