Questão 2. *(2.5 pontos) Aproxime $f(x) = x^{1/3}$ por um polinômio $p(x)$ de grau menor ou igual a dois de forma a minimizar o erro E dado por*

$$E = (f'(-1) - p'(-1))^2 + (f'(1) - p'(1))^2 + \int_{-1}^{1} (f(x) - p(x))^2.$$

$$f(x) = x^{1/3} \qquad f'(x) = \frac{1}{3} x^{-2/3} = \frac{1}{3x^{2/3}}$$

$$p(x) = ax^2 + bx + c$$

$$E = [f'(-1) - p'(-1)]^2 + \int_{-1}^{1} (f(x) - p(x))^2 dx + [f'(1) - p'(1)]^2$$

solução:

$$[f'(-1) - p'(-1)]^2 = \left[\frac{1}{3} - (-2a+b)\right]^2 = \left[\frac{1}{3} + 2a - b\right]^2 = \left(\frac{1}{3} - b\right)^2 + 4a^2 + 4a\left(\frac{1}{3} - b\right)$$

$$[f'(1) - p'(1)]^2 = \left[\frac{1}{3} - (2a+b)\right]^2 = \left[\frac{1}{3} - 2a - b\right]^2 = \left(\frac{1}{3} - b\right)^2 + 4a^2 - 4a\left(\frac{1}{3} - b\right)$$

$$\int_{-1}^{1} (f(x) - p(x))^2 dx = \frac{6}{5} - \frac{12}{7} b + \frac{2a^2}{5} + \frac{2b^2}{3} + 2c^2 + \frac{4ac}{3}$$

$$E = 2\left(\frac{1}{3} - b\right)^2 + 8a^2 + \left(\frac{6}{5} - \frac{12}{7} b + \frac{2a^2}{5} + \frac{2b^2}{3} + 2c^2 + \frac{4ac}{3}\right)$$

$$\nabla E = 0 \iff \boxed{\begin{cases} 16a + \frac{4a}{5} + \frac{4c}{3} = 0 \\ -4\left(\frac{1}{3} - b\right) - \frac{12}{7} + \frac{4b}{3} = 0 \\ 4c + \frac{4}{3} a = 0 \end{cases}} \iff \begin{cases} 4a + \frac{a}{3} = -\frac{c}{3} \\ -\frac{4}{3} + 4b - \frac{12}{7} + \frac{4b}{3} = 0 \\ c = -\frac{a}{3} \end{cases}$$

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$$\iff \begin{cases} 21a = -\frac{5}{3} c \\ \frac{16b}{3} = \frac{12}{7} + \frac{4}{3} = \frac{64}{21} \iff \\ a = -3c \end{cases} \qquad \begin{cases} a = c = 0 \\ b = \frac{3}{\cancel{16}_{8}}\left(\frac{\cancel{36}^{18} + \cancel{28}^{14}}{21}\right) = \frac{96}{168} = \frac{48}{84} \\ = \frac{24}{42} = \frac{12}{21} = \frac{4}{7} \end{cases}$$

ok

Então $p(x) = \frac{4}{7}x$ aproximação de $f(x) = x^{1/3}$

Cálculo detalhado:

$$\int_{-1}^{1} \left(f(x)-p(x)\right)^2 = \int_{-1}^{1} f(x)^2 + p(x)^2 - 2f(x)p(x)$$

a integral destas funções vale 0, pois elas são ímpar

$$= \int_{-1}^{1} x^{\frac{2}{3}} + a^2x^4 + b^2x^2 + c^2 + 2abx^3 + 2acx^2 + 2bcx - 2ax^{7/3} - 2bx^{4/3} - 2cx^{1/3}$$

$$= \left[\frac{x^{1+\frac{2}{3}}}{1+\frac{2}{3}} + a^2\frac{x^5}{5} + b^2\frac{x^3}{3} + c^2x + 2ac\frac{x^3}{3} - 2b\frac{x^{1+\frac{4}{3}}}{1+\frac{4}{3}}\right]_{-1}^{1}$$

$$= \frac{2}{5/3} + \frac{2a^2}{5} + \frac{2b^2}{3} + 2c^2 + \frac{4ac}{3} - \frac{4b}{7/3}$$

$$= \frac{6}{5} + \frac{2a^2}{5} + \frac{2b^2}{3} + 2c^2 + \frac{4ac}{3} - \frac{12b}{7}$$