Fluid Mechanics

Chapter 4:

Analytical solutions to the Navier-Stokes equations

Dr. Naiyer Razmara

Analytic solutions to the Navier-Stokes equations

Some illustrative Examples

A. Couette flow

B. Poiseuille flow

C. Flow in a circular pipe

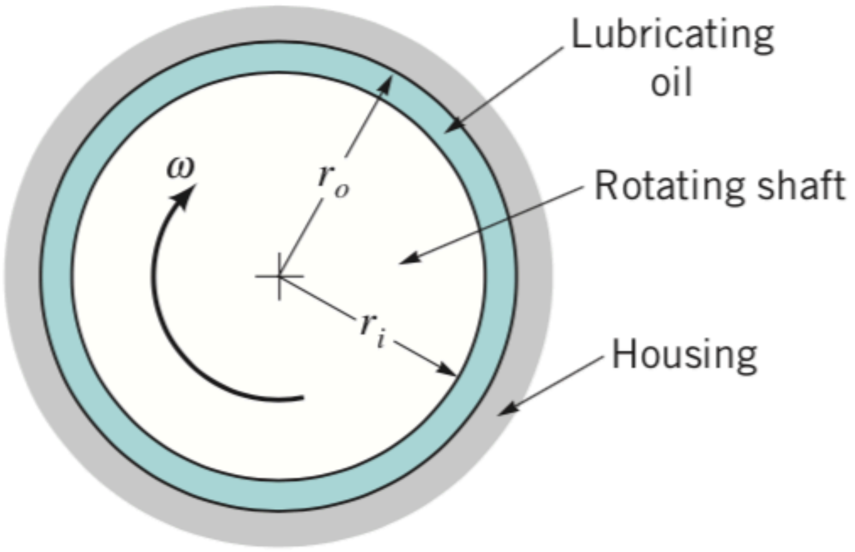
D. Flow near an Oscillating Plate

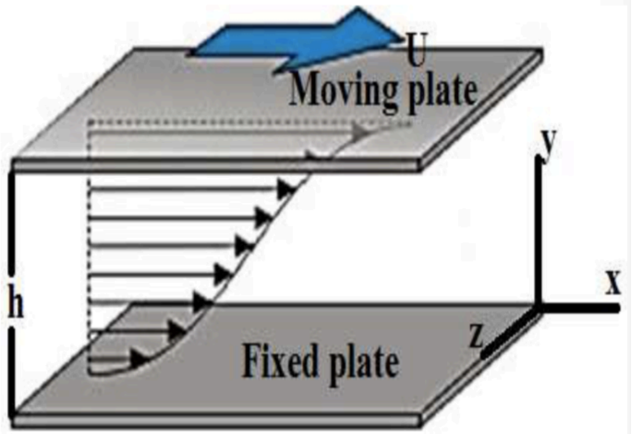
A. Flow between a Fixed and a Moving Plate: Plane Couette Flow

Applications:

- Lubrication
- Geological systems
- Painting, cleaning etc. (thin-film applications)

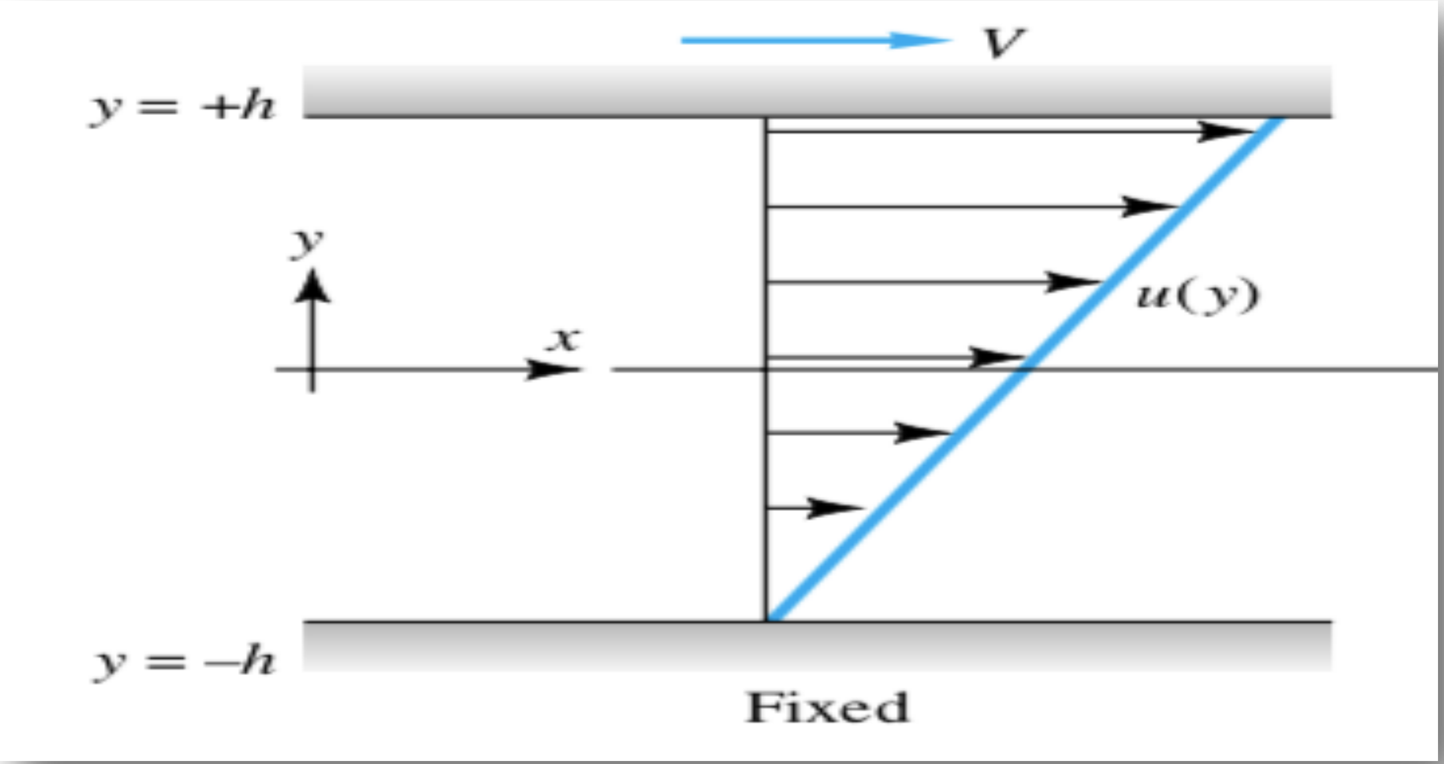
Journal Bearing





A. Flow between a Fixed and a Moving Plate: Plane Couette Flow

- Two-dimensional
- Incompressible
- Plane
- Viscous flow
- Between parallel plates a distance 2h apart
- Assume that the plates are very wide and very long, so that the flow is essentially axial
- The upper plate moves at velocity V but there is no pressure gradient
- Neglect gravity effects



The continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 = \frac{\partial u}{\partial x} + 0 + 0 \qquad \text{or} \qquad u = u(y) \text{ only}$$

A. Flow between a Fixed and a Moving Plate: Plane Couette Flow

- Thus there is a single nonzero axial-velocity component which varies only across the channel.
- The flow is said to be fully developed (far downstream of the entrance).
- The Navier-Stokes momentum equation for two-dimensional (x, y) flow:

$$\rho\left(u\frac{\partial u}{\partial x} + \upsilon\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\rho(0 + 0) = 0 + 0 + \mu\left(0 + \frac{d^2 u}{dy^2}\right)$$

$$\frac{d^2 u}{dy^2} = 0 \qquad \text{or} \qquad u = C_1 y + C_2$$

The no-slip condition

At $y = +h$: $\quad u = V = C_1 h + C_2$

At $y = -h$: $\quad u = 0 = C_1(-h) + C_2$

$$C_1 = \frac{V}{2h} \qquad \text{and} \qquad C_2 = \frac{V}{2}$$

A. Flow between a Fixed and a Moving Plate: Plane Couette Flow

The solution for the flow between plates with a moving upper wall, is:

$$u = \frac{V}{2h} y + \frac{V}{2} \qquad -h \le y \le +h$$

This is *Couette flow* due to a moving wall: a linear velocity profile with no-slip at each wall, as anticipated.

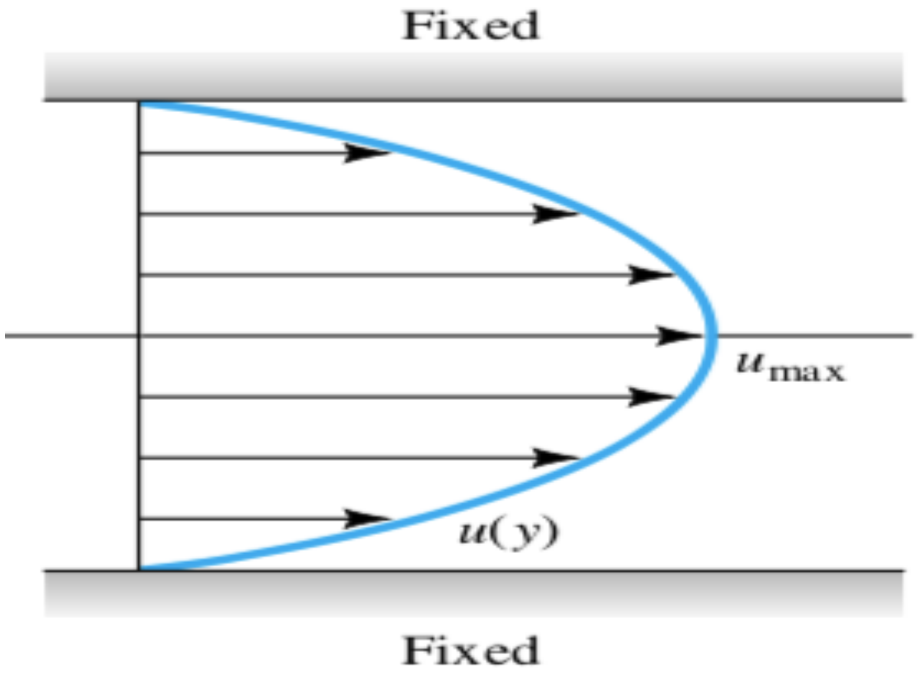
B. Flow due to Pressure Gradient between Two Fixed Plate: Plane Poiseuille Flow (Channel Flow)

- Poiseuille flows are driven by pumps that forces the fluid to flow by modifying the pressure.
- Fluids flow naturally from regions of high pressure to regions of low pressure.
- Typical examples are cylindrical pipe flow and other duct flows.
- Figure below illustrates a fully developed plane channel flow.
- Fully developed Poiseuille flows exists only far from the entrances and exits of the ducts, where the flow is aligned parallel to the duct walls.

Assumptions:

- Both plates are fixed.
- The pressure varies in the x direction.
- The gravity is neglected.
- The x-momentum equation changes only because the pressure is variable:

$$\mu \frac{d^2u}{dy^2} = \frac{\partial p}{\partial x}$$



B. Flow due to Pressure Gradient between Two Fixed Plate: Plane Poiseuille Flow (Channel Flow)

The y- and z-momentum equations lead to: $\frac{\partial p}{\partial y} = 0$ and $\frac{\partial p}{\partial z} = 0$ or $p = p(x)$ only

Thus the pressure gradient is the total and only gradient: $\mu \frac{d^2 u}{dy^2} = \frac{dp}{dx} = \text{const} < 0$

The solution is accomplished by double integration: $u = \frac{1}{\mu} \frac{dp}{dx} \frac{y^2}{2} + C_1 y + C_2$

The constants are found from the no-slip condition at each wall:

At $y = \pm h$: $u = 0$ or $C_1 = 0$ and $C_2 = -\frac{dp}{dx} \frac{h^2}{2\mu}$

B. Flow due to Pressure Gradient between Two Fixed Plate: Plane Poiseuille Flow (Channel Flow)

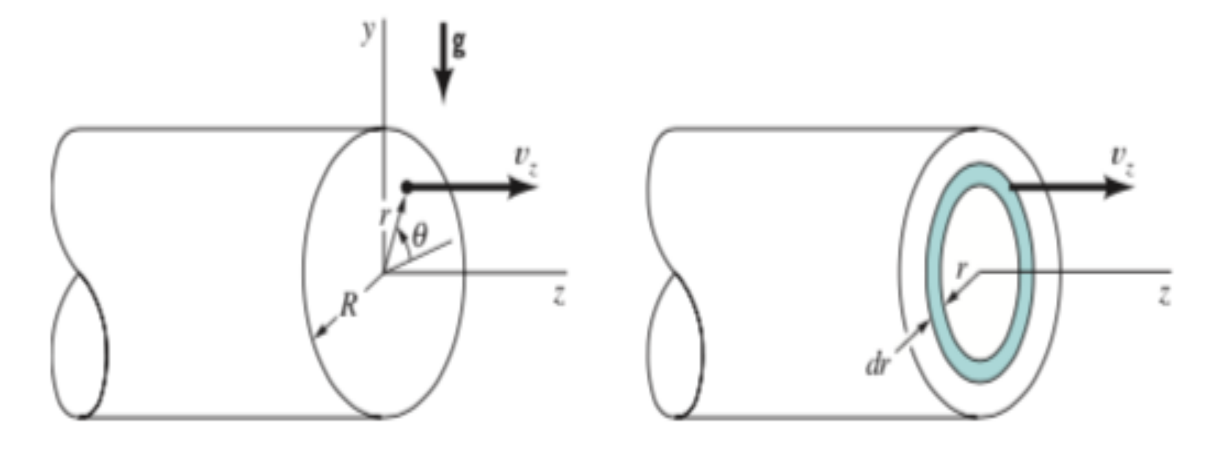
Thus the solution to the flow in a channel due to pressure gradient, is:

$$u = -\frac{dp}{dx}\frac{h^2}{2\mu}\left(1 - \frac{y^2}{h^2}\right)$$

The flow forms a *Poiseuille* parabola of constant negative curvature. The maximum velocity occurs at the centerline:

$$u_{\max} = -\frac{dp}{dx}\frac{h^2}{2\mu}$$

C. Flow in circular pipe



If we assume the flow is in the x-direction, then the velocity vector is u(y, z, t) = (u(y, z, t), 0, 0) and Navier-Stokes equations can be simplified to

$$\frac{\partial u}{\partial t} = \nu \nabla^2 u - \frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x},$$

For a steady cylindrical pipe flow with radius r0 the solution is found simply by integrating twice and applying boundary conditions:

$$\frac{\nu}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u}{\mathrm{d}r}\right) = \frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x},$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u}{\mathrm{d}r}\right) = \frac{r}{\mu}\frac{\mathrm{d}p}{\mathrm{d}x},$$

$$r\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{r^2}{2\mu}\frac{\mathrm{d}p}{\mathrm{d}x} + C_1,$$

$$\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{r}{2\mu}\frac{\mathrm{d}p}{\mathrm{d}x} + \frac{C_1}{r},$$

$$u(r) = \frac{r^2}{4\mu}\frac{\mathrm{d}p}{\mathrm{d}x} + C_1\ln(r) + C_2,$$

C. Flow in circular pipe

where C_1 and C_2 are integration constants. The constant C_1 must be zero for the flow at the center of the pipe to remain finite. The condition at the wall $u(r_0) = 0$ gives

$$C_2 = -\frac{r_0^2}{4\mu}\frac{\partial p}{\partial x},$$

which leads to the final expression for the pipe flow

$$u(r) = -\frac{1}{4\mu}\frac{\partial p}{\partial x}\left(r_0^2 - r^2\right)$$

D. Flow near an Oscillating Plate

Consider that special case of a viscous fluid near a wall that is set suddenly in motion as shown in Figure 1. The unsteady Navier-Stokes reduces to:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

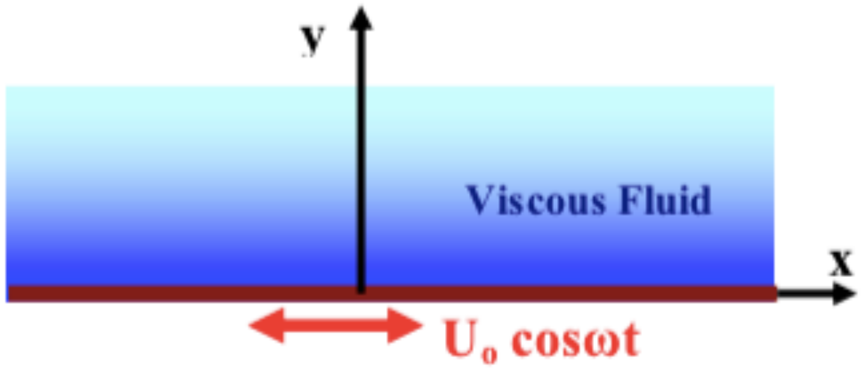


Figure 2. Schematics of flow near an oscillating wall.

The boundary conditions are:

$$u = U_0 \cos \omega t \qquad \text{at } y = 0$$

$$u = 0 \qquad \text{at } y = \infty$$

D. Flow near an Oscillating Plate

Let

$$u = U_0 e^{-ky} \cos(\omega t - ay).$$

Then

$$\frac{\partial u}{\partial t} = -\omega U_0 e^{-ky} \sin(\omega t - ay)$$

$$\frac{\partial u}{\partial y} = U_0 e^{-ky} \left(-k \cos(\omega t - ay) + a \sin(\omega t - ay)\right)$$

$$\frac{\partial^2 u}{\partial y^2} = U_0 e^{-ky} \left(k^2 \cos\theta - 2ka \sin\theta - a^2 \cos\theta\right), \;\; \theta = \omega t - ay$$

D. Flow near an Oscillating Plate

Substituting the last two equations into the reduced Navier-stokes equation, it follows that

$$-\omega \sin\theta = \nu\left(\left(k^2 - a^2\right)\cos\theta - 2ak\sin\theta\right)$$

$$a^2 = k^2$$

$$\omega = 2ak\nu = 2k^2\nu$$

$$k = \sqrt{\frac{\omega}{2\nu}} = a$$

Thus, the velocity profile is given as

$$u = U_0 e^{-ky}\cos(\omega t - ky), \qquad k = \sqrt{\frac{\omega}{2\nu}}.$$