Finite element for beams

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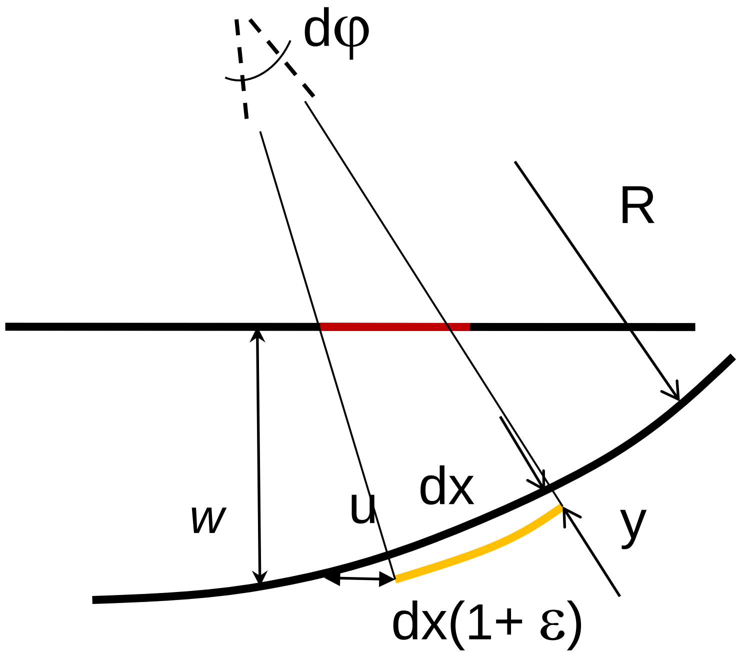
Beam theory: Kinematics

Consider an Euler-Bernoulli beam under some quasi-static distributed load

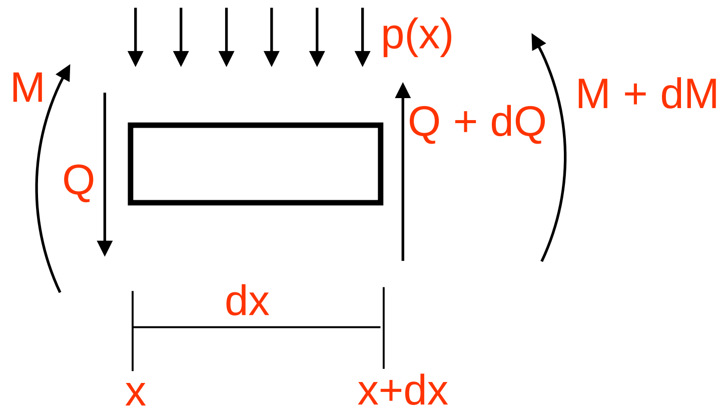
- σ = 0 in the neutral axis
- Plane sections remains plain
- Shear strain not considered
- Small strains and displacement



$$\left.\begin{aligned} \mathrm{d}x &= R\,\mathrm{d}\varphi \\ \mathrm{d}x(1+\varepsilon) &= (R+y)\,\mathrm{d}\varphi \end{aligned}\right\} \therefore \kappa = \frac{\mathrm{d}\varphi}{\mathrm{d}x} = \frac{\varepsilon}{y} = -w''$$

$$\kappa = \frac{w''}{(1+w'^2)^{3/2}}$$

Equilibrium



$$Q(x) = -\frac{\mathrm{d}M}{\mathrm{d}x} \qquad \therefore \qquad p(x) = \frac{\mathrm{d}Q}{\mathrm{d}x}$$

Material behaviour law

$$\sigma = E\varepsilon$$

$$M = \int_A \sigma \; y \, \mathrm{d}A = \int_A E\varepsilon y \, \mathrm{d}A = \int_A E k y^2 \, \mathrm{d}A = EIk$$

$$\therefore M = -EIw''$$

Governing equation

$$p \quad x \quad = \frac{\mathrm{d}Q}{\mathrm{d}x} \qquad \therefore Q \quad x \quad = EIw'''$$

$$p \quad x \quad = EI\,\frac{\partial^4 w}{\partial x^4}$$

The solution of the *consistent* governing equations, using boundary and initial conditions, gives a complete picture of the beam behaviour.

The kinematic and static field used here can be proved to be consistent by using the principle of virtual work or the principle of virtual velocity.

Consistent set of equilibrium equation and geometrical relation for beams

$$\int_l M\kappa \mathrm{d}x = \int_l pw\mathrm{d}x + [Qw - Mw']$$

$$\kappa = -w''$$

$$-\int_l Mw''\mathrm{d}x = \int_l pw\mathrm{d}x + [Qw - Mw']$$

$$-[Mw'] + [wM'] - \int_l M''w\mathrm{d}x = \int_l pw\mathrm{d}x + [Qw - Mw']$$

$$\int_l \quad M'' + p \quad w\,\mathrm{d}x = 0$$

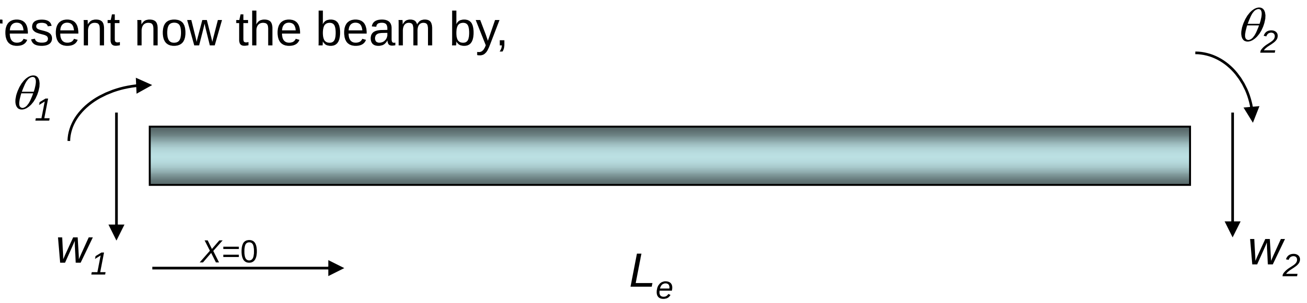
$$M'' + p = 0 \qquad \text{and} \qquad Q = M'$$

Discretisation of the solution

$$p \quad x \quad = EI\,\frac{\partial^4 w}{\partial x^4}$$

$$w = a_1 + a_2 x + a_3 x^2 + a_4 x^3 = [1 \;\; x \;\; x^2 \;\; x^3][a_1 \;\; a_2 \;\; a_3 \;\; a_4]^T = [X]\{a\}$$

Represent now the beam by,

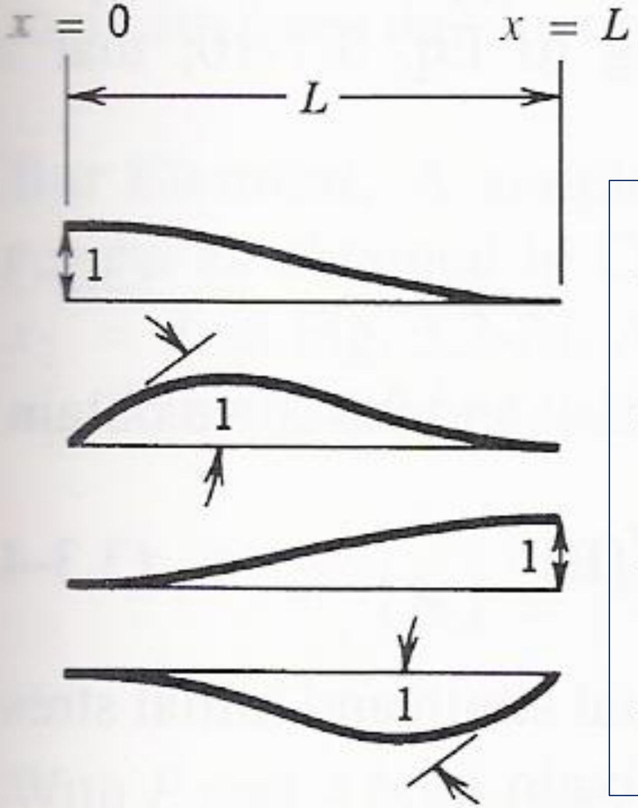


where w and θ are displacement and slope. Since $\theta = \dfrac{dw}{dx} = w'$

$$\begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L_e & L_e^2 & L_e^3 \\ 0 & 1 & 2L_e & 3L_e^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = [\mathrm{A}]\{\mathrm{a}\}$$

Exercise in class

$$w = [X]\{a\}$$
$$w_e = [A]\{a\}$$
$$w = [X][A]^{-1} w_e$$
$$w = [N] w_e$$



Hermite's polynomium.

	At $x = 0$		At $x = L$	
	N_i	$N_{i,x}$	N_i	$N_{i,x}$
$N_1 = 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3}$	1	0	0	0
$N_2 = x - \frac{2x^2}{L} + \frac{x^3}{L^2}$	0	1	0	0
$N_3 = \frac{3x^2}{L^2} - \frac{2x^3}{L^3}$	0	0	1	0
$N_4 = -\frac{x^2}{L} + \frac{x^3}{L^2}$	0	0	0	1

In class

Let us now return to the beam equilibrium equation,

$$M'' + p = 0$$

Multiply by a function δw, which represents a virtual displacement, and integrate the product along the beam length

$$-\int_{L_e} M''\delta w \,\mathrm{d}x = \int_{L_e} p\,\delta w \,\mathrm{d}x$$

$$-\int_{\mathrm{L_e}} M''\delta w \;\mathrm{d}x = \int_{\mathrm{L_e}} p\delta w \;\mathrm{d}x$$

$$1 -> u = \delta w,\ dv = M''dx,$$

$$2 -> u = \delta w',\ dv = M'dx$$

$$(uv)' = u'v + uv'$$

$$\int (uv)'\mathrm{d}x = \int u'v\mathrm{d}x + \int uv'\mathrm{d}x$$

$$uv = \int v\mathrm{d}u + \int u\mathrm{d}v$$

$$\int u\ \mathrm{d}v = [uv] - \int v\ \mathrm{d}u$$

$$\int_{L_e} M\,\delta\kappa\,\mathrm{d}x = \int_{L_e} p\,\delta w\,\mathrm{d}x + \left[Q\delta w - M\frac{\mathrm{d}\delta w}{\mathrm{d}x}\right]$$

Internal energy due to change in the curvature: KINEMATICS

External work due to distributed loads

External work due to concentrated loads: STATICS

Functional (no concentrated loads)

$$\Pi = \int_{L_e} M.\delta\kappa \, dx - \int_{L_e} p.\delta w \, dx$$

$$\kappa = -w'' \qquad \delta\kappa = -\mathbf{N}''\delta\mathbf{w}_e \qquad M = EI\kappa$$

$$\Pi = \int_{L_e} (-EIN''w_e).(-N''\delta w_e)\, dx - \int_{L_e} (p).(N\delta w_e)\, dx = 0$$

$$\Pi = \int_{L_e} (-N''\delta w_e)^T(-EIN''w_e)\, dx - \int_{L_e} (N\delta w_e)^T p\, dx = 0$$

$$\Pi = \int_{L_e} \delta w_e^T N''^T EIN'' w_e\, dx - \int_{L_e} \delta w_e^T N^T p\, dx = 0$$

$$\int_{L_e} N''^T EIN''\, dx\, w_e - \int_{L_e} N^T p\, dx = 0$$

$$k_e w_e = p_e \qquad \text{Compare with } p\ \ x\ = EI\frac{\partial^4 w}{\partial x^4}$$

$$(A).(B) = (B)^T(A) \quad \therefore \quad (AB)^T = B^T A^T$$

Remarks

- The integral $\int_{L_e} \boldsymbol{N}^T p \, \mathrm{d}x$

  transforms the distributed load into nodal ones.

- The differential equation

$$M'' + p = 0$$

  was transformed in an algebraic system of equations.

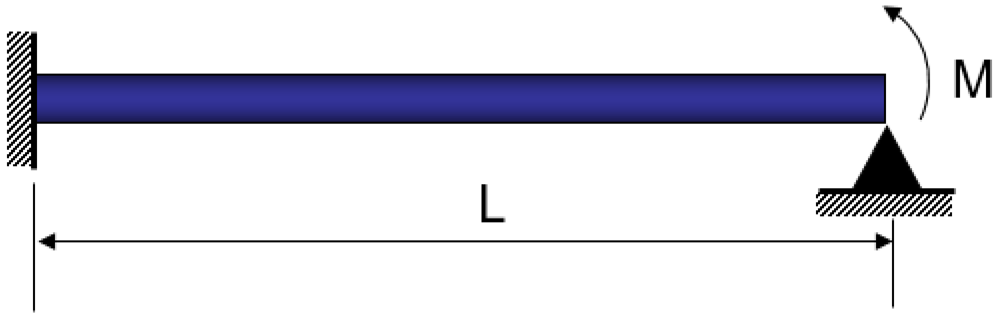
- The above transformation can be seen
  as the greatest advantage of FE method.

Exercise in class

$$\mathbf{k}_e = \int_{L_e} N''^T E I N'' \, dx$$

$$\mathbf{k}_e = \frac{EI}{L_e^3}\begin{bmatrix} 12 & 6L_e & -12 & 6L_e \\ \text{symm.} & 4L_e^2 & -6L_e & 2L_e^2 \\ & & 12 & -6L_e \\ & & & 4L_e^2 \end{bmatrix}$$

Let us now solve the problem below using ONE finite element.



How many FE one needs?

$$\mathbf{Kw} = \mathbf{0}$$

no distributed load

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ & & & 4L^2 \end{bmatrix}\begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Geometrical (essential) boundary conditions ↔displacements and rotations
Natural boundary conditions ↔ forces and moments

$w_1=0 \longrightarrow K_{11}=1$ e $K_{1j}=K_{j1}=0$ para $j\neq 1$

$$\frac{EI}{L^3}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4L^2 & -6L & 2L^2 \\ 0 & -6L & 12 & -6L \\ 0 & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \boxed{w_1 = 0}$$

The same for θ_1 e w_2,

$$\frac{EI}{L^3}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4L^2 \end{bmatrix}\begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Concentrated loads in the functional are,

$$\left[Qw - M\frac{dw}{dx} \right] = -M\mathbf{N}'\mathbf{w}_e$$

so that the load vector is,

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ -M \end{bmatrix}$$

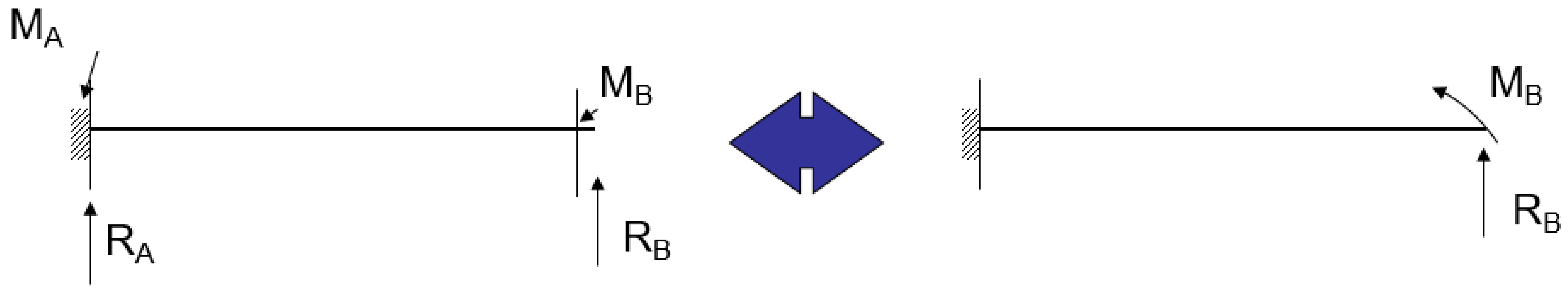
Solving for θ_2 gives,

$$\boxed{\theta_2 = -\frac{ML}{4EI}}$$

The final beam profile is

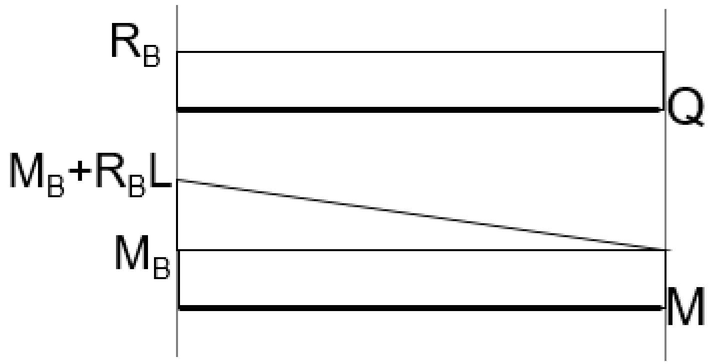
$$w = \mathbf{N}\mathbf{w}_e = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \theta_2 \end{bmatrix} = N_4\theta_2 = \frac{M}{4EI}\left(x^2 - \frac{x^3}{L} \right)$$

Analytical solution (hyperstatic)



$$R_A = -R_B, \quad M_A = -M_B - R_B L$$

Moment-Area Method:



$$\theta_B - \theta_A = \frac{1}{EI}\left(M_B L + \frac{1}{2} R_B L^2\right)$$

0

$$Q_1 = \frac{2}{3} L \frac{1}{2} \frac{R_B L^2}{EI} + \frac{L}{2} \frac{M_B L}{EI} = 0$$

$$R_B = -\frac{3M_B}{2L} \qquad M_A = -M_B - R_B L = \frac{M_B}{2}$$

$$EIw^{iv} = 0$$

$$EIw = c_1 \frac{x^3}{6} + c_2 \frac{x^2}{2} + c_3 x + c_4$$

$$EIw'(0) = 0$$

$$EIw'(L) = c_1 \frac{L^2}{2} + c_2 L = \theta_B$$

$$c_1 = \frac{3M_B}{2L} \qquad c_2 = -\frac{M_B}{2}$$

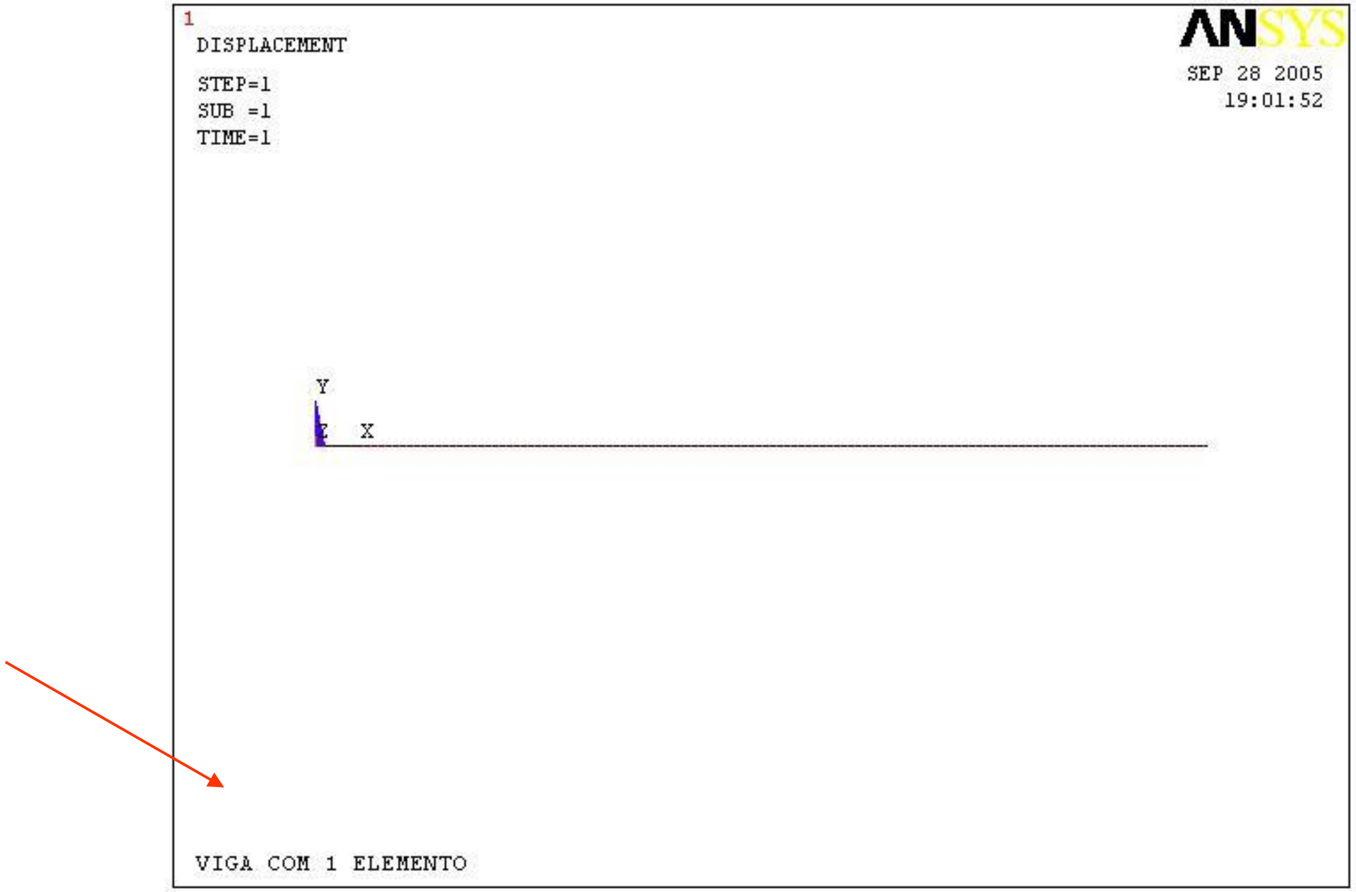
$$w = \frac{M_B}{EIL} x^2 (x - L)$$

$$w' = \theta_2 = \frac{M}{4EIL}\left(3x^2 - 2Lx\right)$$

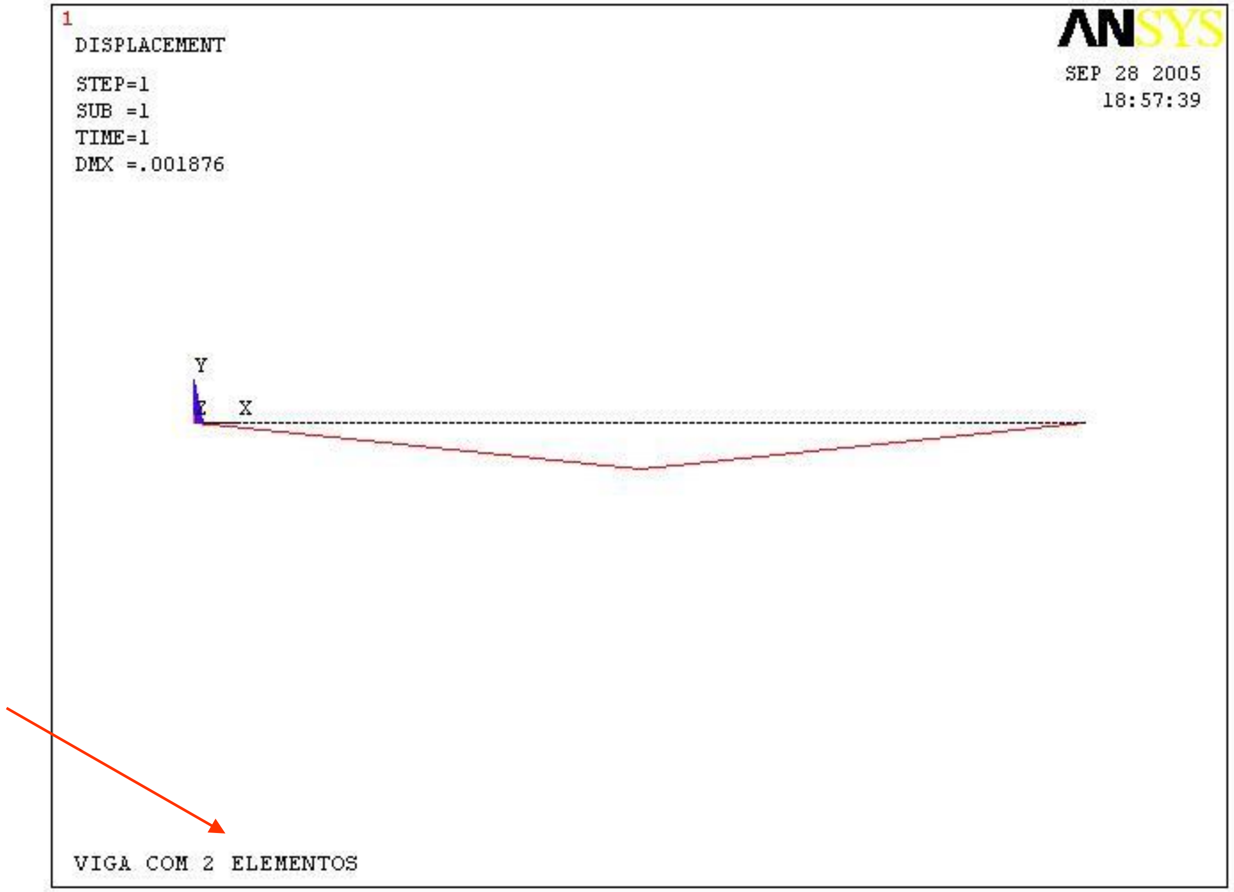
which is identical to the FE solution

Note that we obtained the exact solution using only one element...

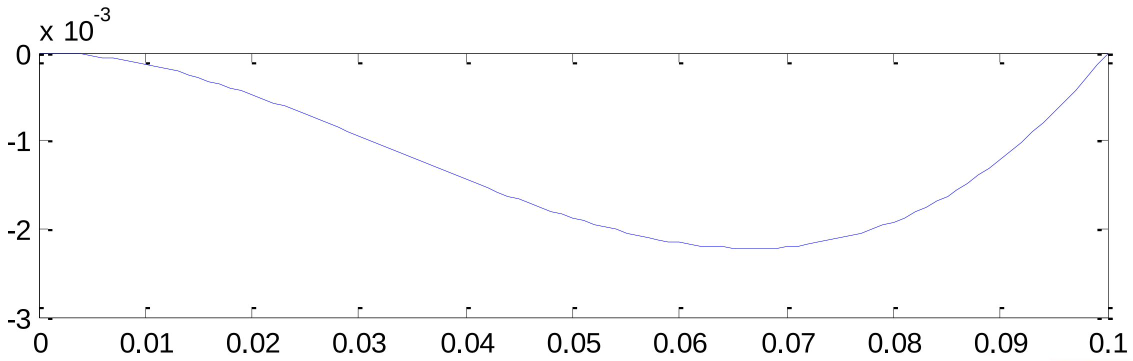
ANSYS


1
DISPLACEMENT
STEP=1
SUB =1
TIME=1
ANSYS
SEP 28 2005
19:01:52
Y
X
VIGA COM 1 ELEMENTO

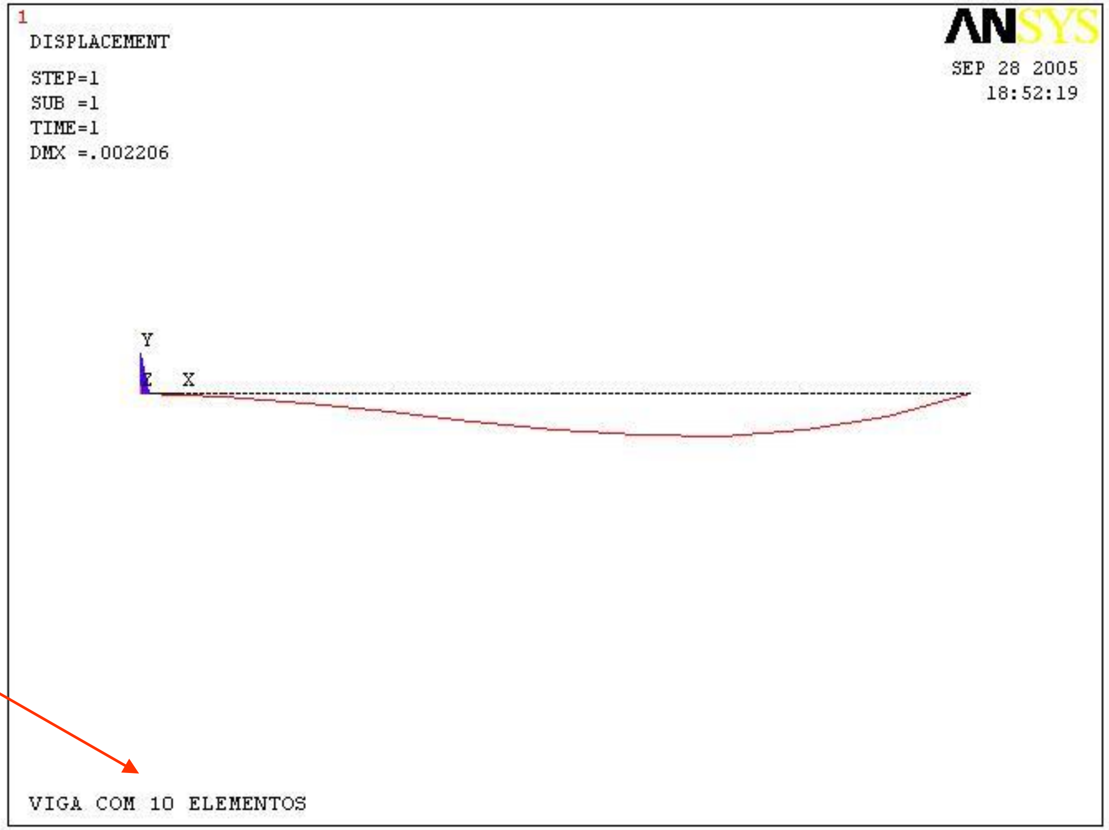
ANSYS


1
DISPLACEMENT
STEP=1
SUB =1
TIME=1
DMX =.001876
ANSYS
SEP 28 2005
18:57:39
Y
X
VIGA COM 2 ELEMENTOS

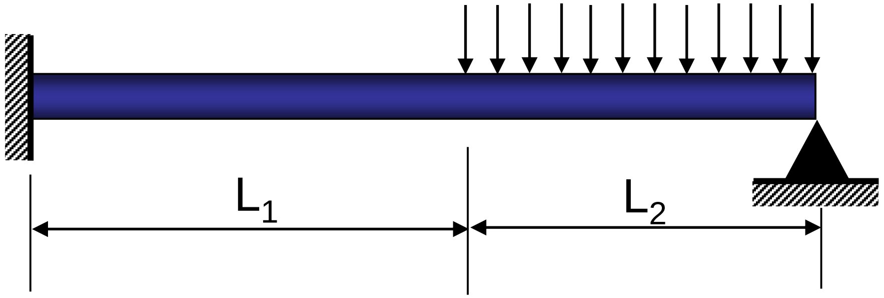
beam.m


x 10^-3
0
-1
-2
-3
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.1

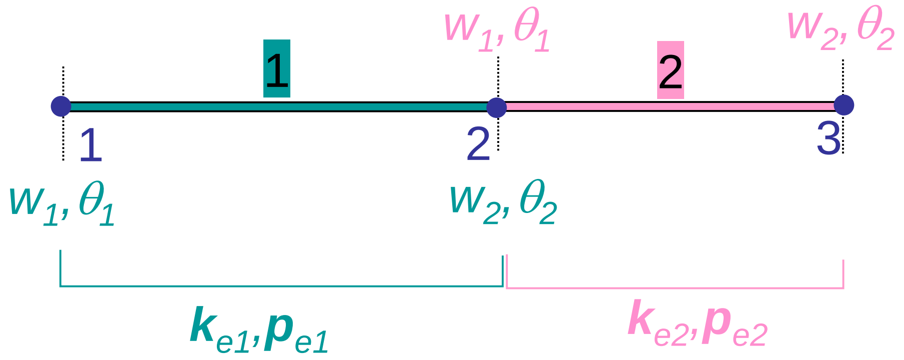
ANSYS


1
DISPLACEMENT
STEP=1
SUB =1
TIME=1
DMX =.002206
ANSYS
SEP 28 2005
18:52:19
Y
Z X
VIGA COM 10 ELEMENTOS

Consider now:

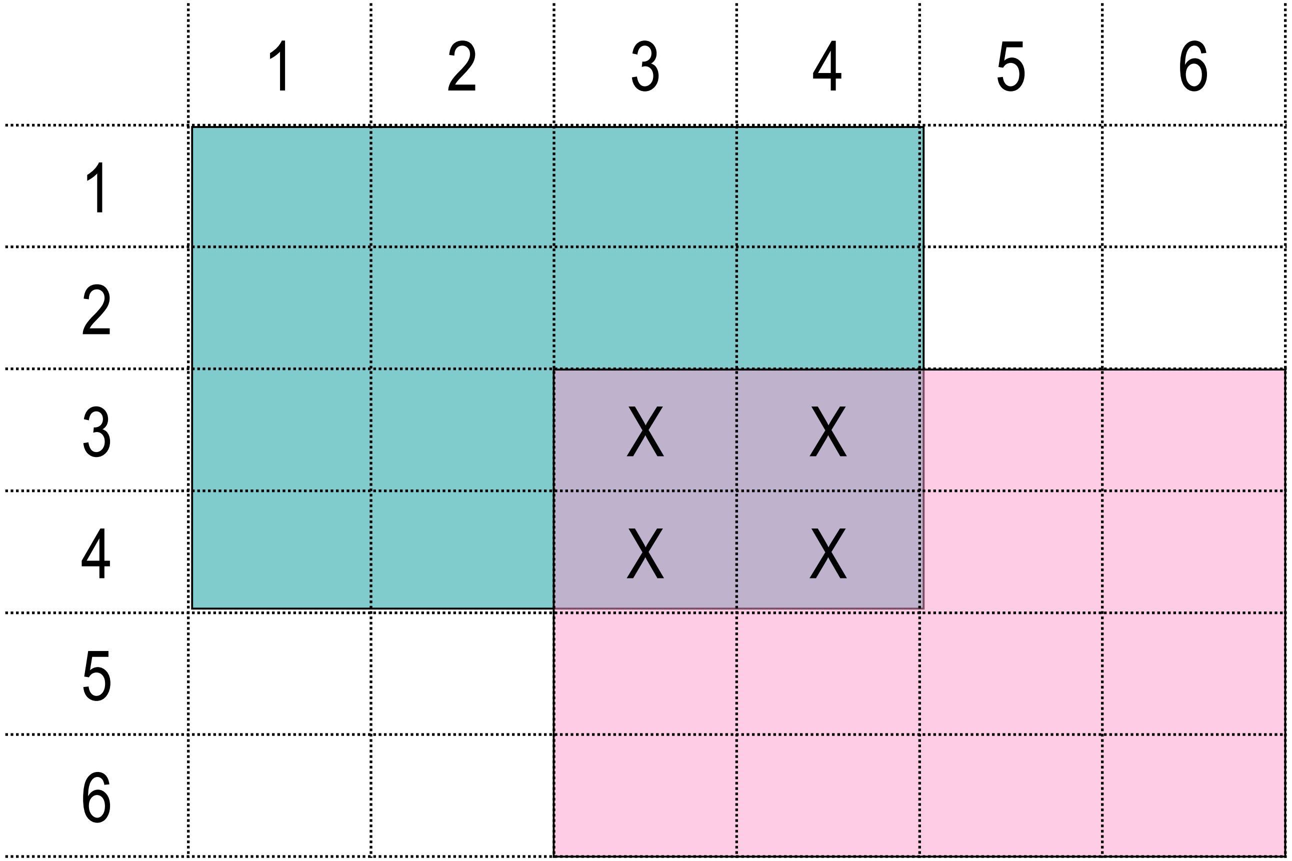


One element yields a poor solution.



In the previous example, the stiffness matrix was of order:
DOFN x NN = 2 x 2 = 4
now: DOFN x NN = 2 x 3 = 6

DOFN: degrees of freedom per node
NN: number of nodes



Element 1

Element 2

The local d.o.f. of element 1, w_2, θ_2, should be added to the local d.o.f. of element 2, w_1, θ_1

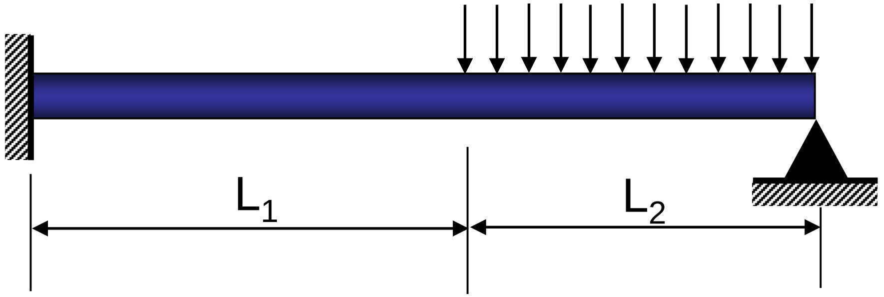
$$\boldsymbol{K} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 12+12 & 6L+6L & -12 & 6L \\ 6L & 2L^2 & 6L+6L & 4L^2+4L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

Assuming $L_1=L_2=L$ and $E_1=E_2$, $I_1=I_2$.

$$\boldsymbol{p}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \boldsymbol{p}_2 = \int \boldsymbol{N}^T \boldsymbol{p} \,\mathrm{d}x = \int_0^L \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \boldsymbol{p} \,\mathrm{d}x = \begin{bmatrix} pL/2 \\ pL^2/12 \\ pL/2 \\ -pL^2/12 \end{bmatrix}$$

$$\boldsymbol{p} = \begin{bmatrix} 0 \\ 0 \\ 0 + pL/2 \\ 0 + pL^2/12 \\ pL/2 \\ -pL^2/12 \end{bmatrix} = \frac{pL}{2} \begin{bmatrix} 0 \\ 0 \\ 1 \\ L/6 \\ 1 \\ -L/6 \end{bmatrix}$$

Geometrical boundary conditions: *w(0)=0, w(L)=0* and *w’(0)=0*



$$\mathbf{K}\mathbf{w} = \mathbf{p}$$

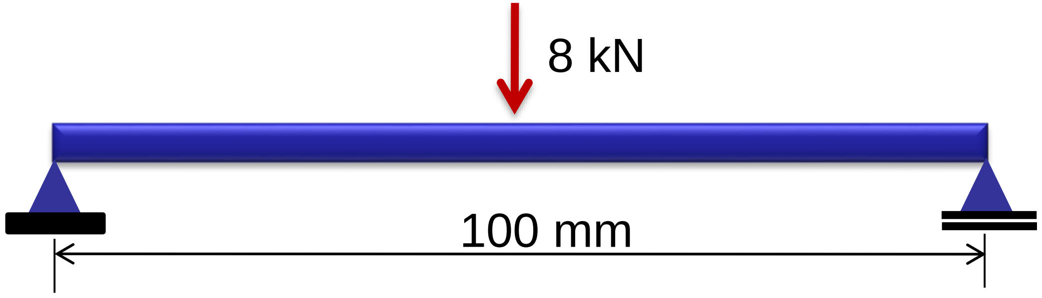
$$\frac{EI}{L^3}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 \\ & & 12+12 & 6L+6L & -12 & 6L \\ & & & 4L^2+4L^2 & -6L & 2L^2 \\ & & & & 12 & -6L \\ & & & & & 4L^2 \end{bmatrix}\begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \end{bmatrix} = \frac{pL}{2}\begin{bmatrix} 0 \\ 0 \\ 1 \\ L/6 \\ 1 \\ -L/6 \end{bmatrix}$$

$$\mathbf{w}^T = \frac{pL^4}{EI}\begin{bmatrix} 0 & 0 & \frac{L+1}{24L} & \frac{L-1}{12L^2} & \frac{7L-2}{24L} & \frac{2L-1}{6L^2} \end{bmatrix}$$

Is this correct?

Example

Obtain the mid displacement of the aluminium beam shown in the figure. Assume a diameter of 19 mm and E=72 GPa.



Analytical solution

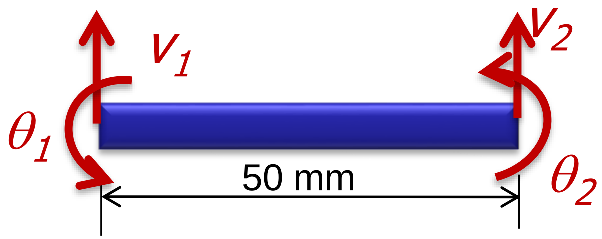
$$y_{max} = \frac{PL^3}{48EI}$$

$$EI = 72000\frac{\pi 19^4}{64} = 460592433 \text{Nmm}^2$$

$$y_{max} = \frac{8000\times 100^3}{48\times 460592433} \cong 0.362 \text{ mm}$$

Solution by Finite Elements

2 elements,



$$\mathbf{K}_1 = \mathbf{K}_2 = 460592433 \begin{bmatrix} 12/50^3 & 6/50^2 & -12/50^3 & 6/50^2 \\ 6/50^2 & 4/50 & -6/50^2 & 2/50 \\ -12/50^3 & -6/50^2 & 12/50^3 & -6/50^2 \\ 6/50^2 & 2/50 & -6/50^2 & 4/50 \end{bmatrix}$$

Global stiffeness matrix

Boundary conditions: v_1=0 e v_3=0

K=

~~44216,87~~	~~1105422~~	~~44216,9~~	~~1105421,84~~	~~0~~	~~0~~
1105422	36847395	-1105422	18423697,3	0	0
-44216,9	-1105422	88433,75	0	-44216,9	1105422
1105422	18423697	0	73694789,3	-1105422	18423697
~~0~~	~~0~~	~~-44216,9~~	~~-1105421,84~~	~~44216,87~~	~~-1105422~~
0	0	1105422	18423697,3	-1105422	36847395

Another way to apply the BC

$$K^{-1}=\begin{bmatrix} 7{,}24E\text{-}08 & 1{,}36E\text{-}06 & -9E\text{-}09 & -3{,}6E\text{-}08 \\ 1{,}36E\text{-}06 & 4{,}52E\text{-}05 & -1{,}6E\text{-}21 & -1{,}4E\text{-}06 \\ -9E\text{-}09 & 0 & 1{,}81E\text{-}08 & -9E\text{-}09 \\ -3{,}6E\text{-}08 & -1{,}4E\text{-}06 & -9E\text{-}09 & 7{,}24E\text{-}08 \end{bmatrix} \qquad F=\begin{bmatrix} 0 \\ -8000 \\ 0 \\ 0 \end{bmatrix}$$

$$u=\begin{bmatrix} -0{,}01086 \\ -0{,}36185 \\ 0 \\ 0{,}010856 \end{bmatrix} \begin{matrix} \leftarrow \theta_1 \\ \leftarrow v_2 \\ \leftarrow \theta_2 \\ \leftarrow \theta_3 \end{matrix}$$

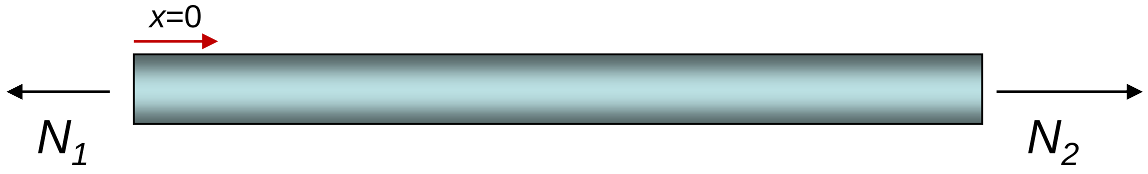
To compare with

$$y_{max} \cong 0.362\ \text{mm}$$

Normal force consideration: frame elements

Axial displacement:

Unidimensional-linear field



$$u = a_5 + a_6 x = [X]\{a\}$$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix} \begin{bmatrix} a5 \\ a6 \end{bmatrix} = [A]\{a\}$$

$$u = [X][A]^{-1}[u_1 \ u_2]^T$$

$$A^{-1} = \begin{bmatrix} 1 & 0 \\ -1/L & 1/L \end{bmatrix}$$

$$N = [X][A]^{-1} = [1 - x/L \quad x/L]$$

$$u = [N]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\Pi = \int_0^L \sigma \varepsilon A dx$$

$$\varepsilon = \frac{\partial u}{\partial x} = [N]' \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\Pi = EA \int_0^L [u_1 \ \mathrm{u}_2][N]'^T [N]' [u_1 \ u_2] \, \mathrm{d}x$$

$$\delta \Pi = \delta W$$

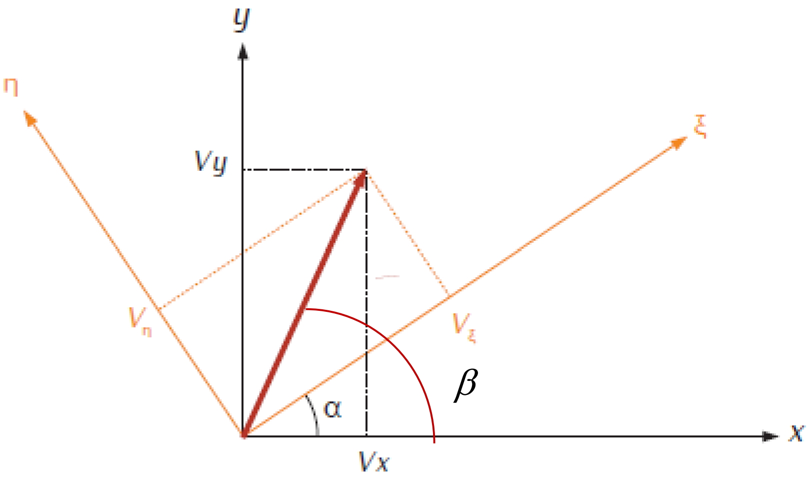
$$EA \begin{bmatrix} 1/L & -1/L \\ -1/L & 1/L \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} -N_1 \\ N_2 \end{Bmatrix}$$

$$[K]\{u\} = \{F\}$$

Axial force influence on K

$$
\begin{bmatrix} F_{x1} \\ F_{y1} \\ M_{z1} \\ F_{x1} \\ F_{y2} \\ M_{z2} \end{bmatrix}
=
\begin{bmatrix}
EA/L & 0 & 0 & -EA/L & 0 & 0 \\
0 & 12EI/L^3 & 6EI/L^2 & 0 & -12EI/L^3 & 6EI/L^2 \\
0 & 6EI/L^2 & 4EI/L & 0 & -6EI/L^2 & 2EI/L \\
-EA/L & 0 & 0 & EA/L & 0 & 0 \\
0 & -12EI/L^3 & -6EI/L^2 & 0 & 12EI/L^3 & -6EI/L^2 \\
0 & 6EI/L^2 & 2EI/L & 0 & -6EI/L^2 & 4EI/L
\end{bmatrix}
\begin{bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{bmatrix}
$$

Coordinate transformation



A vector represented by two different coordinate systems.

Consider now the next figure. There, the vector v has the components

$$v_X = v\cos\beta \quad \text{and} \quad v_Y = v\sin\beta.$$

It has also the components

$$v_x = v\cos(\beta-\alpha) \quad \text{and} \quad v_y = v\sin(\beta-\alpha),$$

or

$$v_x = v(\cos\beta\cos\alpha + \sin\beta\sin\alpha)$$

and

$$v_y = v(\sin\beta\cos\alpha - \cos\beta\sin\alpha),$$

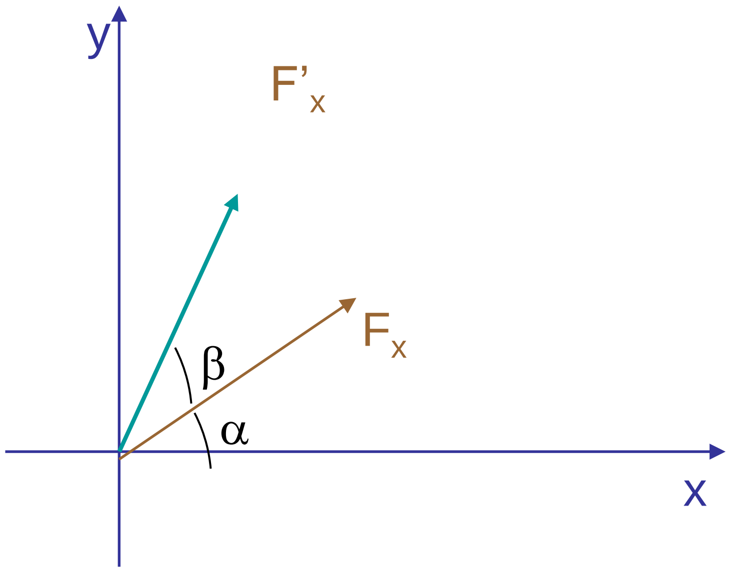
or, in matrix form,

$$\{v_{xy}\} = [T]\{v_{XY}\} \quad \text{and} \quad \{v_{XY}\} = [T]^{-1}\{v_{xy}\},$$

with

$$T = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}.$$

By inspection, $[T]^{-1} = [T]^{\mathrm{T}}$, *ie* the coordinate transformation matrix $[T]$ is orthogonal.



$$F_x = F\cos\alpha$$
$$F_y = F\sin\alpha$$

$$F_x' = F\cos(\alpha+\beta) = F\cos\alpha F\cos\beta - F\sin\alpha F\sin\beta = F_x\cos\beta - F_y\sin\beta$$
$$F_y' = F\sin(\alpha+\beta) = F\sin\alpha F\cos\beta + F\cos\alpha F\sin\beta = F_x\sin\beta + F_y\cos\beta$$

$$\begin{bmatrix} F_x' \\ F_y' \end{bmatrix} = \begin{bmatrix} \cos\beta & -\text{sen}\,\beta \\ \text{sen}\,\beta & \cos\beta \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix}$$

$$\vec{i}\cdot\vec{j}' = \left|\vec{i}\right|\left|\vec{j}'\right|\cos\left(90^{o}+\alpha\right) = -\operatorname{sen}\alpha$$

$$\vec{i}\cdot\vec{i}' = \left|\vec{i}\right|\left|\vec{i}'\right|\cos\alpha = \cos\alpha$$

$$\vec{v} = v_x\vec{i} + v_y\vec{j} = v_\xi\vec{i}' + v_\eta\vec{j}'$$

$$v_x\vec{i}\cdot\vec{i}' + v_y\vec{j}\cdot\vec{i}' = v_\xi\vec{i}\cdot\vec{i}' + v_\eta\vec{j}'\cdot\vec{i}'$$

$$.\ \vec{j}' \qquad\qquad .\ \vec{i}'$$

$$-v_x\operatorname{sen}\alpha + v_y\cos\alpha = v_\eta$$

$$v_x\cos\alpha + v_y\operatorname{sen}\alpha = v_\xi$$

To transform from local to global coordinate system,

$$\boldsymbol{K}\boldsymbol{w} = \boldsymbol{p} \qquad \boldsymbol{K}'\boldsymbol{w}' = \boldsymbol{p}'$$

$$\boldsymbol{K}\boldsymbol{w} = \boldsymbol{T}^T \boldsymbol{p}'$$

$$\boldsymbol{K}\boldsymbol{w} = \boldsymbol{T}^T \boldsymbol{K}'\boldsymbol{w}'$$

$$\boldsymbol{K}\boldsymbol{w} = \boldsymbol{T}^T \boldsymbol{K}'\boldsymbol{T}\,\boldsymbol{w}$$

$$\boxed{\boldsymbol{K} = \boldsymbol{T}^T \boldsymbol{K}'\boldsymbol{T}}$$

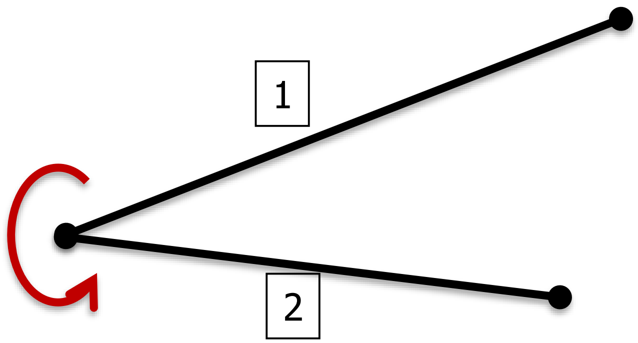
$$\mathbf{k} = \begin{bmatrix} a\lambda^2 + 12b\mu^2 & & & & & \\ (a-12b)\lambda\mu & a\mu^2 + 12b\lambda^2 & & & & \\ -6bL\mu & 6bL\lambda & 4bL^2 & & & \\ -a\lambda^2 - 12b\mu^2 & -(a-12b)\lambda\mu & 6bL\mu & a\lambda^2 + 12b\mu^2 & & \\ -(a-12b)\lambda\mu & -a\mu^2 - 12b\lambda^2 & -6bL\lambda & (a-12b)\lambda\mu & a\mu^2 + 12b\lambda^2 & \\ -6bL\mu & 6bL\lambda & 2bL^2 & 6bL\mu & -6bL\lambda & 4bL^2 \end{bmatrix} \begin{matrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{matrix}$$

$$a = \frac{EA}{L} \qquad b = \frac{EI}{L^3}$$

$$\lambda = \cos\alpha \qquad \mu = \text{sen}\alpha$$

For a 3D portic [spacial portic]

- 12 degrees of freedom
- It should consider:
  - Bending
  - Axial forces
  - Torsion



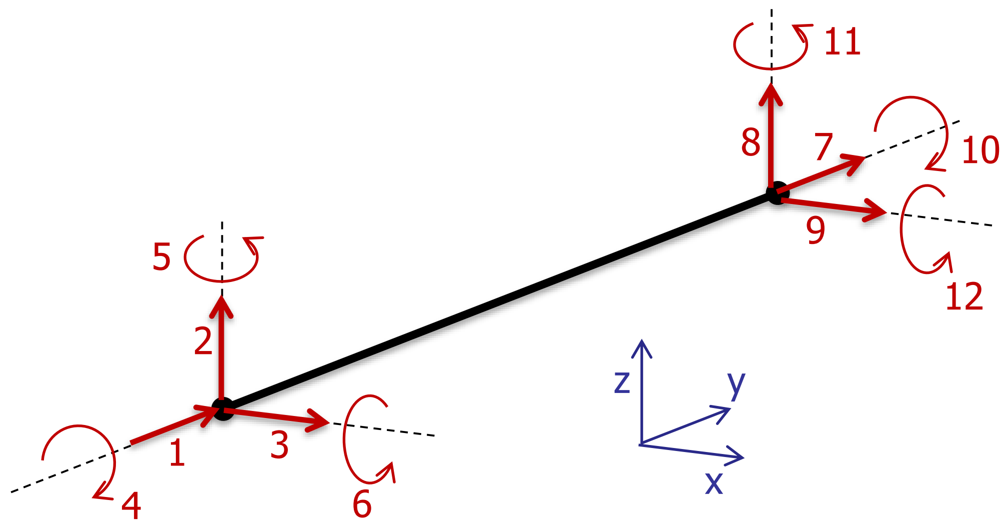
Bending in bar 1 yields torsion in bar 2

Stiffness matrix for torsion

It is similar to the truss matrix:

- the later working in compression or traction, EA/L
- the former in torsion, GJ/L
- G is the shear modulus and J is the polar moment of inertia

$$\mathbf{k}_{torsional} = \frac{GJ}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$



$$a = \frac{EA}{L} \qquad b_y = \frac{EI_y}{L^3} \qquad b_z = \frac{EI_z}{L^3}$$

$$t = \frac{GJ}{L}$$

$$\mathbf{k} = \begin{bmatrix}
a & & & & & & & & & & & \\
0 & 12b_z & & & & & & & & & & \\
0 & 0 & 12b_y & & & & & & & & & \\
0 & 0 & 0 & t & & & & & \text{symmetric} & & & \\
0 & 0 & -6b_yL & 0 & 4b_yL^2 & & & & & & & \\
0 & 6b_zL & 0 & 0 & 0 & 4b_zL^2 & & & & & & \\
-a & 0 & 0 & 0 & 0 & 0 & a & & & & & \\
0 & -12b_z & 0 & 0 & 0 & -6b_zL & 0 & 12b_z & & & & \\
0 & 0 & -12b_y & 0 & -6b_yL & 0 & 0 & 0 & 12b_y & & & \\
0 & 0 & 0 & -t & 0 & 0 & 0 & 0 & 0 & t & & \\
0 & 0 & -6b_yL & 0 & 2b_yL^2 & 0 & 0 & 0 & 6b_yL & 0 & 4b_yL^2 & \\
0 & 6b_zL & 0 & 0 & 0 & 2b_zL^2 & 0 & -6b_zL & 0 & 0 & 0 & 4b_zL^2
\end{bmatrix}$$

Timoshenko beams

- For short beams
- Shear cannot be disregarded

$$a = \frac{EA}{L} \qquad b_y = \frac{EI_y}{(1+\phi_z)L^3} \qquad b_z = \frac{EI_z}{(1+\phi_y)L^3}$$

$$t = \frac{GJ}{L}$$

$$\phi_y = \frac{12EI_z}{GA_{sy}L^2}$$

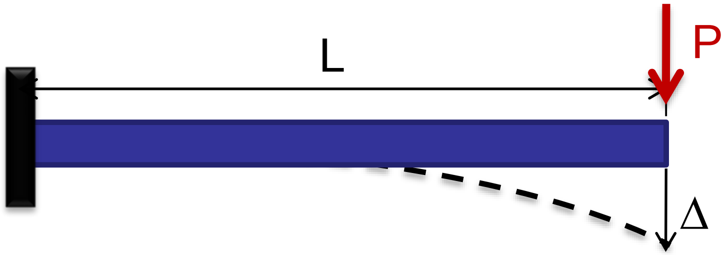
$$\phi_z = \frac{12EI_y}{GA_{sz}L^2}$$

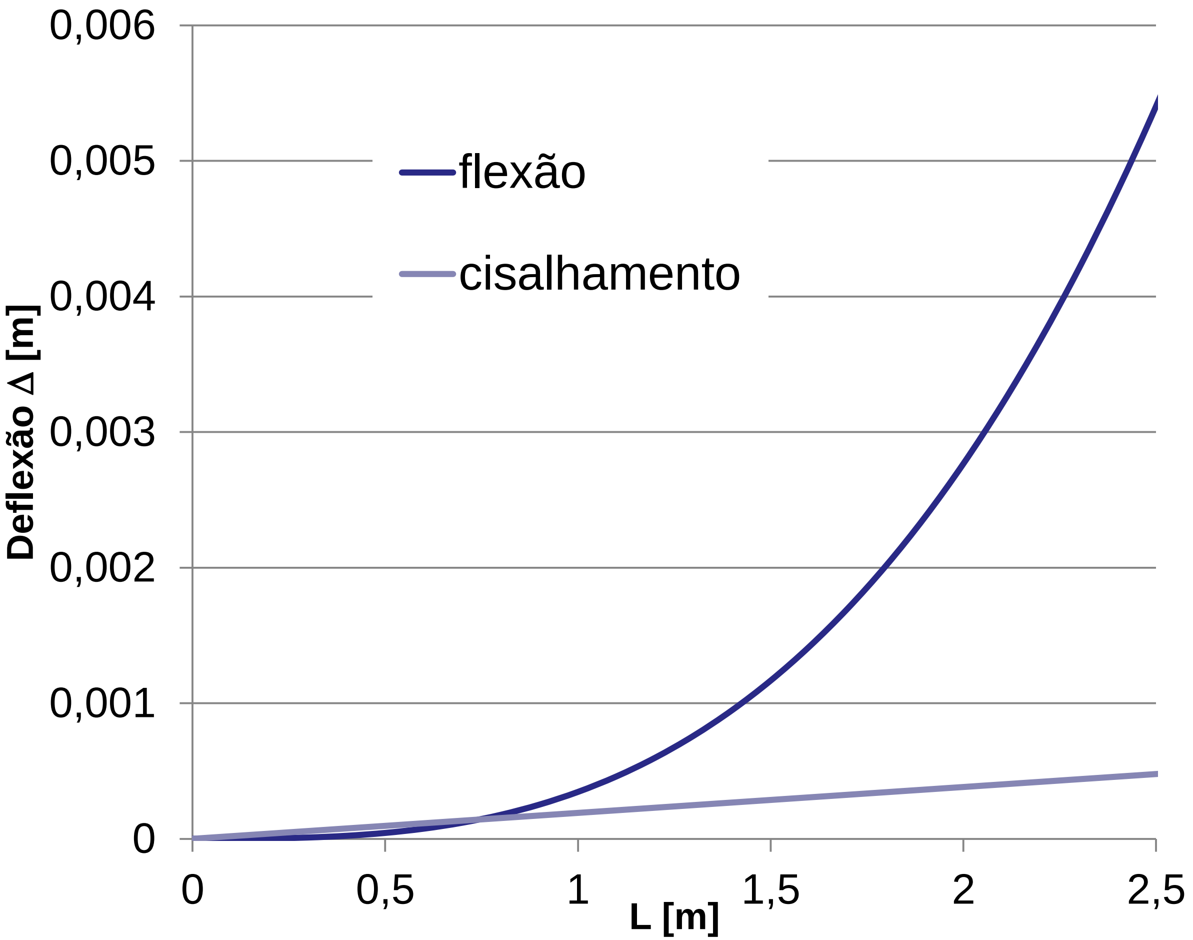
$$\mathbf{k} = \begin{bmatrix}
a &&&&&&&&&&& \\
0 & 12b_z &&&&&&&&&& \\
0 & 0 & 12b_y &&&&&&&&& \\
0 & 0 & 0 & t &&&&&&&& \\
0 & 0 & -6b_yL & 0 & (4+\phi_z)b_yL^2 &&&&&&& \\
0 & 6b_zL & 0 & 0 & 0 & (4+\phi_y)b_zL^2 &&&&&& \\
-a & 0 & 0 & 0 & 0 & 0 & a &&&&& \\
0 & -12b_z & 0 & 0 & 0 & -6b_zL & 0 & 12b_z &&&& \\
0 & 0 & -12b_y & 0 & -6b_yL & 0 & 0 & 0 & 12b_y &&& \\
0 & 0 & 0 & -t & 0 & 0 & 0 & 0 & 0 & t && \\
0 & 0 & -6b_yL & 0 & (2-\phi_z)b_yL^2 & 0 & 0 & 0 & 6b_yL & 0 & (4+\phi_z)b_yL^2 & \\
0 & 6b_zL & 0 & 0 & 0 & (2-\phi_y)b_zL^2 & 0 & -6b_zL & 0 & 0 & 0 & (4+\phi_y)b_zL^2
\end{bmatrix}$$

ϕ_y for different height to length beam ratios

h/L	0.05	0.1	0.2	0.3	0.4	0.5	0.6
ϕ_y	0.0065	0.026	0.104	0.234	0.416	0.65	0.936
$1+\phi_y$	1.0065	1.026	1.104	1.234	1.416	1.65	1.936
$1/(1+\phi_y)$	0.994	0.975	0.906	0.810	0.706	0.606	0.517

From
Elementos Finitos: A base da tecnologia CAE
Avelino Alves Filho
5ª edição – Editora Érica


L
P
Δ


0,006
0,005
0,004
0,003
0,002
0,001
0
flexão
cisalhamento
Deflexão Δ [m]
0
0,5
1
1,5
2
2,5
L [m]

Programming the Finite Element Method

- Fortran
- C++
- Mathematica
- Mapple
- Matlab
- Scilab

FE Programme in MatLab for beams

```
% A FE programme for beams
clear all
clc
format long
rho=7800;        % density
b=0.0283;        % beam depth
h=0.0012;        % beam height
A=b*h;           % area
E=200e9;         % elastic modulus
I=b*h^3/12;      % inercia
L=0.55;          % beam length
nel=3;           % number of elements
nno=nel+1;       % node numbers
dofg=3*nno;      % degrees of freedom
le=L/nel;        % element length   ???
```

Beam material and geometry properties: Pre-processor

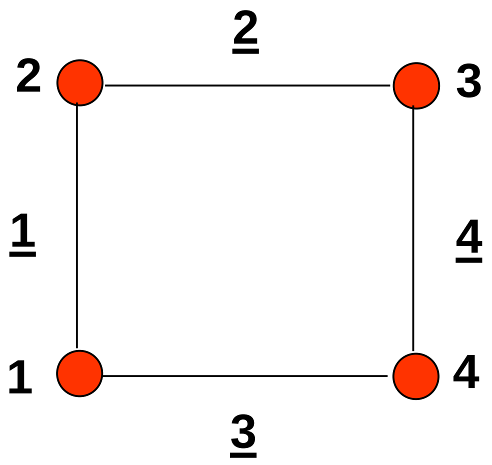
```
%x,y global coordinates
gc(1,1)=0;
for i=2:nno
    gc(i,1)=gc(i-1,1)+le;   % x
    gc(i,2)=0;              % y
end
```

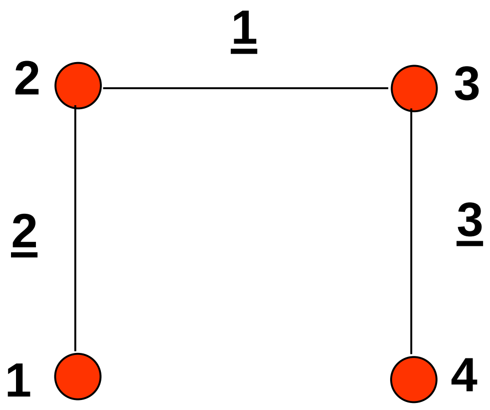
Nodes coordinates: 1D

```
%connective matrix for sequential numbering
%row n stores nodes of element n
cm(1,1)=1;      cm(1,2)=2;
if nel>1
    for i=2:nel
        cm(i,1)=cm(i-1,2);
        cm(i,2)=cm(i,1)+1;
    end
end
```

Connective matrix:
in line elements

Example of connective matrixes





$$\mathbf{cm} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 1 & 4 \\ 3 & 4 \end{bmatrix}$$

$$\mathbf{cm} = \begin{bmatrix} 2 & 3 \\ & \\ & \\ & \end{bmatrix}$$

```
%nodal forces: fx,fy,mz
%f(nno,mz)=moment value
f=zeros(nno,3);
f(1,2)=-5000;
f(2,2)=-5000;
f(1,3)=-100000;
```

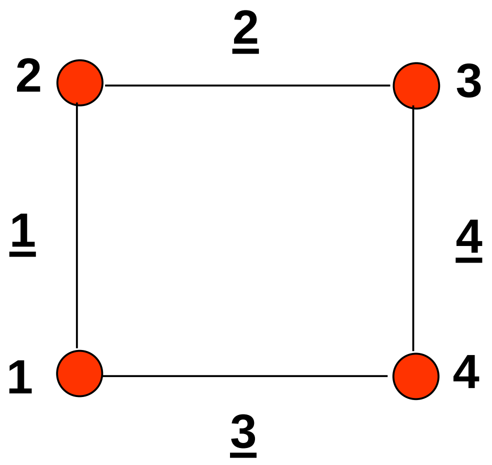
Load matrix

```
%assembly of the local stiffness matrix
k_global=zeros(dofg,dofg);
for el_i=1:nel
nd(1)=cm(el_i,1);%node 1 of element el_1
nd(2)=cm(el_i,2);%node 2 of element el_1
x1=gc(nd(1),1);%x global node 1 of element el_i
……
c1=b*h*Em/le(el_i);
c2=Em*b*h^3/12/le(el_i)^3;
k_e(1,1)=c1;                    k_e(1,4)=-c1;
k_e(2,2)=12*c2;                 k_e(2,3)=6*c2*le(el_i);
…..
k_e=T'*k_e*T;
```

Local stiffness matrix

Local to global matrix

The connective matrix should be used to put the local stiffness elements in the right position in the global stiffness matrix

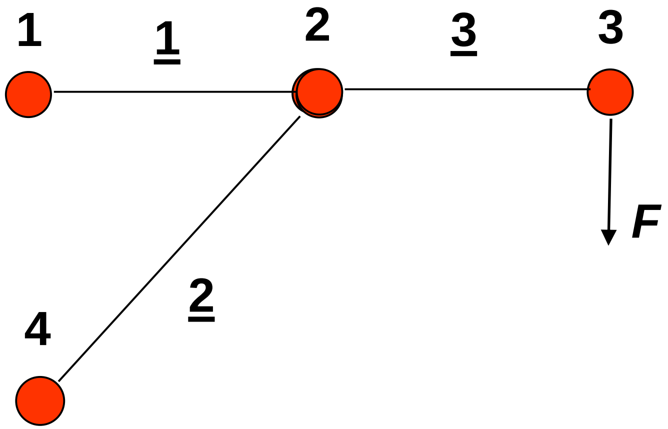


$$\mathbf{cm} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 1 & 4 \\ 3 & 4 \end{bmatrix}$$

Loop n_element:

- -for each element find its position in the global matrix using cm
- -add this contribution, K_g=K_g+K_aux

Example of assembling



$$\mathbf{cm} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 2 & 3 \end{bmatrix} \begin{matrix} \underline{\mathbf{1}} \\ \underline{\mathbf{2}} \\ \underline{\mathbf{3}} \end{matrix}$$

$$\begin{matrix} u_1 & \theta_1 & u_2 & \theta_2 & u_3 & \theta_3 & u_4 & \theta_4 \end{matrix}$$

$$K = \begin{bmatrix} 1 & 1 & 1 & 1 & & & & \\ 1 & 1 & 1 & 1 & & & & \\ 1 & 1 & 123 & 123 & 3 & 3 & & \\ 1 & 1 & 123 & 123 & 3 & 3 & & \\ & & 3 & 3 & 3 & 3 & & \\ & & 3 & 3 & 3 & 3 & & \\ & & & & ? & & 2 & 2 \\ & & & & & & 2 & 2 \end{bmatrix}$$

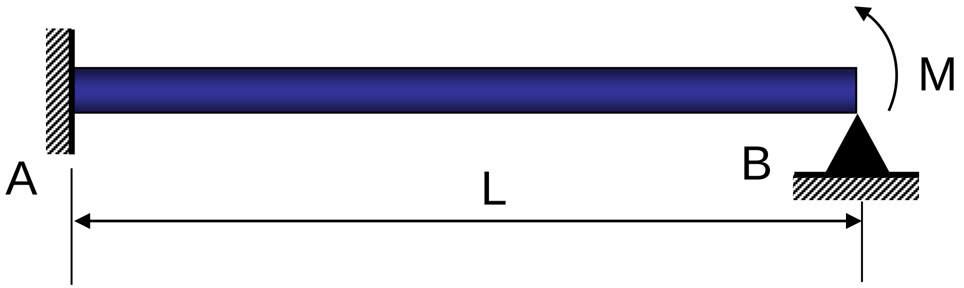
```
%boundary conditions
for i=1:dofg
   if bca(i)==1
      k_global(i,:)=0;
      ….
   end
end
%solving for displacement
u=k_global\fa;
```

Apply bc and solve the system

Explore the results, eg
by plotting the deformed profile and the bending moment distribution

How to calculate M, N and Q

- Use each local stiffness matrix and their corresponding displacement and rotation and operate F = k x
- Note that the 1 and 0 used in BC are now not retained
- Add now the nodal loads
- Be aware that one node may have the contribution of two spans
- Bear in mind the sign convention for internal forces and reactions



$$\frac{EI}{L^3}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4L^2 \end{bmatrix}\begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -M \end{bmatrix}$$

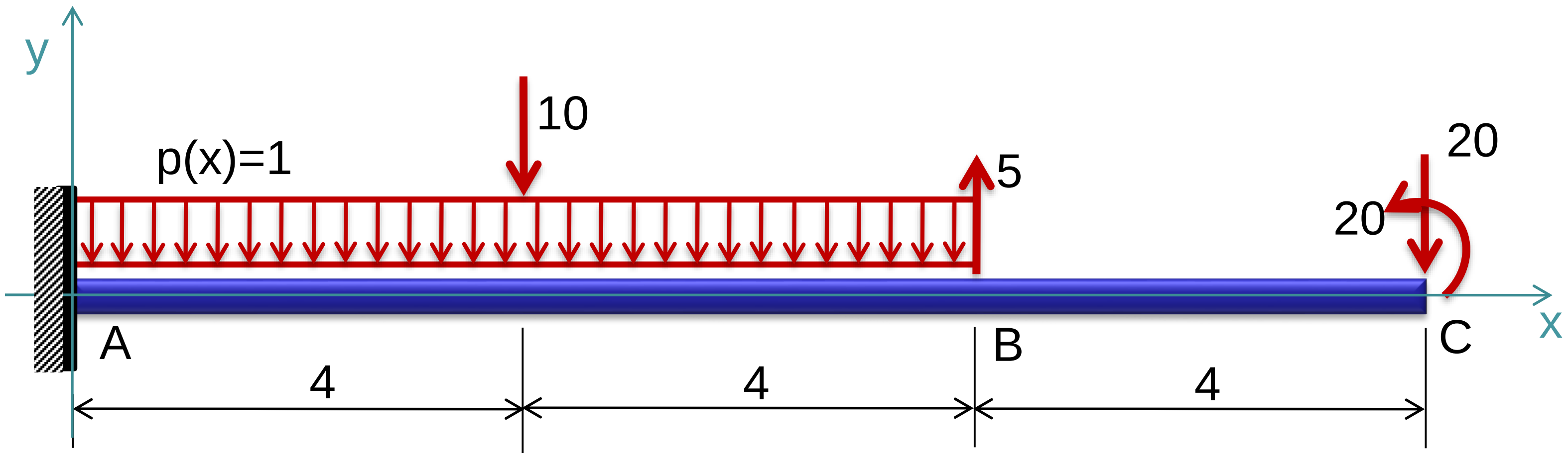
- Are the reaction signals corrected when compared to the analytical solution?
- How about the calculation of the reactions in a node belonging to two elements?
- How can the transverse shear load be calculated?

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ & & & 4L^2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \\ -\dfrac{ML}{4EI} \end{bmatrix} = \begin{bmatrix} R_A \\ M_A \\ R_B \\ -M \end{bmatrix}$$

$$R_A = -\frac{3M}{2L}, \quad R_B = \frac{3M}{2L}, \quad M_A = -\frac{M}{2}$$

Exercises: In a report form!

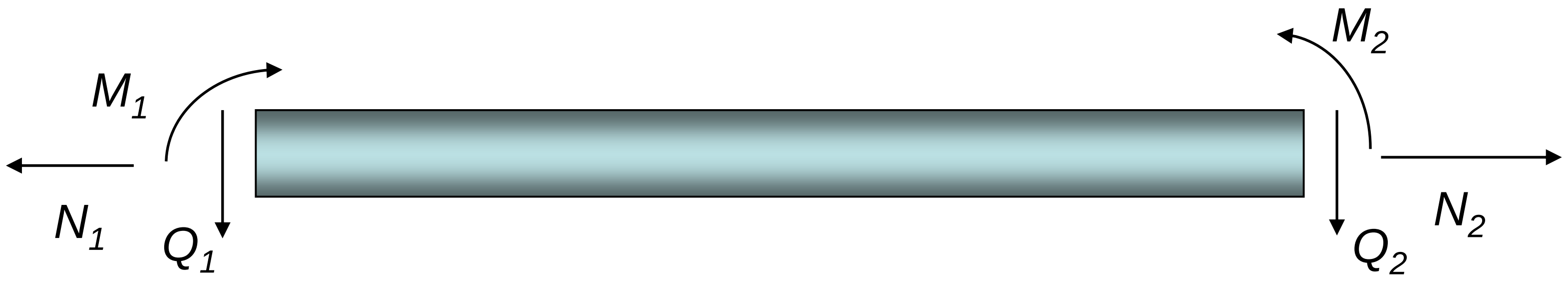
1. Derive in details, using PVW, the stiffness matrix for a truss element
2. Derive in details, using PVW, the weak form of a 2D beam governing equation (M, N, T)
3. Evaluate "by hand", adopting a FE approach, the support reactions and the transverse displacement, bending moment and transverse shear force for the beam shown in the figure. Adopt EI=10^4 N.m^2, with the variables in m and N.



Exercises:

IN A REPORT FORM

4. Write a FE programme to analyse a set of beams [portic] in the plane x-y. Consider the presence of normal forces.



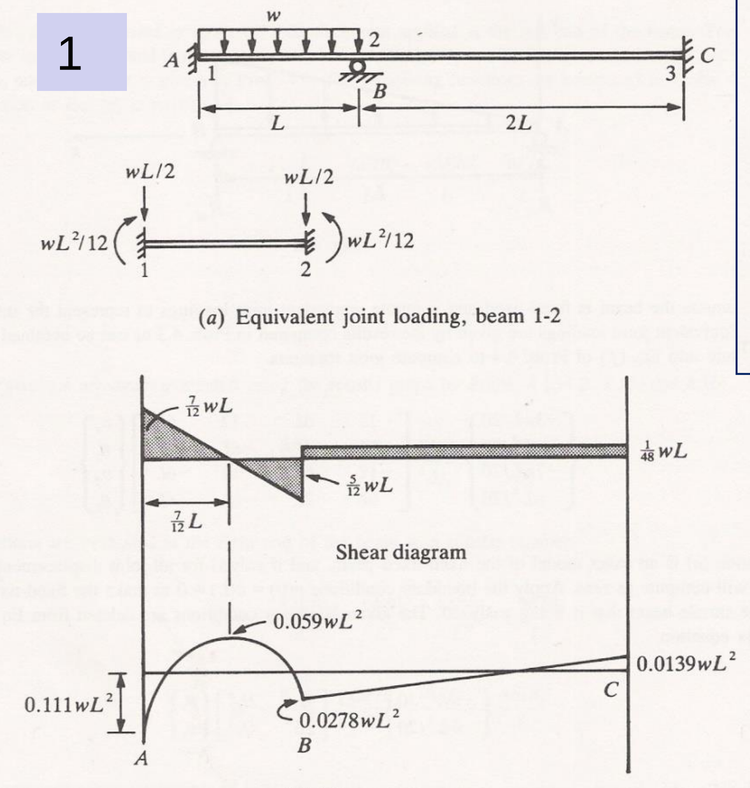
5. Use your programme to solve the next problems; answers below (by G R Buchanan).

6. Plot the displacement, transverse shear, bending moment along the beams for these problems.

Exemplos:

(Finite Element Analysis, G R Buchanan)

1



(a) Equivalent joint loading, beam 1-2

(b) Moment diagram

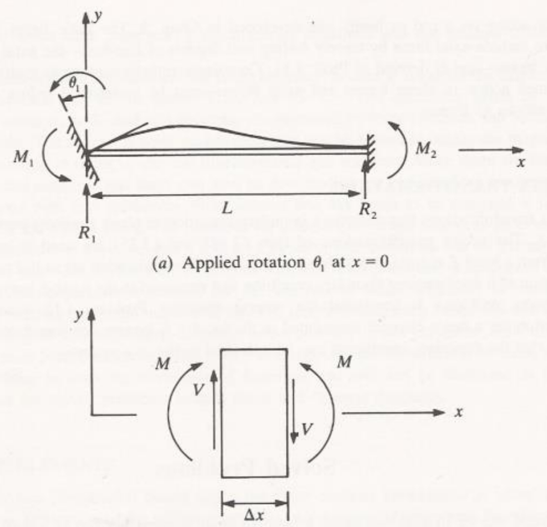
Fig. 4-7

Beam 1-2:

$$\begin{Bmatrix} -wL/2 \\ -wL^2/12 \\ -wL/2 \\ wL^2/12 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} \qquad (a)$$

Beam 2-3:

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \frac{EI}{(2L)^3} \begin{bmatrix} 12 & 6(2L) & -12 & 6(2L) \\ 6(2L) & 4(2L)^2 & -6(2L) & 2(2L)^2 \\ -12 & -6(2L) & 12 & -6(2L) \\ 6(2L) & 2(2L)^2 & -6(2L) & 4(2L)^2 \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} \qquad (b)$$



(a) Applied rotation θ_1 at $x = 0$

(b) Sign convention for shear and moment

The results for each span must be computed individually using the local stiffness matrix for that span. The reactions for span 1-2 are V_A, V_B, M_A, and M_B. The stiffness matrix is the same as Eq. (*a*), and the computation appears as

$$\begin{Bmatrix} V_A \\ M_A \\ V_{B1} \\ M_B \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ wL^3/72EI \end{Bmatrix} + \begin{Bmatrix} wL/2 \\ wL^2/12 \\ wL/2 \\ -wL^2/12 \end{Bmatrix} \qquad (e)$$

Multiplication gives

$$V_A = R_A = \frac{7wL}{12} \qquad V_{B1} = \frac{5wL}{12} \qquad M_A = \frac{wL^2}{9} \qquad M_B = -\frac{wL^2}{36}$$

The notation V_{B1} requires some additional explanation. The total reaction at support B is composed of the end shear at point 2 of beam 1-2 plus the end shear at point 2 of beam 2-3. Then, $R_B = V_{B1} + V_{B2}$.

Similarly, the stiffness matrix of Eq. (*b*) is used to compute the shears and moments for beam 2-3.

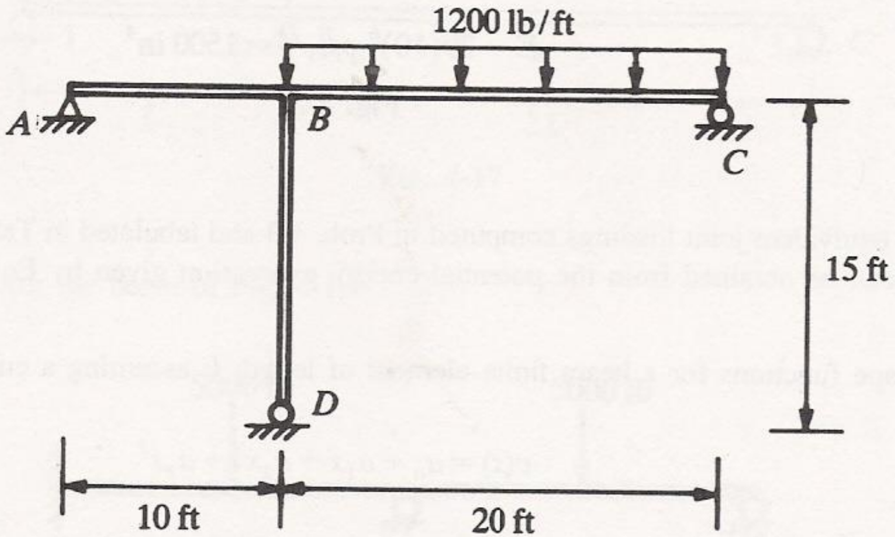
$$\begin{Bmatrix} V_{B2} \\ M_B \\ V_C \\ M_C \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 1.5 & 1.5L & -1.5 & 1.5L \\ 1.5L & 2L^2 & -1.5L & L^2 \\ -1.5 & -1.5L & 1.5 & -1.5L \\ 1.5L & L^2 & -1.5L & 2L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ wL^3/72EI \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (f)$$

$$V_{B2} = \frac{wL}{48} \qquad V_C = R_C = -\frac{wL}{48} \qquad M_B = \frac{wL^2}{36} \qquad M_C = \frac{wL^2}{72} \qquad R_B = V_{B1} + V_{B2}$$

Internal supports such as joint C should always have the same numerical value of bending moment but with opposite signs, thus indicating equilibrium at the joint. The bending moments obtained from the stiffness analysis

2

Compute the reactions at supports A, C, and D and rotations for joints A, B, C, and D along with the horizontal displacements of supports C and D for the frame of Fig. 4-24. Assume $E = 29(10)^6$ psi, $I = 1800$ in^4, and $A = 20$ in^2 for all members.



$E = 29(10)^6$ psi, $I = 1800$ in^4, $A = 20$ in^2

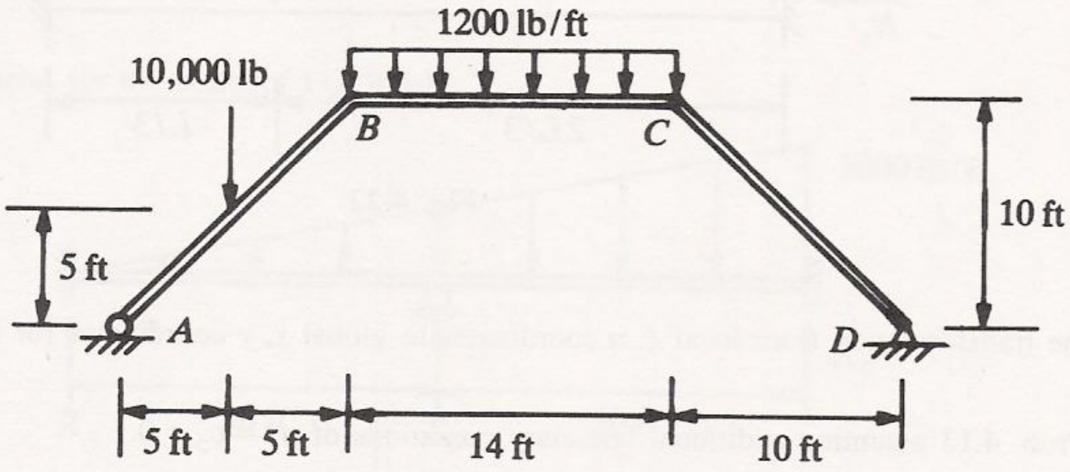
Fig. 4-24

2

$M_A = M_C = M_D = 0$, $M_B = 37{,}520$ ft·lb (acting on member B-C); vertical reactions $V_A = -3750$ lb, $V_C = 10{,}120$, $V_D = 17{,}630$; $\theta_A = 1.296(10)^{-4}$, $\theta_B = \theta_D = -3.906(10)^{-4}$, $\theta_C = 7.812(10)^{-4}$; horizontal displacement at D, $u_D = -0.0703$ in, $u_C = 0$.

3

Compute reactions, joint rotations, and the horizontal displacement of joint A for the frame of Fig. 4-25. Assume $E = 29(10)^6$ psi, $I = 1800$ in^4, and $A = 20$ in^2 for all members.



$E = 29(10)^6$ psi, $I = 1800$ in^4, $A = 20$ in^2

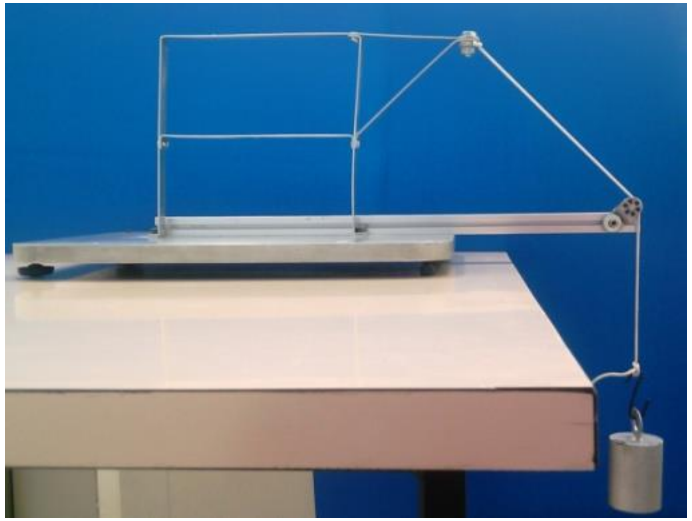
Fig. 4-25

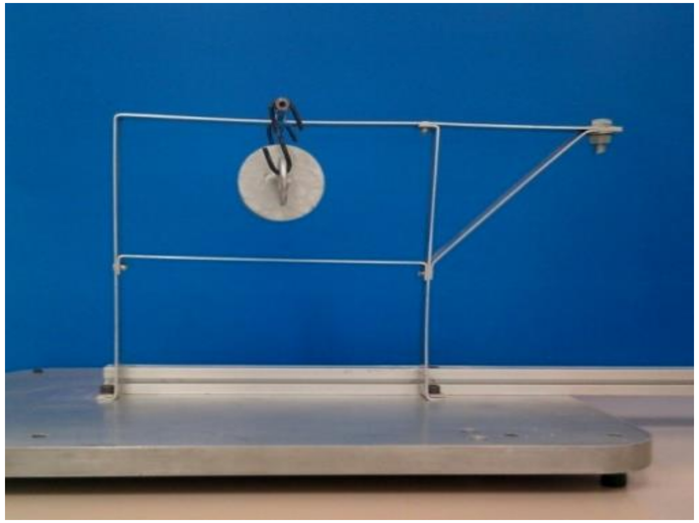
3

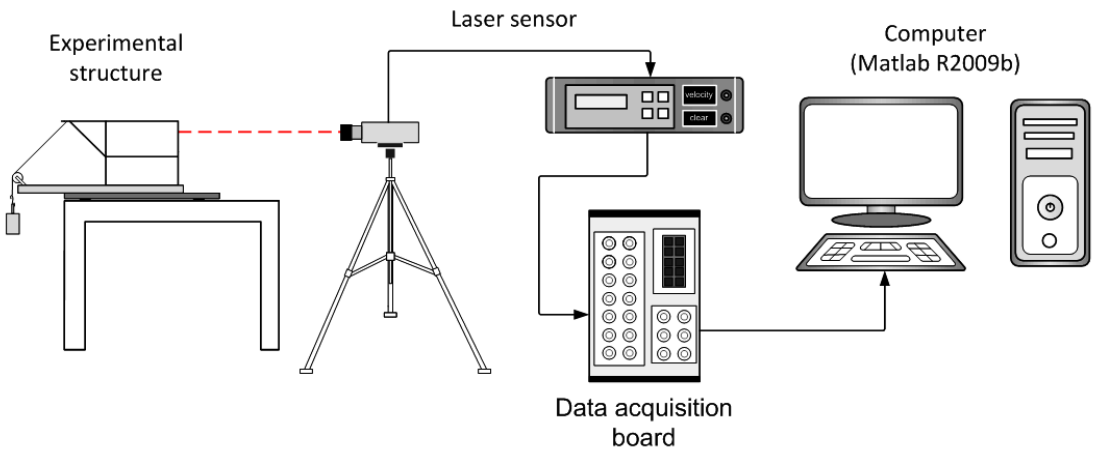
$M_A = M_D = 0$, $M_B = -108{,}960$ ft·lb (acting on member B-C), $M_C = 94{,}400$ (acting on member B-C), $V_A =$ 17,370 lb, $V_D = 9450$, $\theta_A = -4.187(10)^{-3}$, $\theta_B = -2.246(10)^{-3}$, $\theta_C = 2.438(10)^{-3}$, $\theta_D = 4.279(10)^{-3}$, $u_A =$ -0.90 in.

Experiments

Group of Solid Mechanics & Structural Impact
University of São Paulo







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