CHAPTER

7 The Root Locus Method

PREVIEW

The performance of a feedback system can be described in terms of the location of the roots of the characteristic equation in the s-plane. A graph showing how the roots of the characteristic equation move around the s-plane as a single parameter varies is known as a root locus plot. The root locus is a powerful tool for designing and analyzing feedback control systems. We will discuss practical techniques for obtaining a sketch of a root locus plot by hand. We also consider computer-generated root locus plots and illustrate their effectiveness in the design process. We will show that it is possible to use root locus methods for controller design when more than one parameter varies. This is important because we know that the response of a closed-loop feedback system can be adjusted to achieve the desired performance by judicious selection of one or more controller parameters. The popular PID controller is introduced as a practical controller structure. We will also define a measure of sensitivity of a specified root to a small incremental change in a system parameter. The chapter concludes with a controller design based on root locus methods for the Sequential Design Example: Disk Drive Read System.

DESIRED OUTCOMES

Upon completion of Chapter 7, students should:

- ❑ Understand the powerful concept of the root locus and its role in control system design.
- ❑ Know how to obtain a root locus plot by sketching or using computers.
- ❑ Be familiar with the PID controller as a key element of many feedback systems.
- ❑ Recognize the role of root locus plots in parameter design and system sensitivity analysis.
- ❑ Be able to design controllers to meet desired specifications using root locus methods.

7.1 INTRODUCTION

The relative stability and the transient performance of a closed-loop control system are directly related to the location of the closed-loop roots of the characteristic equation in the s-plane. It is frequently necessary to adjust one or more system parameters in order to obtain suitable root locations. Therefore, it is worthwhile to determine how the roots of the characteristic equation of a given system migrate about the s-plane as the parameters are varied; that is, it is useful to determine the **locus** of roots in the s-plane as a parameter is varied. The **root locus method** was introduced by Evans in 1948 and has been developed and utilized extensively in control engineering practice [1–3]. The root locus technique is a graphical method for sketching the locus of roots in the s-plane as a parameter is varied. In fact, the root locus method provides the engineer with a measure of the sensitivity of the roots of the system to a variation in the parameter being considered. The root locus technique may be used to great advantage in conjunction with the Routh–Hurwitz criterion.

The root locus method provides graphical information, and therefore an approximate sketch can be used to obtain qualitative information concerning the stability and performance of the system. Furthermore, the locus of roots of the characteristic equation of a multiloop system may be investigated as readily as for a single-loop system. If the root locations are not satisfactory, the necessary parameter adjustments often can be readily ascertained from the root locus [4].

7.2 THE ROOT LOCUS CONCEPT

The dynamic performance of a closed-loop control system is described by the closed-loop transfer function

$$T(s) = \frac{Y(s)}{R(s)} = \frac{p(s)}{q(s)}, \tag{7.1}$$

where $p(s)$ and $q(s)$ are polynomials in s. The roots of the characteristic equation $q(s)$ determine the modes of response of the system. In the case of the simple single-loop system shown in Figure 7.1, we have the characteristic equation

$$\boxed{1 + KG(s) = 0,} \tag{7.2}$$

where K is a variable parameter and $0 \leq K < \infty$. The characteristic roots of the system must satisfy Equation (7.2), where the roots lie in the s-plane. Because s is a complex variable, Equation (7.2) may be rewritten in polar form as

$$|KG(s)|\underline{/KG(s)} = -1 + j0, \tag{7.3}$$

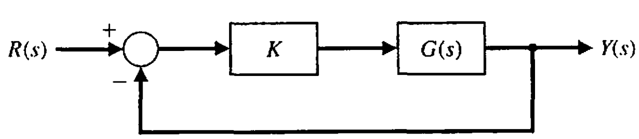


FIGURE 7.1 Closed-loop control system with a variable parameter K.

and therefore it is necessary that

$$\boxed{|KG(s)| = 1}$$

and

$$\boxed{\angle KG(s) = 180° + k360°,} \tag{7.4}$$

where $k = 0, \pm 1, \pm 2, \pm 3, \ldots$.

The root locus is the path of the roots of the characteristic equation traced out in the s-plane as a system parameter varies from zero to infinity.

The simple second-order system considered in the previous chapters is shown in Figure 7.2. The characteristic equation representing this system is

$$\Delta(s) = 1 + KG(s) = 1 + \frac{K}{s(s+2)} = 0,$$

or, alternatively,

$$\Delta(s) = s^2 + 2s + K = s^2 + 2\zeta\omega_n s + \omega_n^2 = 0. \tag{7.5}$$

The locus of the roots as the gain K is varied is found by requiring that

$$|KG(s)| = \left|\frac{K}{s(s+2)}\right| = 1 \tag{7.6}$$

and

$$\angle KG(s) = \pm 180°, \pm 540°, \ldots. \tag{7.7}$$

The gain K may be varied from zero to an infinitely large positive value. For a second-order system, the roots are

$$s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}, \tag{7.8}$$

and for $\zeta < 1$, we know that $\theta = \cos^{-1}\zeta$. Graphically, for two open-loop poles as shown in Figure 7.3, the locus of roots is a vertical line for $\zeta \leq 1$ in order to satisfy the angle requirement, Equation (7.7). For example, as shown in Figure 7.4, at a root s_1, the angles are

$$\left.\angle \frac{K}{s(s+2)}\right|_{s=s_1} = -\angle s_1 - \angle(s_1 + 2) = -[(180° - \theta) + \theta] = -180°. \tag{7.9}$$

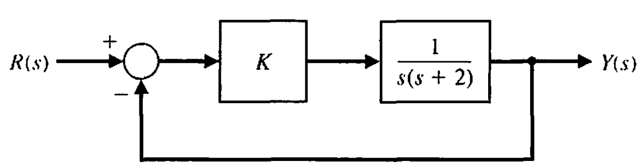


FIGURE 7.2 Unity feedback control system. The gain K is a variable parameter.

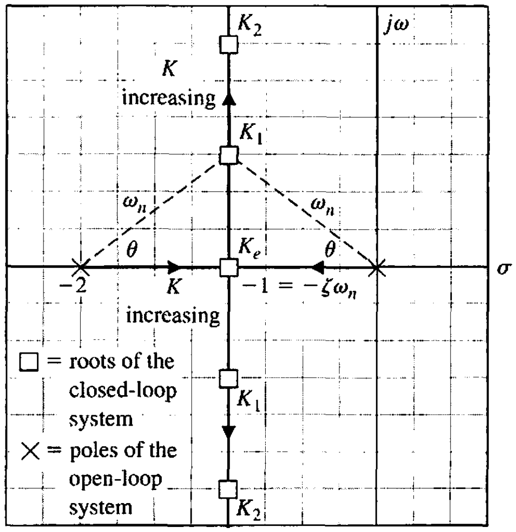


FIGURE 7.3 Root locus for a second-order system when $K_e < K_1 < K_2$. The locus is shown as heavy lines, with arrows indicating the direction of increasing K. Note that roots of the characteristic equation are denoted by "□" on the root locus.

This angle requirement is satisfied at any point on the vertical line that is a perpendicular bisector of the line 0 to −2. Furthermore, the gain K at the particular points is found by using Equation (7.6) as

$$\left|\frac{K}{s(s+2)}\right|_{s=s_1} = \frac{K}{|s_1||s_1+2|} = 1, \tag{7.10}$$

and thus

$$K = |s_1||s_1 + 2|, \tag{7.11}$$

where $|s_1|$ is the magnitude of the vector from the origin to s_1, and $|s_1 + 2|$ is the magnitude of the vector from −2 to s_1.

For a multiloop closed-loop system, we found in Section 2.7 that by using Mason's signal-flow gain formula, we had

$$\Delta(s) = 1 - \sum_{n=1}^{N} L_n + \sum_{\substack{n,m \\ \text{nontouching}}} L_n L_m - \sum_{\substack{n,m,p \\ \text{nontouching}}} L_n L_m L_p + \cdots, \tag{7.12}$$

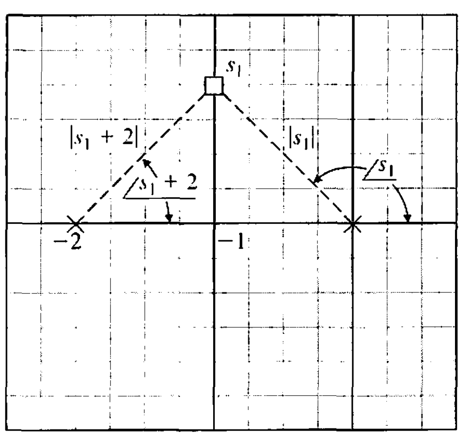


FIGURE 7.4 Evaluation of the angle and gain at s_1 for gain $K = K_1$.

where L_n equals the value of the nth self-loop transmittance. Hence, we have a characteristic equation, which may be written as

$$q(s) = \Delta(s) = 1 + F(s). \tag{7.13}$$

To find the roots of the characteristic equation, we set Equation (7.13) equal to zero and obtain

$$1 + F(s) = 0. \tag{7.14}$$

Equation (7.14) may be rewritten as

$$F(s) = -1 + j0, \tag{7.15}$$

and the roots of the characteristic equation must also satisfy this relation.

In general, the function $F(s)$ may be written as

$$F(s) = \frac{K(s + z_1)(s + z_2)(s + z_3)\cdots(s + z_M)}{(s + p_1)(s + p_2)(s + p_3)\cdots(s + p_n)}.$$

Then the magnitude and angle requirement for the root locus are

$$|F(s)| = \frac{K|s + z_1||s + z_2|\cdots}{|s + p_1||s + p_2|\cdots} = 1 \tag{7.16}$$

and

$$\angle F(s) = \angle s + z_1 + \angle s + z_2 + \cdots$$
$$- (\angle s + p_1 + \angle s + p_2 + \cdots) = 180^\circ + k360^\circ, \tag{7.17}$$

where k is an integer. The magnitude requirement, Equation (7.16), enables us to determine the value of K for a given root location s_1. A test point in the s-plane, s_1, is verified as a root location when Equation (7.17) is satisfied. All angles are measured in a counterclockwise direction from a horizontal line.

To further illustrate the root locus procedure, let us consider again the second-order system of Figure 7.5(a). The effect of varying the parameter a can

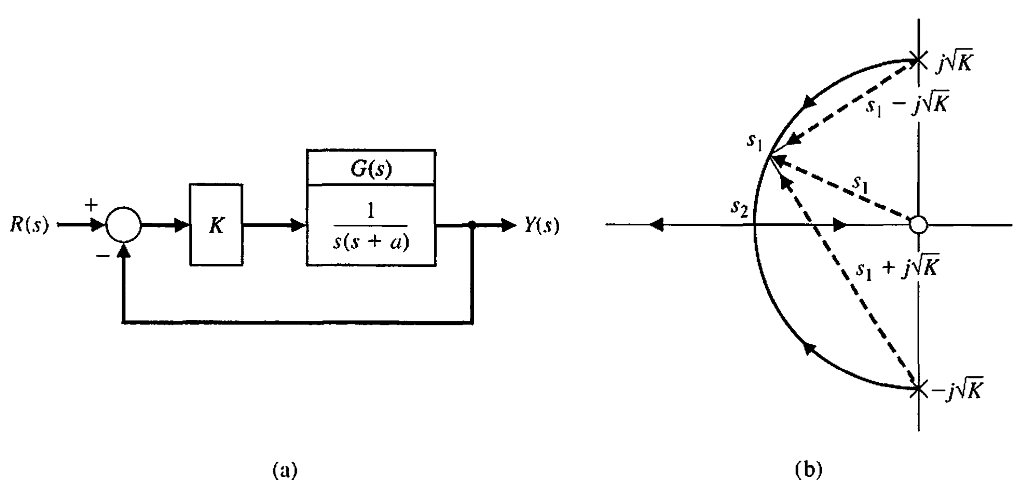


FIGURE 7.5 (a) Single-loop system. (b) Root locus as a function of the parameter a, where $a > 0$.

be effectively portrayed by rewriting the characteristic equation for the root locus form with a as the multiplying factor in the numerator. Then the characteristic equation is

$$1 + KG(s) = 1 + \frac{K}{s(s + a)} = 0,$$

or, alternatively,

$$s^2 + as + K = 0.$$

Dividing by the factor $s^2 + K$, we obtain

$$1 + \frac{as}{s^2 + K} = 0. \tag{7.18}$$

Then the magnitude criterion is satisfied when

$$\frac{a|s_1|}{|s_1^2 + K|} = 1 \tag{7.19}$$

at the root s_1. The angle criterion is

$$\angle s_1 - \left(\angle s_1 + j\sqrt{K} + \angle s_1 - j\sqrt{K}\right) = \pm 180^\circ, \pm 540^\circ, \ldots.$$

In principle, we could construct the root locus by determining the points in the s-plane that satisfy the angle criterion. In the next section, we will develop a multistep procedure to sketch the root locus. The root locus for the characteristic equation in Equation (7.18) is shown in Figure 7.5(b). Specifically at the root s_1, the magnitude of the parameter a is found from Equation (7.19) as

$$a = \frac{|s_1 - j\sqrt{K}||s_1 + j\sqrt{K}|}{|s_1|}. \tag{7.20}$$

The roots of the system merge on the real axis at the point s_2 and provide a critically damped response to a step input. The parameter a has a magnitude at the critically damped roots, $s_2 = \sigma_2$, equal to

$$a = \frac{|\sigma_2 - j\sqrt{K}||\sigma_2 + j\sqrt{K}|}{\sigma_2} = \frac{1}{\sigma_2}(\sigma_2^2 + K) = 2\sqrt{K}, \tag{7.21}$$

where σ_2 is evaluated from the s-plane vector lengths as $\sigma_2 = \sqrt{K}$. As a increases beyond the critical value, the roots are both real and distinct; one root is larger than σ_2, and one is smaller.

In general, we desire an orderly process for locating the locus of roots as a parameter varies. In the next section, we will develop such an orderly approach to sketching a root locus diagram.

7.3 THE ROOT LOCUS PROCEDURE

The roots of the characteristic equation of a system provide a valuable insight concerning the response of the system. To locate the roots of the characteristic equation in a graphical manner on the s-plane, we will develop an orderly procedure of seven steps that facilitates the rapid sketching of the locus.

Step 1: Prepare the root locus sketch. Begin by writing the characteristic equation as

$$1 + F(s) = 0. \tag{7.22}$$

Rearrange the equation, if necessary, so that the parameter of interest, K, appears as the multiplying factor in the form,

$$1 + KP(s) = 0. \tag{7.23}$$

We are usually interested in determining the locus of roots as K varies as

$$0 \le K \le \infty.$$

In Section 7.7, we consider the case when K varies as $-\infty < K \le 0$. Factor $P(s)$, and write the polynomial in the form of poles and zeros as follows:

$$1 + K\frac{\prod_{i=1}^{M}(s + z_i)}{\prod_{j=1}^{n}(s + p_j)} = 0. \tag{7.24}$$

Locate the poles $-p_i$ and zeros $-z_i$ on the s-plane with selected symbols. By convention, we use 'x' to denote poles and 'o' to denote zeros.

Rewriting Equation (7.24), we have

$$\prod_{j=1}^{n}(s + p_j) + K\prod_{i=1}^{M}(s + z_i) = 0. \tag{7.25}$$

Note that Equation (7.25) is another way to write the characteristic equation. When $K = 0$, the roots of the characteristic equation are the poles of $P(s)$. To see this, consider Equation (7.25) with $K = 0$. Then, we have

$$\prod_{j=1}^{n}(s + p_j) = 0.$$

When solved, this yields the values of s that coincide with the poles of $P(s)$. Conversely, as $K \to \infty$, the roots of the characteristic equation are the zeros of $P(s)$. To see this, first divide Equation (7.25) by K. Then, we have

$$\frac{1}{K}\prod_{j=1}^{n}(s + p_j) + \prod_{j=1}^{M}(s + z_j) = 0,$$

which, as $K \to \infty$, reduces to

$$\prod_{j=1}^{M}(s + z_j) = 0.$$

When solved, this yields the values of s that coincide with the zeros of $P(s)$. Therefore, we note that **the locus of the roots of the characteristic equation $1 + KP(s) = 0$ begins at the poles of $P(s)$ and ends at the zeros of $P(s)$ as K increases from zero to infinity.** For most functions $P(s)$ that we will encounter, several of the zeros of $P(s)$ lie at infinity in the s-plane. This is because most of our functions have more poles than zeros. With n poles and M zeros and $n > M$, we have $n - M$ branches of the root locus approaching the $n - M$ zeros at infinity.

Step 2: Locate the segments of the real axis that are root loci. **The root locus on the real axis always lies in a section of the real axis to the left of an odd number of poles and zeros.** This fact is ascertained by examining the angle criterion of Equation (7.17). These two useful steps in plotting a root locus will be illustrated by a suitable example.

EXAMPLE 7.1 **Second-order system**

A single-loop feedback control system possesses the characteristic equation

$$1 + GH(s) = 1 + \frac{K\left(\frac{1}{2}s + 1\right)}{\frac{1}{4}s^2 + s} = 0. \tag{7.26}$$

STEP 1: The characteristic equation can be written as

$$1 + K\frac{2(s + 2)}{s^2 + 4s} = 0,$$

where

$$P(s) = \frac{2(s + 2)}{s^2 + 4s}.$$

The transfer function, $P(s)$, is rewritten in terms of poles and zeros as

$$1 + K\frac{2(s + 2)}{s(s + 4)} = 0, \tag{7.27}$$

and the multiplicative gain parameter is K. To determine the locus of roots for the gain $0 \leq K \leq \infty$, we locate the poles and zeros on the real axis as shown in Figure 7.6(a).

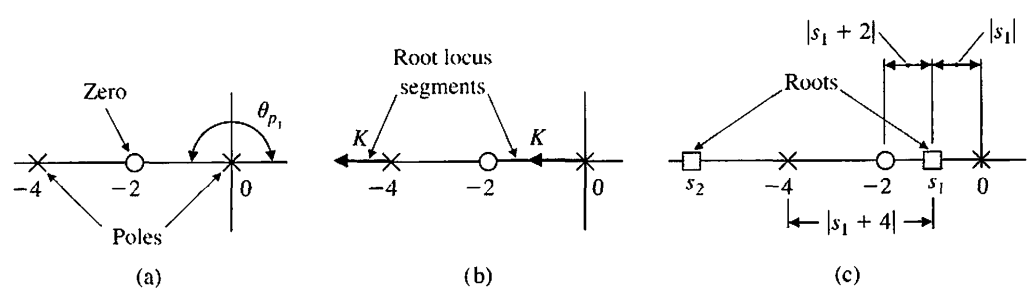


FIGURE 7.6 (a) The zero and poles of a second-order system, (b) the root locus segments, and (c) the magnitude of each vector at s_1.

Step 2: The angle criterion is satisfied on the real axis between the points 0 and -2, because the angle from pole p_1 at the origin is 180°, and the angle from the zero and pole p_2 at $s = -4$ is zero degrees. The locus begins at the pole and ends at the zeros, and therefore the locus of roots appears as shown in Figure 7.6(b), where the direction of the locus as K is increasing ($K\uparrow$) is shown by an arrow. We note that because the system has two real poles and one real zero, the second locus segment ends at a zero at negative infinity. To evaluate the gain K at a specific root location on the locus, we use the magnitude criterion, Equation (7.16). For example, the gain K at the root $s = s_1 = -1$ is found from (7.16) as

$$\frac{(2K)|s_1 + 2|}{|s_1||s_1 + 4|} = 1$$

or

$$K = \frac{|-1||-1 + 4|}{2|-1 + 2|} = \frac{3}{2}. \tag{7.28}$$

This magnitude can also be evaluated graphically, as shown in Figure 7.6(c). For the gain of $K = \frac{3}{2}$, one other root exists, located on the locus to the left of the pole at -4. The location of the second root is found graphically to be located at $s = -6$, as shown in Figure 7.6(c).

Now, we determine the number of separate loci, SL. Because the loci begin at the poles and end at the zeros, the **number of separate loci is equal to the number of poles** since the number of poles is greater than or equal to the number of zeros. Therefore, as we found in Figure 7.6, the number of separate loci is equal to two because there are two poles and one zero.

Note that the **root loci must be symmetrical with respect to the horizontal real axis** because the complex roots must appear as pairs of complex conjugate roots. ■

We now return to developing a general list of root locus steps.

Step 3: The loci proceed to the zeros at infinity along asymptotes centered at σ_A and with angles ϕ_A. When the number of finite zeros of $P(s)$, M, is less than the number of poles n by the number $N = n - M$, then N sections of loci must end at zeros at infinity. These sections of loci proceed to the zeros at infinity along **asymptotes** as K approaches infinity. These linear asymptotes are centered at a point on the real axis given by

$$\boxed{\sigma_A = \frac{\sum \text{poles of } P(s) - \sum \text{zeros of } P(s)}{n - M} = \frac{\sum_{j=1}^{n}(-p_j) - \sum_{i=1}^{M}(-z_i)}{n - M}.} \tag{7.29}$$

The **angle of the asymptotes** with respect to the real axis is

$$\boxed{\phi_A = \frac{2k + 1}{n - M}180°,} \qquad k = 0, 1, 2, \ldots, (n - M - 1), \tag{7.30}$$

where k is an integer index [3]. The usefulness of this rule is obvious for sketching the approximate form of a root locus. Equation (7.30) can be readily derived by considering a point on a root locus segment at a remote distance from the finite poles and zeros in the s-plane. The net phase angle at this remote point is 180°, because it is a point on a root locus segment. The finite poles and zeros of $P(s)$ are a great distance from the remote point, and so the angles from each pole and zero, ϕ, are essentially equal, and therefore the net angle is simply $(n - M)\phi$, where n and M are the number of finite poles and zeros, respectively. Thus, we have

$$(n - M)\phi = 180^\circ,$$

or, alternatively,

$$\phi = \frac{180^\circ}{n - M}.$$

Accounting for all possible root locus segments at remote locations in the s-plane, we obtain Equation (7.30).

The center of the linear asymptotes, often called the **asymptote centroid**, is determined by considering the characteristic equation in Equation (7.24). For large values of s, only the higher-order terms need be considered, so that the characteristic equation reduces to

$$1 + \frac{Ks^M}{s^n} = 0.$$

However, this relation, which is an approximation, indicates that the centroid of $n - M$ asymptotes is at the origin, $s = 0$. A better approximation is obtained if we consider a characteristic equation of the form

$$1 + \frac{K}{(s - \sigma_A)^{n-M}} = 0$$

with a centroid at σ_A.

The centroid is determined by considering the first two terms of Equation (7.24), which may be found from the relation

$$1 + \frac{K\prod_{i=1}^{M}(s + z_i)}{\prod_{j=1}^{n}(s + p_j)} = 1 + K\frac{s^M + b_{M-1}s^{M-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}.$$

From Chapter 6, especially Equation (6.5), we note that

$$b_{M-1} = \sum_{i=1}^{M} z_i \quad \text{and} \quad a_{n-1} = \sum_{j=1}^{n} p_j.$$

Considering only the first two terms of this expansion, we have

$$1 + \frac{K}{s^{n-M} + (a_{n-1} - b_{M-1})s^{n-M-1}} = 0.$$

The first two terms of

$$1 + \frac{K}{(s - \sigma_A)^{n-M}} = 0$$

are

$$1 + \frac{K}{s^{n-M} - (n - M)\sigma_A s^{n-M-1}} = 0.$$

Equating the term for s^{n-M-1}, we obtain

$$a_{n-1} - b_{M-1} = -(n - M)\sigma_A,$$

or

$$\sigma_A = \frac{\sum_{i=1}^{n}(-p_i) - \sum_{i=1}^{M}(-z_i)}{n - M}$$

which is Equation (7.29).

For example, reexamine the system shown in Figure 7.2 and discussed in Section 7.2. The characteristic equation is written as

$$1 + \frac{K}{s(s + 2)} = 0.$$

Because $n - M = 2$, we expect two loci to end at zeros at infinity. The asymptotes of the loci are located at a center

$$\sigma_A = \frac{-2}{2} = -1$$

and at angles of

$$\phi_A = 90° \text{ (for } k = 0\text{)} \quad \text{and} \quad \phi_A = 270° \text{ (for } k = 1\text{)}.$$

The root locus is readily sketched, and the locus shown in Figure 7.3 is obtained. An example will further illustrate the process of using the asymptotes.

EXAMPLE 7.2 **Fourth-order system**

A single-loop feedback control system has a characteristic equation as follows:

$$1 + GH(s) = 1 + \frac{K(s + 1)}{s(s + 2)(s + 4)^2}. \tag{7.31}$$

We wish to sketch the root locus in order to determine the effect of the gain K. The poles and zeros are located in the s-plane, as shown in Figure 7.7(a). The root loci on the real axis must be located to the left of an odd number of poles and zeros; they are shown as heavy lines in Figure 7.7(a). The intersection of the asymptotes is

$$\sigma_A = \frac{(-2) + 2(-4) - (-1)}{4 - 1} = \frac{-9}{3} = -3. \tag{7.32}$$

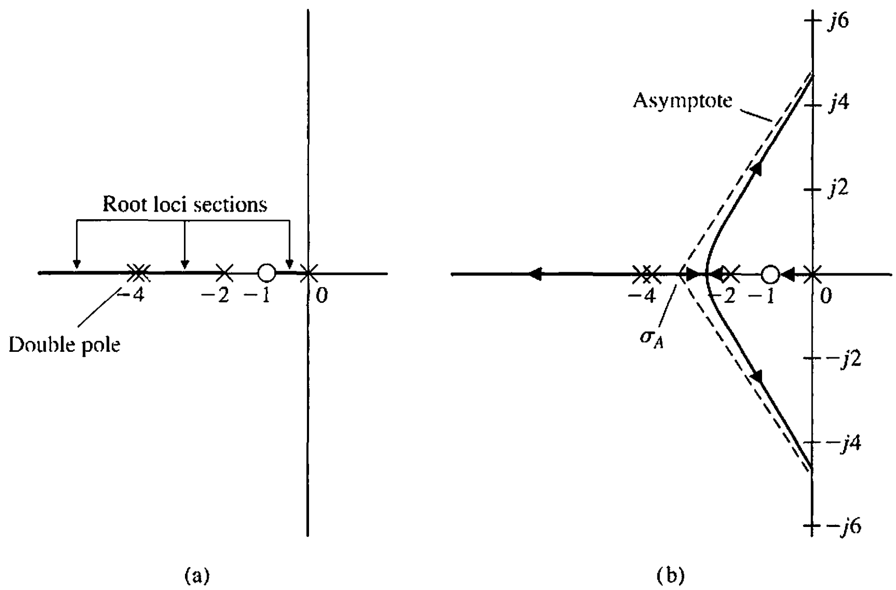


FIGURE 7.7 A fourth-order system with (a) a zero and (b) root locus.

The angles of the asymptotes are

$$\phi_A = +60^\circ \quad (k = 0),$$
$$\phi_A = 180^\circ \quad (k = 1), \text{and}$$
$$\phi_A = 300^\circ \quad (k = 2),$$

where there are three asymptotes, since $n - M = 3$. Also, we note that the root loci must begin at the poles; therefore, two loci must leave the double pole at $s = -4$. Then with the asymptotes sketched in Figure 7.7(b), we may sketch the form of the root locus as shown in Figure 7.7(b). The actual shape of the locus in the area near σ_A would be graphically evaluated, if necessary. ∎

We now proceed to develop more steps for the process of determining the root loci.

Step 4: Determine where the locus crosses the imaginary axis (if it does so), using the Routh–Hurwitz criterion. **The actual point at which the root locus crosses the imaginary axis is readily evaluated by using the criterion.**

Step 5: Determine the breakaway point on the real axis (if any). The root locus in Example 7.2 left the real axis at a **breakaway point**. The locus breakaway from the real axis occurs where the net change in angle caused by a small displacement is zero. The locus leaves the real axis where there is a multiplicity of roots (typically, two). The breakaway point for a simple second-order system is shown in Figure 7.8(a) and, for a special case of a fourth-order system, is shown in Figure 7.8(b). In general, due to the phase criterion, **the tangents to the loci at the breakaway point are equally spaced over 360°. Therefore, in Figure 7.8(a), we find that the two loci at the breakaway point are spaced 180° apart, whereas in Figure 7.8(b), the four loci are spaced 90° apart.**

The breakaway point on the real axis can be evaluated graphically or analytically. The most straightforward method of evaluating the breakaway point involves

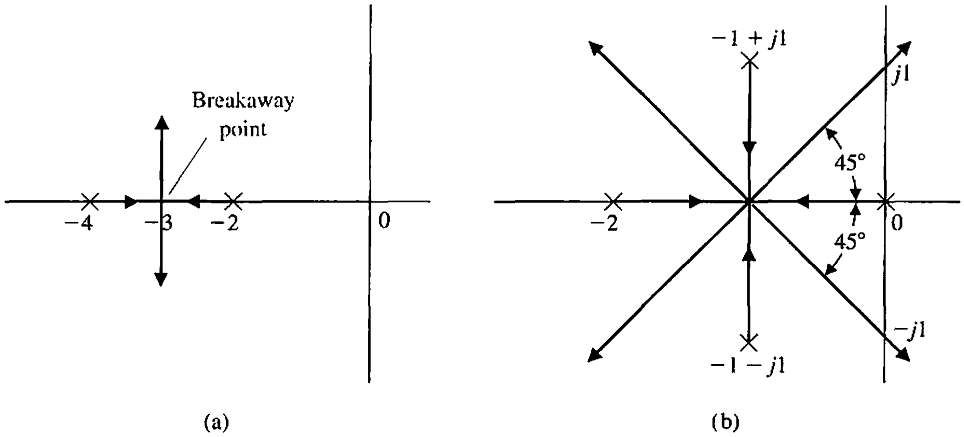


FIGURE 7.8 Illustration of the breakaway point (a) for a simple second-order system and (b) for a fourth-order system.

the rearranging of the characteristic equation to isolate the multiplying factor K. Then the characteristic equation is written as

$$p(s) = K. \tag{7.33}$$

For example, consider a unity feedback closed-loop system with an open-loop transfer function

$$G(s) = \frac{K}{(s + 2)(s + 4)},$$

which has the characteristic equation

$$1 + G(s) = 1 + \frac{K}{(s + 2)(s + 4)} = 0. \tag{7.34}$$

Alternatively, the equation may be written as

$$K = p(s) = -(s + 2)(s + 4). \tag{7.35}$$

The root loci for this system are shown in Figure 7.8(a). We expect the breakaway point to be near $s = \sigma = -3$ and plot $p(s)|_{s=\sigma}$ near that point, as shown in Figure 7.9. In this case, $p(s)$ equals zero at the poles $s = -2$ and $s = -4$. The plot of $p(s)$ versus $s - \sigma$ is symmetrical, and the maximum point occurs at $s = \sigma = -3$, the breakaway point.

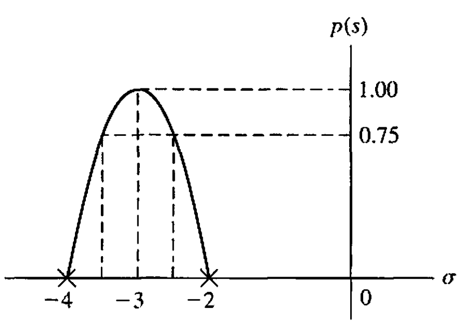


FIGURE 7.9 A graphical evaluation of the breakaway point.

Analytically, the very same result may be obtained by determining the maximum of $K = p(s)$. To find the maximum analytically, we differentiate, set the differentiated polynomial equal to zero, and determine the roots of the polynomial. Therefore, we may evaluate

$$\frac{dK}{ds} = \frac{dp(s)}{ds} = 0 \tag{7.36}$$

in order to find the breakaway point. Equation (7.36) is an analytical expression of the graphical procedure outlined in Figure 7.9 and will result in an equation of only one degree less than the total number of poles and zeros $n + M - 1$.

The proof of Equation (7.36) is obtained from a consideration of the characteristic equation

$$1 + F(s) = 1 + \frac{KY(s)}{X(s)} = 0,$$

which may be written as

$$X(s) + KY(s) = 0. \tag{7.37}$$

For a small increment in K, we have

$$X(s) + (K + \Delta K)Y(s) = 0.$$

Dividing by $X(s) + KY(s)$ yields

$$1 + \frac{\Delta KY(s)}{X(s) + KY(s)} = 0. \tag{7.38}$$

Because the denominator is the original characteristic equation, a multiplicity m of roots exists at a breakaway point, and

$$\frac{Y(s)}{X(s) + KY(s)} = \frac{C_i}{(s - s_i)^m} = \frac{C_i}{(\Delta s)^m}. \tag{7.39}$$

Then we may write Equation (7.38) as

$$1 + \frac{\Delta KC_i}{(\Delta s)^m} = 0, \tag{7.40}$$

or, alternatively,

$$\frac{\Delta K}{\Delta s} = \frac{-(\Delta s)^{m-1}}{C_i}. \tag{7.41}$$

Therefore, as we let Δs approach zero, we obtain

$$\frac{dK}{ds} = 0 \tag{7.42}$$

at the breakaway points.

Now, considering again the specific case where

$$G(s) = \frac{K}{(s + 2)(s + 4)},$$

we obtain

$$p(s) = K = -(s + 2)(s + 4) = -(s^2 + 6s + 8). \tag{7.43}$$

Then, when we differentiate, we have

$$\frac{dp(s)}{ds} = -(2s + 6) = 0, \tag{7.44}$$

or the breakaway point occurs at $s = -3$. A more complicated example will illustrate the approach and demonstrate the use of the graphical technique to determine the breakaway point.

EXAMPLE 7.3 **Third-order system**

A feedback control system is shown in Figure 7.10. The characteristic equation is

$$1 + G(s)H(s) = 1 + \frac{K(s + 1)}{s(s + 2)(s + 3)} = 0. \tag{7.45}$$

The number of poles n minus the number of zeros M is equal to 2, and so we have two asymptotes at $\pm 90°$ with a center at $\sigma_A = -2$. The asymptotes and the sections of loci on the real axis are shown in Figure 7.11(a). A breakaway point occurs between $s = -2$ and $s = -3$. To evaluate the breakaway point, we rewrite the characteristic equation so that K is separated; thus,

$$s(s + 2)(s + 3) + K(s + 1) = 0,$$

or

$$p(s) = \frac{-s(s + 2)(s + 3)}{s + 1} = K. \tag{7.46}$$

Then, evaluating $p(s)$ at various values of s between $s = -2$ and $s = -3$, we obtain the results of Table 7.1, as shown in Figure 7.11(b). Alternatively, we differentiate

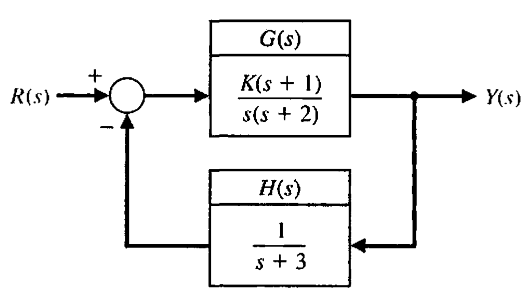


FIGURE 7.10 Closed-loop system.

Table 7.1

$p(s)$	0	0.411	0.419	0.417	+0.390	0
s	−2.00	−2.40	−2.46	−2.50	−2.60	−3.0

Equation (7.46) and set it equal to zero to obtain

$$\frac{d}{ds}\left(\frac{-s(s+2)(s+3)}{(s+1)}\right) = \frac{(s^3+5s^2+6s)-(s+1)(3s^2+10s+6)}{(s+1)^2} = 0$$

$$2s^3 + 8s^2 + 10s + 6 = 0. \tag{7.47}$$

Now to locate the maximum of $p(s)$, we locate the roots of Equation (7.47) to obtain $s = -2.46, -0.77 \pm 0.79j$. The only value of s on the real axis in the interval $s = -2$ to $s = -3$ is $s = -2.46$; hence this must be the breakaway point. It is evident from this one example that the numerical evaluation of $p(s)$ near the expected breakaway point provides an effective method of evaluating the breakaway point. ■

Step 6: Determine the angle of departure of the locus from a pole and the angle of arrival of the locus at a zero, using the phase angle criterion. The **angle of locus departure from a pole is the difference between the net angle due to all other poles and zeros and the criterion angle of ±180° (2k + 1),** and similarly for the locus angle of arrival at a zero. The angle of departure (or arrival) is particularly of interest for complex poles (and zeros) because the information is helpful in completing the root locus. For example, consider the third-order open-loop transfer function

$$F(s) = G(s)H(s) = \frac{K}{(s+p_3)(s^2+2\zeta\omega_n s+\omega_n^2)}. \tag{7.48}$$

The pole locations and the vector angles at one complex pole $-p_1$ are shown in Figure 7.12(a). The angles at a test point s_1, an infinitesimal distance from $-p_1$, must

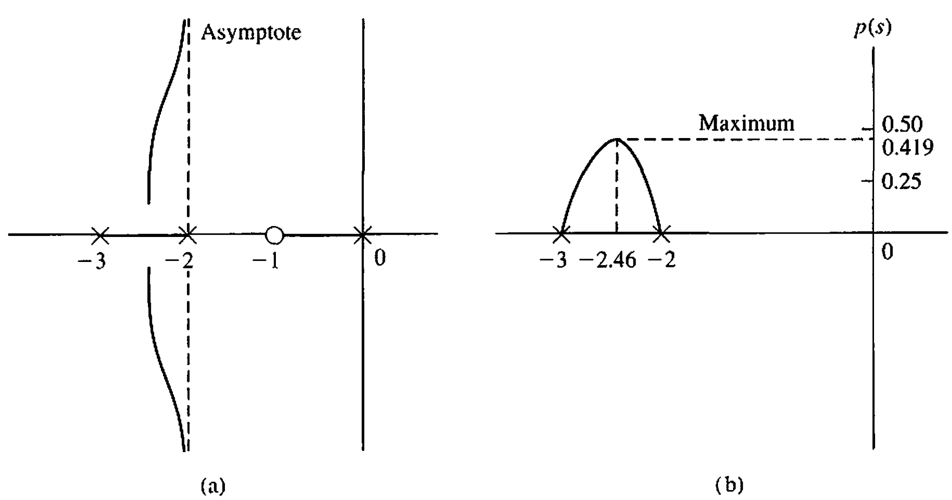


FIGURE 7.11 Evaluation of the (a) asymptotes and (b) breakaway point.

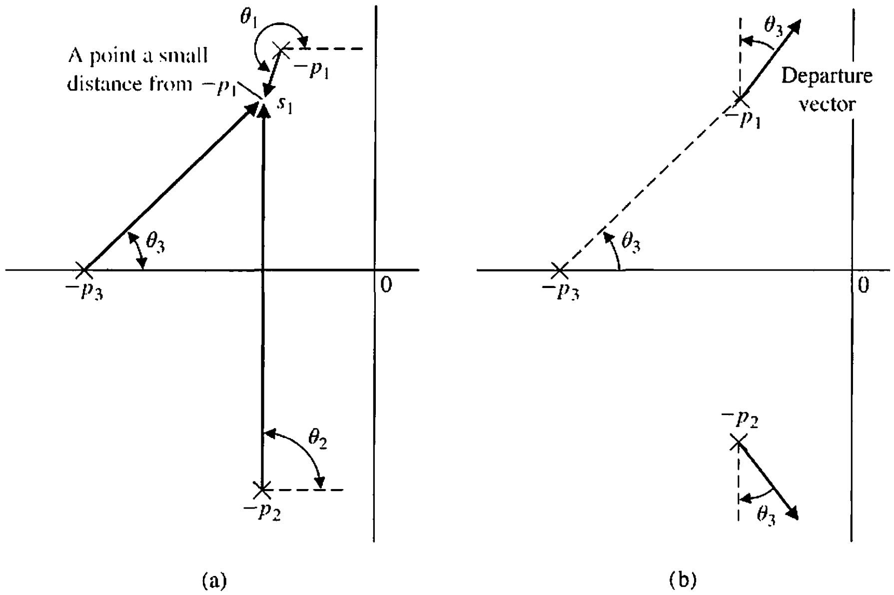


FIGURE 7.12 Illustration of the angle of departure. (a) Test point infinitesimal distance from $-p_1$. (b) Actual departure vector at $-p_1$.

meet the angle criterion. Therefore, since $\theta_2 = 90°$, we have

$$\theta_1 + \theta_2 + \theta_3 = \theta_1 + 90° + \theta_3 = +180°,$$

or the angle of departure at pole p_1 is

$$\theta_1 = 90° - \theta_3,$$

as shown in Figure 7.12(b). The departure at pole $-p_2$ is the negative of that at $-p_1$, because $-p_1$ and $-p_2$ are complex conjugates. Another example of a departure angle is shown in Figure 7.13. In this case, the departure angle is found from

$$\theta_2 - (\theta_1 + \theta_3 + 90°) = 180° + k360°.$$

Since $\theta_2 - \theta_3 = \gamma$ in the diagram, we find that the departure angle is $\theta_1 = 90° + \gamma$.

Step 7: The final step in the root locus sketching procedure is to complete the sketch. This entails sketching in all sections of the locus not covered in the previous

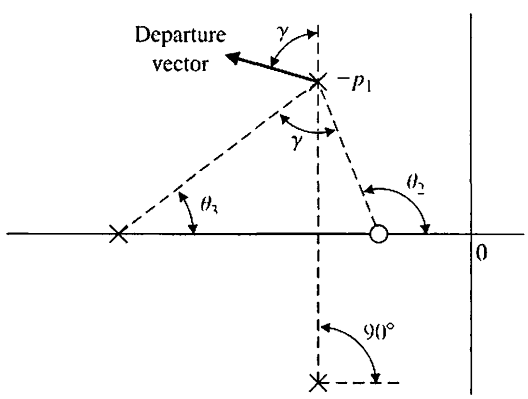


FIGURE 7.13 Evaluation of the angle of departure.

six steps. If a more detailed root locus is required, we recommend using a computer-aided tool. (See Section 7.8.)

In some situation, we may want to determine a root location s_x and the value of the parameter K_x at that root location. Determine the root locations that satisfy the phase criterion at the root s_x, $x = 1, 2, \ldots, n$, using the phase criterion. The phase criterion, given in Equation (7.17), is

$$\angle P(s) = 180° + k360°, \quad \text{and} \quad k = 0, \pm 1, \pm 2, \ldots.$$

To determine the parameter value K_x at a specific root s_x, we use the magnitude requirement (Equation 7.16). The magnitude requirement at s_x is

$$K_x = \left. \frac{\prod_{j=1}^{n} |s + p_j|}{\prod_{i=1}^{M} |s + z_i|} \right|_{s=s_x}.$$

It is worthwhile at this point to summarize the seven steps utilized in the root locus method (Table 7.2) and then illustrate their use in a complete example.

Table 7.2 Seven Steps for Sketching a Root Locus

Step	Related Equation or Rule
1. Prepare the root locus sketch.	
(a) Write the characteristic equation so that the parameter of interest, K, appears as a multiplier.	$1 + KP(s) = 0.$
(b) Factor $P(s)$ in terms of n poles and M zeros.	$1 + K\dfrac{\prod_{i=1}^{M}(s + z_i)}{\prod_{j=1}^{n}(s + p_j)} = 0.$
(c) Locate the open-loop poles and zeros of $P(s)$ in the s-plane with selected symbols.	× = poles, ○ = zeros Locus begins at a pole and ends at a zero.
(d) Determine the number of separate loci, SL.	$SL = n$ when $n \geqslant M$; n = number of finite poles, M = number of finite zeros.
(e) The root loci are symmetrical with respect to the horizontal real axis.	
2. Locate the segments of the real axis that are root loci.	Locus lies to the left of an odd number of poles and zeros.
3. The loci proceed to the zeros at infinity along asymptotes centered at σ_A and with angles ϕ_A.	$\sigma_A = \dfrac{\sum(-p_j) - \sum(-z_i)}{n - M}.$ $\phi_A = \dfrac{2k + 1}{n - M}180°, k = 0, 1, 2, \ldots (n - M - 1).$
4. Determine the points at which the locus crosses the imaginary axis (if it does so).	Use Routh–Hurwitz criterion (see Section 6.2).
5. Determine the breakaway point on the real axis (if any).	a) Set $K = p(s)$. b) Determine roots of $dp(s)/ds = 0$ or use graphical method to find maximum of $p(s)$.
6. Determine the angle of locus departure from complex poles and the angle of locus arrival at complex zeros, using the phase criterion.	$\angle P(s) = 180° + k360°$ at $s = -p_j$ or $-z_i$.
7. Complete the root locus sketch.	

EXAMPLE 7.4 **Fourth-order system**

1. (a) We desire to plot the root locus for the characteristic equation of a system as K varies for $K > 0$ when

$$1 + \frac{K}{s^4 + 12s^3 + 64s^2 + 128s} = 0.$$

(b) Determining the poles, we have

$$1 + \frac{K}{s(s+4)(s+4+j4)(s+4-j4)} = 0 \qquad (7.49)$$

as K varies from zero to infinity. This system has no finite zeros.

(c) The poles are located on the s-plane as shown in Figure 7.14(a).

(d) Because the number of poles n is equal to 4, we have four separate loci.

(e) The root loci are symmetrical with respect to the real axis.

2. A segment of the root locus exists on the real axis between $s = 0$ and $s = -4$.
3. The angles of the asymptotes are

$$\phi_A = \frac{(2k+1)}{4}180°, \qquad k = 0, 1, 2, 3;$$

$$\phi_A = +45°, 135°, 225°, 315°.$$

The center of the asymptotes is

$$\sigma_A = \frac{-4-4-4}{4} = -3.$$

Then the asymptotes are drawn as shown in Figure 7.14(a).

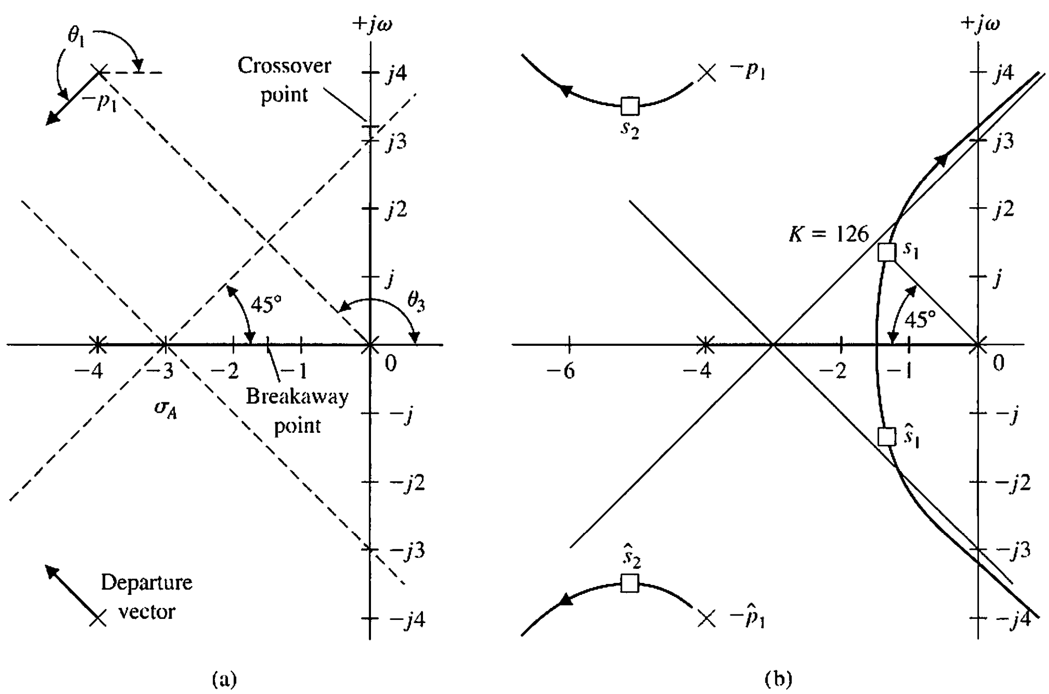


FIGURE 7.14 The root locus for Example 7.4. Locating (a) the poles and (b) the asymptotes.

4. The characteristic equation is rewritten as

$$s(s + 4)(s^2 + 8s + 32) + K = s^4 + 12s^3 + 64s^2 + 128s + K = 0. \qquad (7.50)$$

Therefore, the Routh array is

$$\begin{array}{c|ccc} s^4 & 1 & 64 & K \\ s^3 & 12 & 128 & \\ s^2 & b_1 & K & \\ s^1 & c_1 & & \\ s^0 & K & & \end{array},$$

where

$$b_1 = \frac{12(64) - 128}{12} = 53.33 \quad \text{and} \quad c_1 = \frac{53.33(128) - 12K}{53.33}.$$

Hence, the limiting value of gain for stability is $K = 568.89$, and the roots of the auxiliary equation are

$$53.33s^2 + 568.89 = 53.33(s^2 + 10.67) = 53.33(s + j3.266)(s - j3.266). \qquad (7.51)$$

The points where the locus crosses the imaginary axis are shown in Figure 7.14(a). Therefore, when $K = 568.89$, the root locus crosses the $j\omega$-axis at $s = \pm j3.266$.

5. The breakaway point is estimated by evaluating

$$K = p(s) = -s(s + 4)(s + 4 + j4)(s + 4 - j4)$$

between $s = -4$ and $s = 0$. We expect the breakaway point to lie between $s = -3$ and $s = -1$, so we search for a maximum value of $p(s)$ in that region. The resulting values of $p(s)$ for several values of s are given in Table 7.3. The maximum of $p(s)$ is found to lie at approximately $s = -1.577$, as indicated in the table. A more accurate estimate of the breakaway point is normally not necessary. The breakaway point is then indicated on Figure 7.14(a).

6. The angle of departure at the complex pole p_1 can be estimated by utilizing the angle criterion as follows:

$$\theta_1 + 90° + 90° + \theta_3 = 180° + k360°.$$

Here, θ_3 is the angle subtended by the vector from pole p_3. The angles from the pole at $s = -4$ and $s = -4 - j4$ are each equal to 90°. Since $\theta_3 = 135°$, we find that

$$\theta_1 = -135° \equiv +225°,$$

as shown in Figure 7.14(a).

7. Complete the sketch as shown in Figure 7.14(b).

Table 7.3

$p(s)$	0	51.0	68.44	80.0	83.57	75.0	0
s	−4.0	−3.0	−2.5	−2.0	−1.577	−1.0	0

Using the information derived from the seven steps of the root locus method, the complete root locus sketch is obtained by filling in the sketch as well as possible by visual inspection. The root locus for this system is shown in Figure 7.14(b). When the complex roots near the origin have a damping ratio of $\zeta = 0.707$, the gain K can be determined graphically as shown in Figure 7.14(b). The vector lengths to the root location s_1 from the open-loop poles are evaluated and result in a gain at s_1 of

$$K = |s_1||s_1 + 4||s_1 - p_1||s_1 - \hat{p}_1| = (1.9)(2.9)(3.8)(6.0) = 126. \quad (7.52)$$

The remaining pair of complex roots occurs at s_2 and $\hat{s}_2$, when $K = 126$. The effect of the complex roots at s_2 and $\hat{s}_2$ on the transient response will be negligible compared to the roots s_1 and $\hat{s}_1$. This fact can be ascertained by considering the damping of the response due to each pair of roots. The damping due to s_1 and $\hat{s}_1$ is

$$e^{-\zeta_1 \omega_{n_1} t} = e^{-\sigma_1 t},$$

and the damping factor due to s_2 and $\hat{s}_2$ is

$$e^{-\zeta_2 \omega_{n_2} t} = e^{-\sigma_2 t},$$

where σ_2 is approximately five times as large as σ_1. Therefore, the transient response term due to s_2 will decay much more rapidly than the transient response term due to s_1. Thus, the response to a unit step input may be written as

$$\begin{aligned} y(t) &= 1 + c_1 e^{-\sigma_1 t} \sin(\omega_1 t + \theta_1) + c_2 e^{-\sigma_2 t} \sin(\omega_2 t + \theta_2) \\ &\approx 1 + c_1 e^{-\sigma_1 t} \sin(\omega_1 t + \theta_1). \end{aligned} \quad (7.53)$$

The complex conjugate roots near the origin of the s-plane relative to the other roots of the closed-loop system are labeled the **dominant roots** of the system because they represent or dominate the transient response. The relative dominance of the complex roots, in a third-order system with a pair of complex conjugate roots, is determined by the ratio of the real root to the real part of the complex roots and will result in approximate dominance for ratios exceeding 5.

The dominance of the second term of Equation (7.53) also depends upon the relative magnitudes of the coefficients c_1 and c_2. These coefficients, which are the residues evaluated at the complex roots, in turn depend upon the location of the zeros in the s-plane. Therefore, the concept of dominant roots is useful for estimating the response of a system, but must be used with caution and with a comprehension of the underlying assumptions. ■

EXAMPLE 7.5 **Automatic self-balancing scale**

The analysis and design of a control system can be accomplished by using the Laplace transform, a signal-flow diagram or block diagram, the s-plane, and the root locus method. At this point, it will be worthwhile to examine a control system and select suitable parameter values based on the root locus method.

Figure 7.15 shows an automatic self-balancing scale in which the weighing operation is controlled by the physical balance function through an electrical feedback loop [5]. The balance is shown in the equilibrium condition, and x is the travel of the counterweight W_c from an unloaded equilibrium condition. The weight W to be

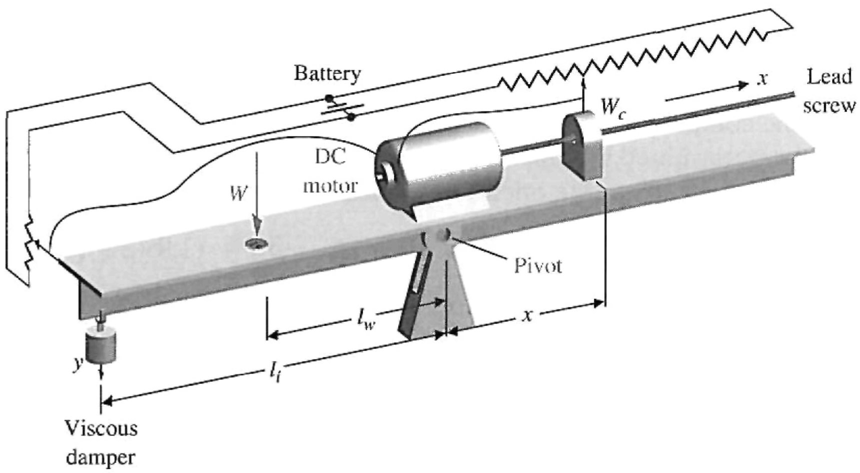


FIGURE 7.15 An automatic self-balancing scale. (Reprinted with permission from J. H. Goldberg, *Automatic Controls*, Allyn and Bacon, Boston, 1964.)

measured is applied 5 cm from the pivot, and the length l_i of the beam to the viscous damper is 20 cm. We desire to accomplish the following:

1. Select the parameters and the specifications of the feedback system.
2. Obtain a model representing the system.
3. Select the gain K based on a root locus diagram.
4. Determine the dominant mode of response.

An inertia of the beam equal to 0.05 kg m^2 will be chosen. We must select a battery voltage that is large enough to provide a reasonable position sensor gain, so we will choose $E_b = 24$ volts. We will use a lead screw of 20 turns/cm and a potentiometer for x equal to 6 cm in length. Accurate balances are required; therefore, an input potentiometer 0.5 cm in length for y will be chosen. A reasonable viscous damper will be chosen with a damping constant $b = 10\sqrt{3}\ N/(m/s)$. Finally, a counterweight W_c is chosen so that the expected range of weights W can be balanced. The parameters of the system are selected as listed in Table 7.4.

Specifications. A rapid and accurate response resulting in a small steady-state weight measurement error is desired. Therefore, we will require that the system be at least a type one so that a zero measurement error is obtained. An underdamped response to a step change in the measured weight W is satisfactory, so a dominant response with $\zeta = 0.5$ will be specified. We want the settling time to be less than

Table 7.4 Self-Balancing Scale Parameters

$W_c = 2$ N	Lead screw gain $K_s = \frac{1}{4000\pi}$ m/rad.
$I = 0.05$ kg m^2	
$l_w = 5$ cm	Input potentiometer gain $K_i = 4800$ V/m.
$l_i = 20$ cm	
$b = 10\sqrt{3}$ N m/s	Feedback potentiometer gain $K_f = 400$ V/m.

Table 7.5 Specifications

Steady-state error	$K_p = \infty$, $e_{ss} = 0$ for a step input
Underdamped response	$\zeta = 0.5$
Settling time (2% criterion)	Less than 2 seconds

2 seconds in order to provide a rapid weight-measuring device. The settling time must be within 2% of the final value of the balance following the introduction of a weight to be measured. The specifications are summarized in Table 7.5.

The derivation of a model of the electromechanical system may be accomplished by obtaining the equations of motion of the balance. For small deviations from balance, the deviation angle is

$$\theta \approx \frac{y}{l_i}. \tag{7.54}$$

The motion of the beam about the pivot is represented by the torque equation

$$I\frac{d^2\theta}{dt^2} = \sum \text{torques}.$$

Therefore, in terms of the deviation angle, the motion is represented by

$$I\frac{d^2\theta}{dt^2} = l_w W - xW_c - l_i^2 b\frac{d\theta}{dt}. \tag{7.55}$$

The input voltage to the motor is

$$v_m(t) = K_i y - K_f x. \tag{7.56}$$

The lead screw motion and transfer function of the motor are described by

$$X(s) = K_s\theta_m(s) \quad \text{and} \quad \frac{\theta_m(s)}{V_m(s)} = \frac{K_m}{s(\tau s + 1)}, \tag{7.57}$$

where τ will be negligible with respect to the time constants of the overall system, and θ_m is the output shaft rotation. A signal-flow graph and block diagram representing Equations (7.54) through (7.57) is shown in Figure 7.16. Examining the forward path from W to $X(s)$, we find that the system is a type one due to the integration preceding $Y(s)$. Therefore, the steady-state error of the system is zero.

The closed-loop transfer function of the system is obtained by utilizing Mason's signal-flow gain formula and is found to be

$$\frac{X(s)}{W(s)} = \frac{l_w l_i K_i K_m K_s/(Is^3)}{1 + l_i^2 b/(Is) + (K_m K_s K_f/s) + l_i K_i K_m K_s W_c/(Is^3) + l_i^2 b K_m K_s K_f/(Is^2)}, \tag{7.58}$$

where the numerator is the path factor from W to X, the second term in the denominator is the loop L_1, the third term is the loop factor L_2, the fourth term is the loop

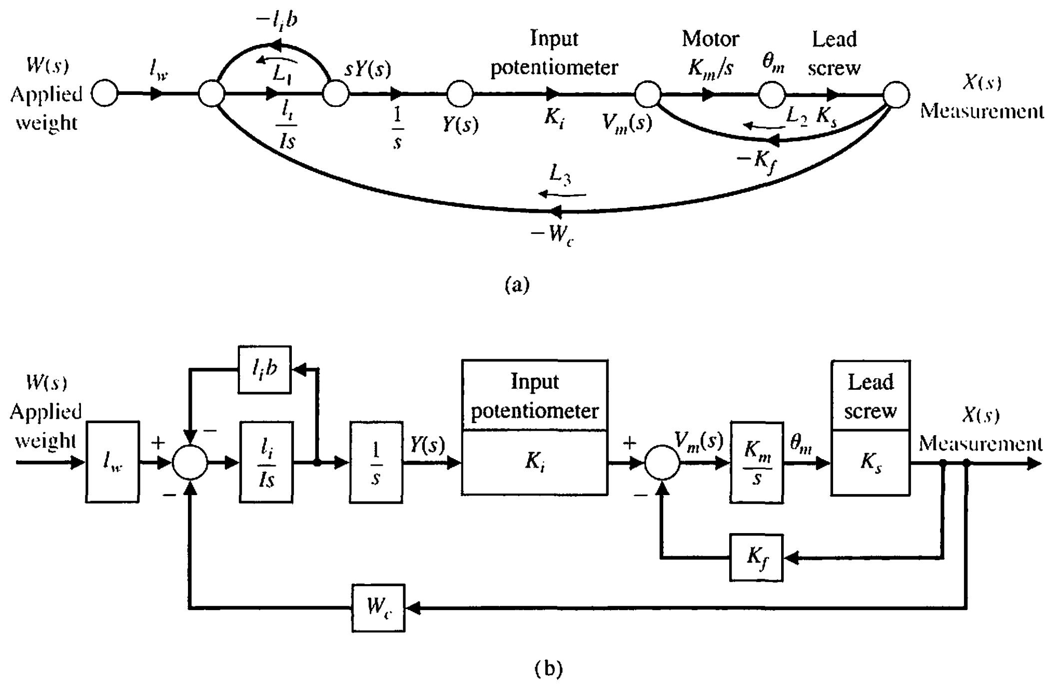


FIGURE 7.16 Model of the automatic self-balancing scale. (a) Signal-flow graph. (b) Block diagram.

L_3, and the fifth term is the two nontouching loops L_1L_2. Therefore, the closed-loop transfer function is

$$\frac{X(s)}{W(s)} = \frac{l_w l_i K_i K_m K_s}{s(Is + l_i^2 b)(s + K_m K_s K_f) + W_c K_m K_s K_i l_i}. \tag{7.59}$$

The steady-state gain of the system is then

$$\lim_{t\to\infty} \frac{x(t)}{|W|} = \lim_{s\to 0} \frac{X(s)}{W(s)} = \frac{l_w}{W_c} = 2.5 \text{ cm/kg} \tag{7.60}$$

when $W(s) = |W|/s$. To obtain the root locus as a function of the motor constant K_m, we substitute the selected parameters into the characteristic equation, which is the denominator of Equation (7.59). Therefore, we obtain the following characteristic equation:

$$s\left(s + 8\sqrt{3}\right)\left(s + \frac{K_m}{10\pi}\right) + \frac{96K_m}{10\pi} = 0. \tag{7.61}$$

Rewriting the characteristic equation in root locus form, we first isolate K_m as follows:

$$s^2\left(s + 8\sqrt{3}\right) + s\left(s + 8\sqrt{3}\right)\frac{K_m}{10\pi} + \frac{96K_m}{10\pi} = 0. \tag{7.62}$$

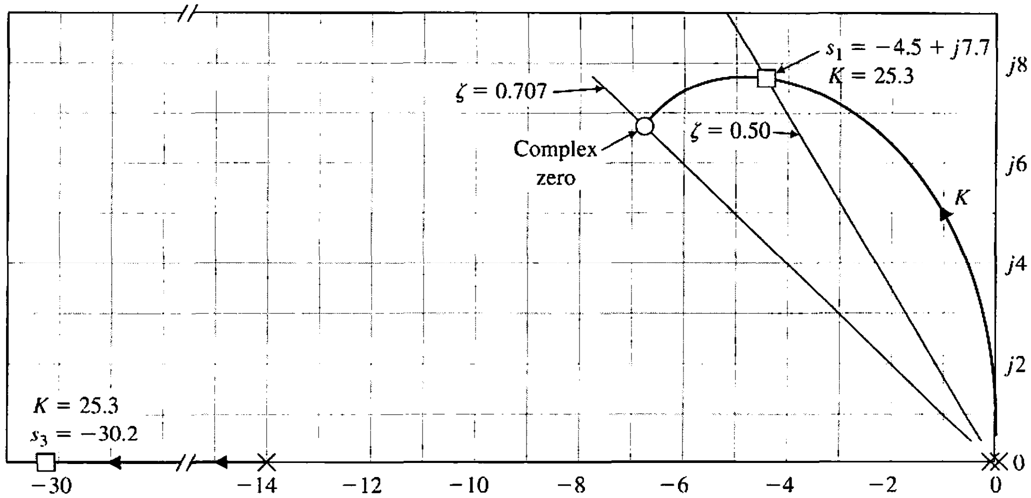


FIGURE 7.17 Root locus as K_m varies (only upper halfplane shown). One locus leaves the two poles at the origin and goes to the two complex zeros as K increases. The other locus is to the left of the pole at $s = -14$.

Then, rewriting Equation (7.62) in root locus form, we have

$$1 + KP(s) = 1 + \frac{K_m/(10\pi)\left[s\left(s + 8\sqrt{3}\right) + 96\right]}{s^2\left(s + 8\sqrt{3}\right)} = 0$$

$$= 1 + \frac{K_m/(10\pi)(s + 6.93 + j6.93)(s + 6.93 - j6.93)}{s^2\left(s + 8\sqrt{3}\right)}. \quad (7.63)$$

The root locus as K_m varies is shown in Figure 7.17. The dominant roots can be placed at $\zeta = 0.5$ when $K = 25.3 = K_m/10\pi$. To achieve this gain,

$$K_m = 795\frac{\text{rad/s}}{\text{volt}} = 7600\frac{\text{rpm}}{\text{volt}}, \quad (7.64)$$

an amplifier would be required to provide a portion of the required gain. The real part of the dominant roots is less than −4; therefore, the settling time, $4/\sigma$, is less than 1 second, and the settling time requirement is satisfied. The third root of the characteristic equation is a real root at $s = -30.2$, and the underdamped roots clearly dominate the response. Therefore, the system has been analyzed by the root locus method and a suitable design for the parameter K_m has been achieved. The efficiency of the s-plane and root locus methods is clearly demonstrated by this example. ■

7.4 PARAMETER DESIGN BY THE ROOT LOCUS METHOD

Originally, the root locus method was developed to determine the locus of roots of the characteristic equation as the system gain, K, is varied from zero to infinity. However, as we have seen, the effect of other system parameters may be readily investigated by using the root locus method. Fundamentally, the root locus method is concerned with a characteristic equation (Equation 7.22), which may be written as

$$1 + F(s) = 0. \tag{7.65}$$

Then the standard root locus method we have studied may be applied. The question arises: How do we investigate the effect of two parameters, α and β? It appears that the root locus method is a single-parameter method; fortunately, it can be readily extended to the investigation of two or more parameters. This method of **parameter design** uses the root locus approach to select the values of the parameters.

The characteristic equation of a dynamic system may be written as

$$a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0. \tag{7.66}$$

Hence, the effect of the coefficient a_1 may be ascertained from the root locus equation

$$1 + \frac{a_1 s}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_2 s^2 + a_0} = 0. \tag{7.67}$$

If the parameter of interest, α, does not appear solely as a coefficient, the parameter may be isolated as

$$a_n s^n + a_{n-1} s^{n-1} + \cdots + (a_{n-q} - \alpha) s^{n-q} + \alpha s^{n-q} + \cdots + a_1 s + a_0 = 0. \tag{7.68}$$

For example, a third-order equation of interest might be

$$s^3 + (3 + \alpha) s^2 + 3s + 6 = 0. \tag{7.69}$$

To ascertain the effect of the parameter α, we isolate the parameter and rewrite the equation in root locus form, as shown in the following steps:

$$s^3 + 3s^2 + \alpha s^2 + 3s + 6 = 0; \tag{7.70}$$

$$1 + \frac{\alpha s^2}{s^3 + 3s^2 + 3s + 6} = 0. \tag{7.71}$$

Then, to determine the effect of two parameters, we must repeat the root locus approach twice. Thus, for a characteristic equation with two variable parameters, α and β, we have

$$a_n s^n + a_{n-1} s^{n-1} + \cdots + (a_{n-q} - \alpha) s^{n-q} + \alpha s^{n-q} + \cdots + (a_{n-r} - \beta) s^{n-r} + \beta s^{n-r} + \cdots + a_1 s + a_0 = 0. \tag{7.72}$$

The two variable parameters have been isolated, and the effect of α will be determined. Then, the effect of β will be determined. For example, for a certain third-order characteristic equation with α and β as parameters, we obtain

$$s^3 + s^2 + \beta s + \alpha = 0. \tag{7.73}$$

In this particular case, the parameters appear as the coefficients of the characteristic equation. The effect of varying β from zero to infinity is determined from the root locus equation

$$1 + \frac{\beta s}{s^3 + s^2 + \alpha} = 0. \tag{7.74}$$

We note that the denominator of Equation (7.74) is the characteristic equation of the system with $\beta = 0$. Therefore, we must first evaluate the effect of varying α from zero to infinity by using the equation

$$s^3 + s^2 + \alpha = 0,$$

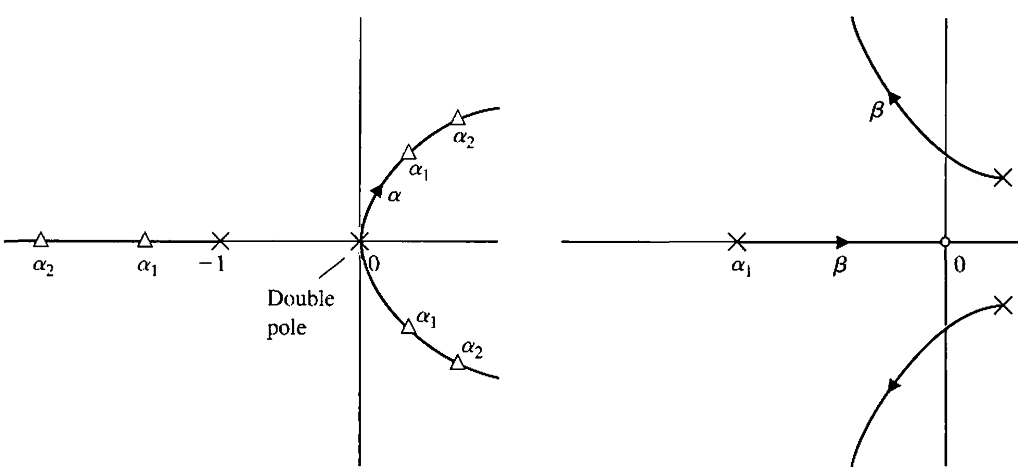
rewritten as

$$1 + \frac{\alpha}{s^2(s + 1)} = 0, \tag{7.75}$$

where β has been set equal to zero in Equation (7.73). Then, upon evaluating the effect of α, a value of α is selected and used with Equation (7.74) to evaluate the effect of β. This two-step method of evaluating the effect of α and then β may be carried out as two root locus procedures. First, we obtain a locus of roots as α varies, and we select a suitable value of α; the results are satisfactory root locations. Then, we obtain the root locus for β by noting that the poles of Equation (7.74) are the roots evaluated by the root locus of Equation (7.75). A limitation of this approach is that we will not always be able to obtain a characteristic equation that is linear in the parameter under consideration (for example, α).

To illustrate this approach effectively, let us obtain the root locus for α and then β for Equation (7.73). A sketch of the root locus as α varies for Equation (7.75) is shown in Figure 7.18(a), where the roots for two values of gain α are shown. If the gain α is selected as α_1, then the resultant roots of Equation (7.75) become the poles of Equation (7.74). The root locus of Equation (7.74) as β varies is shown in Figure 7.18(b), and a suitable β can be selected on the basis of the desired root locations.

Using the root locus method, we will further illustrate this parameter design approach by a specific design example.



(a) (b)

FIGURE 7.18 Root loci as a function of α and β. (a) Loci as α varies. (b) Loci as β varies for one value of $\alpha = \alpha_1$.

EXAMPLE 7.6 **Welding head control**

A welding head for an auto body requires an accurate control system for positioning the welding head [4]. The feedback control system is to be designed to satisfy the following specifications:

1. Steady-state error for a ramp input $\leq 35\%$ of input slope
2. Damping ratio of dominant roots ≥ 0.707
3. Settling time to within 2% of the final value ≤ 3 seconds

The structure of the feedback control system is shown in Figure 7.19, where the amplifier gain K_1 and the derivative feedback gain K_2 are to be selected. The steady-state error specification can be written as

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s(|R|/s^2)}{1 + G_2(s)}, \tag{7.76}$$

where $G_2(s) = G(s)/(1 + G(s)H_1(s))$. Therefore, the steady-state error requirement is

$$\frac{e_{ss}}{|R|} = \frac{2 + K_1K_2}{K_1} \leq 0.35. \tag{7.77}$$

Thus, we will select a small value of K_2 to achieve a low value of steady-state error. The damping ratio specification requires that the roots of the closed-loop system be below the line at 45° in the left-hand s-plane. The settling time specification can be rewritten in terms of the real part of the dominant roots as

$$T_s = \frac{4}{\sigma} \leq 3 \text{ s}. \tag{7.78}$$

Therefore, it is necessary that $\sigma \geq 4/3$; this area in the left-hand s-plane is indicated along with the ζ-requirement in Figure 7.20. Note that $\sigma \geq 4/3$ implies that we want the dominant roots to lie to the left of the line defined by $\sigma = -4/3$. To satisfy the specifications, all the roots must lie within the shaded area of the left-hand plane.

The parameters to be selected are $\alpha = K_1$ and $\beta = K_2K_1$. The characteristic equation is

$$1 + GH(s) = s^2 + 2s + \beta s + \alpha = 0. \tag{7.79}$$

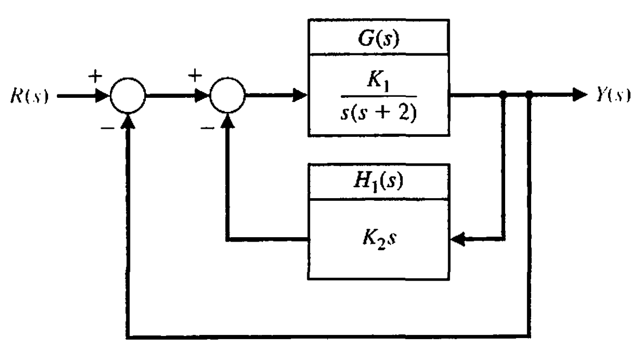


FIGURE 7.19 Block diagram of welding head control system.

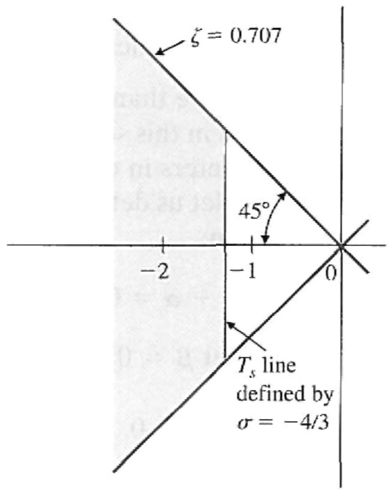


FIGURE 7.20 A region in the s-plane for desired root location.

The locus of roots as $\alpha = K_1$ varies (set $\beta = 0$) is determined from the equation

$$1 + \frac{\alpha}{s(s + 2)} = 0, \tag{7.80}$$

as shown in Figure 7.21(a). For a gain of $K_1 = \alpha = 20$, the roots are indicated on the locus. Then the effect of varying $\beta = 20K_2$ is determined from the locus equation

$$1 + \frac{\beta s}{s^2 + 2s + \alpha} = 0, \tag{7.81}$$

where the poles of this root locus are the roots of the locus of Figure 7.21(a). The root locus for Equation (7.81) is shown in Figure 7.21(b), and roots with $\zeta = 0.707$ are obtained when $\beta = 4.3 = 20K_2$ or when $K_2 = 0.215$. The real part of these roots is

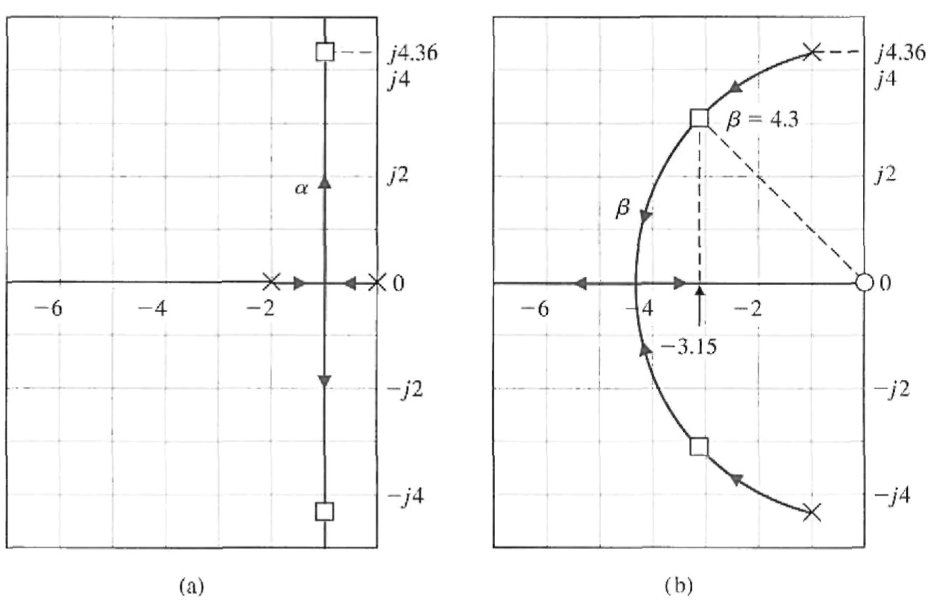


FIGURE 7.21 Root loci as a function of (a) α and (b) β.

$\sigma = -3.15$; therefore, the time to settle (to within 2% of the final value) is equal to 1.27 seconds, which is considerably less than the specification of 3 seconds. ∎

We can extend the root locus method to more than two parameters by extending the number of steps in the method outlined in this section. Furthermore, a family of root loci can be generated for two parameters in order to determine the total effect of varying two parameters. For example, let us determine the effect of varying α and β of the following characteristic equation:

$$s^3 + 3s^2 + 2s + \beta s + \alpha = 0. \tag{7.82}$$

The root locus equation as a function of α is (set $\beta = 0$)

$$1 + \frac{\alpha}{s(s+1)(s+2)} = 0. \tag{7.83}$$

The root locus as a function of β is

$$1 + \frac{\beta s}{s^3 + 3s^2 + 2s + \alpha} = 0. \tag{7.84}$$

The root locus for Equation (7.83) as a function of α is shown in Figure 7.22 (unbroken lines). The roots of this locus, indicated by slashes, become the poles for the locus of Equation (7.84). Then the locus of Equation (7.84) is continued on Figure 7.22 (dotted lines), where the locus for β is shown for several selected values of α. This family of loci, often called **root contours**, illustrates the effect of α and β on the roots of the characteristic equation of a system [3].

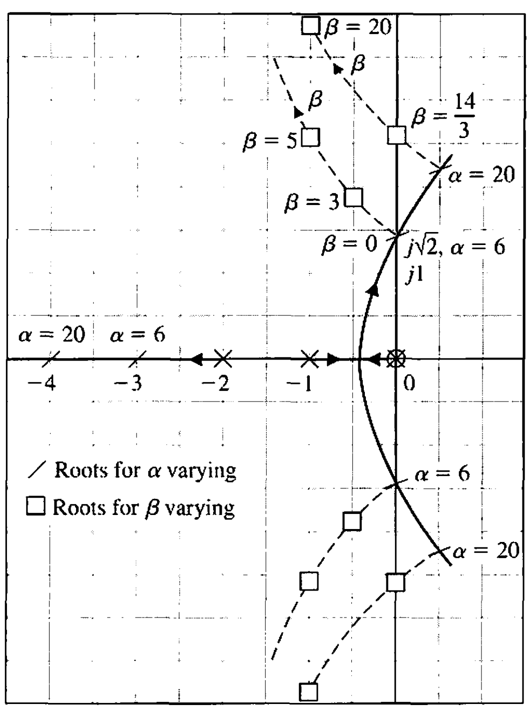


FIGURE 7.22 Two-parameter root locus. The loci for α varying are solid; the loci for β varying are dashed.

7.5 SENSITIVITY AND THE ROOT LOCUS

One of the prime reasons for the use of negative feedback in control systems is the reduction of the effect of parameter variations. The effect of parameter variations, as we found in Section 4.3, can be described by a measure of the sensitivity of the system performance to specific parameter changes. In Section 4.3, we defined the **logarithmic sensitivity** originally suggested by Bode as

$$S_K^T = \frac{\partial \ln T}{\partial \ln K} = \frac{\partial T/T}{\partial K/K}, \tag{7.85}$$

where the system transfer function is $T(s)$ and the parameter of interest is K.

In recent years, there has been an increased use of the pole–zero (s-plane) approach. Therefore, it has become useful to define a sensitivity measure in terms of the positions of the roots of the characteristic equation [7–9]. Because these roots represent the dominant modes of transient response, the effect of parameter variations on the position of the roots is an important and useful measure of the sensitivity. The **root sensitivity** of a system $T(s)$ can be defined as

$$\boxed{S_K^{r_i} = \frac{\partial r_i}{\partial \ln K} = \frac{\partial r_i}{\partial K/K},} \tag{7.86}$$

where r_i equals the ith root of the system, so that

$$T(s) = \frac{K_1 \prod_{j=1}^{M}(s + z_j)}{\prod_{i=1}^{n}(s + r_i)} \tag{7.87}$$

and K is a parameter affecting the roots. The root sensitivity relates the changes in the location of the root in the s-plane to the change in the parameter. The root sensitivity is related to the logarithmic sensitivity by the relation

$$S_K^T = \frac{\partial \ln K_1}{\partial \ln K} - \sum_{i=1}^{n} \frac{\partial r_i}{\partial \ln K} \cdot \frac{1}{s + r_i} \tag{7.88}$$

when the zeros of $T(s)$ are independent of the parameter K, so that

$$\frac{\partial z_j}{\partial \ln K} = 0.$$

This logarithmic sensitivity can be readily obtained by determining the derivative of $T(s)$, Equation (7.87), with respect to K. For this particular case, when the gain of the system is independent of the parameter K, we have

$$S_K^T = -\sum_{i=1}^{n} S_K^{r_i} \cdot \frac{1}{s + r_i}, \tag{7.89}$$

and the two sensitivity measures are directly related.

The evaluation of the root sensitivity for a control system can be readily accomplished by utilizing the root locus methods of the preceding section. The root sensitivity $S_K^{r_i}$ may be evaluated at root $-r_i$ by examining the root contours for the parameter K. We can change K by a small finite amount ΔK and determine the modified root $-(r_i + \Delta r_i)$ at $K + \Delta K$. Then, using Equation (7.86), we have

$$S_K^{r_i} \approx \frac{\Delta r_i}{\Delta K/K}. \tag{7.90}$$

Equation (7.90) is an approximation that approaches the actual value of the sensitivity as $\Delta K \rightarrow 0$. An example will illustrate the process of evaluating the root sensitivity.

EXAMPLE 7.7 **Root sensitivity of a control system**

The characteristic equation of the feedback control system shown in Figure 7.23 is

$$1 + \frac{K}{s(s+\beta)} = 0,$$

or, alternatively,

$$s^2 + \beta s + K = 0. \tag{7.91}$$

The gain K will be considered to be the parameter α. Then the effect of a change in each parameter can be determined by utilizing the relations

$$\alpha = \alpha_0 \pm \Delta\alpha \quad \text{and} \quad \beta = \beta_0 \pm \Delta\beta,$$

where α_0 and β_0 are the nominal or desired values for the parameters α and β, respectively. We shall consider the case when the nominal pole value is $\beta_0 = 1$ and the desired gain is $\alpha_0 = K = 0.5$. Then the root locus can be obtained as a function of $\alpha = K$ by utilizing the root locus equation

$$1 + \frac{K}{s(s+\beta_0)} = 1 + \frac{K}{s(s+1)} = 0, \tag{7.92}$$

as shown in Figure 7.24. The nominal value of gain $K = \alpha_0 = 0.5$ results in two complex roots, $-r_1 = -0.5 + j0.5$ and $-r_2 = -\hat{r}_1$, as shown in Figure 7.24. To evaluate the effect of unavoidable changes in the gain, the characteristic equation with $\alpha = \alpha_0 \pm \Delta\alpha$ becomes

$$s^2 + s + \alpha_0 \pm \Delta\alpha = s^2 + s + 0.5 \pm \Delta\alpha. \tag{7.93}$$

Therefore, the effect of changes in the gain can be evaluated from the root locus of Figure 7.24. For a 20% change in α, we have $\Delta\alpha = \pm 0.1$. The root locations for a

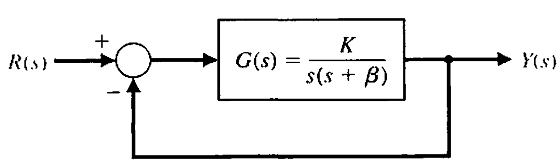


FIGURE 7.23
A feedback control system.

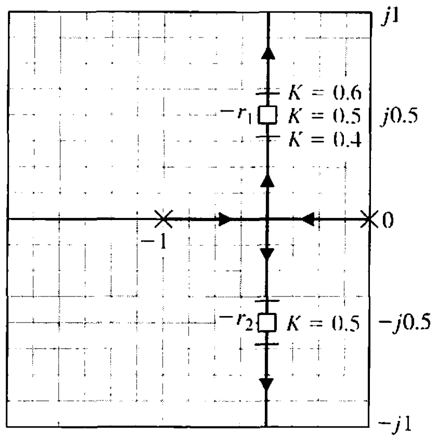


FIGURE 7.24 The root locus for K.

gain $\alpha = 0.4$ and $\alpha = 0.6$ are readily determined by root locus methods, and the root locations for $\Delta\alpha = \pm 0.1$ are shown in Figure 7.24. When $\alpha = K = 0.6$, the root in the second quadrant of the s-plane is

$$(-r_1) + \Delta r_1 = -0.5 + j0.59,$$

and the change in the root is $\Delta r_1 = +j0.09$. When $\alpha = K = 0.4$, the root in the second quadrant is

$$-(r_1) + \Delta r_1 = -0.5 + j0.387,$$

and the change in the root is $-\Delta r_1 = -j0.11$. Thus, the root sensitivity for r_1 is

$$S_{K+}^{r_1} = \frac{\Delta r_1}{\Delta K/K} = \frac{+j0.09}{+0.2} = j0.45 = 0.45\angle{+90^\circ} \tag{7.94}$$

for positive changes of gain. For negative increments of gain, the sensitivity is

$$S_{K-}^{r_1} = \frac{\Delta r_1}{\Delta K/K} = \frac{-j0.11}{+0.2} = -j0.55 = 0.55\angle{-90^\circ}.$$

For infinitesimally small changes in the parameter K, the sensitivity will be equal for negative or positive increments in K. The angle of the root sensitivity indicates the direction the root moves as the parameter varies. The angle of movement for $+\Delta\alpha$ is always 180° from the angle of movement for $-\Delta\alpha$ at the point $\alpha = \alpha_0$.

The pole β varies due to environmental changes, and it may be represented by $\beta = \beta_0 + \Delta\beta$, where $\beta_0 = 1$. Then the effect of variation of the poles is represented by the characteristic equation

$$s^2 + s + \Delta\beta s + K = 0,$$

or, in root locus form,

$$1 + \frac{\Delta\beta s}{s^2 + s + K} = 0. \tag{7.95}$$

The denominator of the second term is the unchanged characteristic equation when $\Delta\beta = 0$. The root locus for the unchanged system ($\Delta\beta = 0$) is shown in Figure 7.24 as a function of K. For a design specification requiring $\zeta = 0.707$, the complex roots lie at

$$-r_1 = -0.5 + j0.5 \quad \text{and} \quad -r_2 = -\hat{r}_1 = -0.5 - j0.5.$$

Then, because the roots are complex conjugates, the root sensitivity for r_1 is the conjugate of the root sensitivity for $\hat{r}_1 = r_2$. Using the parameter root locus techniques discussed in the preceding section, we obtain the root locus for $\Delta\beta$ as shown in Figure 7.25. We are normally interested in the effect of a variation for the parameter so that $\beta = \beta_0 \pm \Delta\beta$, for which the locus as β decreases is obtained from the root locus equation

$$1 + \frac{-(\Delta\beta)s}{s^2 + s + K} = 0.$$

We note that the equation is of the form

$$1 - \Delta\beta P(s) = 0.$$

Comparing this equation with Equation (7.23) in Section 7.3, we find that the sign preceding the gain $\Delta\beta$ is negative in this case. In a manner similar to the development of the root locus method in Section 7.3, we require that the root locus satisfy the equations

$$|\Delta\beta P(s)| = 1 \quad \text{and} \quad \angle P(s) = 0° \pm k360°,$$

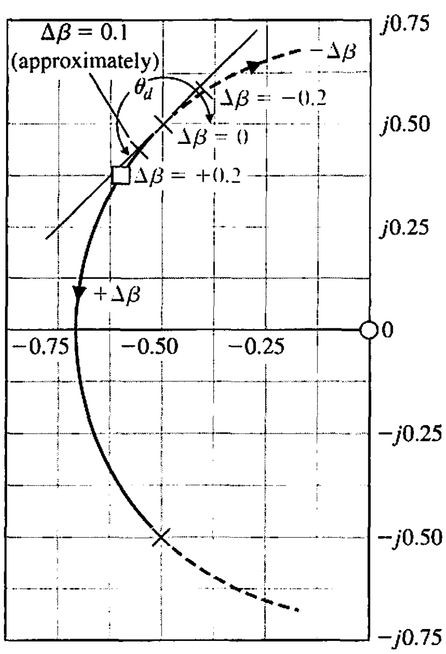


FIGURE 7.25 The root locus for the parameter β.

where k is an integer. The locus of roots follows a zero-degree locus in contrast with the 180° locus considered previously. However, the root locus rules of Section 7.3 may be altered to account for the zero-degree phase angle requirement, and then the root locus may be obtained as in the preceding sections. Therefore, to obtain the effect of reducing β, we determine the zero-degree locus in contrast to the 180° locus, as shown by a dotted locus in Figure 7.25. To find the effect of a 20% change of the parameter β, we evaluate the new roots for $\Delta\beta = \pm 0.20$, as shown in Figure 7.25. The root sensitivity is readily evaluated graphically and, for a positive change in β, is

$$S_{\beta+}^{r_1} = \frac{\Delta r_1}{\Delta\beta/\beta} = \frac{0.16\underline{/-128^\circ}}{0.20} = 0.80\underline{/-128^\circ}.$$

The root sensitivity for a negative change in β is

$$S_{\beta-}^{r_1} = \frac{\Delta r_1}{\Delta\beta/\beta} = \frac{0.125\underline{/39^\circ}}{0.20} = 0.625\underline{/+39^\circ}.$$

As the percentage change $\Delta\beta/\beta$ decreases, the sensitivity measures $S_{\beta+}^{r_1}$ and $S_{\beta-}^{r_1}$ will approach equality in magnitude and a difference in angle of 180°. Thus, for small changes when $\Delta\beta/\beta \leq 0.10$, the sensitivity measures are related as

$$|S_{\beta+}^{r_1}| = |S_{\beta-}^{r_1}|$$

and

$$\underline{/S_{\beta+}^{r_1}} = 180^\circ + \underline{/S_{\beta-}^{r_1}}.$$

Often, the desired root sensitivity measure is desired for small changes in the parameter. When the relative change in the parameter is of the order $\Delta\beta/\beta = 0.10$, we can estimate the increment in the root change by approximating the root locus with the line at the angle of departure θ_d. This approximation is shown in Figure 7.25 and is accurate for only relatively small changes in $\Delta\beta$. However, the use of this approximation allows the analyst to avoid sketching the complete root locus diagram. Therefore, for Figure 7.25, the root sensitivity may be evaluated for $\Delta\beta/\beta = 0.10$ along the departure line, and we obtain

$$S_{\beta+}^{r_1} = \frac{0.075\underline{/-132^\circ}}{0.10} = 0.75\underline{/-132^\circ}. \tag{7.96}$$

The root sensitivity measure for a parameter variation is useful for comparing the sensitivity for various design parameters and at different root locations. Comparing Equation (7.96) for β with Equation (7.94) for α, we find (a) that the sensitivity for β is greater in magnitude by approximately 50% and (b) that the angle for $S_{\beta-}^{r_1}$ indicates that the approach of the root toward the $j\omega$-axis is more sensitive for changes in β. Therefore, the tolerance requirements for β would be more stringent than for α. This information provides the designer with a comparative measure of the required tolerances for each parameter. ■

EXAMPLE 7.8 **Root sensitivity to a parameter**

A unity feedback control system has a forward transfer function

$$G(s) = \frac{20.7(s + 3)}{s(s + 2)(s + \beta)},$$

where $\beta = \beta_0 + \Delta\beta$ and $\beta_0 = 8$. The characteristic equation, as a function of $\Delta\beta$, is

$$s(s + 2)(s + 8 + \Delta\beta) + 20.7(s + 3) = 0,$$

or

$$s(s + 2)(s + 8) + \Delta\beta s(s + 2) + 20.7(s + 3) = 0.$$

When $\Delta\beta = 0$, the roots are

$$-r_1 = -2.36 + j2.48, \qquad -r_2 = \hat{r}_1, \quad \text{and} \quad -r_3 = -5.27.$$

The root locus for $\Delta\beta$ is determined by using the root locus equation

$$1 + \frac{\Delta\beta s(s + 2)}{(s + r_1)(s + \hat{r}_1)(s + r_3)} = 0. \tag{7.97}$$

The roots and zeros of Equation (7.97) are shown in Figure 7.26. The angle of departure at r_1 is evaluated from the angles as follows:

$$\begin{aligned} 180° &= -(\theta_d + 90° + \theta_{p_3}) + (\theta_{z_1} + \theta_{z_2}) \\ &= -(\theta_d + 90° + 40°) + (133° + 98°). \end{aligned}$$

Therefore, $\theta_d = -80°$ and the locus is approximated near $-r_1$ by the line at an angle of θ_d. For a change of $\Delta r_1 = 0.2\underline{/-80°}$ along the departure line, the $+\Delta\beta$ is evaluated by determining the vector lengths from the poles and zeros. Then we have

$$+\Delta\beta = \frac{4.8(3.75)(0.2)}{(3.25)(2.3)} = 0.48.$$

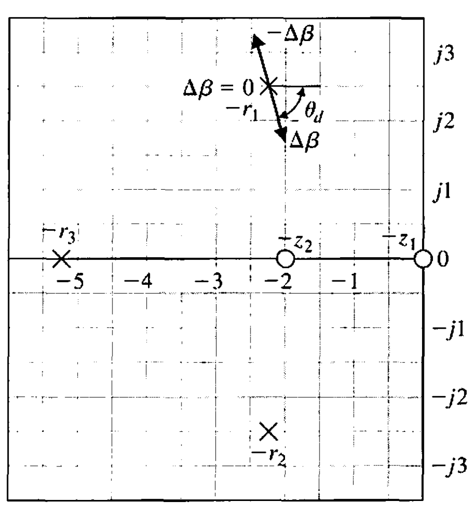


FIGURE 7.26 Pole and zero diagram for the parameter β.

Therefore, the sensitivity at r_1 is

$$S_\beta^{r_1} = \frac{\Delta r_1}{\Delta\beta/\beta} = \frac{0.2\angle{-80^\circ}}{0.48/8} = 3.34\angle{-80^\circ},$$

which indicates that the root is quite sensitive to this 6% change in the parameter β. For comparison, it is worthwhile to determine the sensitivity of the root $-r_1$ to a change in the zero $s = -3$. Then the characteristic equation is

$$s(s + 2)(s + 8) + 20.7(s + 3 + \Delta\gamma) = 0,$$

or

$$1 + \frac{20.7\,\Delta\gamma}{(s + r_1)(s + \hat{r}_1)(s + r_3)} = 0. \tag{7.98}$$

The pole–zero diagram for Equation (7.98) is shown in Figure 7.27. The angle of departure at root $-r_1$ is $180^\circ = -(\theta_d + 90^\circ + 40^\circ)$, or

$$\theta_d = +50^\circ.$$

For a change of $\Delta r_1 = 0.2\angle{+50^\circ}$, the $\Delta\gamma$ is positive. Obtaining the vector lengths, we find that

$$|\Delta\gamma| = \frac{5.22(4.18)(0.2)}{20.7} = 0.21.$$

Therefore, the sensitivity at r_1 for $+\Delta\gamma$ is

$$S_\gamma^{r_1} = \frac{\Delta r_1}{\Delta\gamma/\gamma} = \frac{0.2\angle{+50^\circ}}{0.21/3} = 2.84\angle{+50^\circ}.$$

Thus, we find that the magnitude of the root sensitivity for the pole β and the zero γ is approximately equal. However, the sensitivity of the system to the pole can be considered to be less than the sensitivity to the zero because the angle of the sensitivity, $S_\gamma^{r_1}$, is equal to $+50^\circ$ and the direction of the root change is toward the $j\omega$-axis.

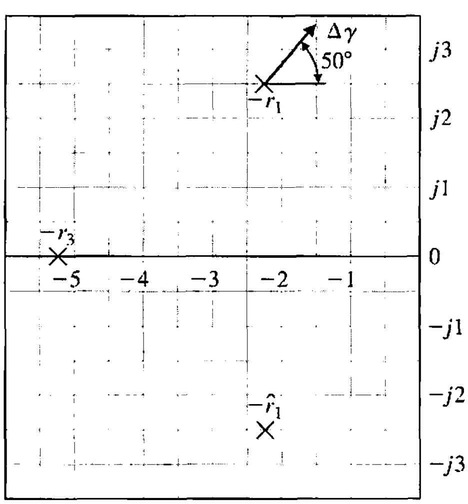


FIGURE 7.27 Pole–zero diagram for the parameter γ.

Evaluating the root sensitivity in the manner of the preceding paragraphs, we find that the sensitivity for the pole $s = -\delta_0 = -2$ is

$$S_{\delta-}^{r_1} = 2.1\angle{+27^\circ}.$$

Thus, for the parameter δ, the magnitude of the sensitivity is less than for the other parameters, but the direction of the change of the root is more important than for β and γ. ■

To utilize the root sensitivity measure for the analysis and design of control systems, a series of calculations must be performed; they will determine the various selections of possible root configurations and the zeros and poles of the open-loop transfer function. Therefore, the root sensitivity measure as a design technique is somewhat limited by two things: the relatively large number of calculations required and the lack of an obvious direction for adjusting the parameters in order to provide a minimized or reduced sensitivity. However, the root sensitivity measure can be utilized as an analysis measure, which permits the designer to compare the sensitivity for several system designs based on a suitable method of design. The root sensitivity measure is a useful index of the system's sensitivity to parameter variations expressed in the s-plane. The weakness of the sensitivity measure is that it relies on the ability of the root locations to represent the performance of the system. As we have seen in the preceding chapters, the root locations represent the performance quite adequately for many systems, but due consideration must be given to the location of the zeros of the closed-loop transfer function and the dominancy of the pertinent roots. The root sensitivity measure is a suitable measure of system performance sensitivity and can be used reliably for system analysis and design.

7.6 PID CONTROLLERS

One form of controller widely used in industrial process control is the three-term, **PID controller** [4, 10]. This controller has a transfer function

$$\boxed{G_c(s) = K_p + \frac{K_I}{s} + K_D s.}$$

The equation for the output in the time domain is

$$u(t) = K_p e(t) + K_I \int e(t)\,dt + K_D \frac{de(t)}{dt}.$$

The three-term controller is called a PID controller because it contains a proportional, an integral, and a derivative term represented by K_p, K_I, and K_D, respectively. The transfer function of the derivative term is actually

$$G_d(s) = \frac{K_D s}{\tau_d s + 1},$$

but τ_d is usually much smaller than the time constants of the process itself, so it is neglected.

If we set $K_D = 0$, then we have the **proportional plus integral (PI) controller**

$$\boxed{G_c(s) = K_p + \frac{K_I}{s}.}$$

When $K_I = 0$, we have

$$\boxed{G_c(s) = K_p + K_D s,}$$

which is called a **proportional plus derivative (PD) controller.**

The PID controller can also be viewed as a cascade of the PI and the PD controllers. Consider the PI controller

$$G_{PI}(s) = \hat{K}_P + \frac{\hat{K}_I}{s}$$

and the PD controller

$$G_{PD}(s) = \overline{K}_P + \overline{K}_D s,$$

where $\hat{K}_P$ and $\hat{K}_I$ are the PI controller gains and $\overline{K}_P$ and $\overline{K}_D$ are the PD controller gains. Cascading the two controllers (that is, placing them in series) yields

$$\begin{aligned} G_c(s) &= G_{PI}(s)G_{PD}(s) \\ &= \left(\hat{K}_P + \frac{\hat{K}_I}{s}\right)(\overline{K}_P + \overline{K}_D s) \\ &= (\overline{K}_P\hat{K}_P + \hat{K}_I\overline{K}_D) + \hat{K}_P\overline{K}_D s + \frac{\hat{K}_I\overline{K}_D}{s} \\ &= K_P + K_D s + \frac{K_I}{s}, \end{aligned}$$

where we have the following relationships between the PI and PD controller gains and the PID controller gains

$$\begin{aligned} K_P &= \overline{K}_P\hat{K}_P + \hat{K}_I\overline{K}_D \\ K_D &= \hat{K}_P\overline{K}_D \\ K_I &= \hat{K}_I\overline{K}_D. \end{aligned}$$

Consider the PID controller

$$\begin{aligned} G_c(s) &= K_P + \frac{K_I}{s} + K_D s = \frac{K_D s^2 + K_P s + K_I}{s} \\ &= \frac{K_D(s^2 + as + b)}{s} = \frac{K_D(s + z_1)(s + z_2)}{s}, \end{aligned}$$

where $a = K_P/K_D$ and $b = K_I/K_D$. Therefore, a PID controller introduces a transfer function with one pole at the origin and two zeros that can be located anywhere in the s-plane.

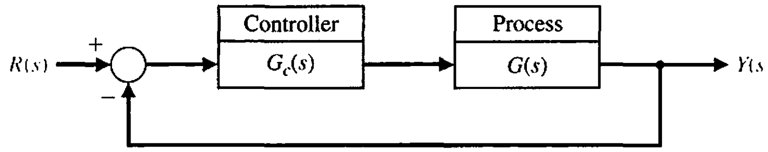


FIGURE 7.28 Closed-loop system with a controller.

Recall that a root locus begins at the poles and ends at the zeros. If we have a system, as shown in Figure 7.28, with

$$G(s) = \frac{1}{(s + 2)(s + 3)},$$

and we use a PID controller with complex zeros $-z_1$ and $-z_2$, where $-z_1 = -3 + j1$ and $-z_2 = -\hat{z}_1$, we can plot the root locus as shown in Figure 7.29. As the gain, K_D, of the controller is increased, the complex roots approach the zeros. The closed-loop transfer function is

$$T(s) = \frac{G(s)G_c(s)}{1 + G(s)G_c(s)}$$
$$= \frac{K_D(s + z_1)(s + \hat{z}_1)}{(s + r_2)(s + r_1)(s + \hat{r}_1)}.$$

The response of this system will be attractive. The percent overshoot to a step will be less than 2%, and the steady-state error for a step input will be zero. The settling time will be approximately 1 second. If a shorter settling time is desired, then we select z_1 and z_2 to lie further left in the left-hand s-plane and set K_D to drive the roots near the complex zeros.

Many industrial processes are controlled using PID controllers. The popularity of PID controllers can be attributed partly to their good performance in a wide range of operating conditions and partly to their functional simplicity that allows

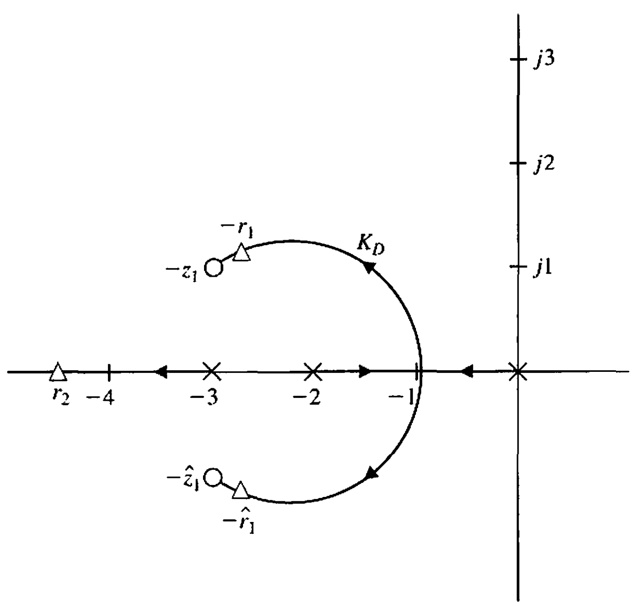


FIGURE 7.29 Root locus for plant with a PID controller with complex zeros.

Table 7.6 Effect of Increasing the PID Gains K_p, K_D, and K_I on the Step Response

PID Gain	Percent Overshoot	Settling Time	Steady-State Error
Increasing K_P	Increases	Minimal impact	Decreases
Increasing K_I	Increases	Increases	Zero steady-state error
Increasing K_D	Decreases	Decreases	No impact

engineers to operate them in a simple, straightforward manner. To implement the PID controller, three parameters must be determined, the proportional gain, denoted by K_P, integral gain, denoted by K_I, and derivative gain denoted by K_D [10].

There are many methods available to determine acceptable values of the PID gains. The process of determining the gains is often called **PID tuning**. A common approach to tuning is to use **manual PID tuning** methods, whereby the PID control gains are obtained by trial-and-error with minimal analytic analysis using step responses obtained via simulation, or in some cases, actual testing on the system and deciding on the gains based on observations and experience. A more analytic method is known as the **Ziegler-Nichols tuning** method. The Ziegler-Nichols tuning method actually has several variations. We discuss in this section a Ziegler-Nichols tuning method based on open-loop responses to a step input and a related a Ziegler-Nichols tuning method based on closed-loop response to a step input.

One approach to manual tuning is to first set $K_I = 0$ and $K_D = 0$. This is followed by slowly increasing the gain K_P until the output of the closed-loop system oscillates just on the edge of instability. This can be done either in simulation or on the actual system if it cannot be taken off-line. Once the value of K_P (with $K_I = 0$ and $K_D = 0$) is found that brings the closed-loop system to the edge of stability, you reduce the value of gain K_P to achieve what is known as the **quarter amplitude decay**. That is, the amplitude of the closed-loop response is reduced approximately to one-fourth of the maximum value in one oscillatory period. A rule-of-thumb is to start by reducing the proportional gain K_P by one-half. The next step of the design process is to increase K_I and K_D manually to achieve a desired step response. Table 7.6 describes in general terms the effect of increasing K_I and K_D.

EXAMPLE 7.9 **Manual PID tuning**

Consider the closed-loop system in Figure 7.30 with

$$G(s) = \frac{1}{s(s + b)(s + 2\zeta\omega_n)},$$

where $b = 10$, $\zeta = 0.707$, and $\omega_n = 4$.

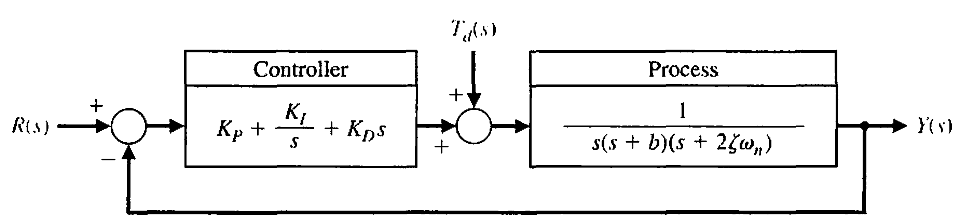


FIGURE 7.30 Unity feedback control system with PID controller.

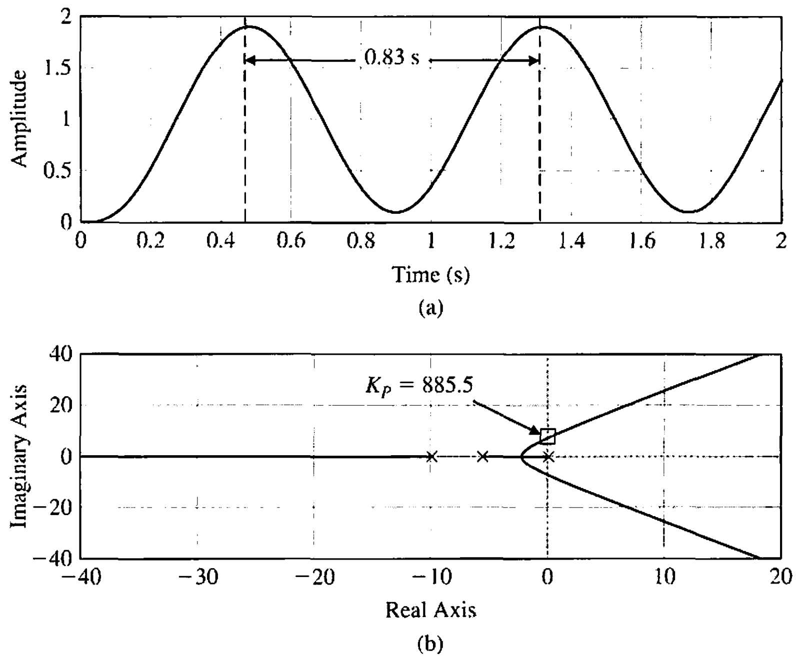


FIGURE 7.31 (a) Step response with $K_P = 885.5$, $K_D = 0$, and $K_I = 0$. (b) Root locus showing $K_P = 885.5$ results in marginal stability with $s = \pm 7.5j$.

To begin the manual tuning process, set $K_I = 0$ and $K_D = 0$ and increase K_P until the closed-loop system has sustained oscillations. As can be seen in Figure 7.31a, when $K_P = 885.5$, we have a sustained oscillation of magnitude $A = 1.9$ and period $P = 0.83$ s. The root locus shown in Figure 7.31b corresponds to the characteristic equation

$$1 + K_P\left[\frac{1}{s(s + 10)(s + 5.66)}\right] = 0.$$

The root locus shown in Figure 7.31b illustrates that when $K_P = 885.5$, we have closed-loop poles at $s = \pm 7.5j$ leading to the oscillatory behavior in the step response in Figure 7.31a.

Reduce $K_P = 885.5$ by half as a first step to achieving a step response with approximately a quarter amplitude decay. You may have to iterate on the value $K_P = 442.75$. The step response is shown in Figure 7.32 where we note that the peak amplitude is reduced to one-fourth of the maximum value in one period, as desired. To accomplish this reduction, we refined the value of K_P by slowly reducing the value from $K_P = 442.75$ to $K_P = 370$.

The root locus for $K_P = 370$, $K_I = 0$, and $0 \leq K_D < \infty$ is shown in Figure 7.33. In this case, the characteristic equation is

$$1 + K_D\left[\frac{s}{(s + 10)(s + 5.66) + K_P}\right] = 0.$$

We see in Figure 7.33 that as K_D increases, the root locus shows that the closed-loop complex poles move left, and in doing so, increases the associated damping ratio and thereby decreases the percent overshoot. The movement of the complex poles to the left also increases the associated $\zeta\omega_n$, thereby reducing the settling time. These

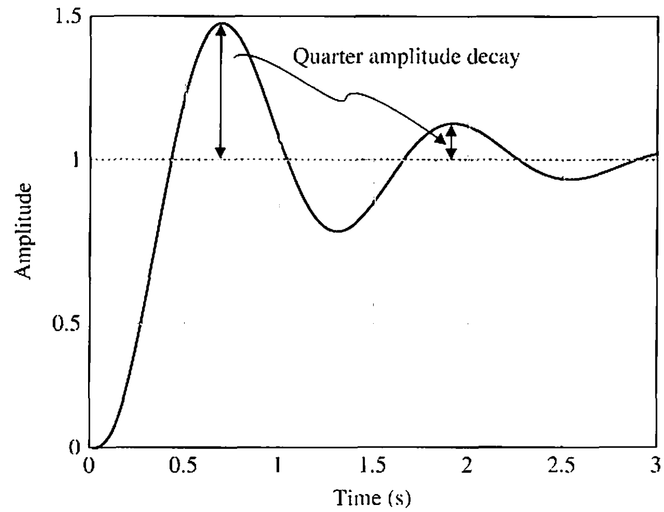


FIGURE 7.32 Step response with $K_P = 370$ showing the quarter amplitude decay.

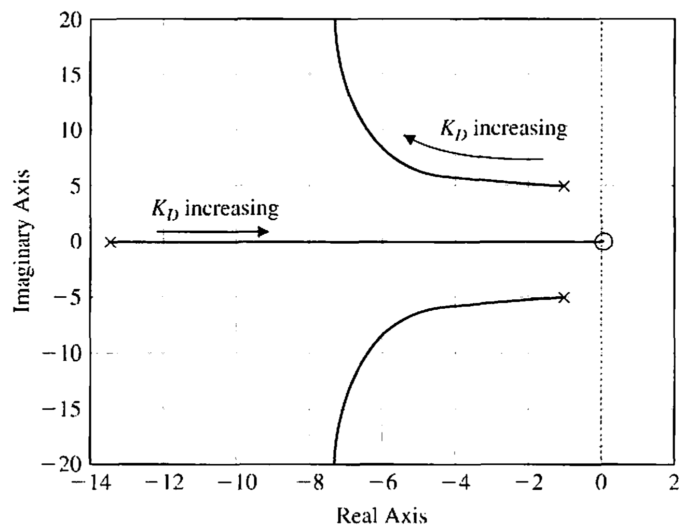


FIGURE 7.33 Root locus for $K_P = 370$, $K_I = 0$, and $0 \leq K_D < \infty$.

effects of varying K_D are consistent with information provided in Table 7.6. As K_D increases (when $K_D > 75$), the real root begins to dominant the response and the trends described in Table 7.6 become less accurate. The percent overshoot and settling time as a function of K_D are shown in Figure 7.34.

The root locus for $K_P = 370$, $K_D = 0$, and $0 \leq K_I < \infty$ is shown in Figure 7.35. The characteristic equation is

$$1 + K_I\left[\frac{1}{s\left(s(s+10)(s+5.66)+K_P\right)}\right] = 0.$$

We see in Figure 7.35 that as K_I increases, the root locus shows that the closed-loop complex pair poles move right. This decreases the associated damping ratio and thereby increasing the percent overshoot. In fact, when $K_I = 778.2$, the system is marginally stable with closed-loop poles at $s = \pm 4.86j$. The movement of the

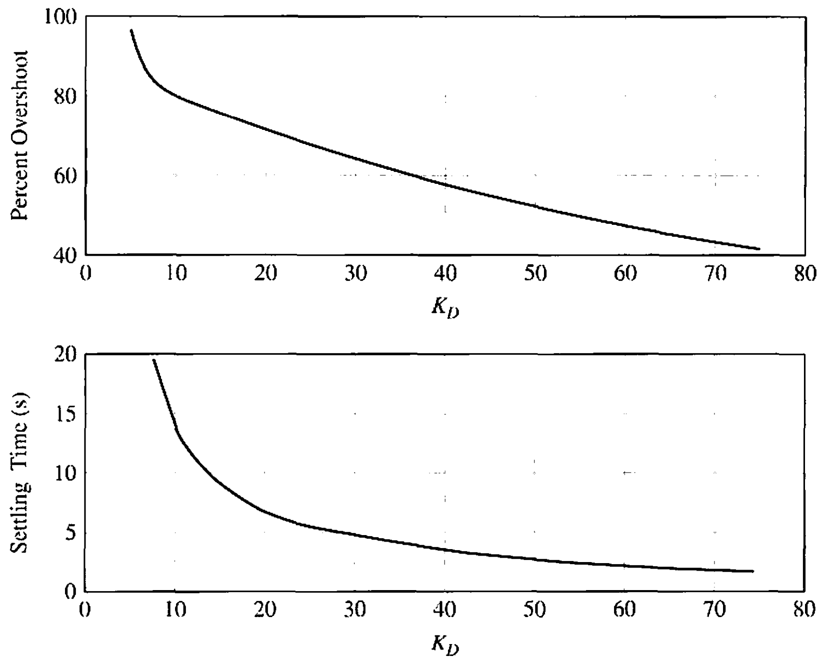


FIGURE 7.34 Percent overshoot and settling time with $K_P = 370$, $K_I = 0$, and $5 \le K_D < 75$.

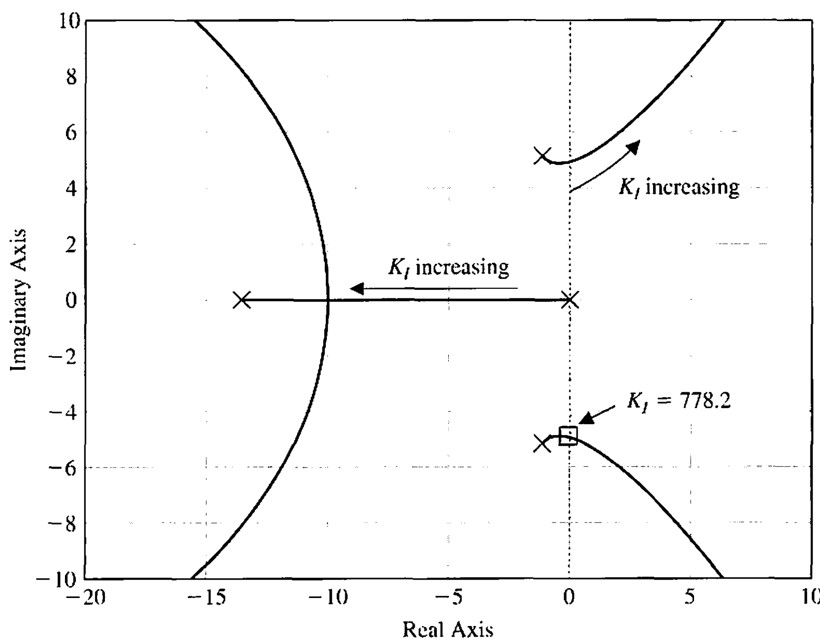


FIGURE 7.35 Root locus for $K_P = 370$, $K_D = 0$, and $0 \le K_I < \infty$.

complex poles to the right also decreases the associated $\zeta\omega_n$, thereby increasing the settling time. The percent overshoot and settling time as a function of K_I are shown in Figure 7.36. The trends in Figure 7.36 are consistent with Table 7.6.

To meet the percent overshoot and settling time specifications, we can select $K_P = 370$, $K_D = 60$, and $K_I = 100$. The step response shown in Figure 7.37 indicates a $T_s = 2.4$ s and $P.O. = 12.8\%$ meeting the specifications. ■

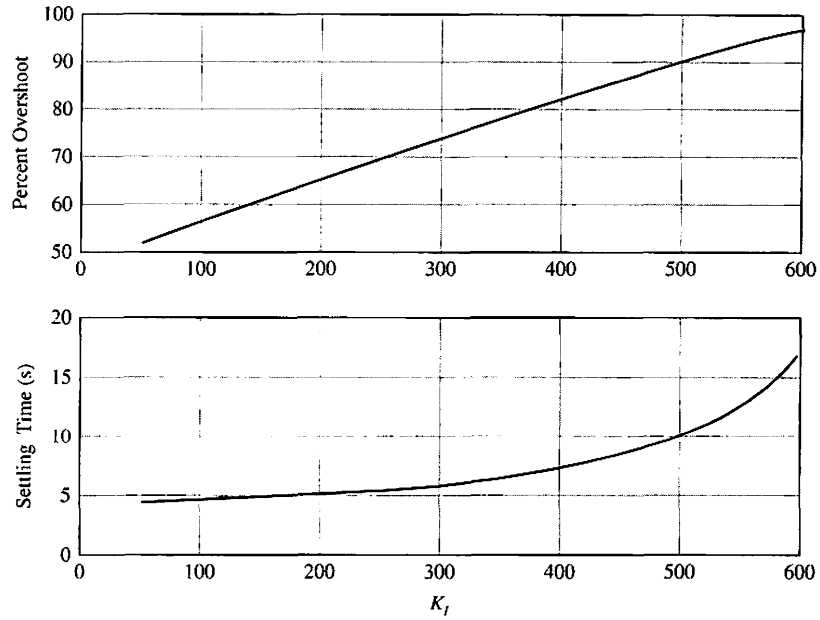


FIGURE 7.36 Percent overshoot and settling time with $K_P = 370$, $K_D = 0$, and $50 \leq K_I < 600$.

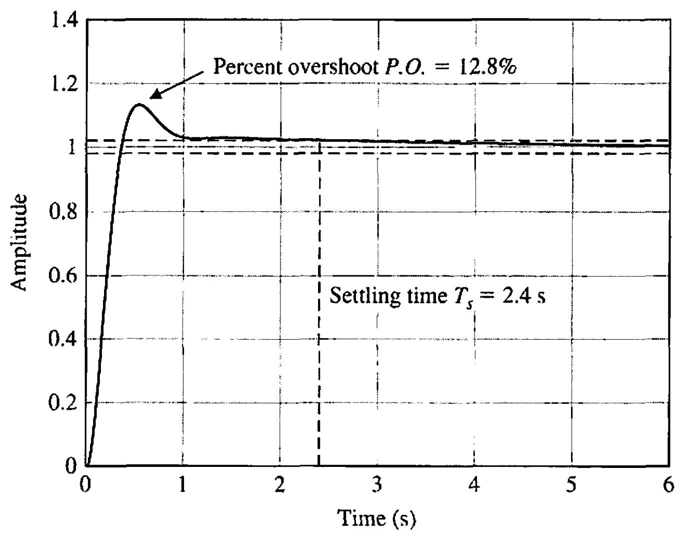


FIGURE 7.37 Percent overshoot and settling time with final design $K_P = 370$, $K_D = 60$, and $K_I = 100$.

Two important PID controller gain tuning methods were published in 1942 by John G. Ziegler and Nathaniel B. Nichols intended to achieve a fast closed-loop step response without excessive oscillations and excellent disturbance rejection. The two approaches are classified under the general heading of Ziegler-Nichols tuning methods. The first approach is based on closed-loop concepts requiring the computation of the **ultimate gain** and **ultimate period**. The second approach is based on open-loop concepts relying on **reaction curves**. The Ziegler-Nichols tuning methods are based on assumed forms of the models of the process, but the models do not have to be precisely known. This makes the tuning approach very practical in process

Table 7.7 Ziegler-Nichols PID Tuning Using Ultimate Gain, K_U, and Oscillation Period, P_U

Ziegler-Nichols PID Controller Gain Tuning Using Closed-loop Concepts			
Controller Type	K_P	K_I	K_D
Proportional (P) $G_c(s) = K_P$	$0.5K_U$	-	-
Proportional-plus-integral (PI) $G_c(s) = K_P + \frac{K_I}{s}$	$0.45K_U$	$\frac{0.54K_U}{T_U}$	-
Proportional-plus-integral-plus-derivative (PID) $G_c(s) = K_P + \frac{K_I}{s} + K_D s$	$0.6K_U$	$\frac{1.2K_U}{T_U}$	$\frac{0.6K_U T_U}{8}$

control applications. Our suggestion is to consider the Ziegler-Nichols rules to obtain initial controller designs followed by design iteration and refinement. Remember that the Ziegler-Nichols rules will not work with all plants or processes.

The closed-loop Ziegler-Nichols tuning method considers the closed-loop system response to a step input (or step disturbance) with the PID controller in the loop. Initially the derivative and integral gains, K_D and K_I, respectively, are set to zero. The proportional gain K_P is increased (in simulation or on the actual system) until the closed-loop system reaches the boundary of instability. The gain on the border of instability, denoted by K_U, is called the ultimate gain. The period of the sustained oscillations, denoted by P_U, is called the ultimate period. Once K_U and P_U are determined, the PID gains are computed using the relationships in Table 7.7 according to the Ziegler-Nichols tuning method.

EXAMPLE 7.10 **Closed-loop Ziegler-Nichols PID tuning**

Re-consider the system in Example 7.9. The plant is

$$G(s) = \frac{1}{s(s + b)(s + 2\zeta\omega_n)},$$

where $b = 10$, $\zeta = 0.707$, and $\omega_n = 4$. The controller is a PID controller

$$G_c(s) = K_P + \frac{K_I}{s} + K_D s,$$

where the gains K_P, K_D, and K_I are computed using the formulas in Table 7.7. We found in Example 7.9 that $K_U = 885.5$ and $T_U = 0.83$ s. By using the Ziegler-Nichols formulas we obtain

$$K_P = 0.6K_U = 531.3, \quad K_I = \frac{1.2K_U}{T_U} = 1280.2, \quad \text{and} \quad K_D = \frac{0.6K_U T_U}{8} = 55.1.$$

Comparing the step response in Figures 7.37 and 7.38 we note that the settling time is approximately the same for the manually tuned and the Ziegler-Nichols tuned PID controllers. However, the percent overshoot of the manually tuned controller is less than that of the Ziegler-Nichols tuning. This is due to the fact that the

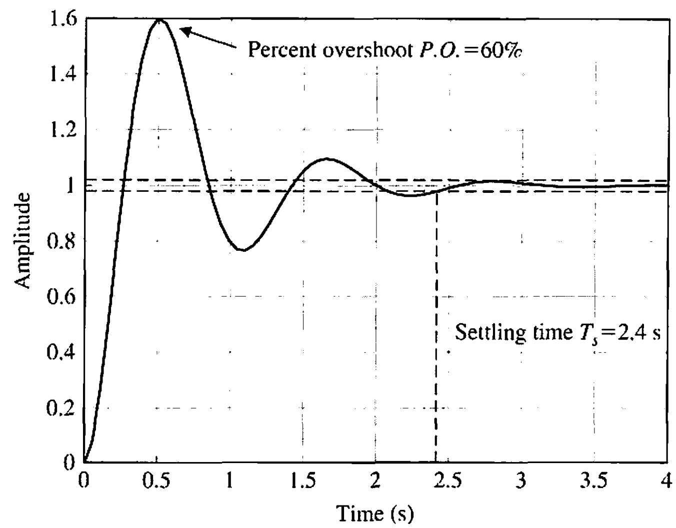


FIGURE 7.38 Time response for the Ziegler-Nichols PID tuning with $K_P = 531.3$, $K_I = 1280.2$, and $K_D = 55.1$.

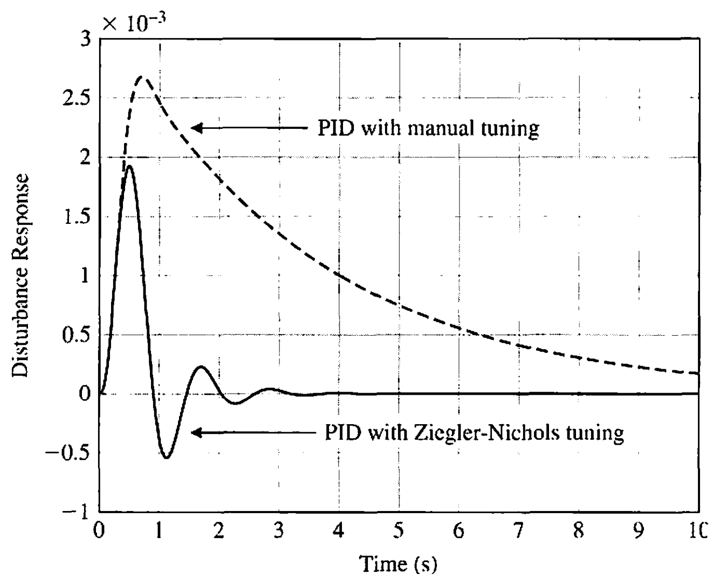


FIGURE 7.39 Disturbance response for the Ziegler-Nichols PID tuning versus the manual tuning in Example 7.9.

Ziegler-Nichols tuning is designed to provide the best disturbance rejection performance rather than the best input response performance.

In Figure 7.39, we see that the step disturbance performance of the Ziegler-Nichols PID controller is indeed better than the manually tuned controller. While Ziegler-Nichols approach provides a structured procedure for obtaining the PID controller gains, the appropriateness of the Ziegler-Nichols tuning depends on the requirements of the problem under investigation. ■

The open-loop Ziegler-Nichols tuning method utilizes a reaction curve obtained by taking the controller off-line (that is, out of the loop) and introducing a step input (or step disturbance). This approach is very commonly used in process control applications. The measured output is the reaction curve and is assumed to

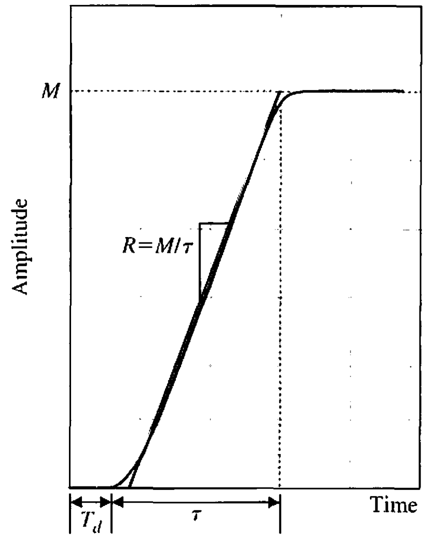


FIGURE 7.40 Reaction curve illustrating parameters R and T_d required for the Ziegler-Nichols open-loop tuning method.

have the general shape shown in Figure 7.40. The response in Figure 7.40 implies that the process is a first-order system with a transport delay. If the actual system does not match the assumed form, then another approach to PID tuning should be considered. However, if the underlying system is linear and lethargic (or sluggish and characterized by delay), the assumed model may suffice to obtain a reasonable PID gain selection using the open-loop Ziegler-Nichols tuning method.

The reaction curve is characterized by the transport delay, T_d, and the reaction rate, R. Generally, the reaction curve is recorded and numerical analysis is performed to obtain estimates of the parameters T_d and R. A system possessing the reaction curve shown in Figure 7.40 can be approximated by a first-order system with a transport delay as

$$G(s) = M\left[\frac{p}{s + p}\right]e^{-T_d s},$$

Table 7.8 Ziegler-Nichols PID Tuning Using Reaction Curve Characterized by Time Delay, T_d, and Reaction Rate, R

Ziegler-Nichols PID Controller Gain Tuning Using Open-loop Concepts

Controller Type	K_P	K_I	K_D
Proportional (P) $G_c(s) = K_P$	$\frac{1}{RT_d}$	–	–
Proportional-plus-integral (PI) $G_c(s) = K_P + \frac{K_I}{s}$	$\frac{0.9}{RT_d}$	$\frac{0.27}{RT_d^2}$	–
Proportional-plus-integral-plus-derivative (PID) $G_c(s) = K_P + \frac{K_I}{s} + K_D s$	$\frac{1.2}{RT_d}$	$\frac{0.6}{RT_d^2}$	$\frac{0.6}{R}$

where M is the magnitude of the response at steady-state, T_d is the transport delay, and p is related to the slope of the reaction curve. The parameters M, τ, and T_d can be estimated from the open-loop step response and then utilized to compute $R = M/\tau$. Once that is accomplished, the PID gains are computed as shown in Table 7.8. You can also use the Ziegler-Nichols open-loop tuning method to design a proportional controller or a proportional-plus-integral controller.

EXAMPLE 7.11 **Open-loop Ziegler-Nichols PI controller tuning**

Consider the reaction curve shown in Figure 7.41. We estimate the transport lag to be $T_d = 0.1$ s and the reaction rate $R = 0.8$.

Using the Ziegler-Nichols tuning for the PI controller gains we have

$$K_P = \frac{0.9}{RT_d} = 11.25 \quad \text{and} \quad K_I = \frac{0.27}{RT_d^2} = 33.75.$$

The closed-loop system step response (assuming unity feedback) is shown in Figure 7.42. The settling time is $T_s = 1.28$ s and the percent overshoot is $P.O. = 78\%$. Since we are using a PI controller, the steady-state is zero, as expected. ∎

The manual tuning method and the two Ziegler-Nichols tuning approaches presented here will not always lead to the desired closed-loop performance. The three methods do provide structured design steps leading to candidate PID gains and should be viewed as first steps in the design iteration. Since the PID (and the related PD and PI) controllers are in wide use today in a variety of applications, it is important to become familiar with various design approaches. We will use the PD controller later in this chapter to control the hard disk drive sequential design problem (see Section 7.10).

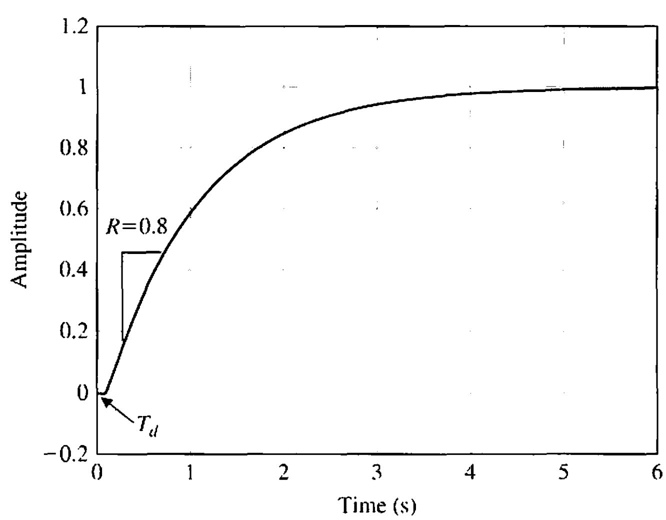


FIGURE 7.41 Reaction curve with $T_d = 0.1$ s and $R = 0.8$.

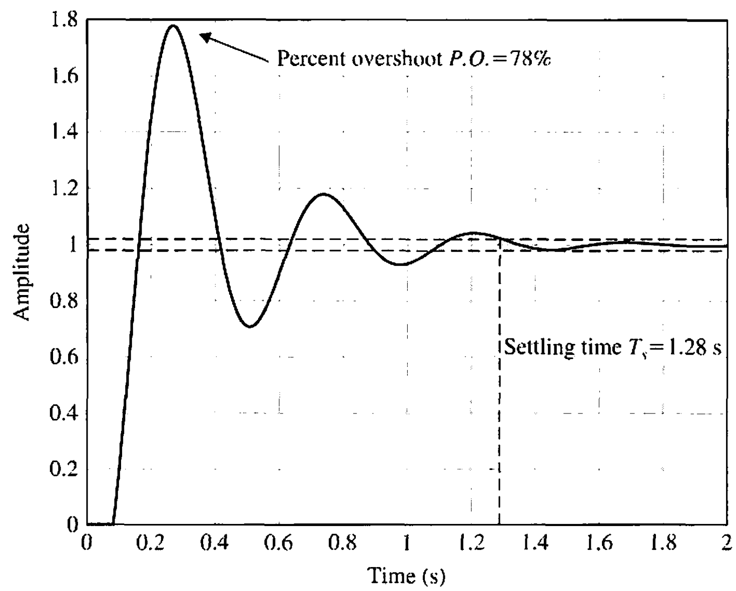


FIGURE 7.42 Time response for the Ziegler-Nichols PI tuning with $K_P = 11.25$ and $K_I = 33.75$.

7.7 NEGATIVE GAIN ROOT LOCUS

As discussed in Section 7.2, the dynamic performance of a closed-loop control system is described by the closed-loop transfer function, that is, by the poles and zeros of the closed-loop system. The root locus is a graphical illustration of the variation of the roots of the characteristic equation as a single parameter of interest varies. We know that the roots of the characteristic equation and the closed-loop poles are one in the same. In the case of the single-loop negative unity feedback system shown in Figure 7.1, the characteristic equation is

$$1 + KG(s) = 0, \tag{7.99}$$

where K is the parameter of interest. The orderly seven-step procedure for sketching the root locus described in Section 7.3 and summarized in Table 7.2 is valid for the case where $0 \leq K < \infty$. Sometimes the situation arises where we are interested in the root locus for negative values of the parameter of interest where $-\infty < K \leq 0$. We refer to this as the **negative gain root locus**. Our objective here is to develop an orderly procedure for sketching the negative gain root locus using familiar concepts from root locus sketching as described in Section 7.2.

Rearranging Equation (7.99) yields

$$G(s) = -\frac{1}{K}.$$

Since K is negative, it follows that

$$|KG(s)| = 1 \quad \text{and} \quad \boxed{\angle KG(s) = 0^\circ + k360^\circ} \tag{7.100}$$

where $k = 0, \pm 1, \pm 2, \pm 3, \ldots$. The magnitude and phase conditions in Equation (7.100) must both be satisfied for all points on the negative gain root locus. Note that the phase condition in Equation (7.100) is different from the phase condition in Equation (7.4). As we will show, the new phase condition leads to several key modifications in the root locus sketching steps from those summarized in Table 7.2.

EXAMPLE 7.12 **Negative gain root locus**

Consider the system shown in Figure 7.43. The loop transfer function is

$$L(s) = KG(s) = K\frac{s - 20}{s^2 + 5s - 50}$$

and the characteristic equation is

$$1 + K\frac{s - 20}{s^2 + 5s - 50} = 0.$$

Sketching the root locus yields the plot shown in Figure 7.44a where it can be seen that the closed-loop system is not stable for any $0 \leq K < \infty$. The negative gain root locus is shown in Figure 7.44b. Using the negative gain root locus in Figure 7.44b we find that the stability is $-5.0 < K < -2.5$. The system in Figure 7.43 can thus be stabilized with only negative gain, K. ∎

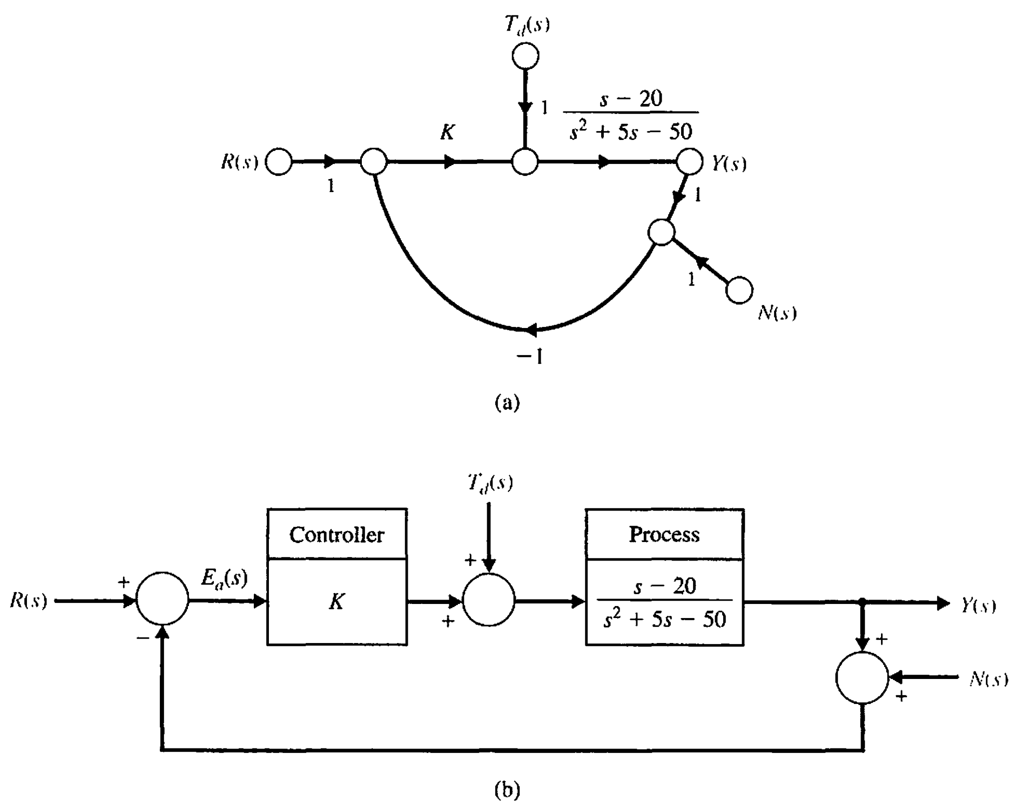


FIGURE 7.43 (a) Signal flow graph and (b) block diagram of unity feedback system with controller gain, K.

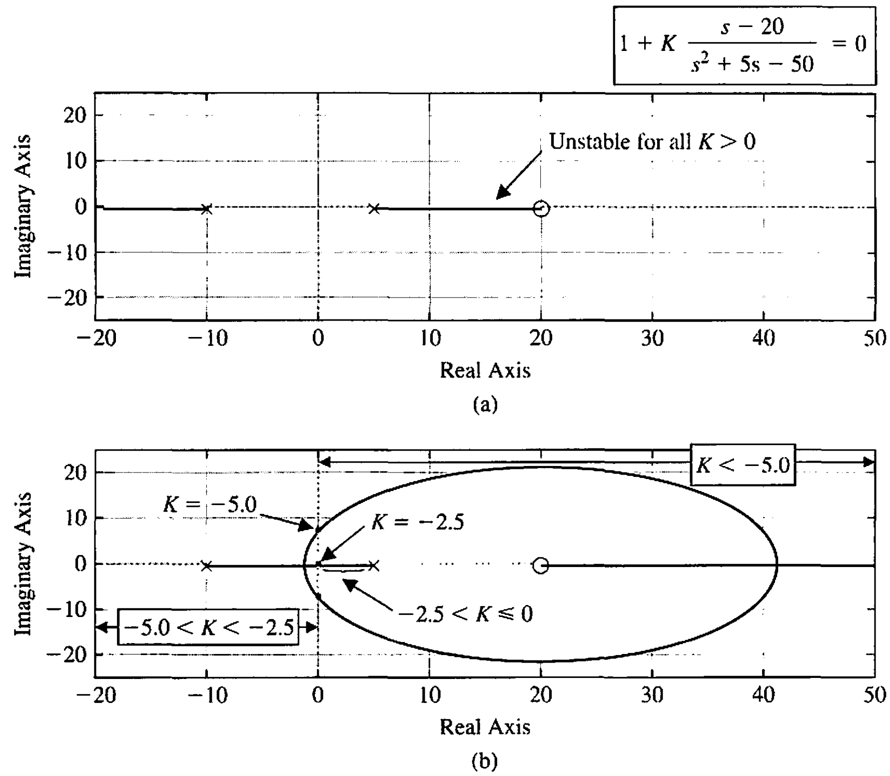


FIGURE 7.44 (a) Root locus for $0 \leq K < \infty$. (b) Negative gain root locus for $-\infty < K \leq 0$.

To locate the roots of the characteristic equation in a graphical manner on the s-plane for negative values of the parameter of interest, we will re-visit the seven steps summarized in Table 7.2 to obtain a similar orderly procedure to facilitate the rapid sketching of the locus.

Step 1: Prepare the root locus sketch. As before, you begin by writing the characteristic equation and rearranging, if necessary, so that the parameter of interest, K, appears as the multiplying factor in the form,

$$1 + KP(s) = 0. \tag{7.101}$$

For the negative gain root locus, we are interested in determining the locus of roots of the characteristic equation in Equation (7.101) for $-\infty < K \leq 0$. As in Equation (7.24), factor $P(s)$ in Equation (7.101) in the form of poles and zeros and locate the poles and zeros on the s-plane with 'x' to denote poles and 'o' to denote zeros.

When $K = 0$, the roots of the characteristic equation are the poles of $P(s)$, and when $K \to -\infty$ the roots of the characteristic equation are the zeros of $P(s)$. Therefore, the locus of the roots of the characteristic equation begins at the poles of $P(s)$ when $K = 0$ and ends at the zeros of $P(s)$ as $K \to -\infty$. If $P(s)$ has n poles and M zeros and $n > M$, we have $n - M$ branches of the root locus approaching the zeros at infinity and the number of separate loci is equal to the number of poles. The root loci are symmetrical with respect to the horizontal real axis because the complex roots must appear as pairs of complex conjugate roots.

Step 2: Locate the segments of the real axis that are root loci. The root locus on the real axis always lies in a section of the real axis to the left of an **even** number of poles and zeros. This follows from the angle criterion of Equation (7.100).

Step 3: When $n > M$, we have $n - M$ branches heading to the zeros at infinity as $K \rightarrow -\infty$ along asymptotes centered at σ_A and with angles ϕ_A. The linear asymptotes are centered at a point on the real axis given by

$$\sigma_A = \frac{\sum \text{poles of } P(s) - \sum \text{zeros of } P(s)}{n - M} = \frac{\sum_{j=1}^{n}(-p_j) - \sum_{i=1}^{M}(-z_i)}{n - M}. \quad (7.102)$$

The angle of the asymptotes with respect to the real axis is

$$\boxed{\phi_A = \frac{2k + 1}{n - M}360^\circ} \quad k = 0, 1, 2, \ldots, (n - M - 1), \quad (7.103)$$

where k is an integer index.

Step 4: Determine where the locus crosses the imaginary axis (if it does so), using the Routh-Hurwitz criterion.

Step 5: Determine the breakaway point on the real axis (if any). In general, due to the phase criterion, the tangents to the loci at the breakaway point are equally spaced over 360°. The breakaway point on the real axis can be evaluated graphically or analytically. The breakaway point can be computed by rearranging the characteristic equation

$$1 + K\frac{n(s)}{d(s)} = 0$$

as

$$p(s) = K,$$

where $p(s) = -d(s)/n(s)$ and finding the values of s that maximize $p(s)$. This is accomplished by solving the equation

$$n(s)\frac{\mathrm{d}[d(s)]}{\mathrm{d}s} - d(s)\frac{\mathrm{d}[n(s)]}{\mathrm{d}s} = 0. \quad (7.104)$$

Equation (7.104) yields a polynomial equation in s of degree $n + M - 1$, where n is the number of poles and M is the number of zeros. Hence the number of solutions is $n + M - 1$. The solutions that exist on the root locus are the breakaway points.

Step 6: Determine the angle of departure of the locus from a pole and the angle of arrival of the locus at a zero using the phase angle criterion. The angle of locus departure from a pole or angle of arrival at a zero is the difference between the net angle due to all other poles and zeros and the criterion angle of $\pm k360^\circ$.

Step 7: The final step is to complete the sketch by drawing in all sections of the locus not covered in the previous six steps.

The seven steps for sketching a negative gain root locus are summarized in Table 7.9.

Table 7.9 Seven Steps for Sketching a Negative Gain Root Locus (color text denotes changes from root locus steps in Table 7.2)

Step	Related Equation or Rule
1. Prepare the root locus sketch.	
(a) Write the characteristic equation so that the parameter of interest, K, appears as a multiplier.	(a) $1 + KP(s) = 0$
(b) Factor $P(s)$ in terms of n poles and M zeros	(b) $1 + K\frac{\prod_{i=1}^{M}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)} = 0$
(c) Locate the open-loop poles and zeros of $P(s)$ in the s-plane with selected symbols.	(c) $\times$ = poles, $\bigcirc$ = zeros
(d) Determine the number of separate loci, SL.	(d) Locus begins at a pole and ends at a zero. $SL = n$ when $n \geq M$; n = number of finite poles, M = number of finite zeros.
(e) The root loci are symmetrical with respect to the horizontal real axis.	
2. Locate the segments of the real axis that are root loci.	Locus lies to the left of an even number of poles and zeros.
3. The loci proceed to the zeros at infinity along asymptotes centered at σ_A and with angles ϕ_A.	$\sigma_A = \frac{\sum_{j=1}^{n}(-p_j) - \sum_{i=1}^{M}(-z_i)}{n-M}$. $\phi_A = \frac{2k+1}{n-M}360^\circ, k = 0, 1, 2, \ldots (n - M - 1)$
4. Determine the points at which the locus crosses the imaginary axis (if it does so).	Use Routh-Hurwitz criterion (see Section 6.2).
5. Determine the breakaway point on the real axis (if any).	a) Set $K = p(s)$ b) Determine roots of $dp(s)/ds = 0$ or use graphical method to find maximum of $p(s)$.
6. Determine the angle of locus departure from complex at or poles and the angle of locus arrival at complex zeros using the phase criterion.	$\angle P(s) = \pm k360^\circ$ at $s = -p_j$ or $-z_i$
7. Complete the negative gain root locus sketch.	

7.8 DESIGN EXAMPLES

In this section we present four illustrative examples. The first example is a wind turbine control system. The feedback control system uses a PI controller to achieve a fast settling time and rise time while limiting the percent overshoot to a step input. The second example is a laser manipulator control system. Here the root locus method is used to show how the closed-loop system poles move in the s-plane as the proportional controller amplifier gain varies. The second example considers a simplified robotic replication facility. In the example, the system is represented by a fifth-order transfer function model. The feedback control strategy employs a velocity feedback coupled with a controller in the forward loop. Root locus design methods are used to select the two feedback controller gains. In the final example, the automatic control of the velocity of an automobile is considered. In this example, the root locus method is extended from one parameter to three parameters as the three gains of a PID controller are determined. The design process is emphasized, including considering the control goals and associated variables to be controlled, the design specifications, and the PID controller design using root locus methods.

EXAMPLE 7.13 Wind turbine speed control

Wind energy conversion to electric power is achieved by wind energy turbines connected to electric generators. Of particular interest are wind turbines, as shown in Figure 7.45, that are located offshore [33]. The new concept is to allow the wind turbine to float rather than positioning the structure on a tower tied deep into the ocean floor. This allows the wind turbine structure to be placed in deeper waters up to 100 miles offshore far enough not to burden the landscape with unsightly structures [34]. Moreover, the wind is generally stronger on the open ocean potentially leading to the production of 5 MW versus the more typical 1.5 MW for wind turbines onshore. However, the irregular character of wind direction and power results in the need for reliable, steady electric energy by using control systems for the wind turbines. The goal of these control devices is to reduce the effects of wind intermittency and of wind direction change. The rotor and generator speed control can be achieved by adjusting the pitch angle of the blades.

A basic model of the generator speed control system is shown in Figure 7.46 [35]. A linearized model from the collective pitch to the generator speed is given by[1]

$$G(s) = \frac{4.2158(s - 827.1)(s^2 - 5.489s + 194.4)}{(s + 0.195)(s^2 + 0.101s + 482.6)}. \tag{7.105}$$

The model corresponds to a 600 KW turbine with hub height = 36.6 m, rotor diameter = 40 m, rated rotor speed = 41.7 rpm, rated generator speed = 1800 rpm,

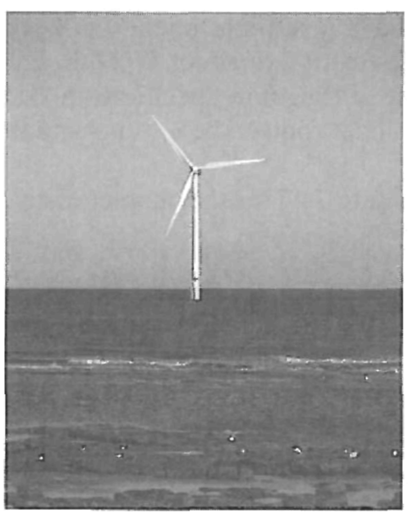


FIGURE 7.45 Wind turbine placed offshore can help alleviate the energy needs. (Photo courtesy of Alamy Images.)

[1] Provided by Dr. Lucy Pao and Jason Laks in private correspondence.

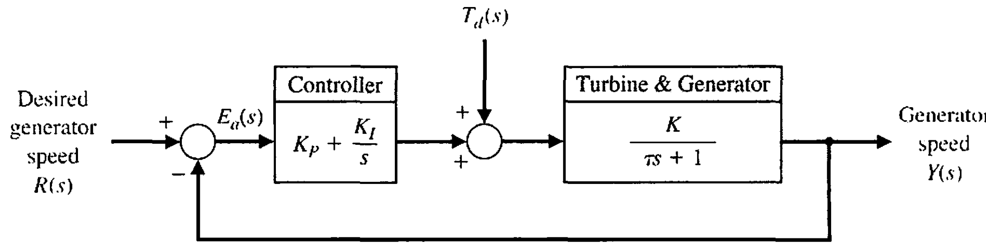


FIGURE 7.46 Wind turbine generator speed control system.

and maximum pitch rate = 18.7 deg/sec. Note that the linearized model in Equation (7.105) has zeros in the right half-plane at $s_1 = 827.1$ and $s_{2,3} = 0.0274 \pm 0.1367j$ making this a nonminimum phase system (see Chapter 8 for more information on nonminimum phase systems).

A simplified version of the model in Equation (7.105) is given by the transfer function

$$G(s) = \frac{K}{\tau s + 1}, \tag{7.106}$$

where $\tau = 5$ seconds and $K = -7200$. We will design a PI controller to control the speed of the turbine generator using the simplified first-order model in Equation (7.106) and confirm that the design specifications are satisfied for both the first-order model and the third-order model in Equation (7.105). The PI controller, denoted by $G_c(s)$, is given by

$$G_c(s) = K_P + \frac{K_I}{s} = K_P\left[\frac{s + \tau_c}{s}\right],$$

where $\tau_c = K_I/K_P$ and the gains K_P and K_I are to be determined. A stability analysis indicates that negative gains $K_I < 0$ and $K_P < 0$ will stabilize the system. The main design specification is to have a settling time $T_s < 4$ seconds to a unit step input. We also desire a limited percent overshoot ($P.O. < 25\%$) and a short rise time ($T_r < 1$ s) while meeting the settling time specification. To this end, we will target the damping ratio of the dominant roots to be $\zeta > 0.4$ and the natural frequency $\omega_n > 2.5$ rad/s.

The root locus is shown in Figure 7.47 for the characteristic equation

$$1 + \hat{K}_P\left[\frac{s + \tau_c}{s}\,\frac{7200}{5s + 1}\right] = 0,$$

where $\tau_c = 2$ and $\hat{K}_P = -K_p > 0$. The placement of the controller zero at $s = -\tau_c = -2$ is a design parameter. We select the value of $\hat{K}_P$ such that the damping ratio of the closed-loop complex poles is $\zeta = 0.707$. Selecting $\hat{K}_P = 0.0025$ yields $K_P = -0.0025$ and $K_I = -0.005$. The PI controller is

$$G_c(s) = K_P + \frac{K_I}{s} = -0.0025\left[\frac{s + 2}{s}\right].$$

The step response is shown in Figure 7.48 using the simplified first-order model in Equation (7.106). The step response has $T_s = 1.8$ seconds, $T_r = 0.34$ seconds, and $\zeta = 0.707$ which translates to $P.O. = 19\%$. The PI controller is able to meet all the control specifications. The step response using the third-order model in Equation (7.105) is shown in Figure 7.49 where we see the effect of the neglected components in the design as small oscillations in the speed response. The closed-loop impulse disturbance response in Figure 7.50 shows fast and accurate rejection of the disturbance in less than 3 seconds due to a 1° pitch angle change. ■

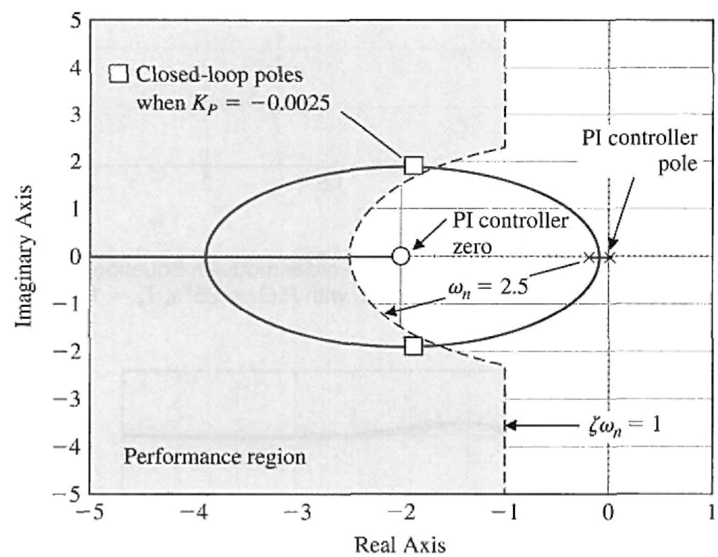


FIGURE 7.47 Wind turbine generator speed control root locus with a PI controller.

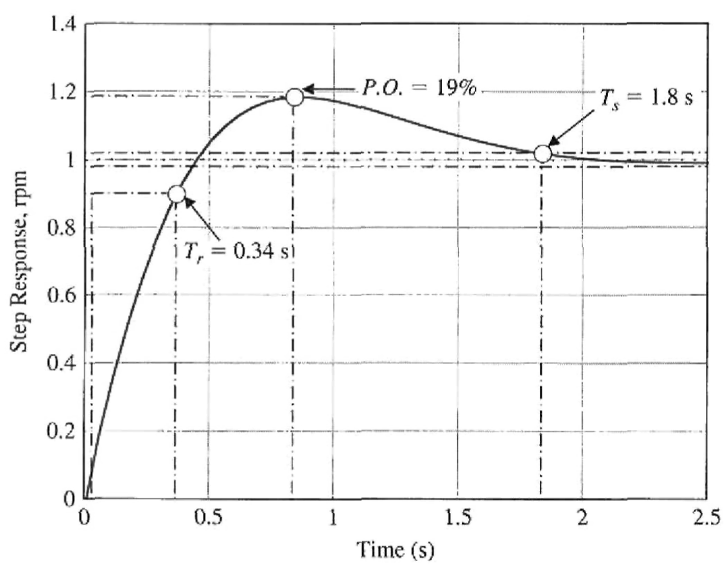


FIGURE 7.48 Step response of the wind turbine generator speed control system using the first-order model in Equation (7.106) with the designed PI controller showing all specifications are satisfied with $P.O. = 19\%$, $T_s = 1.8$ s, and $T_r = 0.34$ s.

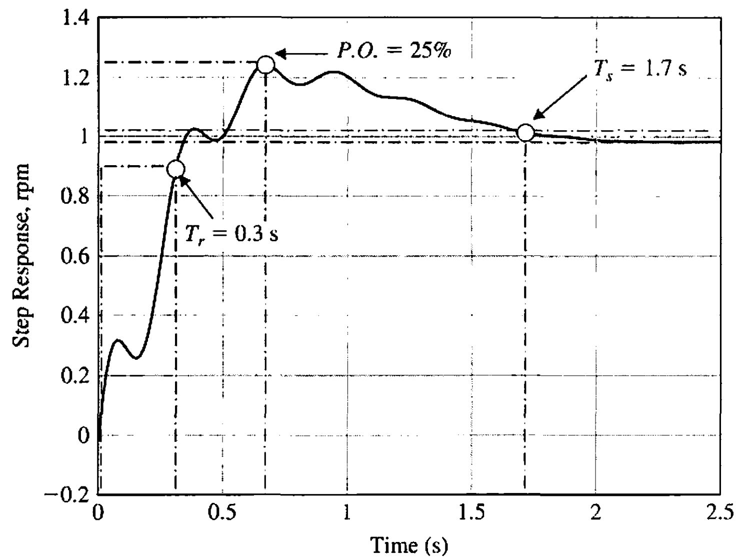


FIGURE 7.49 Step response of the third-order model in Equation (7.105) with the PI controller showing that all specifications are satisfied with $P.O. = 25\%$, $T_s = 1.7$ s, and $T_r = 0.3$ s.

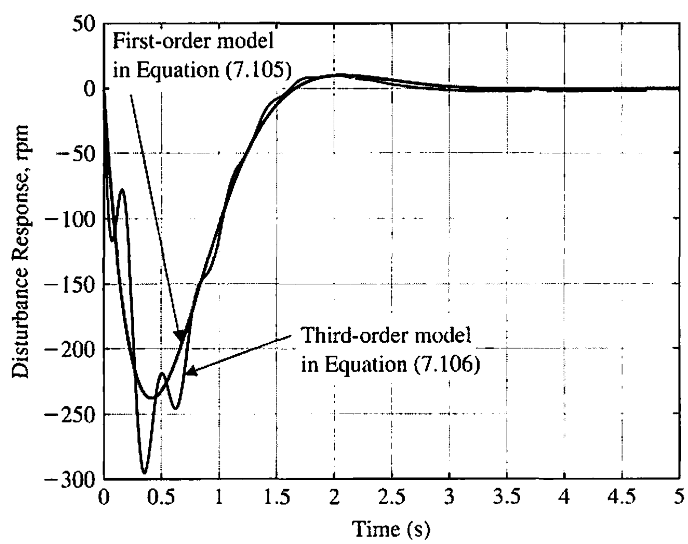


FIGURE 7.50 Disturbance response of the wind turbine generator speed control system with a PI controller shows excellent disturbance rejection characteristics.

EXAMPLE 7.14 Laser manipulator control system

Lasers can be used to drill the hip socket for the appropriate insertion of an artificial hip joint. The use of lasers for surgery requires high accuracy for position and velocity response. Let us consider the system shown in Figure 7.51, which uses a DC motor manipulator for the laser. The amplifier gain K must be adjusted so that the steady-state error for a ramp input, $r(t) = At$ (where $A = 1$ mm/s), is less than or equal to 0.1 mm, while a stable response is maintained.

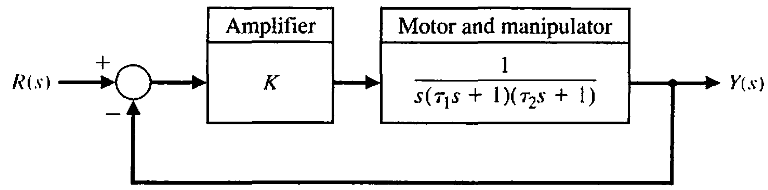


FIGURE 7.51 Laser manipulator control system.

To obtain the steady-state error required and a good response, we select a motor with a field time constant $\tau_1 = 0.1$ s and a motor-plus-load time constant $\tau_2 = 0.2$ s. We then have

$$T(s) = \frac{KG(s)}{1 + KG(s)} = \frac{K}{s(\tau_1 s + 1)(\tau_2 s + 1) + K}$$
$$= \frac{K}{0.02s^3 + 0.3s^2 + s + K} = \frac{50K}{s^3 + 15s^2 + 50s + 50K}. \quad (7.107)$$

The steady-state error for a ramp, $R(s) = A/s^2$, from Equation (5.29), is

$$e_{ss} = \frac{A}{K_v} = \frac{A}{K}.$$

Since we desire $e_{ss} = 0.1$ mm (or less) and $A = 1$ mm, we require $K = 10$ (or greater).

To ensure a stable system, we obtain the characteristic equation from Equation (7.107) as

$$s^3 + 15s^2 + 50s + 50K = 0.$$

Establishing the Routh array, we have

$$\begin{array}{c|cc} s^3 & 1 & 50 \\ s^2 & 15 & 50K \\ s^1 & b_1 & 0 \\ s_0 & 50K & \end{array},$$

where

$$b_1 = \frac{750 - 50K}{15}.$$

Therefore, the system is stable for

$$0 \leq K \leq 15.$$

The characteristic equation can be written as

$$1 + K\frac{50}{s^3 + 15s^2 + 50s} = 0.$$

The root locus for $K > 0$ is shown in Figure 7.52. Using $K = 10$ results in a stable system that also satisfies the steady-state tracking error specification. The roots at $K = 10$ are $-r_2 = -13.98$, $-r_1 = -0.51 + j5.96$, and $-\hat{r}_1$. The ζ of the complex

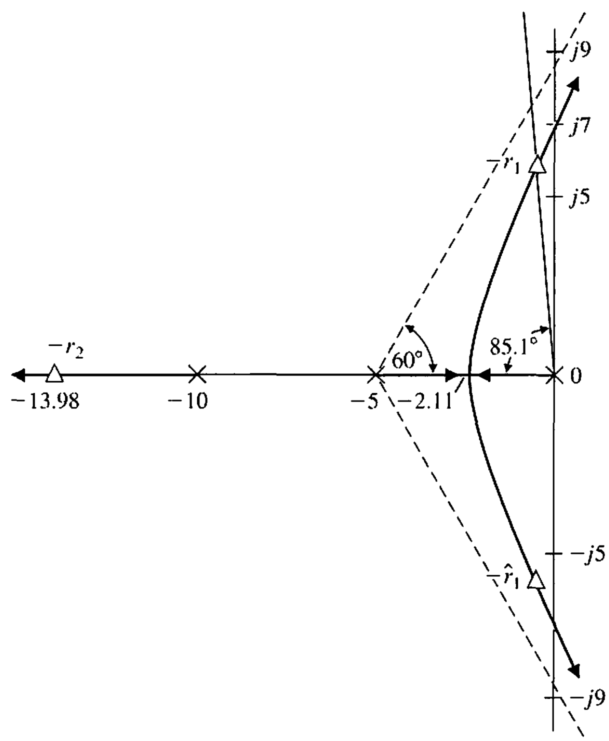


FIGURE 7.52 Root locus for a laser control system.

roots is 0.085 and $\zeta\omega_n = 0.51$. Thus, assuming that the complex roots are dominant, we expect (using Equation 5.16 and 5.13) a step input to have an overshoot of 76% and a settling time (to within 2% of the final value) of

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{0.51} = 7.8 \text{ s}.$$

Plotting the actual system response, we find that the overshoot is 70% and the settling time is 7.5 seconds. Thus, the complex roots are essentially dominant. The system response to a step input is highly oscillatory and cannot be tolerated for laser surgery. The command signal must be limited to a low-velocity ramp signal. The response to a ramp signal is shown in Figure 7.53. ■

EXAMPLE 7.15 **Robot control system**

The concept of robot replication is relatively easy to grasp. The central idea is that robots replicate themselves and develop a factory that automatically produces robots. An example of a robot replication facility is shown in Figure 7.54. To achieve the rapid and accurate control of a robot, it is important to keep the robotic arm stiff and yet lightweight [6].

The specifications for controlling the motion of the arm are (1) a settling time to within 2% of the final value of less than 2 seconds, (2) a percent overshoot of less than 10% for a step input, and (3) a steady-state error of zero for a step input.

The block diagram of the proposed system with a controller is shown in Figure 7.55. The configuration proposes the use of velocity feedback as well as

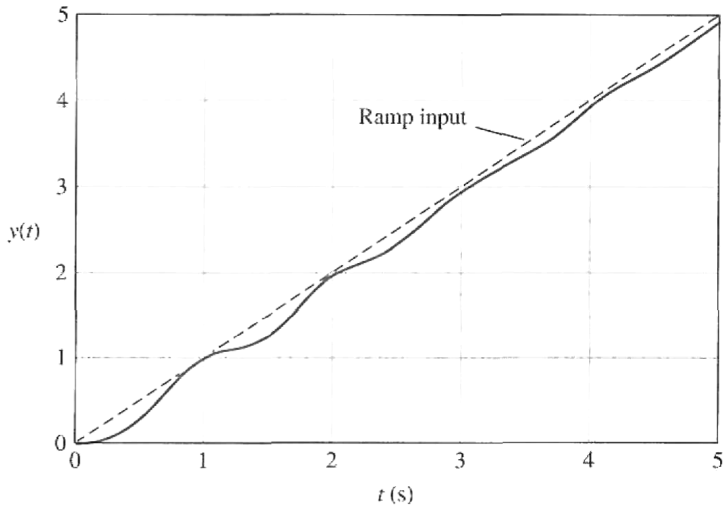


FIGURE 7.53 The response to a ramp input for a laser control system.

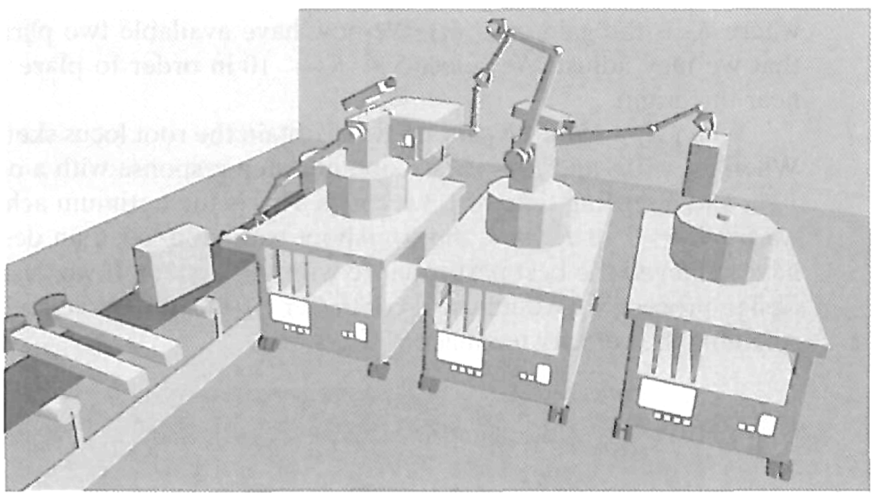


FIGURE 7.54 A robot replication facility.

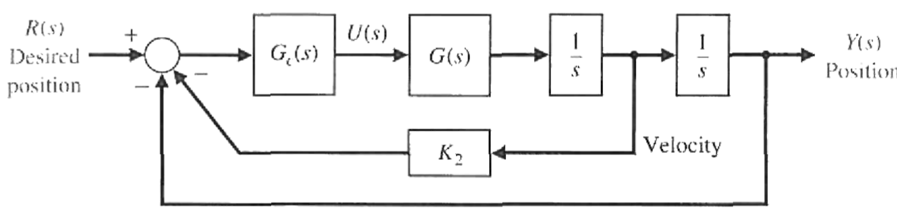


FIGURE 7.55 Proposed configuration for control of the lightweight robotic arm.

the use of a controller $G_c(s)$. The transfer function of the arm is

$$\frac{Y(s)}{U(s)} = \frac{1}{s^2}G(s)$$

where

$$G(s) = \frac{(s^2 + 4s + 10004)(s^2 + 12s + 90036)}{(s + 10)(s^2 + 2s + 2501)(s^2 + 6s + 22509)}.$$

The complex zeros are located at

$$s = -2 \pm j100 \quad \text{and} \quad s = -6 \pm j300.$$

The complex poles are located at

$$s = -1 \pm j50 \quad \text{and} \quad s = -3 \pm j150.$$

A sketch of the root locus when $K_2 = 0$ and the controller is an adjustable gain, $G_c(s) = K_1$, is shown in Figure 7.56. The system is unstable since two roots of the characteristic equation appear in the right-hand s-plane for $K_1 > 0$.

It is clear that we need to introduce the use of velocity feedback by setting K_2 to a positive magnitude. Then we have $H(s) = 1 + K_2 s$; therefore, the loop transfer function is

$$\frac{1}{s^2}G_c(s)G(s)H(s) = \frac{K_1K_2\left(s + \frac{1}{K_2}\right)(s^2 + 4s + 10004)(s^2 + 12s + 90036)}{s^2(s + 10)(s^2 + 2s + 2501)(s^2 + 6s + 22509)},$$

where K_1 is the gain of $G_c(s)$. We now have available two parameters, K_1 and K_2, that we may adjust. We select $5 < K_2 < 10$ in order to place the adjustable zero near the origin.

When $K_2 = 5$ and K_1 is varied, we obtain the root locus sketched in Figure 7.57. When $K_1 = 0.8$ and $K_2 = 5$, we obtain a step response with a percent overshoot of 12% and a settling time of 1.8 seconds. This is the optimum achievable response. If we try $K_2 = 7$ or $K_2 = 4$, the overshoot will be larger than desired. Therefore, we have achieved the best performance with this system. If we desired to continue the design process, we would use a controller $G_c(s)$ with a pole and zero in addition to retaining the velocity feedback with $K_2 = 5$.

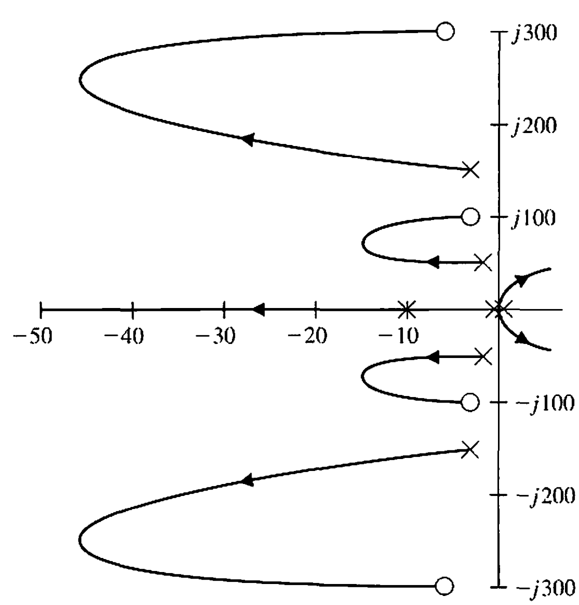


FIGURE 7.56 Root locus of the system if $K_2 = 0$, K_1 is varied from $K_1 = 0$ to $K_1 = \infty$, and $G_c(s) = K_1$.

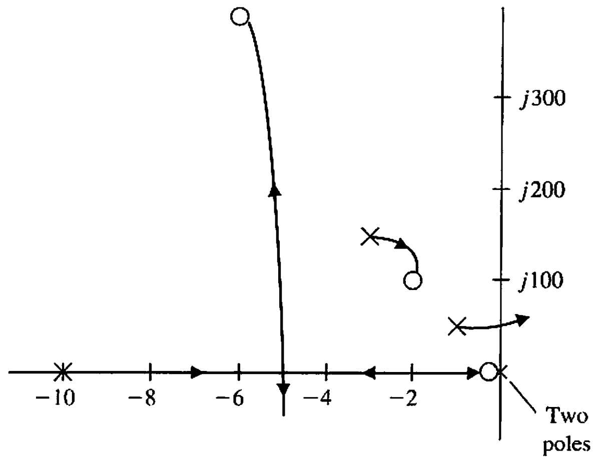


FIGURE 7.57 Root locus for the robot controller with a zero inserted at $s = -0.2$ with $G_c(s) = K_1$.

One possible selection of a controller is

$$G_c(s) = \frac{K_1(s + z)}{s + p}.$$

If we select $z = 1$ and $p = 5$, then, when $K_1 = 5$, we obtain a step response with an overshoot of 8% and a settling time of 1.6 seconds. ■

EXAMPLE 7.16 Automobile velocity control

The automotive electronics market is expected to reach $243 billion by 2015. It is predicted that there will be growth of about 6.4% up to the year 2015 in electronic braking, steering, and driver information. Much of the additional computing power will be used for new technology for smart cars and smart roads, such as IVHS (intelligent vehicle/highway systems) [14, 30, 31]. New systems on-board the automobile will support semi-autonomous automobiles, safety enhancements, emission reduction, and other features including intelligent cruise control, and brake by wire systems eliminating the hydraulics [32].

The term IVHS refers to a varied assortment of electronics that provides real-time information on accidents, congestion, and roadside services to drivers and traffic controllers. IVHS also encompasses devices that make vehicles more autonomous: collision-avoidance systems and lane-tracking technology that alert drivers to impending disasters and allow a car to drive itself.

An example of an automated highway system is shown in Figure 7.58. A velocity control system for maintaining the velocity between vehicles is shown in Figure 7.59. The output $Y(s)$ is the relative velocity of the two automobiles; the input $R(s)$ is the desired relative velocity between the two vehicles. Our design goal is to develop a controller that can maintain the prescribed velocity between the vehicles and maneuver the active vehicle (in this case the rearward automobile) as commanded. The elements of the design process emphasized in this example are depicted in Figure 7.60.

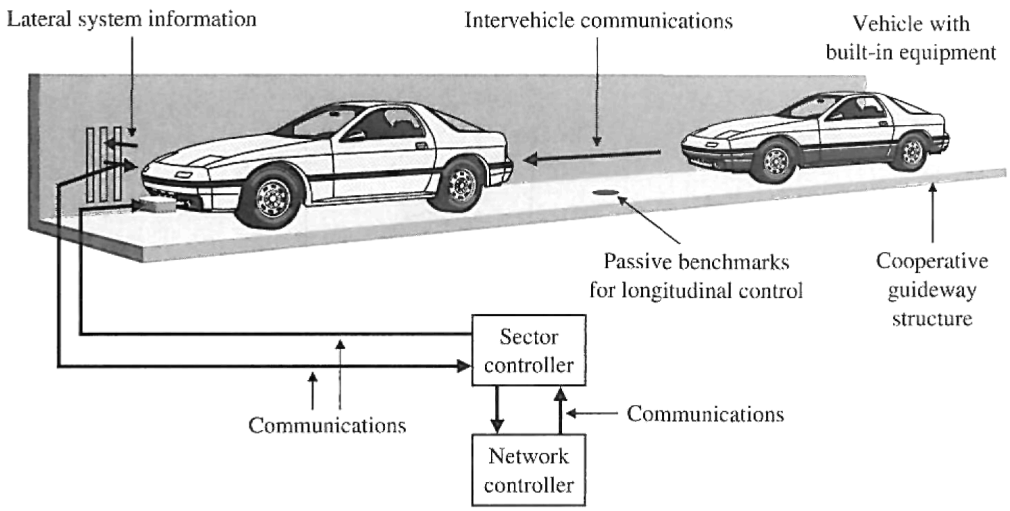


FIGURE 7.58 Automated highway system.

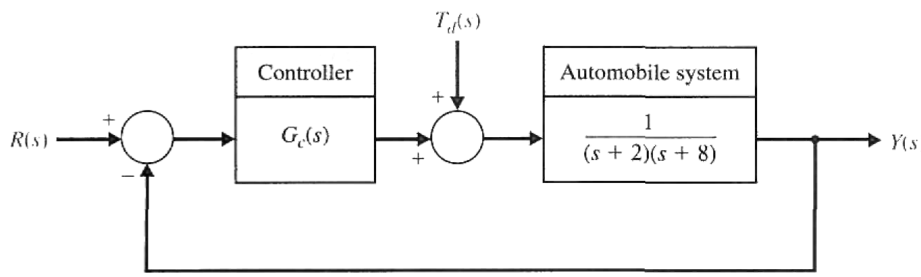


FIGURE 7.59 Vehicle velocity control system.

The control goal is

Control Goal

Maintain the prescribed velocity between the two vehicles, and maneuver the active vehicle as commanded.

The variable to be controlled is the relative velocity between the two vehicles:

Variable to Be Controlled

The relative velocity between vehicles, denoted by $y(t)$.

The design specifications are

Design Specifications

DS1 Zero steady-state error to a step input.

DS2 Steady-state error due to a ramp input of less than 25% of the input magnitude.

DS3 Percent overshoot less than 5% to a step input.

DS4 Settling time less than 1.5 seconds to a step input (using a 2% criterion to establish settling time).

From the design specifications and knowledge of the open-loop system, we find that we need a type 1 system to guarantee a zero steady-state error to a step input. The open-loop system transfer function is a type 0 system; therefore, the controller

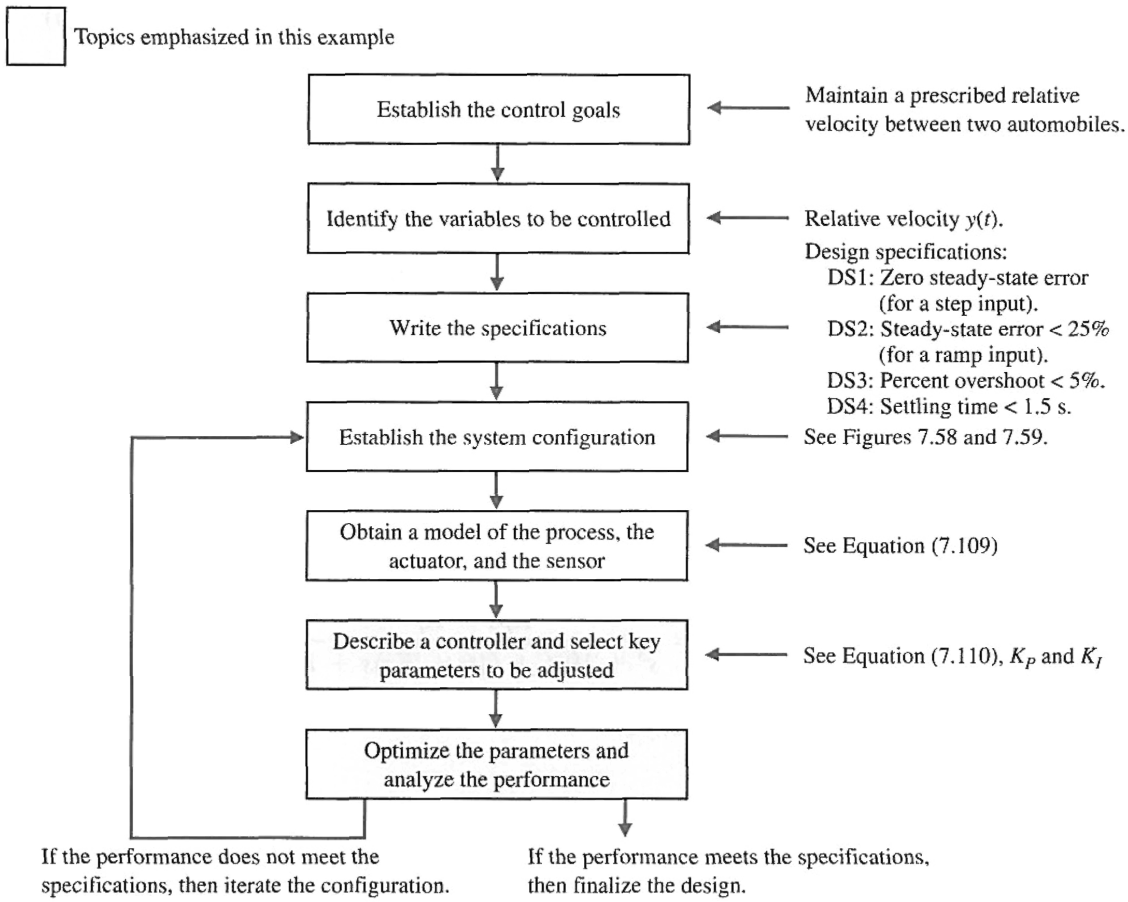


FIGURE 7.60 Elements of the control system design process emphasized in the automobile velocity control example.

needs to increase the system type by at least 1. A type 1 controller (that is, a controller with one integrator) satisfies DS1. To meet DS2 we need to have the velocity error constant (see Equation (5.29))

$$K_v = \lim_{s \to 0} sG_c(s)G(s) \geq \frac{1}{0.25} = 4, \tag{7.108}$$

where

$$G(s) = \frac{1}{(s+2)(s+8)}, \tag{7.109}$$

and $G_c(s)$ is the controller (yet to be specified).

The percent overshoot specification DS3 allows us to define a target damping ratio (see Figure 5.8):

$$P.O. \leq 5\% \quad \text{implies} \quad \zeta \geq 0.69.$$

Similarly from the settling time specification DS4 we have (see Equation (5.13))

$$T_s \approx \frac{4}{\zeta\omega_n} \leq 1.5.$$

Solving for $\zeta\omega_n$ yields $\zeta\omega_n \geq 2.6$.

The desired region for the poles of the closed-loop transfer function is shown in Figure 7.61. Using a proportional controller $G_c(s) = K_P$, is not reasonable, because DS2 cannot be satisfied. We need at least one pole at the origin to track a ramp input. Consider the PI controller

$$G_c(s) = \frac{K_P s + K_I}{s} = K_P \frac{s + \dfrac{K_I}{K_P}}{s}. \tag{7.110}$$

The question is where to place the zero at $s = -K_I/K_P$.

We ask for what values of K_P and K_I is the system stable. The closed-loop transfer function is

$$T(s) = \frac{K_P s + K_I}{s^3 + 10s^2 + (16 + K_P)s + K_I}.$$

The corresponding Routh array is

$$\begin{array}{c|cc} s^3 & 1 & 16 + K_P \\ s^2 & 10 & K_I \\ s & \dfrac{10(K_P + 16) - K_I}{10} & 0 \\ 1 & K_I & \end{array}$$

The first requirement for stability (from column one, row four) is

$$K_I > 0. \tag{7.111}$$

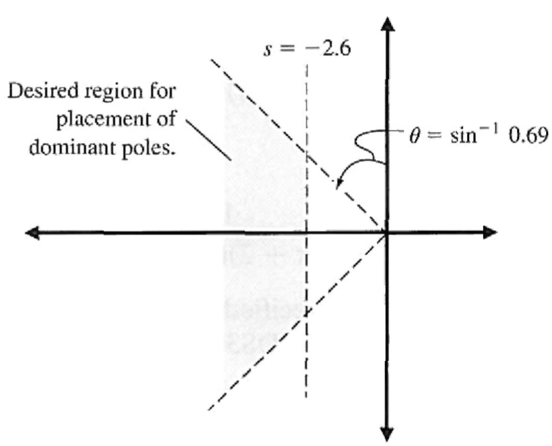


FIGURE 7.61 Desired region in the complex plane for locating the dominant system poles.

From the first column, third row, we have the inequality

$$K_P > \frac{K_I}{10} - 16. \tag{7.112}$$

It follows from DS2 that

$$K_v = \lim_{s\to 0} sG_c(s)G(s) = \lim_{s\to 0} s\frac{K_P\left(s + \frac{K_I}{K_P}\right)}{s}\frac{1}{(s+2)(s+8)} = \frac{K_I}{16} > 4.$$

Therefore, the integral gain must satisfy

$$K_I > 64. \tag{7.113}$$

If we select $K_I > 64$, then the inequality in Equation (7.103) is satisfied. The valid region for K_P is then given by Equation (7.112), where $K_I > 64$.

We need to consider DS4. Here we want to have the dominant poles to the left of the $s = -2.6$ line. We know from our experience sketching the root locus that since we have three poles (at $s = 0, -2$, and -8) and one zero (at $s = -K_I/K_P$), we expect two branches of the loci to go to infinity along two asymptotes at $\phi = -90°$ and $+90°$ centered at

$$\sigma_A = \frac{\sum(-p_i) - \sum(-z_i)}{n_p - n_z},$$

where $n_p = 3$ and $n_z = 1$. In our case

$$\sigma_A = \frac{-2 - 8 - \left(-\frac{K_I}{K_P}\right)}{2} = -5 + \frac{1}{2}\frac{K_I}{K_P}.$$

We want to have $\alpha < -2.6$ so that the two branches will bend into the desired regions. Therefore,

$$-5 + \frac{1}{2}\frac{K_I}{K_P} < -2.6,$$

or

$$\frac{K_I}{K_P} < 4.7. \tag{7.114}$$

So as a first design, we can select K_P and K_I such that

$$K_I > 64, K_P > \frac{K_I}{10} - 16, \quad \text{and} \quad \frac{K_I}{K_P} < 4.7.$$

Suppose we choose $K_I/K_P = 2.5$. Then the closed-loop characteristic equation is

$$1 + K_P\frac{s + 2.5}{s(s+2)(s+8)} = 0.$$

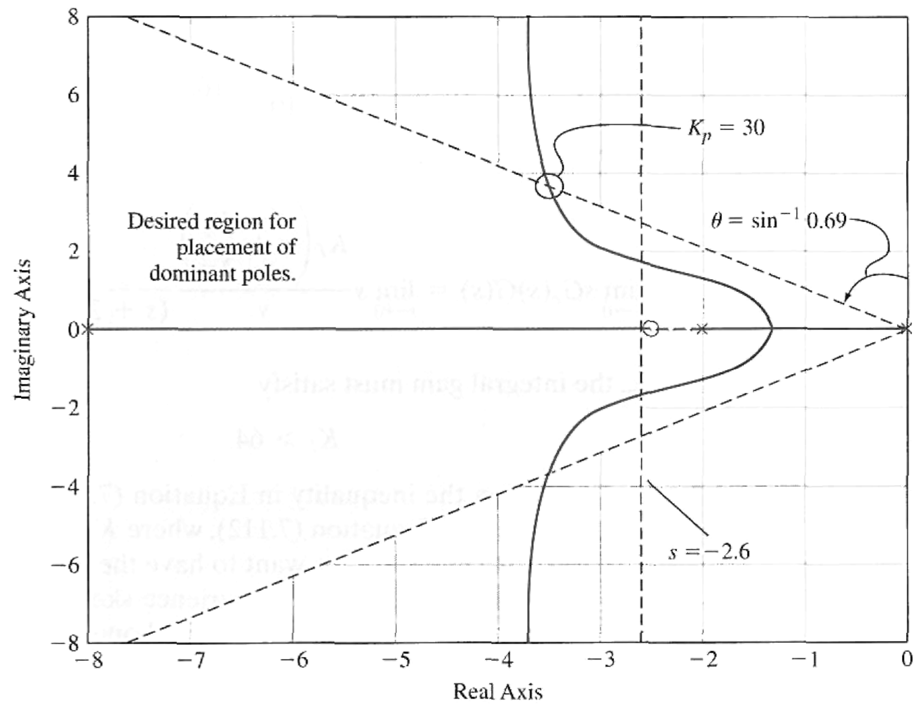


FIGURE 7.62 Root locus for $K_I/K_P = 2.5$.

The root locus is shown in Figure 7.62. To meet the $\zeta = 0.69$ (which evolved from DS3), we need to select $K_P < 30$. We selected the value at the boundary of the performance region (see Figure 7.62) as carefully as possible.

Selecting $K_P = 26$, we have $K_I/K_P = 2.5$ which implies $K_I = 65$. This satisfies the steady-state tracking error specification (DS2) since $K_I = 65 > 64$.

The resulting PI controller is

$$G_c(s) = 26 + \frac{65}{s}. \tag{7.115}$$

The step response is shown in Figure 7.63.

The percent overshoot is $P.O. = 8\%$, and the settling time is $T_s = 1.45$ s. The percent overshoot specification is not precisely satisfied, but the controller in Equation (7.115) represents a very good first design. We can iteratively refine it. Even though the closed-loop poles lie in the desired region, the response does not exactly meet the specifications because the controller zero influences the response. The closed-loop system is a third-order system and does not have the performance of a second-order system. We might consider moving the zero to $s = -2$ (by choosing $K_I/K_P = 2$) so that the pole at $s = -2$ is cancelled and the resulting system is a second-order system. ■

7.9 THE ROOT LOCUS USING CONTROL DESIGN SOFTWARE

An approximate root locus sketch can be obtained by applying the orderly procedure summarized in Table 7.2. Alternatively, we can use control design software to obtain an accurate root locus plot. However, we should not be tempted to rely solely on the computer for obtaining root locus plots while neglecting the manual steps in developing an

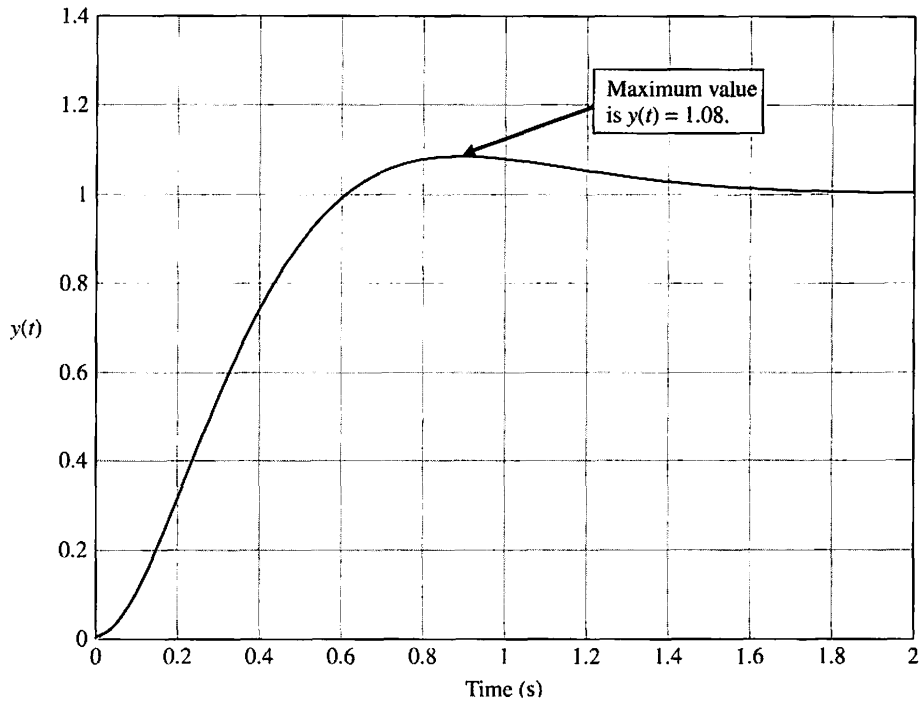


FIGURE 7.63 Automobile velocity control using the PI controller in Equation (7.107).

approximate root locus. The fundamental concepts behind the root locus method are embedded in the manual steps, and it is essential to understand their application fully.

The section begins with a discussion on obtaining a computer-generated root locus plot. This is followed by a discussion of the connections between the partial fraction expansion, dominant poles, and the closed-loop system response. Root sensitivity is covered in the final paragraphs.

The functions covered in this section are rlocus, rlocfind, and residue. The functions rlocus and rlocfind are used to obtain root locus plots, and the residue function is utilized for partial fraction expansions of rational functions.

Obtaining a Root Locus Plot. Consider the closed-loop control system in Figure 7.10. The closed-loop transfer function is

$$T(s) = \frac{Y(s)}{R(s)} = \frac{K(s+1)(s+3)}{s(s+2)(s+3) + K(s+1)}.$$

The characteristic equation can be written as

$$1 + K\frac{s+1}{s(s+2)(s+3)} = 0. \tag{7.116}$$

The form of the characteristic equation in Equation (7.116) is necessary to use the rlocus function for generating root locus plots. The general form of the characteristic equation necessary for application of the rlocus function is

$$1 + KG(s) = 1 + K\frac{p(s)}{q(s)} = 0, \tag{7.117}$$

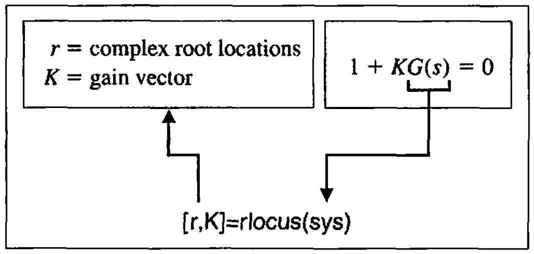


FIGURE 7.64 The **rlocus** function.

where K is the parameter of interest to be varied from $0 < K < \infty$. The rlocus function is shown in Figure 7.64, where we define the transfer function object sys $= G(s)$. The steps to obtaining the root locus plot associated with Equation (7.116), along with the associated root locus plot, are shown in Figure 7.65. Invoking the rlocus function without left-hand arguments results in an automatic generation of the root locus plot. When invoked with left-hand arguments, the rlocus function returns a matrix of root locations and the associated gain vector.

The steps to obtain a computer-generated root locus plot are as follows:

1. Obtain the characteristic equation in the form given in Equation (7.117), where K is the parameter of interest.
2. Use the rlocus function to generate the plots.

Referring to Figure 7.65, we can see that as K increases, two branches of the root locus break away from the real axis. This means that, for some values of K, the closed-loop system characteristic equation will have two complex roots. Suppose we want to find the value of K corresponding to a pair of complex roots. We can use

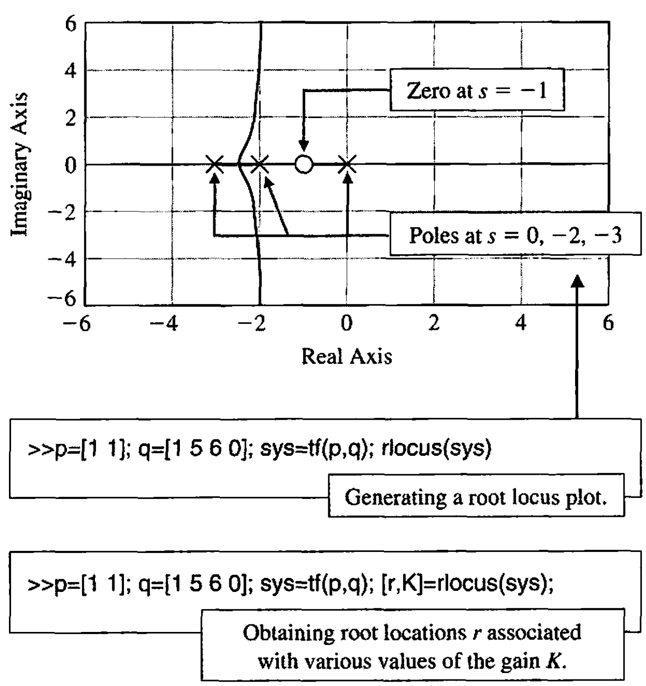


FIGURE 7.65 The root locus for the characteristic equation, Equation (7.116).

the rlocfind function to do this, but only after a root locus has been obtained with the rlocus function. Executing the rlocfind function will result in a cross-hair marker appearing on the root locus plot. We move the cross-hair marker to the location on the locus of interest and hit the enter key. The value of the parameter K and the value of the selected point will then be displayed in the command display. The use of the rlocfind function is illustrated in Figure 7.66.

Control design software packages may respond differently when interacting with plots, such as with the rlocfind function on the root locus. The response of rlocfind in Figure 7.66 corresponds to MATLAB. Refer to the companion website for more information on other control design software applications.

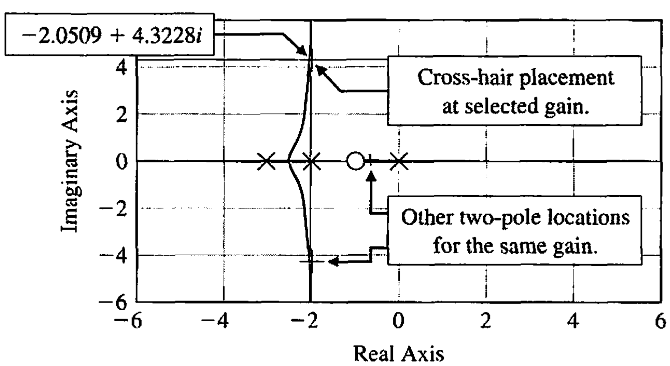
Continuing our third-order root locus example, we find that when $K = 20.5775$, the closed-loop transfer function has three poles and two zeros, at

$$\text{poles: } s = \begin{pmatrix} -2.0505 + j4.3227 \\ -2.0505 - j4.3227 \\ -0.8989 \end{pmatrix}; \qquad \text{zeros: } s = \begin{pmatrix} -1 \\ -3 \end{pmatrix}.$$

Considering the closed-loop pole locations only, we would expect that the real pole at $s = -.8989$ would be the dominant pole. To verify this, we can study the closed-loop system response to a step input, $R(s) = 1/s$. For a step input, we have

$$Y(s) = \frac{20.5775(s + 1)(s + 3)}{s(s + 2)(s + 3) + 20.5775(s + 1)} \cdot \frac{1}{s}. \tag{7.118}$$

Generally, the first step in computing $y(t)$ is to expand Equation (7.118) in a partial fraction expansion. The residue function can be used to expand Equation (7.118), as shown in Figure 7.67. The residue function is described in Figure 7.68.




```
>>p=[1 1]; q=[1 5 6 0]; sys=tf(p,q); rlocus(sys)
>>rlocfind(sys)

Select a point in the graphics window

 selected_point =
   -2.0509 + 4.3228i
 ans =
    20.5775
```

rlocfind follows the rlocus function.

Value of K at selected point

FIGURE 7.66 Using the **rlocfind** function.

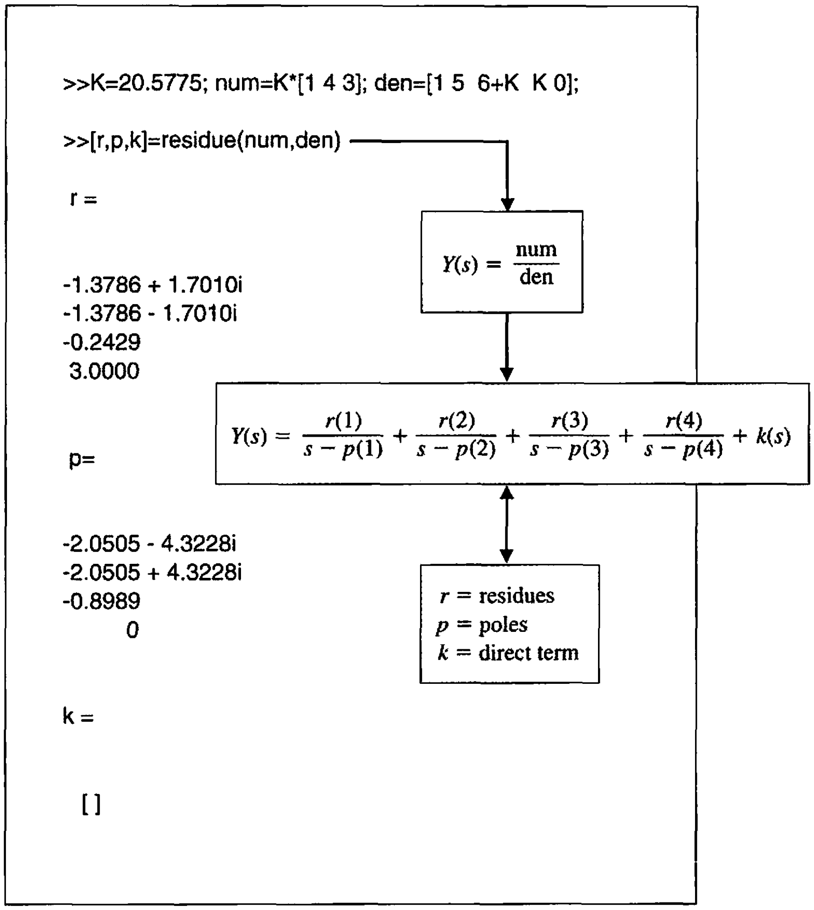


FIGURE 7.67 Partial fraction expansion of Equation (7.118).

The partial fraction expansion of Equation (7.118) is

$$Y(s) = \frac{-1.3786 + j1.7010}{s + 2.0505 + j4.3228} + \frac{-1.3786 - j1.7010}{s + 2.0505 - j4.3228} + \frac{-0.2429}{s + 0.8989} + \frac{3}{s}.$$

Comparing the residues, we see that the coefficient of the term corresponding to the pole at $s = -0.8989$ is considerably smaller than the coefficient of the terms corresponding to the complex-conjugate poles at $s = -2.0505 \pm j4.3227$. From this, we expect that the influence of the pole at $s = -0.8989$ on the output response $y(t)$ is not dominant. The settling time (to within 2% of the final value) is then predicted by considering the complex-conjugate poles. The poles at $s = -2.0505 \pm j4.3227$ correspond to a damping of $\zeta = 0.4286$ and a natural frequency of $\omega_n = 4.7844$. Thus, the settling time is predicted to be

$$T_s \simeq \frac{4}{\zeta\omega_n} = 1.95 \text{ s}.$$

Using the step function, as shown in Figure 7.69, we find that $T_s = 1.6$ s. Hence, our approximation of settling time $T_s \simeq 1.95$is a fairly good approximation. The percent overshoot can be predicted using Figure 5.13 since the zero of $T(s)$ at $s = -3$ will impact the system response. Using Figure 5.13, we predict an overshoot of 60%. As can be seen in Figure 7.48, the actual overshoot is 50%.

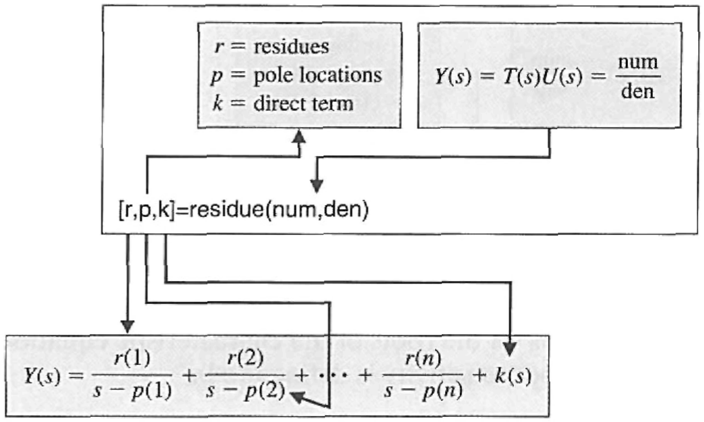
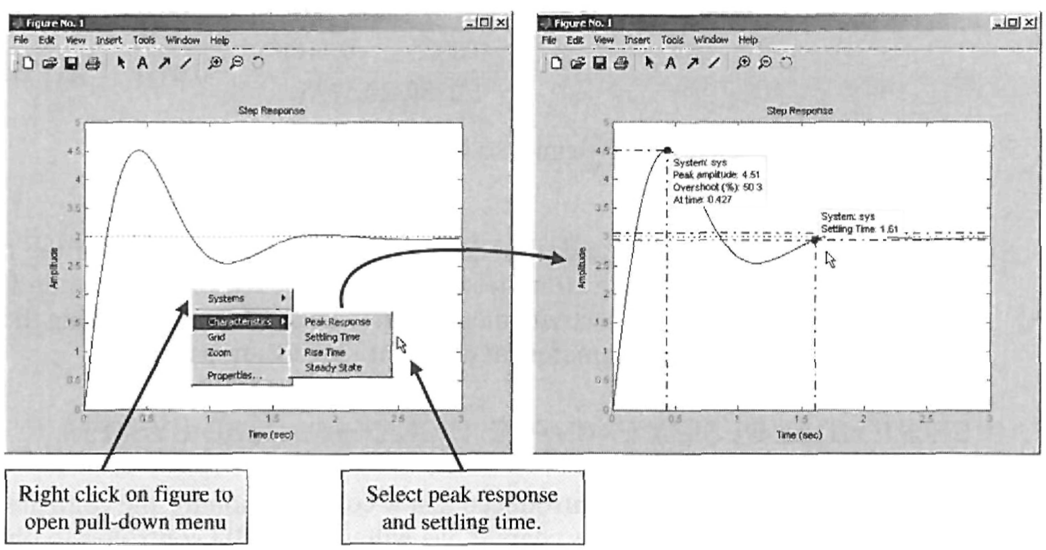


FIGURE 7.68 The **residue** function.

When using the step function, we can right-click on the figure to access the pull-down menu, which allows us to determine the step response settling time and peak response, as illustrated in Figure 7.69. On the pull-down menu select "Characteristics" and select "Settling Time." A dot will appear on the figure at the settling point. Place the cursor over the dot to determine the settling time.

In this example, the role of the system zeros on the transient response is illustrated. The proximity of the zero at $s = -1$ to the pole at $s = -0.8989$ reduces the impact of that pole on the transient response. The main contributors to the transient response are the complex-conjugate poles at $s = -2.0505 \pm j4.3228$ and the zero at $s = -3$.

There is one final point regarding the residue function: We can convert the partial fraction expansion back to the polynomials num/den, given the residues r, the pole locations p, and the direct terms k, with the command shown in Figure 7.70.




```
>>K=20.5775;num=k*[1 4 3]; den=[1 5 6+K K]; sys=tf(num,den);
>>step(sys)
```

FIGURE 7.69 Step response for the closed-loop system in Figure 7.10 with $K = 20.5775$.

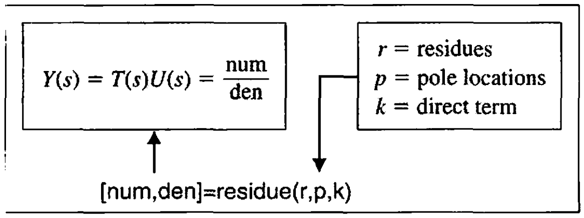


FIGURE 7.70 Converting a partial fraction expansion back to a rational function.

Sensitivity and the Root Locus. The roots of the characteristic equation play an important role in defining the closed-loop system transient response. The effect of parameter variations on the roots of the characteristic equation is a useful measure of sensitivity. The root sensitivity is defined to be

$$\frac{\partial r_i}{\partial K/K}. \tag{7.119}$$

We can use Equation (7.119) to investigate the sensitivity of the roots of the characteristic equation to variations in the parameter K. If we change K by a small finite amount ΔK, and evaluate the modified root $r_i + \Delta r_i$, it follows that

$$S_K^{r_i} \approx \frac{\Delta r_i}{\Delta K/K}. \tag{7.120}$$

The quantity $S_K^{r_i}$ is a complex number. Referring back to the third-order example of Figure 7.10 (Equation 7.116), if we change K by a factor of 5%, we find that the dominant complex-conjugate pole at $s = -2.0505 + j4.3228$ changes by

$$\Delta r_i = -0.0025 - j0.1168$$

when K changes from $K = 20.5775$ to $K = 21.6064$. From Equation (7.120), it follows that

$$S_K^{r_i} = \frac{-0.0025 - j0.1168}{1.0289/20.5775} = -0.0494 - j2.3355.$$

The sensitivity $S_K^{r_i}$ can also be written in the form

$$S_K^{r_i} = 2.34\underline{/268.79^\circ}.$$

The magnitude and direction of $S_K^{r_i}$ provides a measure of the root sensitivity. The script used to perform these sensitivity calculations is shown in Figure 7.71.

The root sensitivity measure may be useful for comparing the sensitivity for various system parameters at different root locations.

7.10 SEQUENTIAL DESIGN EXAMPLE: DISK DRIVE READ SYSTEM

In Chapter 6, we introduced a new configuration for the control system using velocity feedback. In this chapter, we will use the PID controller to obtain a desirable response. We will proceed with our model and then select a controller. Finally, we will optimize the parameters and analyze the performance. In this chapter, we will use the root locus method in the selection of the controller parameters.

```
% Compute the system sensitivity to a parameter
% variation
%
K=20.5775, den=[1 5 6+K K]; r1=roots(den);
%
dK=1.0289;   <---- 5% change in K
%
Km=K+dK; denm=[1 5 6+Km Km]; r2=roots(denm);
dr=r1-r2;    <---- Δr
%
S=dr/(dK/K); <---- Sensitivity formula
```

FIGURE 7.71 Sensitivity calculations for the root locus for a 5% change in $K = 20.5775$.

We use the root locus to select the controller gains. The PID controller introduced in this chapter is

$$G_c(s) = K_P + \frac{K_I}{s} + K_D s.$$

Since the process model $G_1(s)$ already possesses an integration, we set $K_I = 0$. Then we have the PD controller

$$G_c(s) = K_P + K_D s,$$

and our goal is to select K_P and K_D in order to meet the specifications. The system is shown in Figure 7.72. The closed-loop transfer function of the system is

$$\frac{Y(s)}{R(s)} = T(s) = \frac{G_c(s)G_1(s)G_2(s)}{1 + G_c(s)G_1(s)G_2(s)H(s)},$$

where $H(s) = 1$.

In order to obtain the root locus as a function of a parameter, we write $G_c(s)G_1(s)G_2(s)H(s)$ as

$$G_c(s)G_1(s)G_2(s)H(s) = \frac{5000(K_P + K_D s)}{s(s+20)(s+1000)} = \frac{5000K_D(s+z)}{s(s+20)(s+1000)},$$

where $z = K_P/K_D$. We use K_P to select the location of the zero z and then sketch the locus as a function of K_D. Based on the insight developed in Section 6.7, we select $z = 1$ so that

$$G_c(s)G_1(s)G_2(s)H(s) = \frac{5000K_D(s+1)}{s(s+20)(s+1000)}.$$

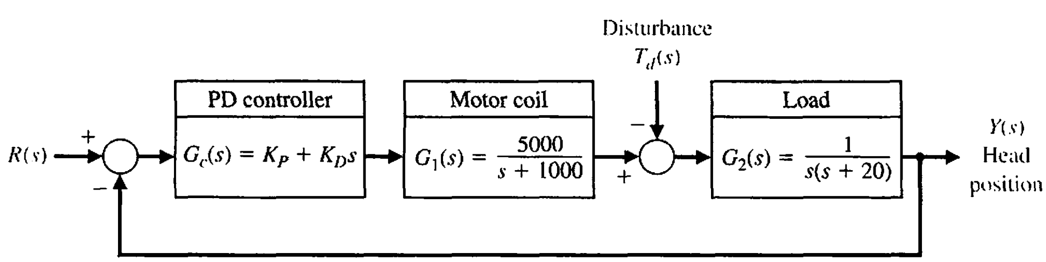


FIGURE 7.72 Disk drive control system with a PD controller.

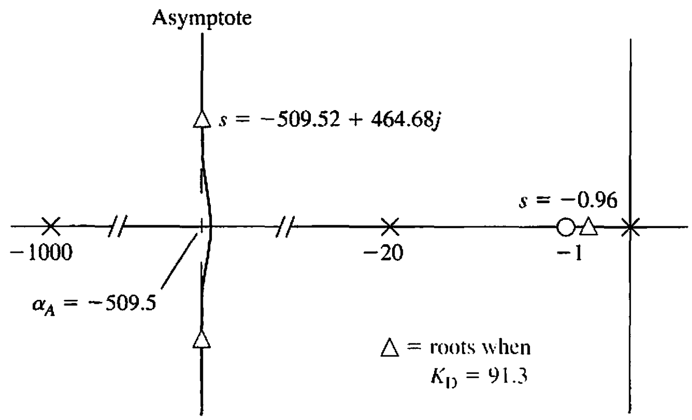


FIGURE 7.73 Sketch of the root locus.

Table 7.10 Disk Drive Control System Specifications and Actual Design Performance

Performance Measure	Desired Value	Actual Response
Percent overshoot	Less than 5%	0%
Settling time	Less than 250 ms	20 ms
Maximum response to a unit disturbance	Less than 5×10^{-3}	2×10^{-3}

The number of poles minus the number of zeros is 2, and we expect asymptotes at $\phi_A = \pm 90°$ with a centroid

$$\sigma_A = \frac{-1020 + 1}{2} = -509.5,$$

as shown in Figure 7.73. We can quickly sketch the root locus, as shown in Figure 7.73. We use the computer-generated root locus to determine the root values for various values of K_D. When $K_D = 91.3$, we obtain the roots shown in Figure 7.73. Then, obtaining the system response, we achieve the actual response measures as listed in Table 7.10. As designed, the system meets all the specifications. It takes the system a settling time of 20 ms to "practically" reach the final value. In reality, the system drifts very slowly toward the final value after quickly achieving 97% of the final value.

7.11 SUMMARY

The relative stability and the transient response performance of a closed-loop control system are directly related to the location of the closed-loop roots of the characteristic equation. We investigated the movement of the characteristic roots on the s-plane as key system parameters (such as controller gains) are varied. The root locus and the negative gain root locus are graphical representations of the variation of the system closed-loop poles as one parameter varies. The plots can be sketched by hand using a given set of rules in order to analyze the initial design of a system and determine suitable alterations of the system structure and the parameter values. A computer is then commonly used to obtain the accurate root locus for use in the final design and analysis. A summary of fifteen typical root locus diagrams is shown in Table 7.11.

Table 7.11 Root Locus Plots for Typical Transfer Functions

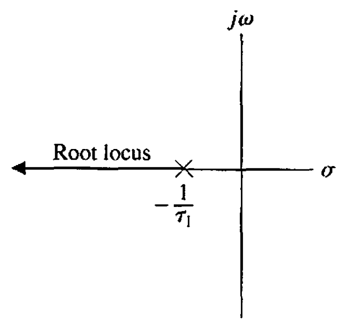
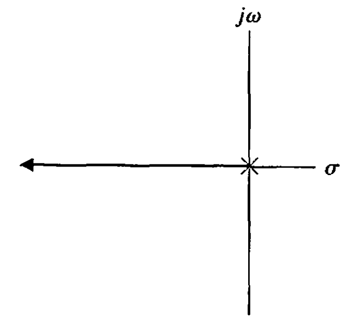
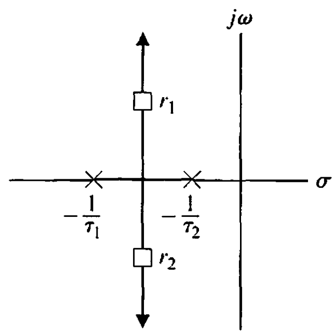
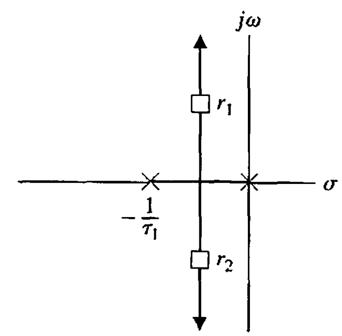
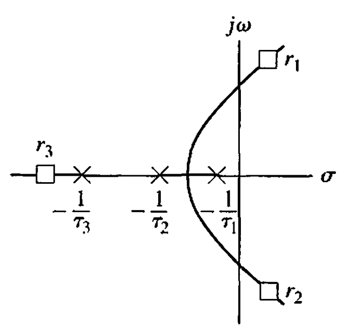
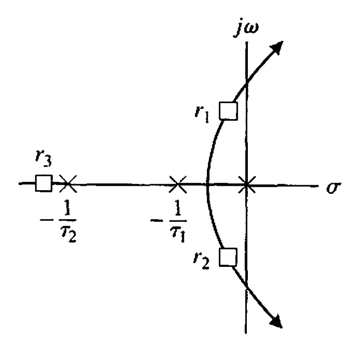
G(s)	Root Locus	G(s)	Root Locus
1. $\dfrac{K}{s\tau_1 + 1}$		4. $\dfrac{K}{s}$	
2. $\dfrac{K}{(s\tau_1 + 1)(s\tau_2 + 1)}$		5. $\dfrac{K}{s(s\tau_1 + 1)}$	
3. $\dfrac{K}{(s\tau_1 + 1)(s\tau_2 + 1)(s\tau_3 + 1)}$		6. $\dfrac{K}{s(s\tau_1 + 1)(s\tau_2 + 1)}$	

Table 7.11 *(continued)*

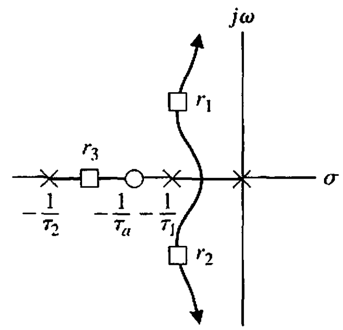
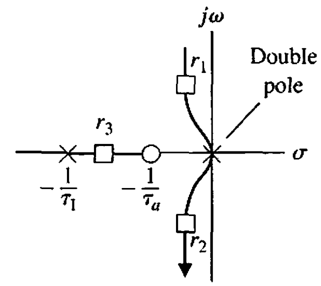
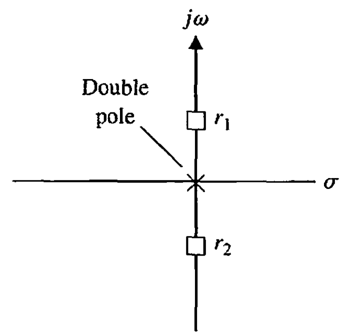
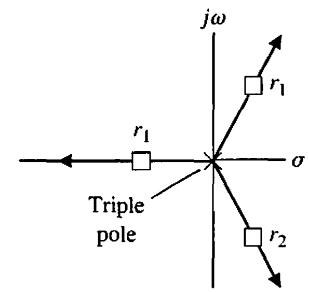
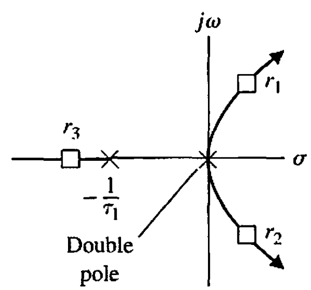
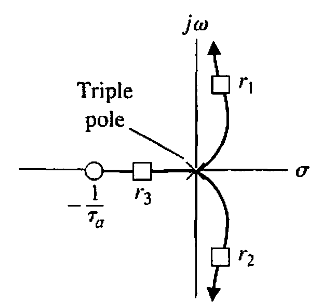
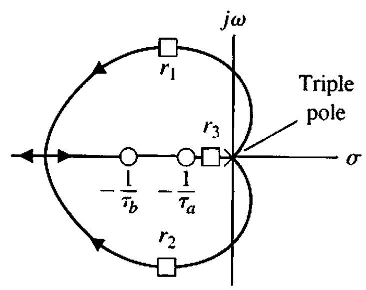
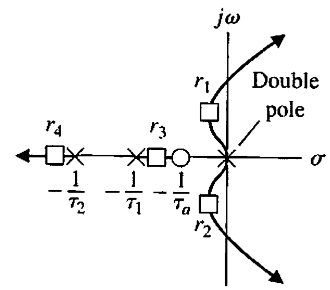
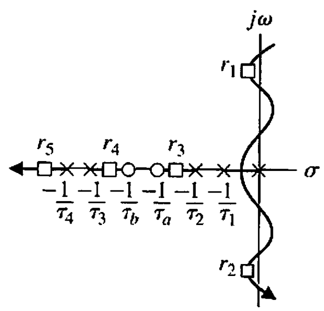
<table>
<tr><th>G(s)</th><th>Root Locus</th><th>G(s)</th><th>Root Locus</th></tr>
<tr><td>7. $\dfrac{K(s\tau_a + 1)}{s(s\tau_1 + 1)(s\tau_2 + 1)}$</td><td>
</td><td>10. $\dfrac{K(s\tau_a + 1)}{s^2(s\tau_1 + 1)}$
$\tau_a > \tau_1$</td><td>
</td></tr>
<tr><td>8. $\dfrac{K}{s^2}$</td><td>
</td><td>11. $\dfrac{K}{s^3}$</td><td>
</td></tr>
<tr><td>9. $\dfrac{K}{s^2(s\tau_1 + 1)}$</td><td>
</td><td>12. $\dfrac{K(s\tau_a + 1)}{s^3}$</td><td>
</td></tr>
</table>

Table 7.11 *(continued)*

G(s)	Root Locus	G(s)	Root Locus
13. $\dfrac{K(s\tau_a + 1)(s\tau_b + 1)}{s^3}$		15. $\dfrac{K(s\tau_a + 1)}{s^2(s\tau_1 + 1)(s\tau_2 + 1)}$	
14. $\dfrac{K(s\tau_a + 1)(s\tau_b + 1)}{s(s\tau_1 + 1)(s\tau_2 + 1)(s\tau_3 + 1)(s\tau_4 + 1)}$			

Furthermore, we extended the root locus method for the design of several parameters for a closed-loop control system. Then the sensitivity of the characteristic roots was investigated for undesired parameter variations by defining a root sensitivity measure. It is clear that the root locus method is a powerful and useful approach for the analysis and design of modern control systems and will continue to be one of the most important procedures of control engineering.

SKILLS CHECK

In this section, we provide three sets of problems to test your knowledge: True or False, Multiple Choice, and Word Match. To obtain direct feedback, check your answers with the answer key provided at the conclusion of the end-of-chapter problems. Use the block diagram in Figure 7.74 as specified in the various problem statements.

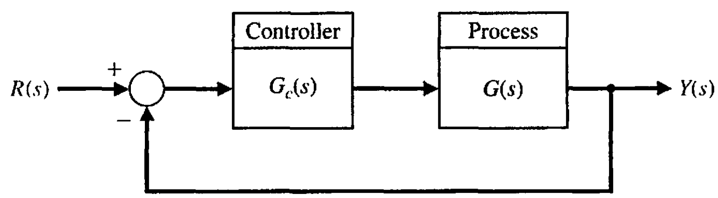


FIGURE 7.74 Block diagram for the Skills Check.

In the following **True or False** and **Multiple Choice** problems, circle the correct answer.

1. The root locus is the path the roots of the characteristic equation (given by $1 + KG(s) = 0$) trace out on the s-plane as the system parameter $0 \le K < \infty$ varies. *True or False*
2. On the root locus plot, the number of separate loci is equal to the number of poles of $G(s)$. *True or False*
3. The root locus always starts at the zeros and ends at the poles of $G(s)$. *True or False*
4. The root locus provides the control system designer with a measure of the sensitivity of the poles of the system to variations of a parameter of interest. *True or False*
5. The root locus provides valuable insight into the response of a system to various test inputs. *True or False*
6. Consider the control system in Figure 7.74, where the loop transfer function is

$$L(s) = G_c(s)G(s) = \frac{K(s^2 + 5s + 9)}{s^2(s + 3)}.$$

Using the root locus method, determine the value of K such that the dominant roots have a damping ratio $\zeta = 0.5$.

a. $K = 1.2$
b. $K = 4.5$
c. $K = 9.7$
d. $K = 37.4$

In Problems 7 and 8, consider the unity feedback system in Figure 7.74 with

$$L(s) = G_c(s)G(s) = \frac{K(s+1)}{s^2 + 5s + 17.33}.$$

7. The approximate angles of departure of the root locus from the complex poles are

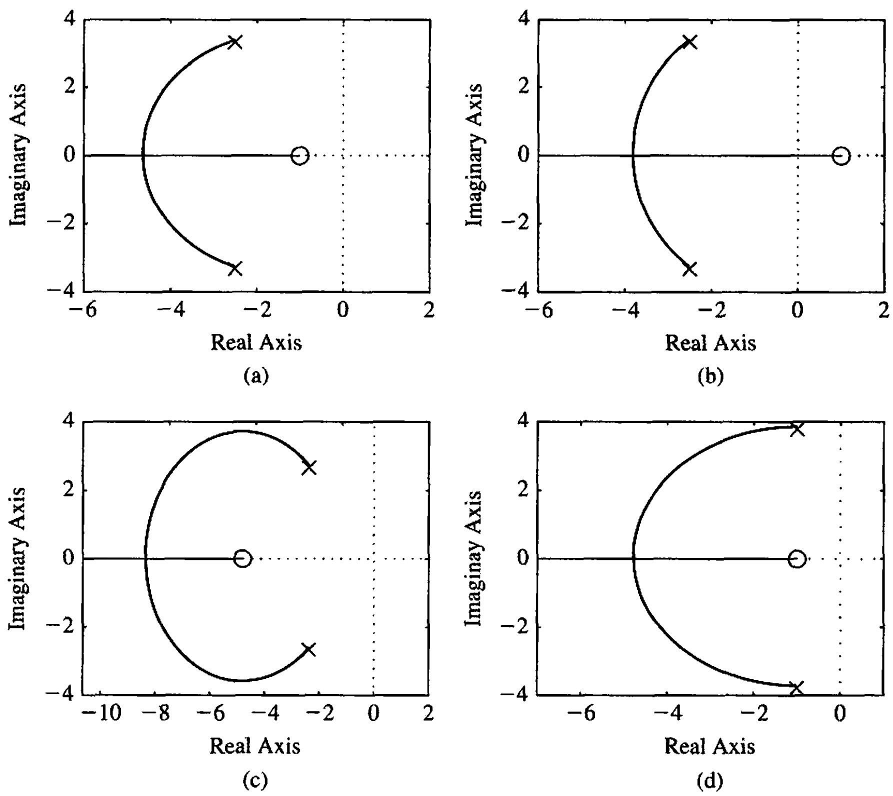
a. $\phi_d = \pm 180°$

b. $\phi_d = \pm 115°$

c. $\phi_d = \pm 205°$

d. None of the above

8. The root locus of this system is given by which of the following



9. A unity feedback system has the closed-loop transfer function given by

$$T(s) = \frac{K}{(s+45)^2 + K}.$$

Using the root locus method, determine the value of the gain K so that the closed-loop system has a damping ratio $\zeta = \sqrt{2}/2$.

a. $K = 25$

b. $K = 1250$

c. $K = 2025$

d. $K = 10500$

10. Consider the unity feedback control system in Figure 7.74 where

$$L(s) = G_c(s)G(s) = \frac{10(s + z)}{s(s^2 + 4s + 8)}.$$

Using the root locus method, determine that maximum value of z for closed-loop stability.

a. $z = 7.2$

b. $z = 12.8$

c. Unstable for all $z > 0$

d. Stable for all $z > 0$

In Problems 11 and 12, consider the control system in Figure 7.74 where the model of the process is

$$G(s) = \frac{7500}{(s + 1)(s + 10)(s + 50)}.$$

11. Suppose that the controller is

$$G_c(s) = \frac{K(1 + 0.2s)}{1 + 0.025s}.$$

Using the root locus method, determine the maximum value of the gain K for closed-loop stability.

a. $K = 2.13$

b. $K = 3.88$

c. $K = 14.49$

d. Stable for all $K > 0$

12. Suppose that a simple proportional controller is utilized, that is, $G_c(s) = K$. Using the root locus method, determine the maximum controller gain K for closed-loop stability.

a. $K = 0.50$

b. $K = 1.49$

c. $K = 4.49$

d. Unstable for $K > 0$

13. Consider the unity feedback system in Figure 7.74 where

$$L(s) = G_c(s)G(s) = \frac{K}{s(s + 5)(s^2 + 6s + 17.76)}.$$

Determine the breakaway point on the real axis and the respective gain, K.

a. $s = -1.8, K = 58.75$

b. $s = -2.5, K = 4.59$

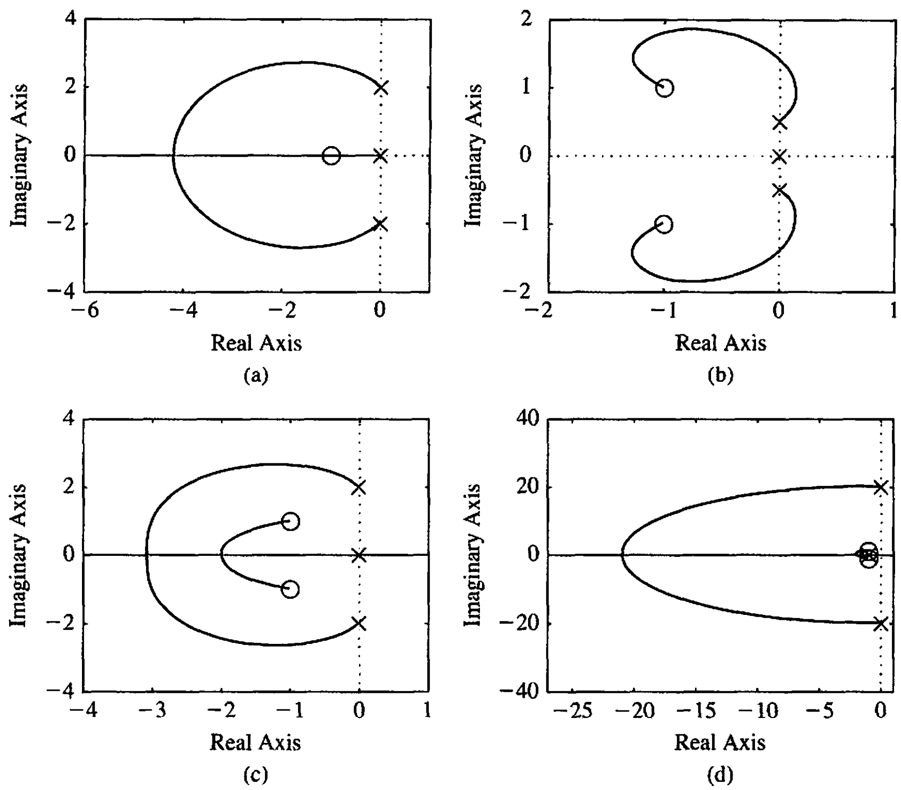
c. $s = 1.4, K = 58.75$

d. None of the above

In Problems 14 and 15, consider the feedback system in Figure 7.74, where

$$L(s) = G_c(s)G(s) = \frac{K(s + 1 + j)(s + 1 - j)}{s(s + 2j)(s - 2j)}.$$

14. Which of the following is the associated root locus?



15. The departure angles from the complex poles and the arrival angles at the complex zeros are:

a. $\phi_D = \pm 180°, \phi_A = 0°$

b. $\phi_D = \pm 116.6°, \phi_A = \pm 198.4°$

c. $\phi_D = \pm 45.8°, \phi_A = \pm 116.6°$

d. None of the above

In the following **Word Match** problems, match the term with the definition by writing the correct letter in the space provided.

a. Parameter design	The amplitude of the closed-loop response is reduced approximately to one-fourth of the maximum value in one oscillatory period.	______
b. Root sensitivity	The path the root locus follows as the parameter becomes very large and approaches ∞.	______
c. Root locus	The center of the linear asymptotes, σ_A.	______
d. Root locus segments on the real axis	The process of determining the PID controller gains using one of several analytic methods based on open-loop and closed-loop responses to step inputs.	______
e. Root locus method	A method of selecting one or two parameters using the root locus method.	______

f. Asymptote centroid	The root locus lying in a section of the real axis to the left of an odd number of poles and zeros.	______
g. Breakaway point	The root locus for negative values of the parameter of interest where $-\infty < K \leq 0$.	______
h. Locus	The angle at which a locus leaves a complex pole in the s-plane.	______
i. Angle of departure	A path or trajectory that is traced out as a parameter is changed.	______
j. Number of separate loci	The locus or path of the roots traced out on the s-plane as a parameter is changed.	______
k. Asymptote	The sensitivity of the roots as a parameter changes from its normal value.	______
l. Negative gain root locus	The method for determining the locus of roots of the characteristic equation $1 + KG(s) = 0$ as $0 \leq K < \infty$.	______
m. PID tuning	The process of determining the PID controller gains.	______
n. Quarter amplitude decay	The point on the real axis where the locus departs from the real axis of the s-plane.	______
o. Ziegler-Nichols PID tuning method	Equal to the number of poles of the transfer function, assuming that the number of poles is greater than or equal to the number of zeros of the transfer function.	______

EXERCISES

E7.1 Let us consider a device that consists of a ball rolling on the inside rim of a hoop [11]. This model is similar to the problem of liquid fuel sloshing in a rocket. The hoop is free to rotate about its horizontal principal axis as shown in Figure E7.1. The angular position of the hoop may be controlled via the torque T applied to the hoop from a torque motor attached to the hoop drive shaft. If negative feedback is used, the system characteristic equation is

$$1 + \frac{Ks(s + 4)}{s^2 + 2s + 2} = 0.$$

(a) Sketch the root locus. (b) Find the gain when the roots are both equal. (c) Find these two equal roots.

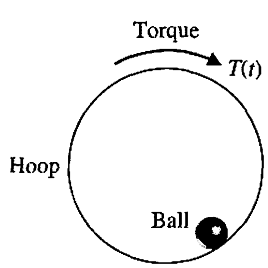


FIGURE E7.1 Hoop rotated by motor.

(d) Find the settling time of the system when the roots are equal.

E7.2 A tape recorder has a speed control system so that $H(s) = 1$ with negative feedback and

$$L(s) = G_c(s)G(s) = \frac{K}{s(s + 2)(s^2 + 4s + 5)}.$$

(a) Sketch a root locus for K, and show that the dominant roots are $s = -0.35 \pm j0.80$ when $K = 6.5$.
(b) For the dominant roots of part (a), calculate the settling time and overshoot for a step input.

E7.3 A control system for an automobile suspension tester has negative unity feedback and a process [12]

$$L(s) = G_c(s)G(s) = \frac{K(s^2 + 4s + 8)}{s^2(s + 4)}.$$

We desire the dominant roots to have a ζ equal to 0.5. Using the root locus, show that $K = 7.35$ is required and the dominant roots are $s = -1.3 \pm j2.2$.

E7.4 Consider a unity feedback system with

$$L(s) = G_c(s)G(s) = \frac{K(s + 1)}{s^2 + 4s + 5}.$$

(a) Find the angle of departure of the root locus from the complex poles. (b) Find the entry point for the root locus as it enters the real axis.

Answers: $\pm 225°$; -2.4

E7.5 Consider a unity feedback system with a loop transfer function

$$G_c(s)G(s) = \frac{s^2 + 2s + 10}{s^4 + 38s^3 + 515s^2 + 2950s + 6000}.$$

(a) Find the breakaway points on the real axis. (b) Find the asymptote centroid. (c) Find the values of K at the breakaway points.

E7.6 One version of a space station is shown in Figure E7.6 [28]. It is critical to keep this station in the proper orientation toward the Sun and the Earth for generating power and communications. The orientation controller may be represented by a unity feedback system with an actuator and controller, such as

$$G_c(s)G(s) = \frac{15K}{s(s^2 + 15s + 75)}.$$

Sketch the root locus of the system as K increases. Find the value of K that results in an unstable system.

Answers: $K = 75$

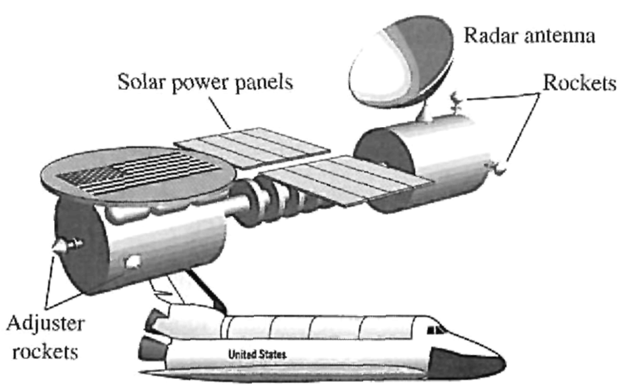


FIGURE E7.6 Space station.

E7.7 The elevator in a modern office building travels at a top speed of 25 feet per second and is still able to stop within one-eighth of an inch of the floor outside. The loop transfer function of the unity feedback elevator position control is

$$L(s) = G_c(s)G(s) = \frac{K(s + 8)}{s(s + 4)(s + 6)(s + 9)}.$$

Determine the gain K when the complex roots have a ζ equal to 0.8.

E7.8 Sketch the root locus for a unity feedback system with

$$L(s) = G_c(s)G(s) = \frac{K(s + 1)}{s^2(s + 9)}.$$

(a) Find the gain when all three roots are real and equal. (b) Find the roots when all the roots are equal as in part (a).

Answers: $K = 27$; $s = -3$

E7.9 The world's largest telescope is located in Hawaii. The primary mirror has a diameter of 10 m and consists of a mosaic of 36 hexagonal segments with the orientation of each segment actively controlled. This unity feedback system for the mirror segments has the loop transfer function

$$L(s) = G_c(s)G(s) = \frac{K}{s(s^2 + 2s + 5)}.$$

(a) Find the asymptotes and draw them in the s-plane.
(b) Find the angle of departure from the complex poles.
(c) Determine the gain when two roots lie on the imaginary axis.
(d) Sketch the root locus.

E7.10 A unity feedback system has the loop transfer function

$$L(s) = KG(s) = \frac{K(s + 2)}{s(s + 1)}.$$

(a) Find the breakaway and entry points on the real axis.
(b) Find the gain and the roots when the real part of the complex roots is located at -2.
(c) Sketch the locus.

Answers: (a) $-0.59, -3.41$; (b) $K = 3, s = -2 \pm j\sqrt{2}$

E7.11 A robot force control system with unity feedback has a loop transfer function [6]

$$L(s) = KG(s) = \frac{K(s + 2.5)}{(s^2 + 2s + 2)(s^2 + 4s + 5)}.$$

(a) Find the gain K that results in dominant roots with a damping ratio of 0.707. Sketch the root locus.
(b) Find the actual percent overshoot and peak time for the gain K of part (a).

E7.12 A unity feedback system has a loop transfer function

$$L(s) = KG(s) = \frac{K(s + 1)}{s(s^2 + 6s + 18)}.$$

(a) Sketch the root locus for $K > 0$. (b) Find the roots when $K = 10$ and 20. (c) Compute the rise time, percent overshoot, and settling time (with a 2% criterion) of the system for a unit step input when $K = 10$ and 20.

E7.13 A unity feedback system has a loop transfer function

$$L(s) = G_c(s)G(s) = \frac{4(s + z)}{s(s + 1)(s + 3)}.$$

(a) Draw the root locus as z varies from 0 to 100. (b) Using the root locus, estimate the percent overshoot and settling time (with a 2% criterion) of the system at $z = 0.6$, 2, and 4 for a step input. (c) Determine the actual overshoot and settling time at $z = 0.6$, 2, and 4.

E7.14 A unity feedback system has the loop transfer function

$$L(s) = G_c(s)G(s) = \frac{K(s + 10)}{s(s + 5)}.$$

(a) Determine the breakaway and entry points of the root locus and sketch the root locus for $K > 0$. (b) Determine the gain K when the two characteristic roots have a ζ of $1/\sqrt{2}$. (c) Calculate the roots.

E7.15 (a) Plot the root locus for a unity feedback system with loop transfer function

$$L(s) = G_c(s)G(s) = \frac{K(s + 10)(s + 2)}{s^3}.$$

(b) Calculate the range of K for which the system is stable. (c) Predict the steady-state error of the system for a ramp input.

Answers: (a) $K > 1.67$; (b) $e_{ss} = 0$

E7.16 A negative unity feedback system has a loop transfer function

$$L(s) = G_c(s)G(s) = \frac{Ke^{-sT}}{s + 1},$$

where $T = 0.1$ s. Show that an approximation for the time delay is

$$e^{-sT} \approx \frac{\frac{2}{T} - s}{\frac{2}{T} + s}.$$

Using

$$e^{-0.1s} = \frac{20 - s}{20 + s},$$

obtain the root locus for the system for $K > 0$. Determine the range of K for which the system is stable.

E7.17 A control system, as shown in Figure E7.17, has a process

$$G(s) = \frac{1}{s(s - 1)}.$$

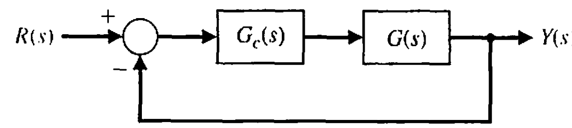


FIGURE E7.17 Feedback system.

(a) When $G_c(s) = K$, show that the system is always unstable by sketching the root locus. (b) When

$$G_c(s) = \frac{K(s + 2)}{s + 20},$$

sketch the root locus and determine the range of K for which the system is stable. Determine the value of K and the complex roots when two roots lie on the $j\omega$-axis.

E7.18 A closed-loop negative unity feedback system is used to control the yaw of the A-6 Intruder attack jet. When the loop transfer function is

$$L(s) = G_c(s)G(s) = \frac{K}{s(s + 3)(s^2 + 2s + 2)},$$

determine (a) the root locus breakaway point and (b) the value of the roots on the $j\omega$-axis and the gain required for those roots. Sketch the root locus.

Answers: (a) Breakaway: $s = -2.29$ (b) $j\omega$-axis: $s = \pm j1.09$, $K = 8$

E7.19 A unity feedback system has a loop transfer function

$$L(s) = G_c(s)G(s) = \frac{K}{s(s + 3)(s^2 + 6s + 64)}.$$

(a) Determine the angle of departure of the root locus at the complex poles. (b) Sketch the root locus. (c) Determine the gain K when the roots are on the $j\omega$-axis and determine the location of these roots.

E7.20 A unity feedback system has a loop transfer function

$$L(s) = G_c(s)G(s) = \frac{K(s + 1)}{s(s - 2)(s + 6)}.$$

(a) Determine the range of K for stability. (b) Sketch the root locus. (c) Determine the maximum ζ of the stable complex roots.

Answers: (a) $K > 16$; (b) $\zeta = 0.25$

E7.21 A unity feedback system has a loop transfer function

$$L(s) = G_c(s)G(s) = \frac{Ks}{s^3 + 5s^2 + 10}.$$

Sketch the root locus. Determine the gain K when the complex roots of the characteristic equation have a ζ approximately equal to 0.66.

E7.22 A high-performance missile for launching a satellite has a unity feedback system with a loop transfer function

$$G_c(s)G(s) = \frac{K(s^2 + 18)(s + 2)}{(s^2 - 2)(s + 12)}.$$

Sketch the root locus as K varies from $0 < K < \infty$.

E7.23 A unity feedback system has a loop transfer function

$$L(s) = G_c(s)G(s) = \frac{4(s^2 + 1)}{s(s + a)}.$$

Sketch the root locus for $0 \leq a < \infty$.

E7.24 Consider the system represented in state variable form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$y = \mathbf{C}\mathbf{x} + \mathbf{D}u,$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -4 & -k \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$\mathbf{C} = [1 \quad 0], \text{ and } \mathbf{D} = [0].$$

Determine the characteristic equation and then sketch the root locus as $0 < k < \infty$.

E7.25 A closed-loop feedback system is shown in Figure E7.25. For what range of values of the parameters K is the system stable? Sketch the root locus as $0 < K < \infty$.

E7.26 Consider the signle-input, single-output system is described by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}\mathbf{x}(t)$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 3 - K & -2 - K \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{C} = [1 \quad -1].$$

Compute the characteristic polynomial and plot the root locus as $0 \leq K < \infty$. For what values of K is the system stable?

E7.27 Consider the unity feedback system in Figure E7.27. Sketch the root locus as $0 \leq p < \infty$.

E7.28. Consider the feedback system in Figure E7.28. Obtain the negative gain root locus as $-\infty < K \leq 0$. For what values of K is the system stable?

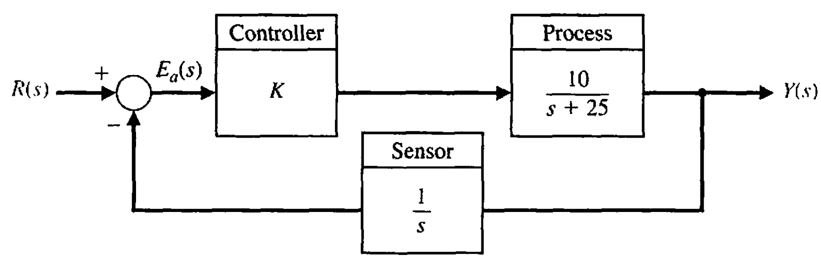


FIGURE E7.25 Nonunity feedback system with parameter K.

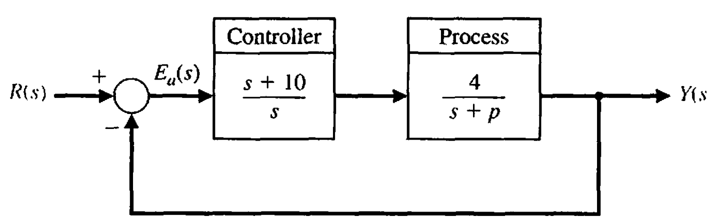


FIGURE E7.27 Unity feedback system with parameter p.

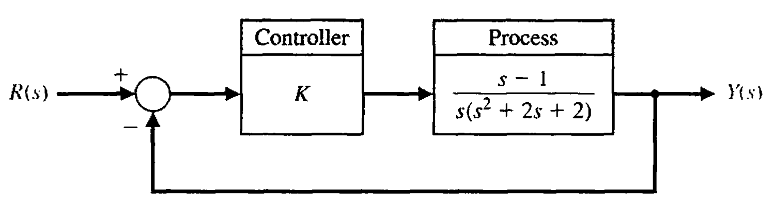


FIGURE E7.28 Feedback system for negative gain root locus.

PROBLEMS

P7.1 Sketch the root locus for the following loop transfer functions of the system shown in Figure P7.1 when $0 < K < \infty$:

(a) $G_c(s)G(s) = \dfrac{K}{s(s+10)(s+8)}$

(b) $G_c(s)G(s) = \dfrac{K}{(s^2+2s+2)(s+2)}$

(c) $G_c(s)G(s) = \dfrac{K(s+5)}{s(s+1)(s+10)}$

(d) $G_c(s)G(s) = \dfrac{K(s^2+4s+8)}{s^2(s+1)}$

P7.2 The linear model of a phase detector was presented in Problem P6.7. Sketch the root locus as a function of the gain $K_v = K_a K$. Determine the value of K_v attained if the complex roots have a damping ratio equal to 0.60 [13].

P7.3 A unity feedback system has the loop transfer function

$$G_c(s)G(s) = \frac{K}{s(s+2)(s+5)}.$$

Find (a) the breakaway point on the real axis and the gain K for this point, (b) the gain and the roots when two roots lie on the imaginary axis, and (c) the roots when $K = 6$. (d) Sketch the root locus.

P7.4 The analysis of a large antenna was presented in Problem P4.5. Sketch the root locus of the system as $0 < k_a < \infty$. Determine the maximum allowable gain of the amplifier for a stable system.

P7.5 Automatic control of helicopters is necessary because, unlike fixed-wing aircraft which possess a fair degree of inherent stability, the helicopter is quite unstable. A helicopter control system that utilizes an automatic control loop plus a pilot stick control is shown in Figure P7.5. When the pilot is not using the control stick, the switch may be considered to be open. The dynamics of the helicopter are represented by the transfer function

$$G(s) = \frac{25(s+0.03)}{(s+0.4)(s^2-0.36s+0.16)}.$$

(a) With the pilot control loop open (hands-off control), sketch the root locus for the automatic stabilization loop. Determine the gain K_2 that results in a damping for the complex roots equal to $\zeta = 0.707$. (b) For the gain K_2 obtained in part (a), determine the steady-state error due to a wind gust $T_d(s) = 1/s$. (c) With the pilot loop added, draw the root locus as K_1 varies from zero to ∞ when K_2 is set at the value calculated in part (a). (d) Recalculate the steady-state error of part (b) when K_1 is equal to a suitable value based on the root locus.

P7.6 An attitude control system for a satellite vehicle within the earth's atmosphere is shown in Figure P7.6. The transfer functions of the system are

$$G(s) = \frac{K(s+0.20)}{(s+0.90)(s-0.60)(s-0.10)}$$

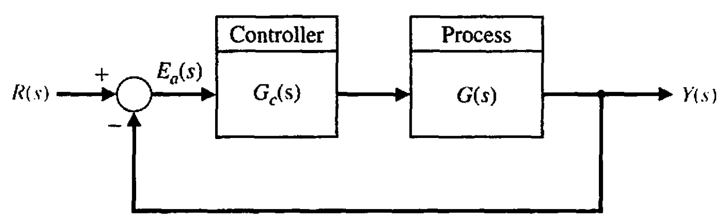


FIGURE P7.1

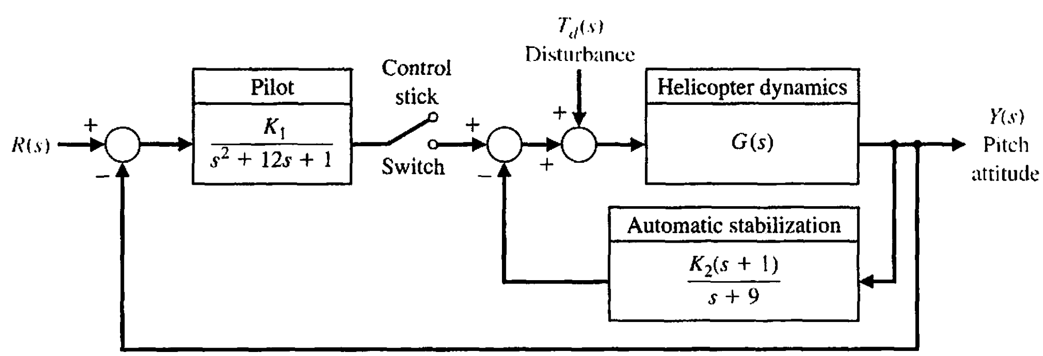


FIGURE P7.5
Helicopter control.

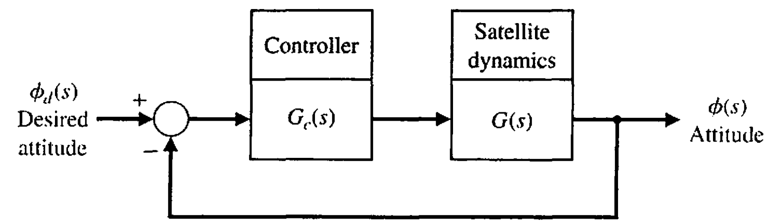


FIGURE P7.6 Satellite attitude control.

and

$$G_c(s) = \frac{(s + 2 + j1.5)(s + 2 - j1.5)}{s + 4.0}.$$

(a) Draw the root locus of the system as K varies from 0 to ∞. (b) Determine the gain K that results in a system with a settling time (with a 2% criterion) less than 12 seconds and a damping ratio for the complex roots greater than 0.50.

P7.7 The speed control system for an isolated power system is shown in Figure P7.7. The valve controls the steam flow input to the turbine in order to account for load changes $\Delta L(s)$ within the power distribution network. The equilibrium speed desired results in a generator frequency equal to 60 cps. The effective rotary inertia J is equal to 4000 and the friction constant b is equal to 0.75. The steady-state speed regulation factor R is represented by the equation $R \approx (\omega_0 - \omega_r)/\Delta L$, where ω_r equals the speed at rated load and ω_0 equals the speed at no load. We want to obtain a very small R, usually less than 0.10. (a) Using root locus techniques, determine the regulation R attainable when the damping ratio of the roots of the system must be greater than 0.60. (b) Verify that the steady-state speed deviation for a load torque change $\Delta L(s) = \Delta L/s$ is, in fact, approximately equal to $R\Delta L$ when $R \leq 0.1$.

P7.8 Consider again the power control system of Problem P7.7 when the steam turbine is replaced by a hydroturbine. For hydroturbines, the large inertia of the water used as a source of energy causes a considerably larger time constant. The transfer function of a hydroturbine may be approximated by

$$G_t(s) = \frac{-\tau s + 1}{(\tau/2)s + 1},$$

where $\tau = 1$ second. With the rest of the system remaining as given in Problem P7.7, repeat parts (a) and (b) of Problem P7.7.

P7.9 The achievement of safe, efficient control of the spacing of automatically controlled guided vehicles is an important part of the future use of the vehicles in a manufacturing plant [14, 15]. It is important that the system eliminate the effects of disturbances (such as oil on the floor) as well as maintain accurate spacing between vehicles on a guideway. The system can be represented by the block diagram of Figure P7.9. The vehicle dynamics can be represented by

$$G(s) = \frac{(s + 0.1)(s^2 + 2s + 289)}{s(s - 0.4)(s + 0.8)(s^2 + 1.45s + 361)}.$$

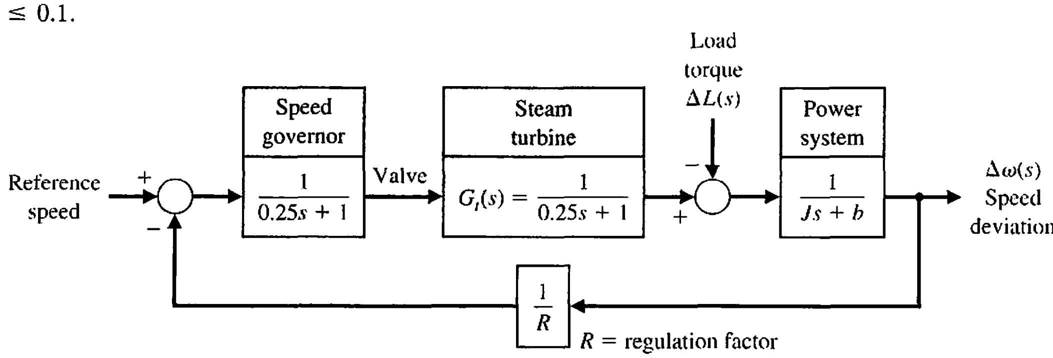


FIGURE P7.7 Power system control.

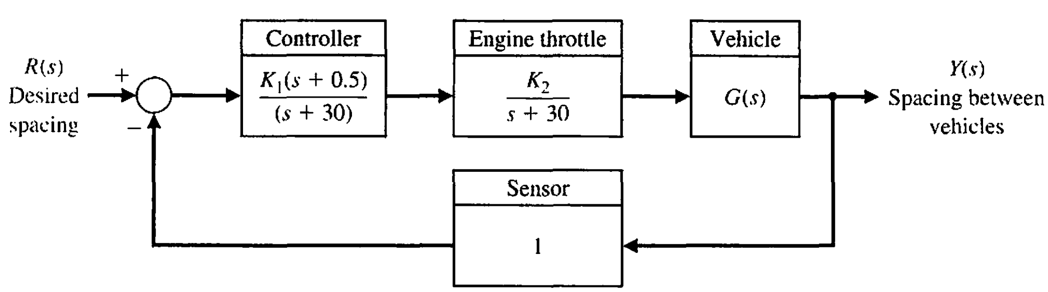


FIGURE P7.9 Guided vehicle control.

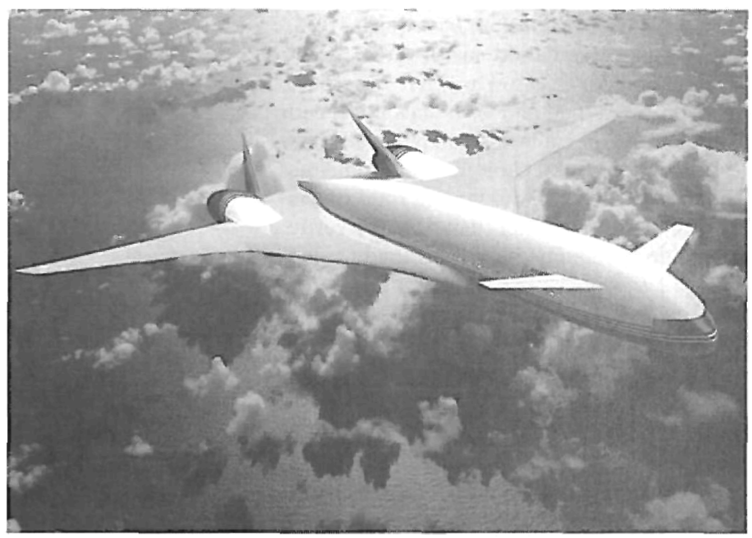
(a) Sketch the root locus of the system. (b) Determine all the roots when the loop gain $K = K_1K_2$ is equal to 4000.

P7.10 New concepts in passenger airliner design will have the range to cross the Pacific in a single flight and the efficiency to make it economical [16, 29]. These new designs will require the use of temperature-resistant, lightweight materials and advanced control systems. Noise control is an important issue in modern aircraft designs since most airports have strict noise level requirements. One interesting concept is the Boeing Sonic Cruiser depicted in Figure P7.10(a). It would seat 200 to 250 passengers and cruise at just below the speed of sound.

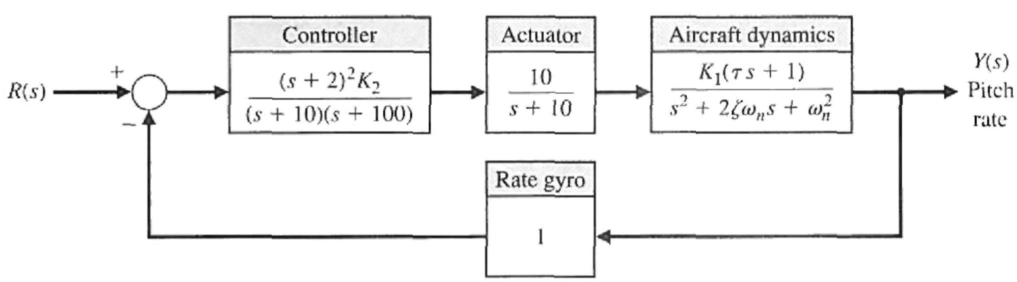
The flight control system must provide good handling characteristics and comfortable flying conditions. An automatic control system can be designed for the next generation passenger aircraft.

The desired characteristics of the dominant roots of the control system shown in Figure P7.10(b) have a $\zeta = 0.707$. The characteristics of the aircraft are $\omega_n = 2.5$, $\zeta = 0.30$, and $\tau = 0.1$. The gain factor K_1, however, will vary over the range 0.02 at medium-weight cruise conditions to 0.20 at lightweight descent conditions. (a) Sketch the root locus as a function of the loop gain K_1K_2. (b) Determine the gain K_2 necessary to yield roots with $\zeta = 0.707$ when the aircraft is in the medium-cruise condition. (c) With the gain K_2 as found in part (b), determine the ζ of the roots when the gain K_1 results from the condition of light descent.

P7.11 A computer system requires a high-performance magnetic tape transport system [17]. The environmental conditions imposed on the system result in a severe test of control engineering design. A direct-drive DC motor system for the magnetic tape reel system is shown in Figure P7.11, where r equals the reel radius, and J equals the reel and rotor inertia. A complete reversal of the tape reel direction is required in 6 ms, and the tape reel must follow a step command in 3 ms or less. The tape is normally operating at a speed of



(a)



(b)

FIGURE P7.10 (a) A passenger jet aircraft of the future. (™ and © Boeing. Used under license.) (b) Control system.

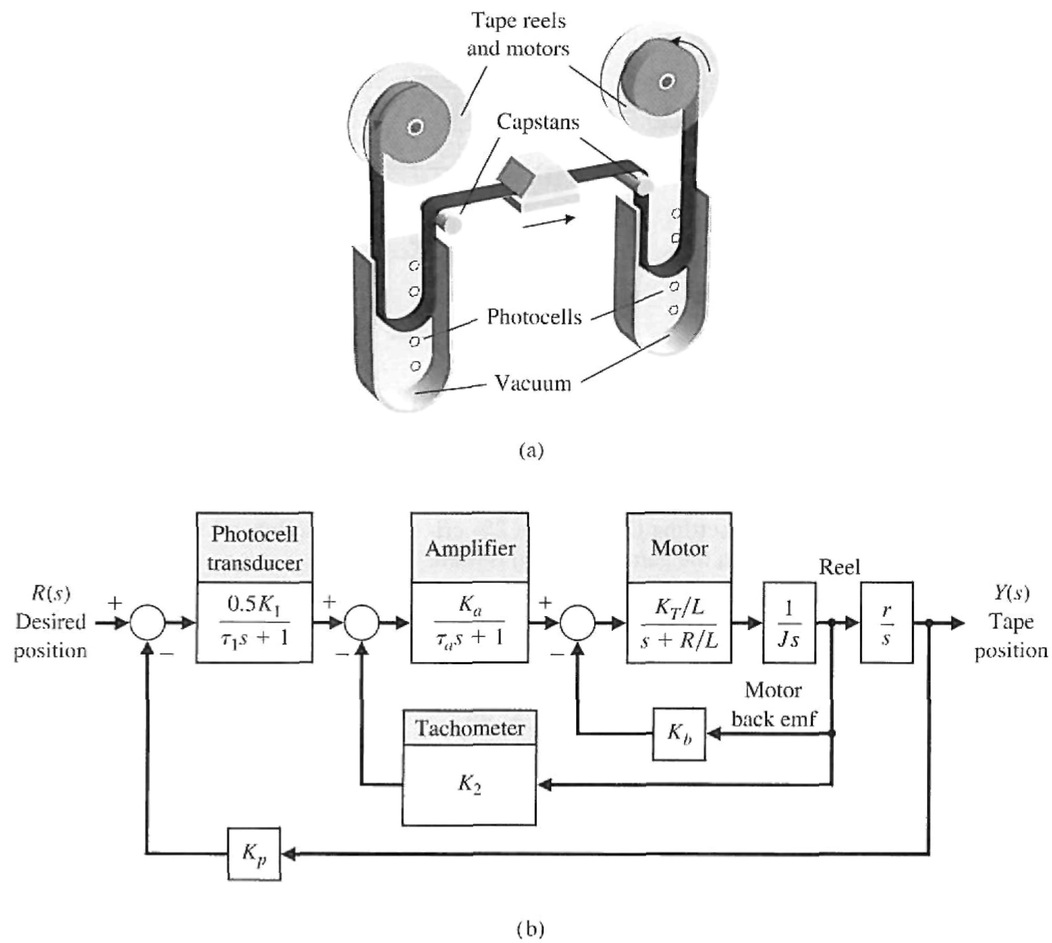


FIGURE P7.11 (a) Tape control system. (b) Block diagram.

100 in/s. The motor and components selected for this system possess the following characteristics:

$K_b = 0.40$	$r = 0.2$
$K_p = 1$	$K_1 = 2.0$
$\tau_1 = \tau_a = 1$ ms	K_2 is adjustable.
$K_T/(LJ) = 2.0$	

The inertia of the reel and motor rotor is 2.5×10^{-3} when the reel is empty, and 5.0×10^{-3} when the reel is full. A series of photocells is used as an error-sensing device. The time constant of the motor is $L/R = 0.5$ ms. (a) Sketch the root locus for the system when $K_2 = 10$ and $J = 5.0 \times 10^{-3}$, $0 < K_a < \infty$. (b) Determine the gain K_a that results in a well-damped system so that the ζ of all the roots is greater than or equal to 0.60. (c) With the K_a determined from part (b), sketch a root locus for $0 < K_2 < \infty$.

P7.12 A precision speed control system (Figure P7.12) is required for a platform used in gyroscope and inertial system testing where a variety of closely controlled speeds is necessary. A direct-drive DC torque motor system was utilized to provide (1) a speed range of 0.01°/s to 600°/s, and (2) 0.1% steady-state error maximum for a step input. The direct-drive DC torque motor avoids the use of a gear train with its attendant backlash and friction. Also, the direct-drive motor has a high-torque capability, high efficiency, and low motor time constants. The motor gain constant is nominally $K_m = 1.8$, but is subject to variations up to 50%. The amplifier gain K_a is normally greater than 10 and subject to a variation of 10%. (a) Determine the minimum loop gain necessary to satisfy the steady-state error requirement. (b) Determine the limiting value of gain for stability. (c) Sketch the root locus as K_a varies from 0 to ∞. (d) Determine the roots when $K_a = 40$, and estimate the response to a step input.

P7.13 A unity feedback system has the loop transfer function

$$L(s) = G_c(s)G(s) = \frac{K}{s(s+3)(s^2+4s+7.84)}.$$

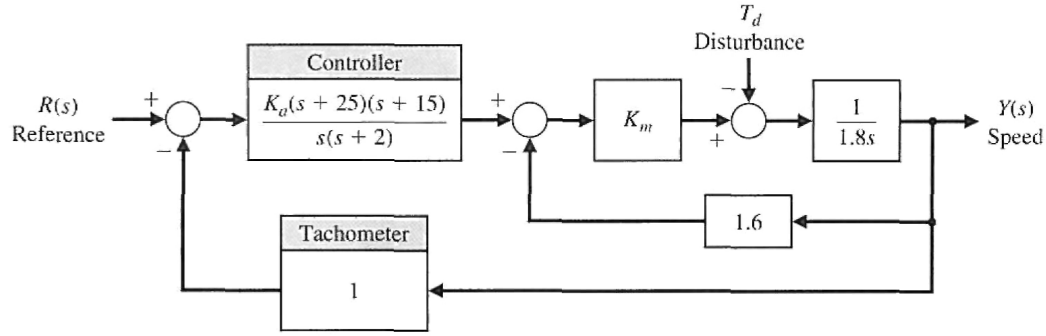


FIGURE P7.12
Speed control.

(a) Find the breakaway point on the real axis and the gain for this point. (b) Find the gain to provide two complex roots nearest the $j\omega$-axis with a damping ratio of 0.707. (c) Are the two roots of part (b) dominant? (d) Determine the settling time (with a 2% criterion) of the system when the gain of part (b) is used.

P7.14 The loop transfer function of a single-loop negative feedback system is

$$L(s) = G_c(s)G(s) = \frac{K(s+2.5)(s+3.2)}{s^2(s+1)(s+10)(s+30)}.$$

This system is called conditionally stable because it is stable only for a range of the gain K such that $k_1 < K < k_2$. Using the Routh–Hurwitz criteria and the root locus method, determine the range of the gain for which the system is stable. Sketch the root locus for $0 < K < \infty$.

P7.15 Let us again consider the stability and ride of a rider and high performance motorcycle as outlined in Problem P6.13. The dynamics of the motorcycle and rider can be represented by the loop transfer function

$$G_c(s)G(s) = \frac{K(s^2+30s+625)}{s(s+20)(s^2+20s+200)(s^2+60s+3400)}.$$

Sketch the root locus for the system. Determine the ζ of the dominant roots when $K = 3 \times 10^4$.

P7.16 Control systems for maintaining constant tension on strip steel in a hot strip finishing mill are called "loopers." A typical system is shown in Figure P7.16. The looper is an arm 2 to 3 feet long with a roller on the end; it is raised and pressed against the strip by a motor [18]. The typical speed of the strip passing the looper is 2000 ft/min. A voltage proportional to the looper

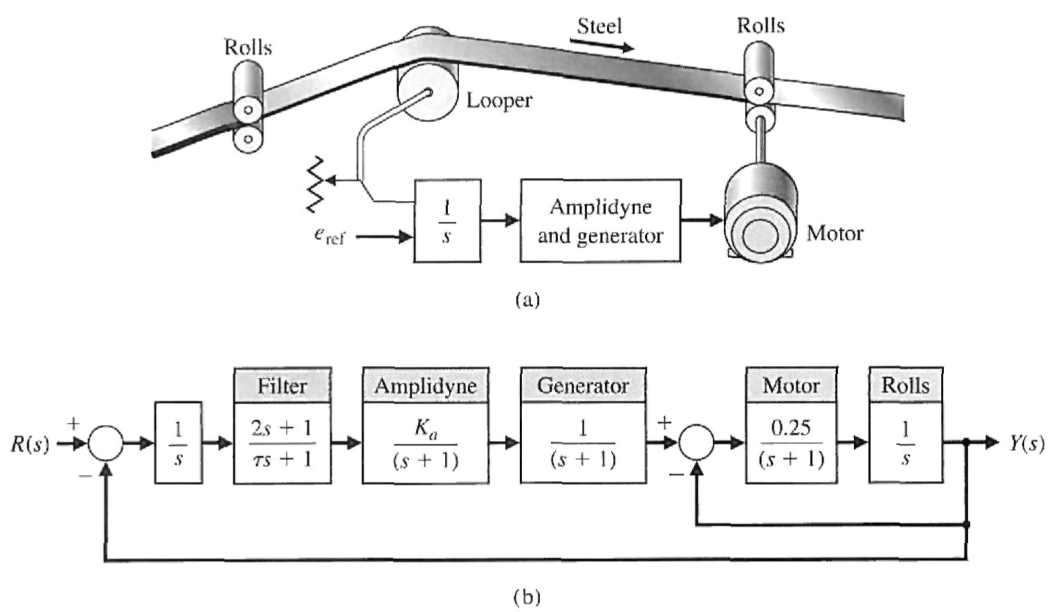


FIGURE P7.16
Steel mill control system.

position is compared with a reference voltage and integrated where it is assumed that a change in looper position is proportional to a change in the steel strip tension. The time constant τ of the filter is negligible relative to the other time constants in the system. (a) Sketch the root locus of the control system for $0 < K_a < \infty$. (b) Determine the gain K_a that results in a system whose roots have a damping ratio of $\zeta = 0.707$ or greater. (c) Determine the effect of τ as τ increases from a negligible quantity.

P7.17 Consider again the vibration absorber discussed in Problems 2.2 and 2.10 as a design problem. Using the root locus method, determine the effect of the parameters M_2 and k_{12}. Determine the specific values of the parameters M_2 and k_{12} so that the mass M_1 does not vibrate when $F(t) = a\sin(\omega_0 t)$. Assume that $M_1 = 1, k_1 = 1$, and $b = 1$. Also assume that $k_{12} < 1$ and that the term k_{12}^2 may be neglected.

P7.18 A feedback control system is shown in Figure P7.18. The filter $G_c(s)$ is often called a compensator, and the design problem involves selecting the parameters α and β. Using the root locus method, determine the effect of varying the parameters. Select a suitable filter so that the time to settle (to within 2% of the final value) is less than 4 seconds and the damping ratio of the dominant roots is greater than 0.60.

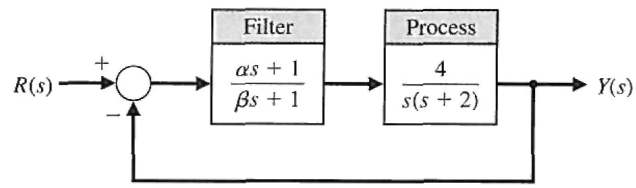


FIGURE P7.18 Filter design.

P7.19 In recent years, many automatic control systems for guided vehicles in factories have been installed. One system uses a magnetic tape applied to the floor to guide the vehicle along the desired lane [10, 15]. Using transponder tags on the floor, the automatically guided vehicles can be tasked (for example, to speed up or slow down) at key locations. An example of a guided vehicle in a factory is shown in Figure P7.19(a). We have

$$G(s) = \frac{s^2 + 4s + 100}{s(s+2)(s+6)}$$

and K_a is the amplifier gain. Sketch a root locus and determine a suitable gain K_a so that the damping ratio of the complex roots is 0.707.

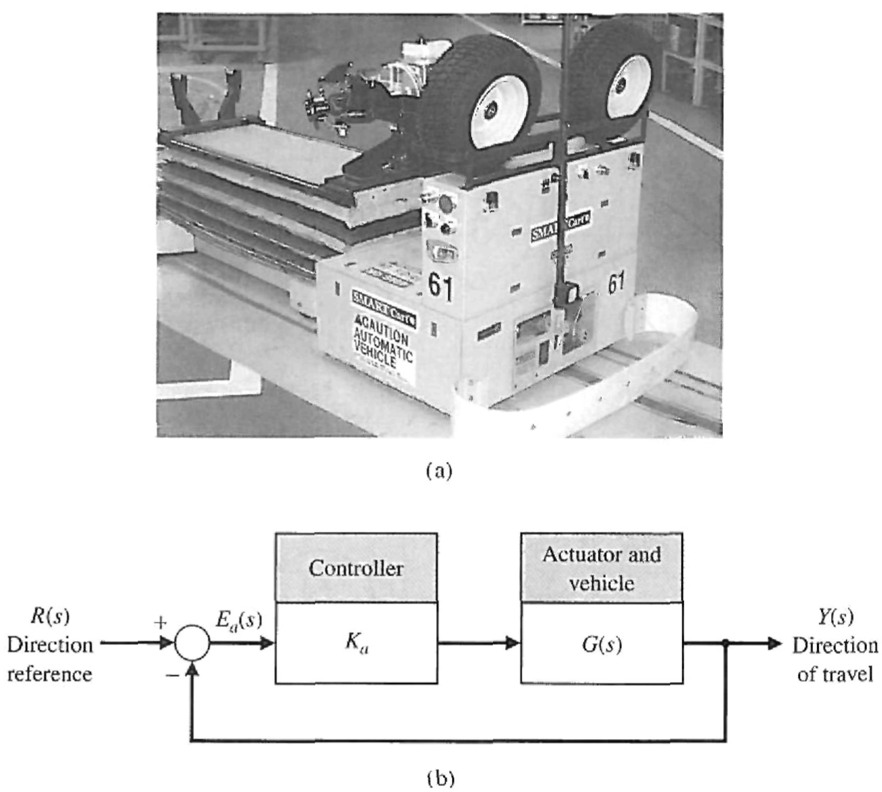


FIGURE P7.19 (a) An automatically guided vehicle. (Photo courtesy of the Jervis B. Webb Company) (b) Block diagram.

P7.20 Determine the root sensitivity for the dominant roots of the design for Problem P7.18 for the gain $K = 4\alpha/\beta$ and the pole $s = -2$.

P7.21 Determine the root sensitivity of the dominant roots of the power system of Problem P7.7. Evaluate the sensitivity for variations of (a) the poles at $s = -4$, and (b) the feedback gain, $1/R$.

P7.22 Determine the root sensitivity of the dominant roots of Problem P7.1(a) when K is set so that the damping ratio of the unperturbed roots is 0.707. Evaluate and compare the sensitivity as a function of the poles and zeros of $G_c(s)G(s)$.

P7.23 Repeat Problem P7.22 for the loop transfer function $G_c(s)G(s)$ of Problem P7.1(c).

P7.24 For systems of relatively high degree, the form of the root locus can often assume an unexpected pattern. The root loci of four different feedback systems of third order or higher are shown in Figure P7.24. The open-loop poles and zeros of $KG(s)$ are shown, and the form of the root loci as K varies from zero to infinity is presented. Verify the diagrams of Figure P7.24 by constructing the root loci.

P7.25 Solid-state integrated electronic circuits are composed of distributed R and C elements. Therefore, feedback electronic circuits in integrated circuit form must be investigated by obtaining the transfer function of the distributed RC networks. It has been shown that the slope of the attenuation curve of a distributed RC network is $10n$ dB/decade, where n is the order of the RC filter [13]. This attenuation is in contrast with the normal $20n$ dB/decade for the lumped parameter circuits. (The concept of the slope of an attenuation curve is considered in Chapter 8. If it is unfamiliar,

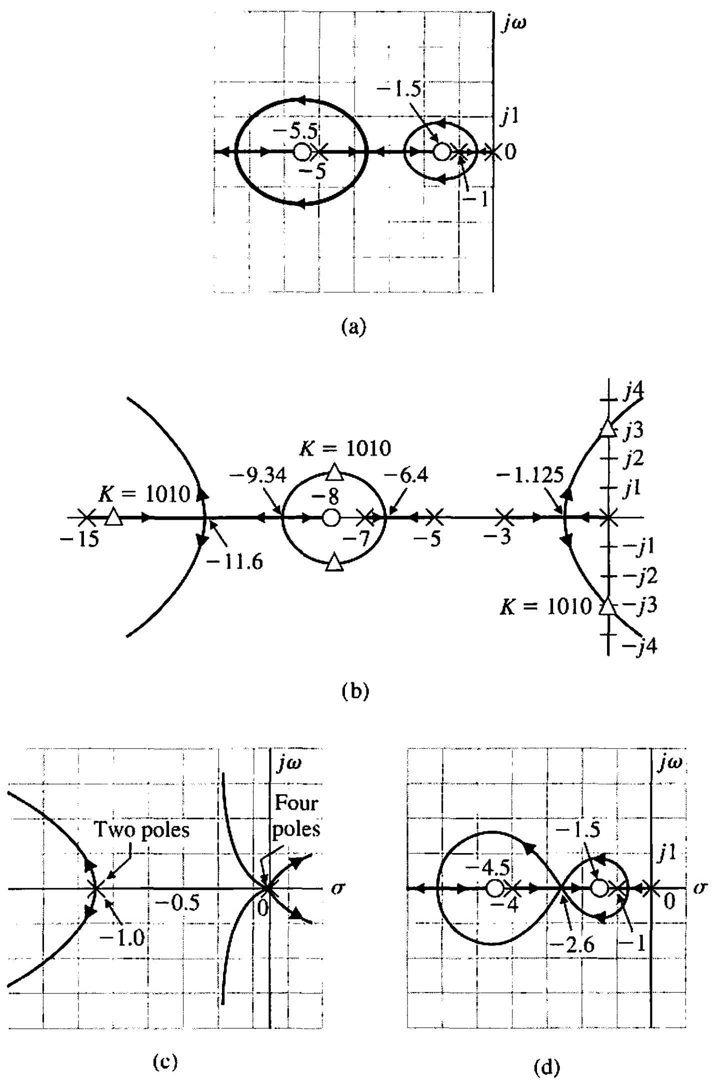


FIGURE P7.24 Root loci of four systems.

reexamine this problem after studying Chapter 8.) An interesting case arises when the distributed RC network occurs in a series-to-shunt feedback path of a transistor amplifier. Then the loop transfer function may be written as

$$L(s) = G_c(s)G(s) = \frac{K(s - 1)(s + 3)^{1/2}}{(s + 1)(s + 2)^{1/2}}.$$

(a) Using the root locus method, determine the locus of roots as K varies from zero to infinity. (b) Calculate the gain at borderline stability and the frequency of oscillation for this gain.

P7.26 A single-loop negative feedback system has a loop transfer function

$$L(s) = G_c(s)G(s) = \frac{K(s + 2)^2}{s(s^2 + 1)(s + 8)}.$$

(a) Sketch the root locus for $0 \le K \le \infty$ to indicate the significant features of the locus. (b) Determine the range of the gain K for which the system is stable. (c) For what value of K in the range $K \ge 0$ do purely imaginary roots exist? What are the values of these roots? (d) Would the use of the dominant roots approximation for an estimate of settling time be justified in this case for a large magnitude of gain ($K > 50$)?

P7.27 A unity negative feedback system has a loop transfer function

$$L(s) = G_c(s)G(s) = \frac{K(s^2 + 0.1)}{s(s^2 + 2)} = \frac{K(s + j0.3162)(s - j0.3162)}{s(s^2 + 1)}.$$

Sketch the root locus as a function of K. Carefully calculate where the segments of the locus enter and leave the real axis.

P7.28 To meet current U.S. emissions standards for automobiles, hydrocarbon (HC) and carbon monoxide (CO) emissions are usually controlled by a catalytic converter in the automobile exhaust. Federal standards for nitrogen oxides (NO_x) emissions are met mainly by exhaust-gas recirculation (EGR) techniques. However, as NO_x emissions standards were tightened from the current limit of 2.0 grams per mile to 1.0 gram per mile, these techniques alone were no longer sufficient.

Although many schemes are under investigation for meeting the emissions standards for all three emissions, one of the most promising employs a three-way catalyst—for HC, CO, and NO_x emissions—in conjunction with a closed-loop engine-control system. The approach is to use a closed-loop engine control, as shown in Figure P7.28 [19, 23]. The exhaust-gas sensor gives an indication of a rich or lean exhaust and compares it to a reference. The difference signal is processed by the controller, and the output of the controller modulates the vacuum level in the carburetor to achieve the best air-fuel ratio for proper operation of the catalytic converter. The loop transfer function is represented by

$$L(s) = \frac{Ks^2 + 12s + 20}{s^3 + 10s^2 + 25s}.$$

Calculate the root locus as a function of K. Carefully calculate where the segments of the locus enter and leave the real axis. Determine the roots when $K = 2$. Predict the step response of the system when $K = 2$.

P7.29 A unity feedback control system has a transfer function

$$L(s) = G_c(s)G(s) = \frac{K(s^2 + 10s + 30)}{s^2(s + 10)}.$$

We desire the dominant roots to have a damping ratio equal to 0.707. Find the gain K when this condition is satisfied. Show that the complex roots are $s = -3.56 \pm j3.56$ at this gain.

P7.30 An RLC network is shown in Figure P7.30. The nominal values (normalized) of the network elements are $L - C = 1$ and $R = 2.5$. Show that the root sensitivity of the two roots of the input impedance $Z(s)$ to a change in R is different by a factor of 4.

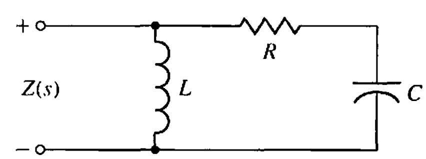


FIGURE P7.30 *RLC* network.

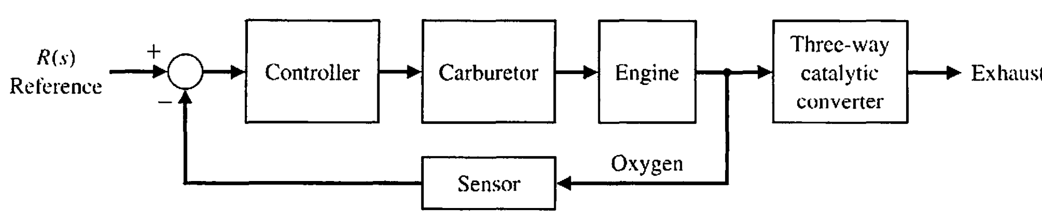


FIGURE P7.28
Auto engine control.

P7.31 The development of high-speed aircraft and missiles requires information about aerodynamic parameters prevailing at very high speeds. Wind tunnels are used to test these parameters. These wind tunnels are constructed by compressing air to very high pressures and releasing it through a valve to create a wind. Since the air pressure drops as the air escapes, it is necessary to open the valve wider to maintain a constant wind speed. Thus, a control system is needed to adjust the valve to maintain a constant wind speed. The loop transfer function for a unity feedback system is

$$L(s) = G_c(s)G(s) = \frac{K(s+4)}{s(s+0.16)(s+p)(s-\bar{p})},$$

where $p = 7.3 + 9.7831j$. Sketch the root locus and show the location of the roots for $K = 326$ and $K = 1350$.

P7.32 A mobile robot suitable for nighttime guard duty is available. This guard never sleeps and can tirelessly patrol large warehouses and outdoor yards. The steering control system for the mobile robot has a unity feedback with the loop transfer function

$$L(s) = G_c(s)G(s) = \frac{K(s+1)(s+5)}{s(s+1.5)(s+2)}.$$

(a) Find K for all breakaway and entry points on the real axis. (b) Find K when the damping ratio of the complex roots is 0.707. (c) Find the minimum value of the damping ratio for the complex roots and the associated gain K. (d) Find the overshoot and the time to settle (to within 2% of the final value) for a unit step input for the gain, K, determined in parts (b) and (c).

P7.33 The Bell-Boeing V-22 Osprey Tiltrotor is both an airplane and a helicopter. Its advantage is the ability to rotate its engines to 90° from a vertical position for takeoffs and landings as shown in Figure P7.33(a), and then to switch the engines to a horizontal position for cruising as an airplane [20]. The altitude control system in the helicopter mode is shown in Figure P7.33(b). (a) Determine the root locus as K varies and determine the range of K for a stable system. (b) For $K = 280$, find the actual $y(t)$ for a unit step input $r(t)$ and the percentage overshoot and settling time (with a 2% criterion). (c) When $K = 280$ and $r(t) = 0$, find $y(t)$ for a unit step disturbance, $T_d(s) = 1/s$. (d) Add a prefilter between $R(s)$ and the summing node so that

$$G_p(s) = \frac{0.5}{s^2 + 1.5s + 0.5},$$

and repeat part (b).

P7.34 The fuel control for an automobile uses a diesel pump that is subject to parameter variations. A unity negative feedback has a loop transfer function

$$G_c(s)G(s) = \frac{K(s+2)}{(s+1)(s+2.5)(s+4)(s+10)}.$$

(a) Sketch the root locus as K varies from 0 to 2000. (b) Find the roots for K equal to 400, 500, and 600. (c) Predict how the percent overshoot to a step will vary for the gain K, assuming dominant roots. (d) Find the actual time response for a step input for all three gains and compare the actual overshoot with the predicted overshoot.

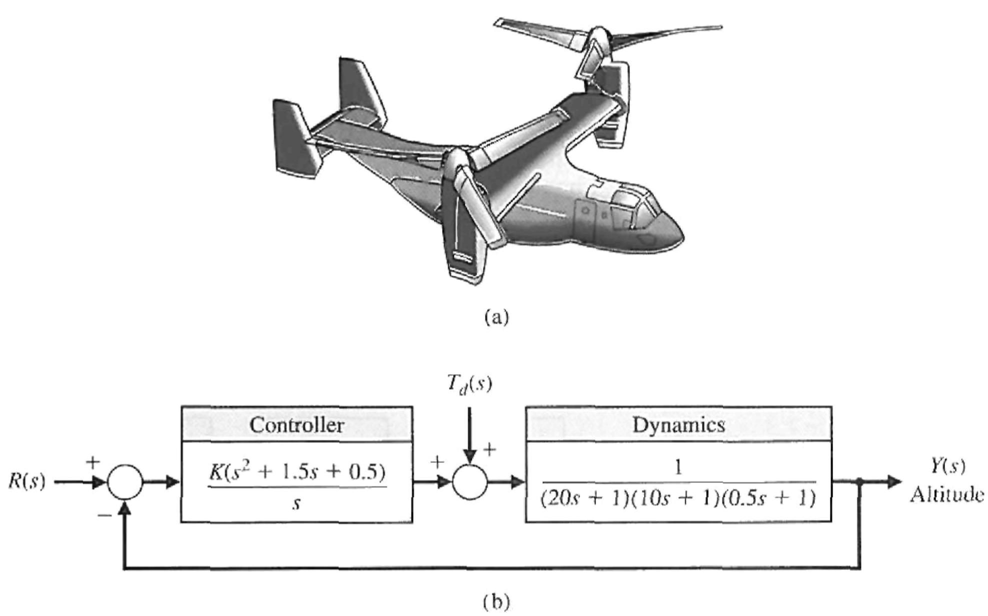


FIGURE P7.33 (a) Osprey Tiltrotor aircraft. (b) Its control system.

P7.35 A powerful electrohydraulic forklift can be used to lift pallets weighing several tons on top of 35-foot scaffolds at a construction site. The negative unity feedback system has a loop transfer function

$$L(s) = G_c(s)G(s) = \frac{K(s+1)^2}{s(s^2+1)}.$$

(a) Sketch the root locus for $K > 0$. (b) Find the gain K when two complex roots have a ζ of 0.707, and calculate all three roots. (c) Find the entry point of the root locus at the real axis. (d) Estimate the expected overshoot to a step input, and compare it with the actual overshoot determined from a computer program.

P7.36 A microrobot with a high-performance manipulator has been designed for testing very small particles, such as simple living cells [6]. The single-loop unity negative feedback system has a loop transfer function

$$L(s) = G_c(s)G(s) = \frac{K(s+1)(s+2)(s+3)}{s^3(s-1)}.$$

(a) Sketch the root locus for $K > 0$. (b) Find the gain and roots when the characteristic equation has two imaginary roots. (c) Determine the characteristic roots when $K = 20$ and $K = 100$. (d) For $K = 20$, estimate the percent overshoot to a step input, and compare the estimate to the actual overshoot determined from a computer program.

P7.37 Identify the parameters K, a, and b of the system shown in Figure P7.37. The system is subject to a unit step input, and the output response has an overshoot but ultimately attains the final value of 1. When the closed-loop system is subjected to a ramp input, the output response follows the ramp input with a finite steady-state error. When the gain is doubled to $2K$, the output response to an impulse input is a pure sinusoid with a period of 0.314 second. Determine K, a, and b.

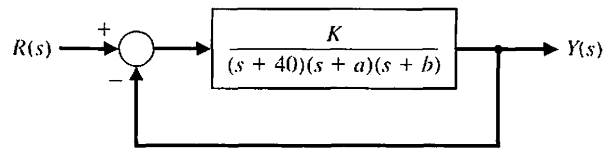


FIGURE P7.37 Feedback system.

P7.38 A unity feedback system has the loop transfer function

$$L(s) = G_c(s)G(s) = \frac{K(s+1)}{s(s-3)}.$$

This system is open-loop unstable. (a) Determine the range of K so that the closed-loop system is stable. (b) Sketch the root locus. (c) Determine the roots for $K = 10$. (d) For $K = 10$, predict the percent overshoot for a step input using Figure 5.13. (e) Determine the actual overshoot by plotting the response.

P7.39 High-speed trains for U.S. railroad tracks must traverse twists and turns. In conventional trains, the axles are fixed in steel frames called trucks. The trucks pivot as the train goes into a curve, but the fixed axles stay parallel to each other, even though the front axle tends to go in a different direction from the rear axle [24]. If the train is going fast, it may jump the tracks. One solution uses axles that pivot independently. To counterbalance the strong centrifugal forces in a curve, the train also has a computerized hydraulic system that tilts each car as it rounds a turn. On-board sensors calculate the train's speed and the sharpness of the curve and feed this information to hydraulic pumps under the floor of each car. The pumps tilt the car up to eight degrees, causing it to lean into the curve like a race car on a banked track.

The tilt control system is shown in Figure P7.39. Sketch the root locus, and determine the value of K when the complex roots have maximum damping. Predict the response of this system to a step input $R(s)$.

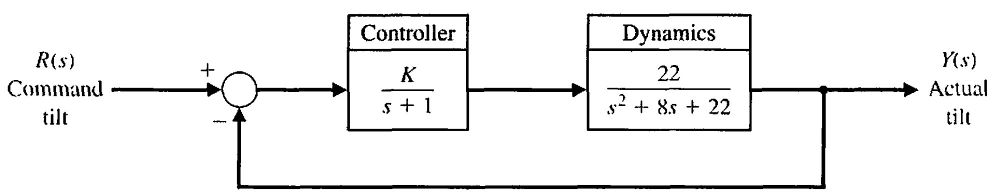


FIGURE P7.39 Tilt control for a high-speed train.

ADVANCED PROBLEMS

AP7.1 The top view of a high-performance jet aircraft is shown in Figure AP7.1(a) [20]. Sketch the root locus and determine the gain K so that the ζ of the complex poles near the $j\omega$-axis is the maximum achievable. Evaluate the roots at this K and predict the response to a step input. Determine the actual response and compare it to the predicted response.

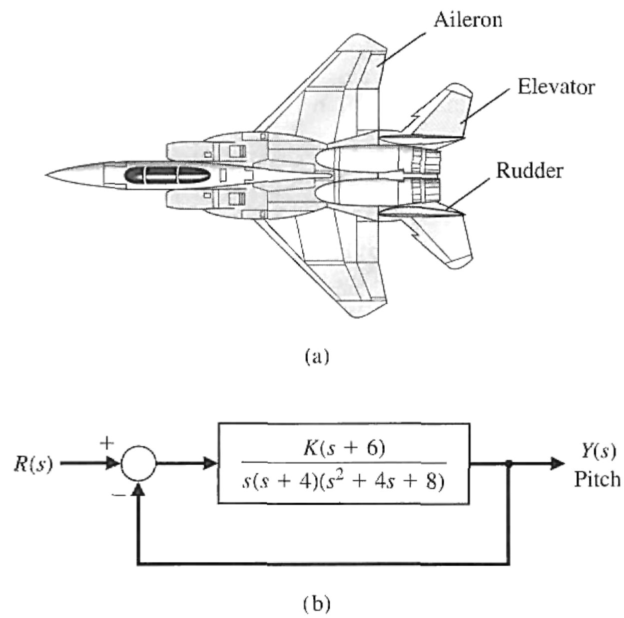


FIGURE AP7.1
(a) High-performance aircraft. (b) Pitch control system.

AP7.2 A magnetically levitated high-speed train "flies" on an air gap above its rail system, as shown in Figure AP7.2(a) [24]. The air gap control system has a unity feedback system with a loop transfer function

$$G_c(s)G(s) = \frac{K(s+1)(s+3)}{s(s-1)(s+4)(s+8)}.$$

The feedback control system is illustrated in Figure AP7.2(b). The goal is to select K so that the response for a unit step input is reasonably damped and the settling time is less than 3 seconds. Sketch the root locus, and select K so that all of the complex roots have a ζ greater than 0.6. Determine the actual response for the selected K and the percent overshoot.

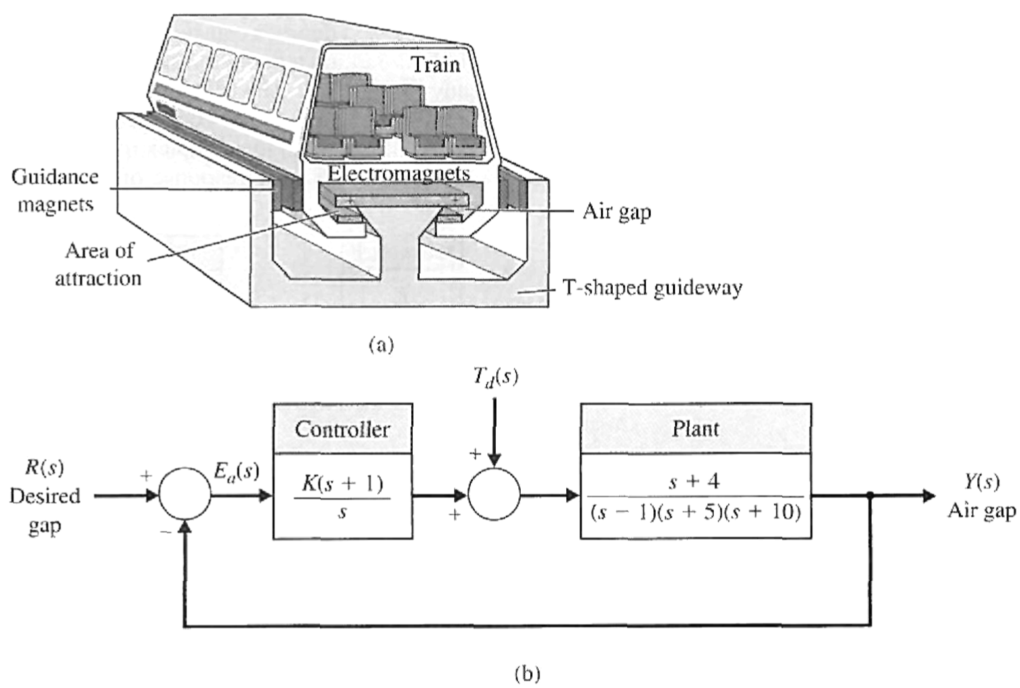


FIGURE AP7.2
(a) Magnetically levitated high-speed train.
(b) Feedback control system.

AP7.3 A compact disc player for portable use requires a good rejection of disturbances and an accurate position of the optical reader sensor. The position control system uses unity feedback and a loop transfer function

$$L(s) = G_c(s)G(s) = \frac{10}{s(s+1)(s+p)}.$$

The parameter p can be chosen by selecting the appropriate DC motor. Sketch the root locus as a function of p. Select p so that the ζ of the complex roots of the characteristic equation is approximately $1/\sqrt{2}$.

AP7.4 A remote manipulator control system has unity feedback and a loop transfer function

$$G_c(s)G(s) = \frac{(s+\alpha)}{s^3 + (1+\alpha)s^2 + (\alpha - 1)s + 1 - \alpha}.$$

We want the steady-state position error for a step input to be less than or equal to 10% of the magnitude of the input. Sketch the root locus as a function of the parameter α. Determine the range of α required for the desired steady-state error. Locate the roots for the allowable value of α to achieve the required steady-state error, and estimate the step response of the system.

AP7.5 A unity feedback system has a loop transfer function

$$L(s) = G_c(s)G(s) = \frac{K}{s^3 + 10s^2 + 7s - 18}.$$

(a) Sketch the root locus and determine K for a stable system with complex roots with ζ equal to $1/\sqrt{2}$.
(b) Determine the root sensitivity of the complex roots of part (a).
(c) Determine the percent change in K (increase or decrease) so that the roots lie on the $j\omega$-axis.

AP7.6 A unity feedback system has a loop transfer function

$$L(s) = G_c(s)G(s) = \frac{K(s^2 + 3s + 6)}{s^3 + 2s^2 + 3s + 1}.$$

Sketch the root locus for $K > 0$, and select a value for K that will provide a closed step response with settling time less than 1 second.

AP7.7 A feedback system with positive feedback is shown in Figure AP7.7. The root locus for $K > 0$ must meet the condition

$$KG(s) = 1\angle \pm k360^\circ \quad \text{for } k = 0, 1, 2, \ldots.$$

Sketch the root locus for $0 < K < \infty$.

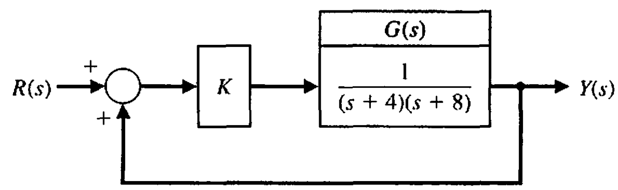


FIGURE AP7.7 A closed-loop system with positive feedback.

AP7.8 A position control system for a DC motor is shown in Figure AP7.8. Obtain the root locus for the velocity feedback constant K, and select K so that all the roots of the characteristic equation are real (two are equal and real). Estimate the step response of the system for the K selected. Compare the estimate with the actual response.

AP7.9 A control system is shown in Figure AP7.9. Sketch the root loci for the following transfer functions $G_c(s)$:
(a) $G_c(s) = K$
(b) $G_c(s) = K(s+3)$

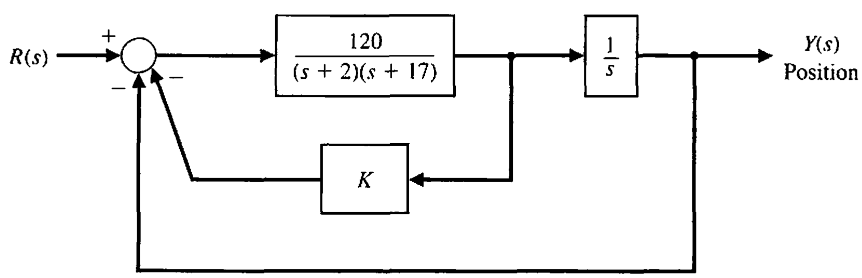


FIGURE AP7.8 A position control system with velocity feedback.

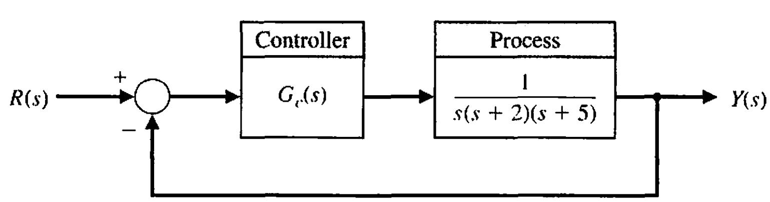


FIGURE AP7.9 A unity feedback control system.

(c) $G_c(s) = \dfrac{K(s+1)}{s+20}$

(d) $G_c(s) = \dfrac{K(s+1)(s+4)}{s+10}$

AP7.10 A feedback system is shown in Figure AP7.10. Sketch the root locus as K varies when $K \geqslant 0$. Determine a value for K that will provide a step response with an overshoot less than 5% and a settling time (with a 2% criterion) less than 2.5 seconds.

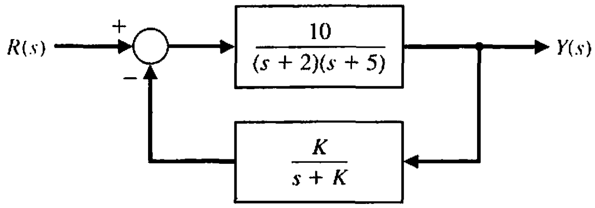


FIGURE AP7.10 A nonunity feedback control system.

AP7.11 A control system is shown in Figure AP7.11. Sketch the root locus, and select a gain K so that the step response of the system has an overshoot of less than 10% and the settling time (with a 2% criterion) is less than 4 seconds.

AP7.12 A control system with PI control is shown in Figure AP7.12. (a) Let $K_I/K_P = 0.2$ and determine K_P so that the complex roots have maximum damping ratio. (b) Predict the step response of the system with K_P set to the value determined in part (a).

AP7.13 The feedback system shown in Figure AP7.13 has two unknown parameters K_1 and K_2. The process transfer function is unstable. Sketch the root locus for $0 \leq K_1, K_2 < \infty$. What is the fastest settling time that you would expect of the closed-loop system in response to a unit step input $R(s) = 1/s$? Explain.

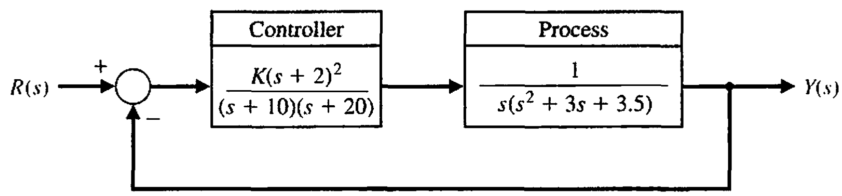


FIGURE AP7.11 A control system with parameter K.

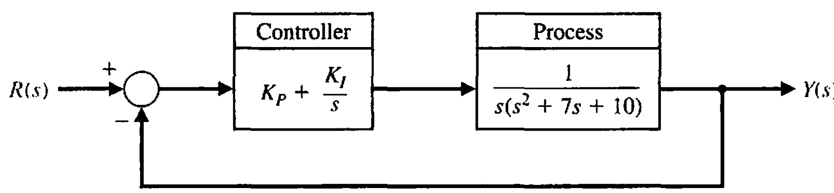


FIGURE AP7.12 A control system with a PI controller.

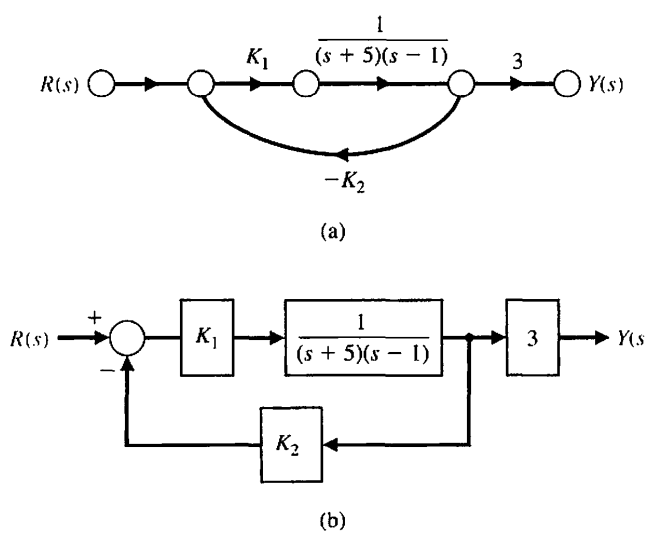


FIGURE AP7.13 An unstable plant with two parameters K_1 and K_2.

AP7.14 A unity feedback control system shown in Figure AP7.14 has the process

$$G(s) = \frac{10}{s(s + 10)(s + 7.5)}.$$

Design a PID controller using Ziegler-Nichols methods. Determine the unit step response and the unit disturbance response. What is the maximum percent overshoot and settling time for the unit step input?

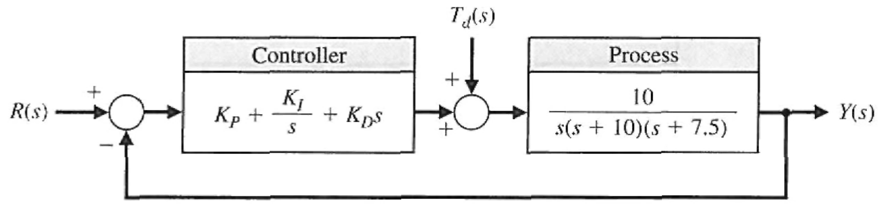


FIGURE AP7.14 Unity feedback loop with PID controller.

DESIGN PROBLEMS

CDP7.1 The drive motor and slide system uses the output of a tachometer mounted on the shaft of the motor as shown in Figure CDP4.1 (switch-closed option). The output voltage of the tachometer is $v_T = K_1\theta$. Use the velocity feedback with the adjustable gain K_1. Select the best values for the gain K_1 and the amplifier gain K_a so that the transient response to a step input has an overshoot less than 5% and a settling time (to within 2% of the final value) less than 300 ms.

DP7.1 A high-performance aircraft, shown in Figure DP7.1(a), uses the ailerons, rudder, and elevator to steer through a three-dimensional flight path [20]. The pitch rate control system for a fighter aircraft at 10,000 m and Mach 0.9 can be represented by the system in Figure DP7.1(b), where

$$G(s) = \frac{-18(s + 0.015)(s + 0.45)}{(s^2 + 1.2s + 12)(s^2 + 0.01s + 0.0025)}.$$

(a) Sketch the root locus when the controller is a gain, so that $G_c(s) = K$, and determine K when ζ for the roots with $\omega_n > 2$ is larger than 0.15 (seek a maximum ζ). (b) Plot the response $q(t)$ for a step input $r(t)$ with K as in (a). (c) A designer suggests an anticipatory controller with $G_c(s) = K_1 + K_2 s = K(s + 2)$. Sketch the root locus for this system as K varies and determine a K so that the ζ of all the closed-loop roots is >0.8. (d) Plot the response $q(t)$ for a step input $r(t)$ with K as in (c).

DP7.2 A large helicopter uses two tandem rotors rotating in opposite directions, as shown in Figure P7.33(a). The controller adjusts the tilt angle of the main rotor and thus the forward motion as shown in Figure DP7.2. The helicopter dynamics are represented by

$$G(s) = \frac{10}{s^2 + 4.5s + 9},$$

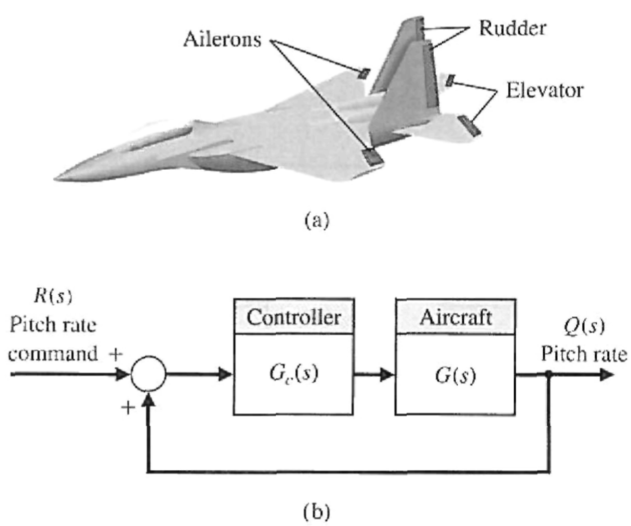


FIGURE DP7.1 (a) High-performance aircraft. (b) Pitch rate control system.

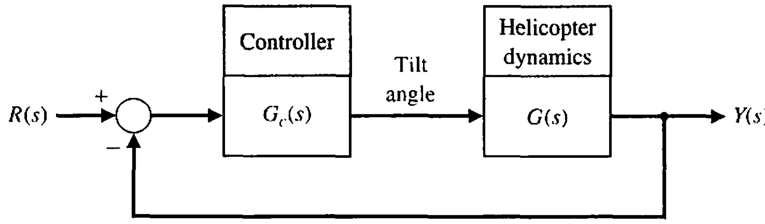


FIGURE DP7.2 Two-rotor helicopter velocity control.

and the controller is selected as

$$G_c(s) = K_1 + \frac{K_2}{s} = \frac{K(s+1)}{s}.$$

(a) Sketch the root locus of the system and determine K when ζ of the complex roots is equal to 0.6. (b) Plot the response of the system to a step input $r(t)$ and find the settling time (with a 2% criterion) and overshoot for the system of part (a). What is the steady-state error for a step input? (c) Repeat parts (a) and (b) when the ζ of the complex roots is 0.41. Compare the results with those obtained in parts (a) and (b).

DP7.3 The vehicle Rover has been designed for maneuvering at 0.25 mph over Martian terrain. Because Mars is 189 million miles from Earth and it would take up to 40 minutes each way to communicate with Earth [22, 27], Rover must act independently and reliably. Resembling a cross between a small flatbed truck and an elevated jeep, Rover is constructed of three articulated sections, each with its own two independent, axle-bearing, one-meter conical wheels. A pair of sampling arms—one for chipping and drilling, the other for manipulating fine objects—extend from its front end like pincers. The control of the arms can be represented by the system shown in Figure DP7.3. (a) Sketch the root locus for K and identify the roots for $K = 4.1$ and 41. (b) Determine the gain K that results in an overshoot to a step of approximately 1%. (c) Determine the gain that minimizes the settling time (with a 2% criterion) while maintaining an overshoot of less than 1%.

DP7.4 A welding torch is remotely controlled to achieve high accuracy while operating in changing and hazardous environments [21]. A model of the welding arm position control is shown in Figure DP7.4, with the disturbance representing the environmental changes. (a) With $T_d(s) = 0$, select K_1 and K to provide high-quality performance of the position control system. Select a set of performance criteria, and examine the results of your design. (b) For the system in part (a), let $R(s) = 0$ and determine the effect of a unit step $T_d(s) = 1/s$ by obtaining $y(t)$.

DP7.5 A high-performance jet aircraft with an autopilot control system has a unity feedback and control system, as shown in Figure DP7.5. Sketch the root locus and select a gain K that leads to dominant poles. With this gain K, predict the step response of the system. Determine the actual response of the system, and compare it to the predicted response.

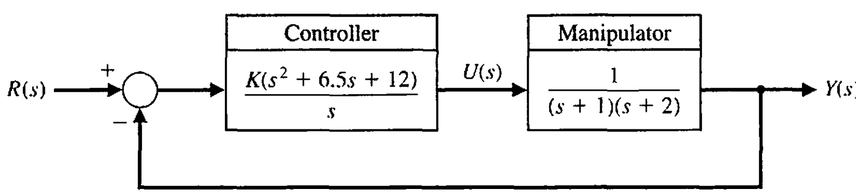


FIGURE DP7.3 Mars vehicle robot control system.

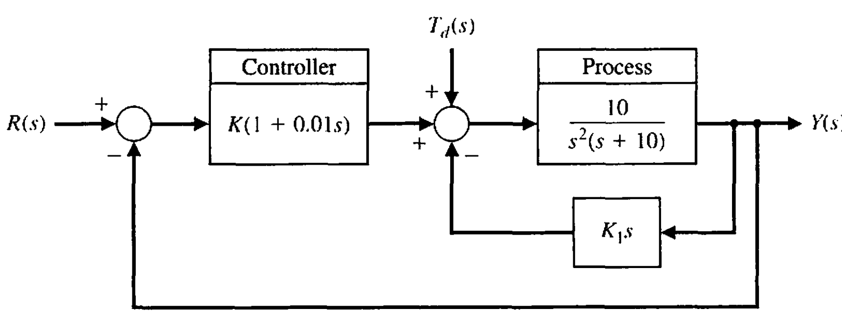


FIGURE DP7.4 Remotely controlled welder.

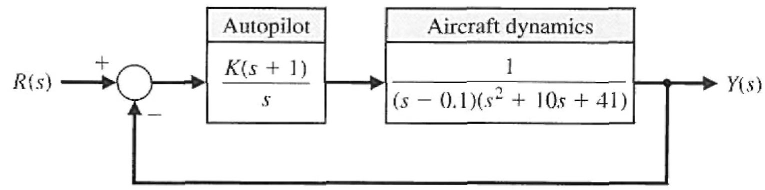


FIGURE DP7.5 High-performance jet aircraft.

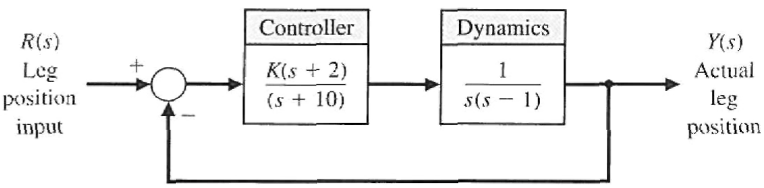


FIGURE DP7.6 Automatic control of walking motion.

DP7.6 A system to aid and control the walk of a partially disabled person could use automatic control of the walking motion [25]. One model of a system that is open-loop unstable is shown in Figure DP7.6. Using the root locus, select K for the maximum achievable ζ of the complex roots. Predict the step response of the system, and compare it with the actual step response.

DP7.7 A mobile robot using a vision system as the measurement device is shown in Figure DP7.7(a) [36]. The control system is shown in Figure DP7.7(b) where

$$G(s) = \frac{1}{(s+1)(0.5s+1)},$$

and $G_c(s)$ is selected as a PI controller so that the steady-state error for a step input is equal to zero. We then have

$$G_c(s) = K_P + \frac{K_I}{s}.$$

Design the PI controller so that (a) the percent overshoot for a step input is $P.O. \leq 5\%$; (b) the settling time (with a 2% criterion) is $T_s \leq 6$ seconds; (c) the system velocity error constant $K_v > 0.9$; and (d) the peak time, T_P, for a step input is minimized.

DP7.8 Most commercial op-amps are designed to be unity-gain stable [26]. That is, they are stable when

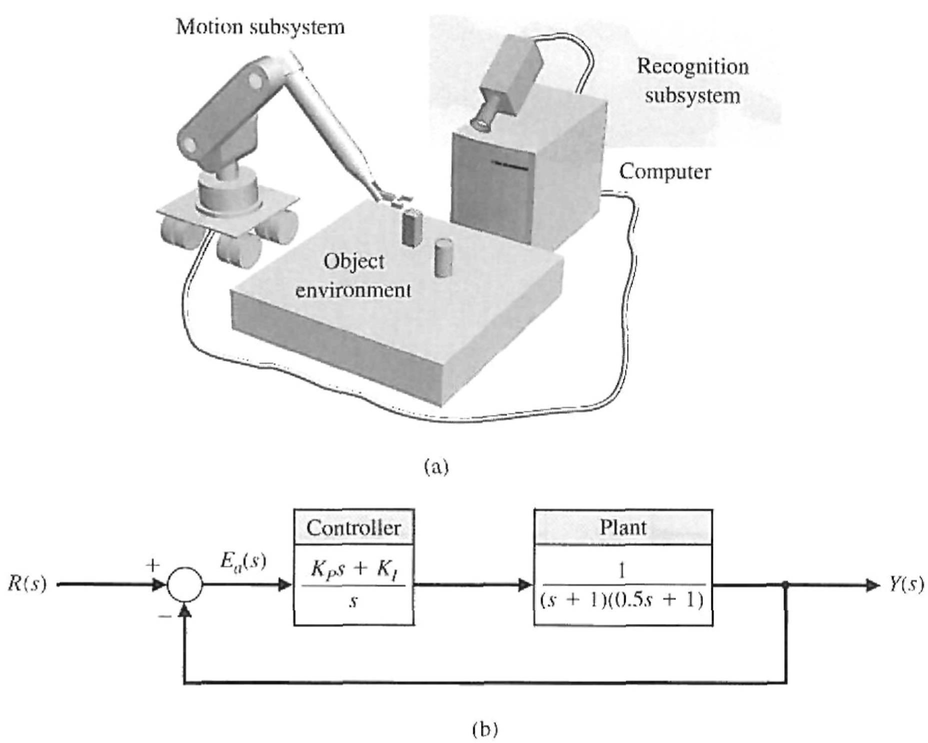


FIGURE DP7.7 (a) A robot and vision system. (b) Feedback control system.

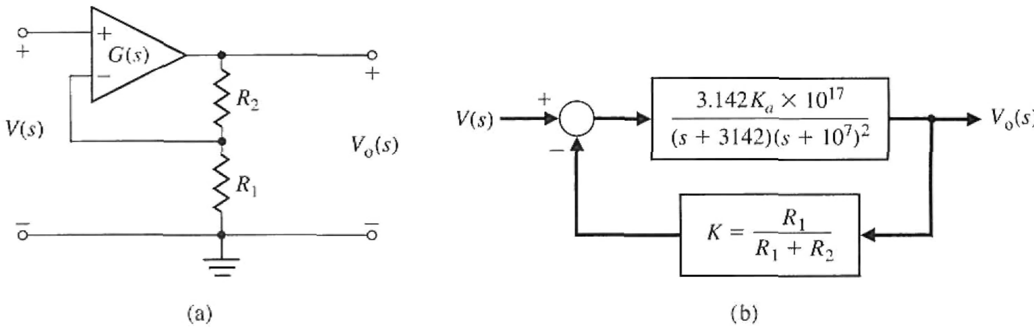


FIGURE DP7.8
(a) Op-amp circuit.
(b) Control system.

used in a unity-gain configuration. To achieve higher bandwidth, some op-amps relax the requirement to be unity-gain stable. One such amplifier has a DC gain of 10^5 and a bandwidth of 10 kHz. The amplifier, $G(s)$, is connected in the feedback circuit shown in Figure DP7.8(a). The amplifier is represented by the model shown in Figure DP7.8(b), where $K_a = 10^5$. Sketch the root locus of the system for K. Determine the minimum value of the DC gain of the closed-loop amplifier for stability. Select a DC gain and the resistors R_1 and R_2.

DP7.9 A robotic arm actuated at the elbow joint is shown in Figure DP7.9(a), and the control system for the actuator is shown in Figure DP7.9(b). Plot the root locus for $K \geq 0$. Select $G_p(s)$ so that the steady-state error for a step input is equal to zero. Using the $G_p(s)$ selected, plot $y(t)$ for K equal to 1, 1.5, and 2.85. Record the rise time, settling time (with a 2% criterion), and percent overshoot for the three gains. We wish to limit the overshoot to less than 6% while achieving the shortest rise time possible. Select the best system for $1 \leq K \leq 2.85$.

DP7.10 The four-wheel-steering automobile has several benefits. The system gives the driver a greater degree of control over the automobile. The driver gets a more forgiving vehicle over a wide variety of conditions.

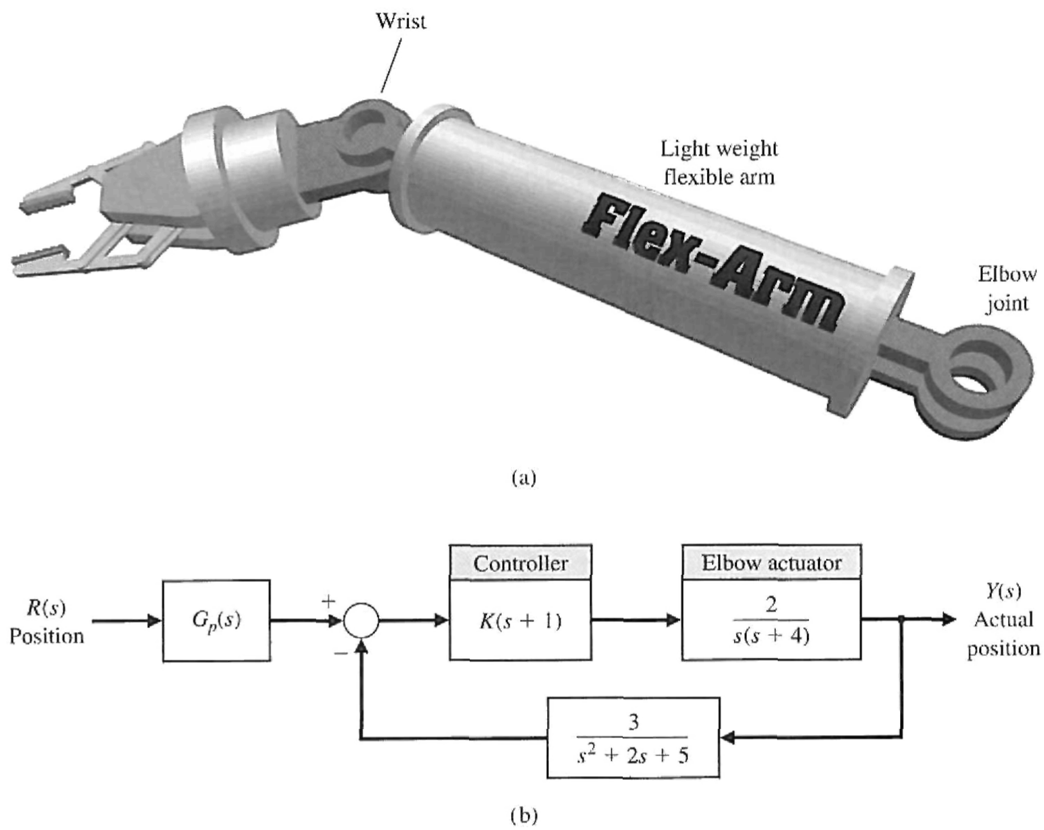


FIGURE DP7.9
(a) A robotic arm actuated at the joint elbow. (b) Its control system.

The system enables the driver to make sharp, smooth lane transitions. It also prevents yaw, which is the swaying of the rear end during sudden movements. Furthermore, the four-wheel-steering system gives a car increased maneuverability. This enables the driver to park the car in extremely tight quarters. With additional closed-loop computer operating systems, a car could be prevented from sliding out of control in abnormal icy or wet road conditions.

The system works by moving the rear wheels relative to the front-wheel-steering angle. The control system takes information about the front wheels' steering angle and passes it to the actuator in the back. This actuator then moves the rear wheels appropriately.

When the rear wheels are given a steering angle relative to the front ones, the vehicle can vary its lateral acceleration response according to the loop transfer function

$$G_c(s)G(s) = K\frac{1 + (1+\lambda)T_1 s + (1+\lambda)T_2 s^2}{s[1 + (2\zeta/\omega_n)s + (1/\omega_n^2)s^2]},$$

where $\lambda = 2q/(1-q)$, and q is the ratio of rear wheel angle to front wheel steering angle [14]. We will assume that $T_1 = T_2 = 1$ second and $\omega_n = 4$. Design a unity feedback system, selecting an appropriate set of parameters (λ, K, ζ) so that the steering control response is rapid and yet will yield modest overshoot characteristics. In addition, q must be between 0 and 1.

DP7.11 A pilot crane control is shown in Figure DP7.11(a). The trolley is moved by an input $F(t)$ in order to control $x(t)$ and $\phi(t)$ [13]. The model of the pilot crane control is shown in Figure DP7.11(b). Design a controller that will achieve control of the desired variables when $G_c(s) = K$.

DP7.12 A rover vehicle designed for use on other planets and moons is shown in Figure DP7.12(a) [21]. The block diagram of the steering control is shown in Figure DP7.12(b), where

$$G(s) = \frac{s + 1.5}{(s+1)(s+2)(s+4)(s+10)}.$$

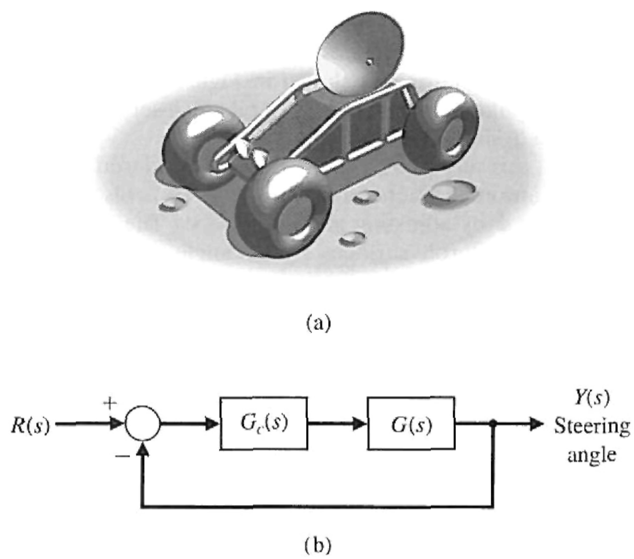


FIGURE DP7.12 (a) Planetary rover vehicle. (b) Steering control system.

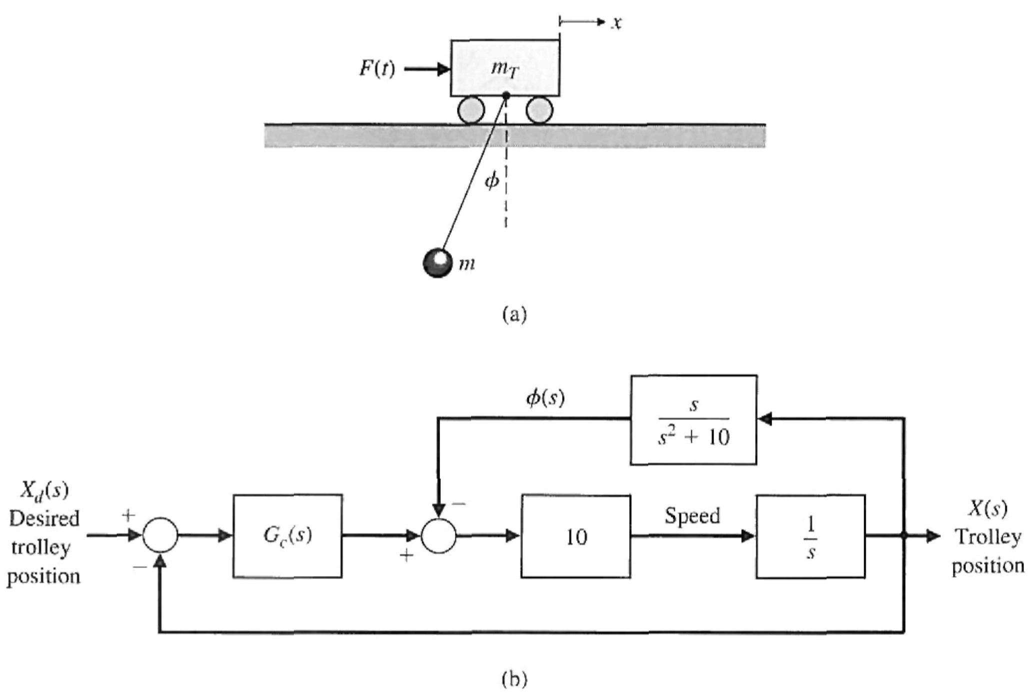


FIGURE DP7.11 (a) Pilot crane control system. (b) Block diagram.

(a) When $G_c(s) = K$, sketch the root locus as K varies from 0 to 1000. Find the roots for K equal to 100, 300, and 600. (b) Predict the overshoot, settling time (with a 2% criterion), and steady-state error for a step input, assuming dominant roots. (c) Determine the actual time response for a step input for the three values of the gain K, and compare the actual results with the predicted results.

DP7.13 The automatic control of an airplane is one example that requires multiple-variable feedback methods. In this system, the attitude of an aircraft is controlled by three sets of surfaces: elevators, a rudder, and ailerons, as shown in Figure DP7.13(a). By manipulating these surfaces, a pilot can set the aircraft on a desired flight path [20].

An autopilot, which will be considered here, is an automatic control system that controls the roll angle ϕ by adjusting aileron surfaces. The deflection of the aileron surfaces by an angle θ generates a torque due to air pressure on these surfaces. This causes a rolling motion of the aircraft. The aileron surfaces are controlled by a hydraulic actuator with a transfer function $1/s$.

The actual roll angle ϕ is measured and compared with the input. The difference between the desired roll angle ϕ_d and the actual angle ϕ will drive the hydraulic actuator, which in turn adjusts the deflection of the aileron surface.

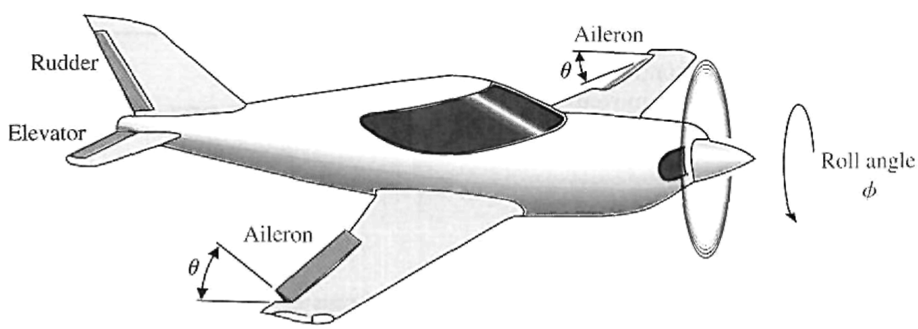
A simplified model where the rolling motion can be considered independent of other motions is assumed, and its block diagram is shown in Figure DP7.13(b). Assume that $K_1 = 1$ and that the roll rate $\dot{\phi}$ is fed back using a rate gyro. The step response desired has an overshoot less than 10% and a settling time (with a 2% criterion) less than 9 seconds. Select the parameters K_a and K_2.

DP7.14 Consider the feedback system shown in Figure DP7.14. The process transfer function is marginally stable. The controller is the proportional-derivative (PD) controller

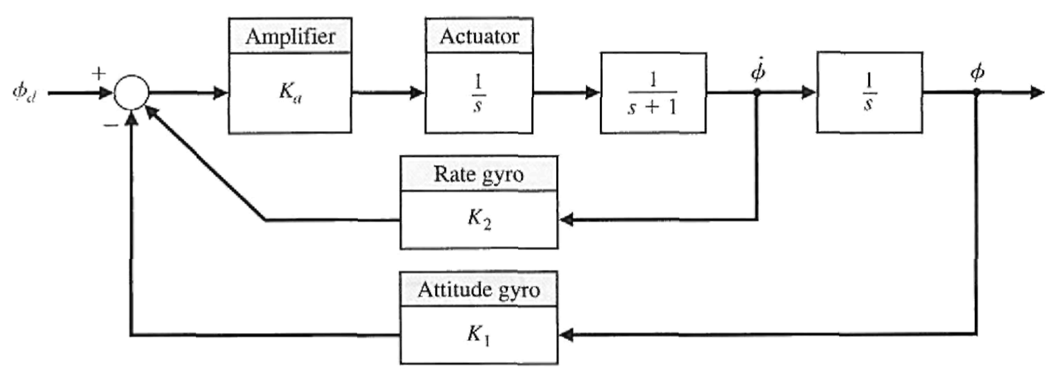
$$G_c(s) = K_P + K_D s.$$

(a) Determine the characteristic equation of the closed-loop system.
(b) Let $\tau = K_P/K_D$. Write the characteristic equation in the form

$$\Delta(s) = 1 + K_D \frac{n(s)}{d(s)}.$$



(a)



(b)

FIGURE DP7.13 (a) An airplane with a set of ailerons. (b) The block diagram for controlling the roll rate of the airplane.

(c) Plot the root locus for $0 \leq K_D < \infty$ when $\tau = 6$.

(d) What is the effect on the root locus when $0 < \tau < \sqrt{10}$?

(e) Design the PD controller to meet the following specifications:

(i) $P.O. < 5\%$

(ii) $T_s < 1$ s

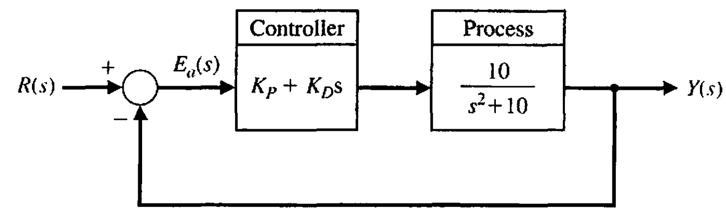


FIGURE DP7.14 A marginally stable plant with a PD controller in the loop.

COMPUTER PROBLEMS

CP7.1 Using the rlocus function, obtain the root locus for the following transfer functions of the system shown in Figure CP7.1 when $0 < K < \infty$:

(a) $G(s) = \dfrac{30}{s^3 + 14s^2 + 43s + 30}$,

(b) $G(s) = \dfrac{s + 20}{s^2 + 4s + 20}$

(c) $G(s) = \dfrac{s^2 + s + 2}{s(s^2 + 6s + 10)}$

(d) $G(s) = \dfrac{s^5 + 4s^4 + 6s^3 + 10s^2 + 6s + 4}{s^6 + 4s^5 + 4s^4 + s^3 + s^2 + 10s + 1}$

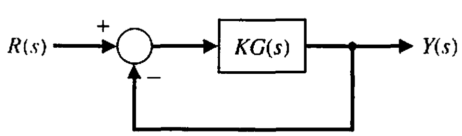


FIGURE CP7.1 A single-loop feedback system with parameter K.

CP7.2 A unity negative feedback system has the loop transfer function

$$KG(s) = K\frac{s^2 - 2s + 2}{s(s^2 + 3s + 2)}.$$

Develop an m-file to plot the root locus and show with the rlocfind function that the maximum value of K for a stable system is $K = 0.79$.

CP7.3 Compute the partial fraction expansion of

$$Y(s) = \frac{s + 6}{s(s^2 + 5s + 4)}$$

and verify the result using the residue function.

CP7.4 A unity negative feedback system has the loop transfer function

$$G_c(s)G(s) = \frac{(1 + p)s - p}{s^2 + 4s + 10}.$$

Develop an m-file to obtain the root locus as p varies; $0 < p < \infty$. For what values of p is the closed-loop stable?

CP7.5 Consider the feedback system shown in Figure CP7.1, where

$$G(s) = \frac{s + 1}{s^2}.$$

For what value of K is $\zeta = 0.707$ for the dominant closed-loop poles?

CP7.6 A large antenna, as shown in Figure CP7.6(a), is used to receive satellite signals and must accurately track the satellite as it moves across the sky. The control system uses an armature-controlled motor and a controller to be selected, as shown in Figure CP7.6(b). The system specifications require a steady-state error for a ramp input $r(t) = Bt$, less than or equal to $0.01B$, where B is a constant. We also seek a percent overshoot to a step input of $P.O. \leq 5\%$ with a settling time (with a 2% criterion) of $T_s \leq 2$ seconds. (a) Using root locus methods, create an m-file to assist in designing the controller. (b) Plot the resulting unit step response and compute the percent overshoot and the settling time and label the plot accordingly. (c) Determine the effect of the disturbance $T_d(s) = Q/s$ (where Q is a constant) on the output $Y(s)$.

CP7.7 Consider the feedback control system in Figure CP7.7. We have three potential controllers for our system:

1. $G_c(s) = K$ (proportional controller)
2. $G_c(s) = K/s$ (integral controller)
3. $G_c(s) = K(1 + 1/s)$ (proportional, integral (PI) controller)

The design specifications are $T_s \leq 10$ seconds and $P.O. \leq 10\%$ for a unit step input.

(a) For the proportional controller, develop an m-file to sketch the root locus for $0 < K < \infty$, and

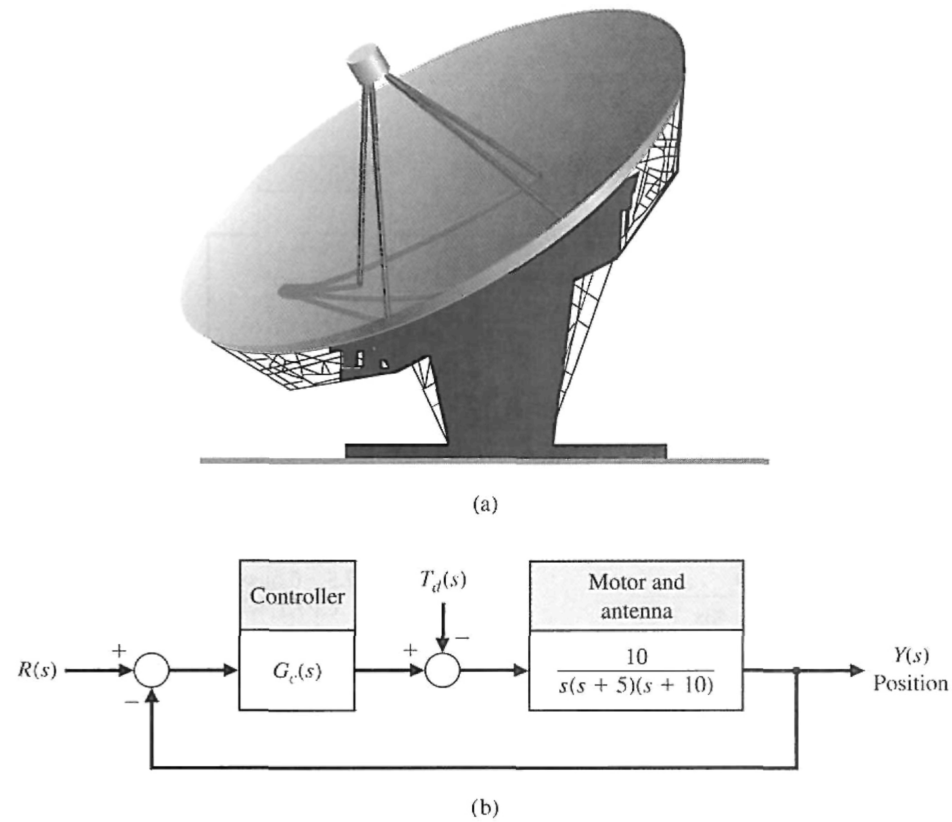


FIGURE CP7.6 Antenna position control.

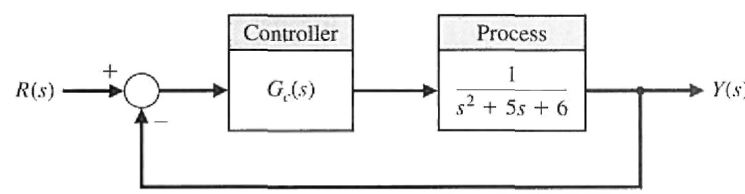


FIGURE CP7.7 A single-loop feedback control system with controller $G_c(s)$.

determine the value of K so that the design specifications are satisfied.

(b) Repeat part (a) for the integral controller.

(c) Repeat part (a) for the PI controller.

(d) Co-plot the unit step responses for the closed-loop systems with each controller designed in parts (a)–(c).

(e) Compare and contrast the three controllers obtained in parts (a)–(c), concentrating on the steady-state errors and transient performance.

CP7.8 Consider the spacecraft single-axis attitude control system shown in Figure CP7.8. The controller is known as a proportional-derivative (PD) controller. Suppose that we require the ratio of $K_p/K_D = 5$. Then, develop

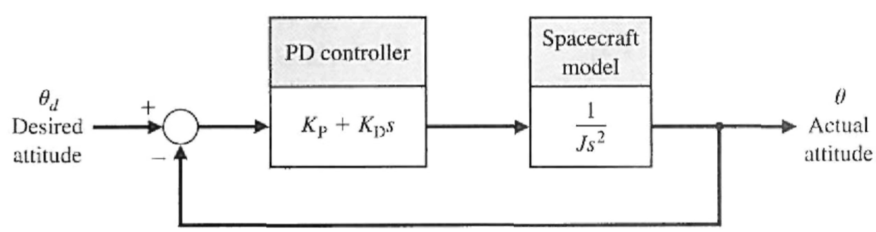


FIGURE CP7.8 A spacecraft attitude control system with a proportional-derivative controller.

an m-file using root locus methods find the values of K_D/J and K_p/J so that the settling time T_s is less than or equal to 4 seconds, and the peak overshoot P.O. is less than or equal to 10% for a unit step input. Use a 2% criterion in determining the settling time.

CP7.9 Consider the feedback control system in Figure CP7.9. Develop an m-file to plot the root locus for $0 < K < \infty$. Find the value of K resulting in a damping ratio of the closed-loop poles equal to 0.707.

CP7.10 Consider the system represented in state variable form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$y = \mathbf{C}\mathbf{x} + \mathbf{D}u,$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -2-k \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix},$$
$$\mathbf{C} = [1 \quad -9 \quad 12], \text{ and } \mathbf{D} = [0].$$

(a) Determine the characteristic equation. (b) Using the Routh–Hurwitz criterion, determine the values of k for which the system is stable. (c) Develop an m-file to plot the root locus and compare the results to those obtained in (b).

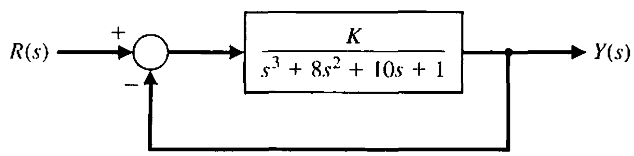


FIGURE CP7.9 Unity feedback system with parameter K.

ANSWERS TO SKILLS CHECK

True or False: (1) True; (2) True; (3) False; (4) True; (5) True

Multiple Choice: (6) b; (7) c; (8) a; (9) c; (10) a; (11) b; (12) c; (13) a; (14) c; (15) b

Word Match (in order, top to bottom): k, f, a, d, i, h, c, b, e, g, j

TERMS AND CONCEPTS

Angle of departure The angle at which a locus leaves a complex pole in the s-plane.

Angle of the asymptotes The angle ϕ_A that the asymptote makes with respect to the real axis.

Asymptote The path the root locus follows as the parameter becomes very large and approaches infinity. The number of asymptotes is equal to the number of poles minus the number of zeros.

Asymptote centroid The center σ_A of the linear asymptotes.

Breakaway point The point on the real axis where the locus departs from the real axis of the s-plane.

Dominant roots The roots of the characteristic equation that represent or dominate the closed-loop transient response.

Locus A path or trajectory that is traced out as a parameter is changed.

Logarithmic sensitivity A measure of the sensitivity of the system performance to specific parameter changes, given by $S_K^T(s) = \dfrac{\partial T(s)/T(s)}{\partial K/K}$, where $T(s)$ is the system transfer function and K is the parameter of interest.

Manual PID tuning methods The process of determining the PID controller gains by trial-and-error with minimal analytic analysis.

Negative gain root locus The root locus for negative values of the parameter of interest, where $-\infty < K \leq 0$.

Number of separate loci Equal to the number of poles of the transfer function, assuming that the number of poles is greater than or equal to the number of zeros of the transfer function.

Parameter design A method of selecting one or two parameters using the root locus method.

PID controller A widely used controller used in industry of the form $G_c(s) = K_p + \frac{K_I}{s} + K_D s$, where K_p is the proportional gain, K_I is the integral gain, and K_D is the derivative gain.

PID tuning The process of determining the PID controller gains.

Proportional plus deriviative (PD) controller A two-term controller of the form $G_c(s) = K_p + K_D s$, where K_p is the proportional gain and K_D is the derivative gain.

Proportional plus integral (PI) controller A two-term controller of the form $G_c(s) = K_p + \frac{K_I}{s}$, where K_p is the proportional gain and K_I is the integral gain.

Quarter amplitude decay The amplitude of the closed-loop response is reduced approximately to one-fourth of the maximum value in one oscillatory period.

Reaction curve The response obtained by taking the controller off-line and introducing a step input. The underlying process is assumed to be a first-order system with a transport delay.

Root contours The family of loci that depict the effect of varying two parameters on the roots of the characteristic equation.

Root locus The locus or path of the roots traced out on the s-plane as a parameter is changed.

Root locus method The method for determining the locus of roots of the characteristic equation $1 + KP(s) = 0$ as K varies from 0 to infinity.

Root locus segments on the real axis The root locus lying in a section of the real axis to the left of an odd number of poles and zeros.

Root sensitivity The sensitivity of the roots as a parameter changes from its normal value. The root sensitivity is given by $S_K^r = \frac{\partial r}{\partial K/K}$, the incremental change in the root divided by the proportional change of the parameter.

Ultimate gain The PD controller proportional gain, K_P, on the border of instability when $K_D = 0$ and $K_I = 0$.

Ultimate period The period of the sustained oscillations when K_P is the ultimate gain and $K_D = 0$ and $K_I = 0$.

Ziegler-Nichols PID tuning method The process of determining the PID controller gains using one of several analytic methods based on open-loop and closed-loop responses to step inputs.