Apêndice H

Integral $\int \frac{d\phi}{a-\rho\cos(\phi)}$

Vamos calcular a integral I abaixo (onde $a > \rho$)

$$I = \int_0^{2\pi} \frac{d\phi}{a - \rho\cos(\phi)} \tag{H.1}$$

Primeiro vamos dividir a integral em duas partes

$$I = \int_0^{\pi} \frac{d\phi}{a - \rho\cos(\phi)} + \int_{\pi}^{2\pi} \frac{d\phi}{a - \rho\cos(\phi)} \tag{H.2}$$

Na segunda integral vamos mudar a variável $\phi \to \phi' = 2\pi - \phi$, ou similarmente $\phi = 2\pi - \phi'$. Temos $d\phi = -d\phi'$ e os limites de integração $\phi = \pi \to \phi' = \pi$ e $\phi = 2\pi \to \phi' = 0$. Além disso $\cos(\phi) = \cos(2\pi - \phi') = \cos(-\phi') = \cos(\phi')$. Assim

$$I = \int_0^{\pi} \frac{d\phi}{a - \rho\cos(\phi)} + \int_{\pi}^{0} \frac{-d\phi'}{a - \rho\cos(\phi')} = 2\int_0^{\pi} \frac{d\phi}{a - \rho\cos(\phi)} \,. \tag{H.3}$$

Agora vamos mudar $\phi \to u = \tan(\phi/2)$. Temos

$$\frac{du}{d\phi} = \frac{d}{du}\left[\frac{\sin(\phi/2)}{\cos(\phi/2)}\right] = \frac{1}{2}\left[1 + \frac{\sin^2(\phi/2)}{\cos^2(\phi/2)}\right] = \frac{1}{2}[1 + \tan^2(\phi/2)] = \frac{1}{2\cos^2(\phi/2)} \,, \tag{H.4}$$

ou seja

$$d\phi = \frac{2du}{1 + u^2} \tag{H.5}$$

Além disso, temos

$$\cos\phi = 2\cos^2(\phi/2) - 1 = \frac{2}{1 + \tan^2(\phi/2)} - 1 = \frac{2}{1 + u^2} - 1 = \frac{1 - u^2}{1 + u^2} \tag{H.6}$$

Assim, a integral fica

$$\begin{aligned} I &= 2\int \left(\frac{2du}{1 + u^2}\right) \frac{1}{a - \rho\left(\frac{1-u^2}{1+u^2}\right)} = 2\int \frac{2du}{a(1 + u^2) - \rho(1 - u^2)} = 4\int \frac{du}{(a - \rho) + (a + \rho)u^2} \\ &= \frac{4}{a - \rho} \int_{\phi=0}^{\phi=\pi} \frac{du}{1 + \left(\frac{a+\rho}{a-\rho}\right)u^2} \end{aligned} \tag{H.7}$$

Agora mudamos de variável novamente $u \to y = \sqrt{\frac{a+\rho}{a-\rho}}u$, com $du = \sqrt{\frac{a-\rho}{a+\rho}}dy$. A integral fica:

$$\begin{aligned} I &= \frac{4}{a-\rho}\int\sqrt{\frac{a-\rho}{a+\rho}}\frac{dy}{1+y^2} = \frac{4}{\sqrt{a-\rho}\sqrt{a+\rho}}\int_{\phi=0}^{\phi=\pi}\frac{dy}{1+y^2} \\ &= \frac{4}{\sqrt{a^2-\rho^2}}\arctan(y)|_{\phi=0}^{\phi=\pi} \end{aligned} \tag{H.8}$$

Voltando às variáveis originais

$$I = \frac{4}{\sqrt{a^2-\rho^2}}\arctan\left(\sqrt{\frac{a+\rho}{a-\rho}}u\right)|_{\phi=0}^{\phi=\pi} = \frac{4}{\sqrt{a^2-\rho^2}}\arctan\left(\sqrt{\frac{a+\rho}{a-\rho}}\tan(\phi/2)\right)|_{\phi=0}^{\phi=\pi} \tag{H.9}$$

Com $\tan(\pi/2) = \infty$ e $\tan(0) = 0$, e portanto $\arctan(\infty) = \pi/2$, temos

$$I = \frac{4}{\sqrt{a^2-\rho^2}}\left[\arctan(\infty) - \arctan(0)\right] = \frac{4}{\sqrt{a^2-\rho^2}}\frac{\pi}{2} \tag{H.10}$$

ou finalmente

$$I = \int_0^{2\pi}\frac{d\phi}{a-\rho\cos(\phi)} = \frac{2\pi}{\sqrt{a^2-\rho^2}} \tag{H.11}$$