An Introduction to Switched Systems

Prof. Rafael F. Q. Magossi[1]
Prof. Vilma A. Oliveira[2]

[1]CEFET/RJ, Brazil
[2]USP, Brazil

June 1, 2020

Note to the reader

This text is based on the following main references

- LIBERZON, Daniel. Switching in systems and control. Springer Science & Business Media, 2003.
- GEROMEL, Jose C.; COLANERI, Patrizio. Stability and stabilization of continuous-time switched linear systems. SIAM Journal on Control and Optimization, v. 45, n. 5, p. 1915-1930, 2006.
- DEAECTO, Grace S. Lecture notes from IM420 - Continuous-Time Switched Dynamical Systems - UNICAMP.
- BOYD, Stephen et al. Linear matrix inequalities in system and control theory. Siam, 1994.

Introduction

There are some simple examples of switched systems

Figure 1: Manual and automatic transmission of a car

Introduction

In Power Electronics the central key for any conversion is regarding on switches. The DC–DC converter uses switching states to convert DC voltage into DC voltage.

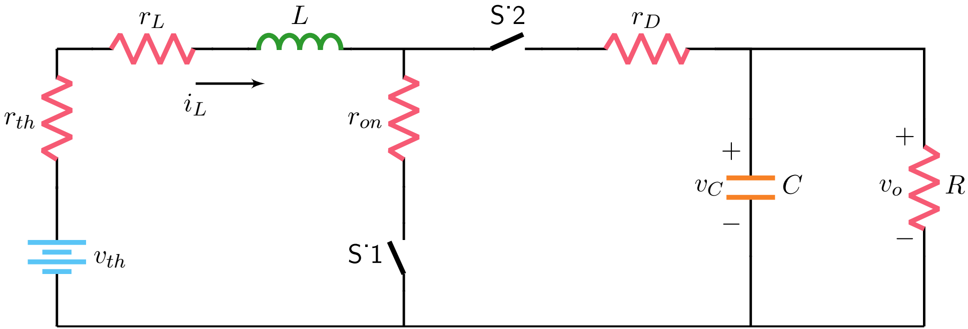


Figure 2: Boost converter.

Introduction

How should I switch ?

Fundamentals

Dynamical systems that are described by an interaction between continuous and discrete dynamics are usually called **hybrid systems**. The hybrid systems can be divided as

1. Mainly discrete with simple continuous dynamic systems - usually dealt with in the field of automation.
2. Mainly continuous with isolated discrete events - such hybrid system is called **switched systems**.

Switched systems can be classified, but not only, into

- State-dependent versus time-dependent;
- Autonomous (uncontrolled) versus controlled.

Fundamentals

- **State-dependent switched system** are systems that use state information to select the subsystem - We use the speed of the car to decide which gear to use.
- **Time-dependent switched system** are systems that use time to select the subsystem - In DC - DC converters, usually a PWM circuit dictates how long to stay in each subsystem.
- **Autonomous (uncontrolled) switched system** are systems in which we do not have the possibility to select which subsystem to use - An airplane that has engine failure does not give the option of not going to this subsystem.
- **Controlled switched system** are systems that we can select the subsystem to achieve an objective - We can choose whether or not to turn on a refrigerator compressor to maintain the temperature.

Fundamentals

We have interest in the following switched system with time-dependent switching

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \tag{1}$$

where

- $x(t) \in \mathbb{R}^{n_x}$ is the state
- $u(t) \in \mathbb{R}^{n_u}$ is the control input
- $\sigma(\cdot) : t \geq 0 \to \mathbb{K} \triangleq \{1, 2, \cdots, m\}$ is the **switching function** that selects one of the m available subsystems at each instant of time.

Suposição 3.1

The switching function σ is continuous from the right everywhere, which means that $\sigma(t) = \lim_{\tau \to t^+} \sigma(\tau)$ for each $\tau \geq 0$.

Fundamentals

For instance, consider that $m = 2$, therefore an example of switching function can be described as follow

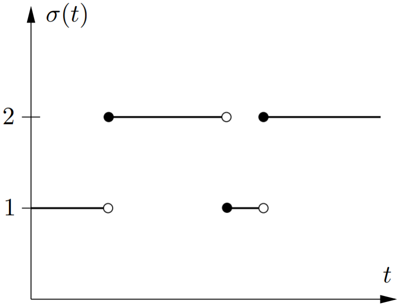


Figure 3: A switching signal. Source: Liberzon, 2003[1]

[1] Daniel Liberzon (2003). *Switching in systems and control*. Springer Science & Business Media.

Fundamentals

Let us focus on linear switched systems as

$$\dot{x}(t) = A_{\sigma(t)}x(t), \quad x(0) = x_0, \quad x_0 \in \mathbb{R}^{n\times n}, \quad A_p \in \mathbb{R}^{n\times n} \quad \forall p \in \mathbb{K}. \tag{2}$$

The following example illustrates the importance of switching signal.
Example Let us consider the following subsystems

$$A_1 = \begin{bmatrix} -1 & 1 \\ 0 & -1.5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 5 \\ 10 \end{bmatrix} \tag{3}$$

Fundamentals

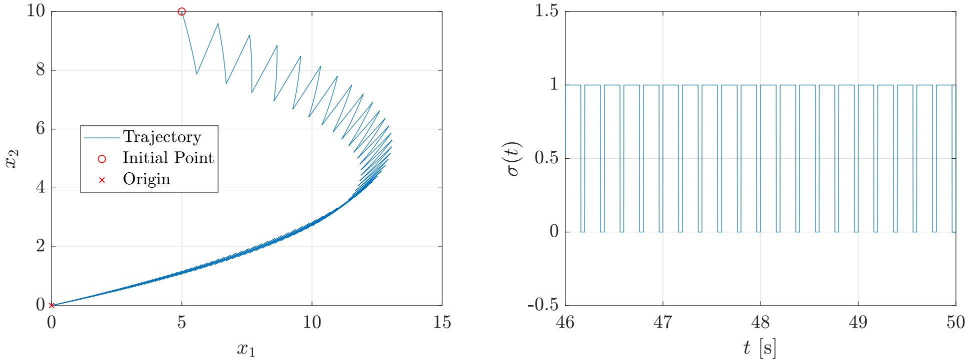


Figure 4: Stabilizing switching signal.

Fundamentals

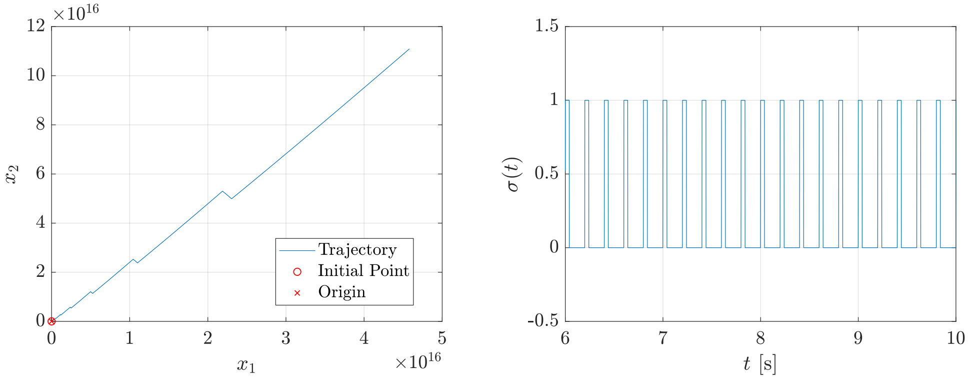


Figure 5: Non stabilizing switching signal.

Some recalls

Consider a $\mathcal{C}^1$ (i.e., continuously differentiable) function $V : \mathbb{R}^n \to \mathbb{R}$.

Definition 1

V is called **positive definite** if $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$.

Definition 2

V is said to be radially unbounded if $V(x) \to \infty$ as $|x| \to \infty$

Let,

$$\dot{x} = f(x) \quad x \in \mathbb{R}^n \tag{4}$$

where$f : \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitz function ($|f(x_1) - f(x_2)| \leq m|x_1 - x_2|$, $m \in \mathbb{R}$).

Stability

Theorem 3

(Lyapunov[a]) Suppose that there exists a positive definite C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ whose derivative along solutions of the system (4) satisfies

$$\dot{V}(x) \leq 0 \quad \forall x$$

Then the system (4) is **stable***. If the derivative of V satisfies*

$$\dot{V}(x) < 0 \quad \forall x \neq 0$$

then (4) is **asymptotically stable***. If in the latter case V is also radially unbounded, then (4) is* **globally asymptotically stable***.*

[a] Daniel Liberzon (2003). *Switching in systems and control*. Springer Science & Business Media.

Stability

Before we deal with stability by itself we need to state an important definitions

Definition 4

(Dwell-time) If $\exists h \in \mathbb{R}_{>0}$ such that

$$\inf_q t_q - t_{q-1} \geq h$$

then h is called dwell-time.

Definition 5

(Hurwitz Matrix) A square matrix A is called a Hurwitz matrix if all eigenvalues of A have strictly negative real part

Stability

Definition 6

Consider $n \times n$ symmetric real matrix M. Then,

- M is positive definite $\iff$ $x^\mathsf{T} M x > 0$ for all $x \in \mathbb{R}^n \setminus \mathbf{0}$
- M is positive semi-definite $\iff$ $x^\mathsf{T} M x \geq 0$ for all $x \in \mathbb{R}^n$
- M is negative definite $\iff$ $x^\mathsf{T} M x < 0$ for all $x \in \mathbb{R}^n \setminus \mathbf{0}$
- M is negative semi-definite $\iff$ $x^\mathsf{T} M x \leq 0$ for all $x \in \mathbb{R}^n$

Stability

Proposition 4.1

Let M be an $n \times n$ Hermitian matrix.

- *M is positive definite if and only if all of its eigenvalues are positive.*
- *M is positive semi-definite if and only if all of its eigenvalues are non-negative.*
- *M is negative definite if and only if all of its eigenvalues are negative*
- *M is negative semi-definite if and only if all of its eigenvalues are non-positive.*
- *M is indefinite if and only if it has both positive and negative eigenvalues.*

Stability

Definition 7

(Linear Matrix Inequality - LMI) An LMI is expressed as

$$\mathcal{A}(x) < 0$$

with

$$\mathcal{A}(x) = A_0 + \sum_{i=1}^{n} A_i x_i$$

where $A_i \in \mathbb{R}^{m \times m}$, $i = 1, \cdots, n$ are symmetric matrices and $x_i \in \mathbb{R}$ is the i-th component of vector x.

Stability

Consider $\dot{x} = Ax$, $A \in \mathbb{R}^{n \times n}$. The following theorem holds.

Theorem 8

(Lyapunov Theorem) Matrix A is Hurwitz stable if and only if for any given $Q > 0$ there exists a positive definite symmetric matrix P satisfying the Lyapunov equation

$$A'P + PA + Q = 0$$

Moreover, matrix P is the unique solution of this equation.

Note that from this theorem, it is suffice that

$$P > 0, \quad A'P + PA < 0$$

Stability

Proof

Consider $V(x) = x'Px$. Note that,

$$V(0) = 0 \quad \text{and} \quad V(x) > 0, \forall x \neq 0 \iff P > 0.$$

Now we should verify that $\dot{V}(x) < 0$, which means by Gateaux differential that

$$\dot{V}(x) = \langle \nabla V(x), \dot{x} \rangle.$$

Using the Leibniz's formula, we have

$$\begin{aligned} \dot{V}(x) &= \dot{x}'Px + x'P\dot{x} < 0 \\ &= (Ax)'Px + x'P(Ax) < 0 \\ &= x'(A'P + PA)x < 0 \iff A'P + PA < 0. \end{aligned}$$

Stability

If there exists $P > 0$ such that

$$A'P + PA < 0$$

then there exists also $Q > 0$ such that

$$A'P + PA + Q = 0$$

which is the Lyapunov equation. Moreover, let λ be the eigenvalue of A with eigenvector $v \neq 0$ and v^* the notation for conjugate transpose. Then,

$$-v^*Qv = v^*(A'P + PA)v = v^*A'Pv + v^*PAv = (\lambda v)^*Pv + v^*P(\lambda v) = (\lambda^* + \lambda)v^*Pv = Re(\lambda)v^*Pv$$

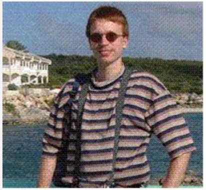
therefore, $Re(\lambda) < 0$. In other words, there exists $P > 0 \iff A$ is Hurwitz.

For a linear system, as $x = 0$ is the only equilibrium point, $P > 0$ and $A'P + PA < 0$ ensures that $x = 0$ is globally asymptotically stable.

How to solve a LMI ?

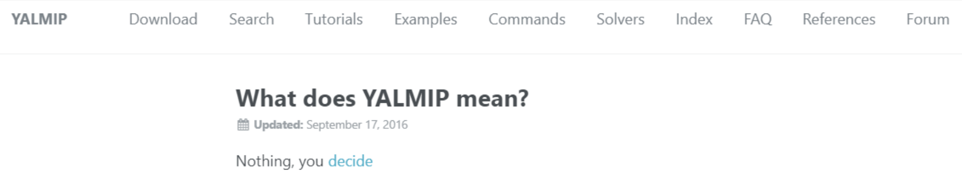
There are several forms and algorithms to solve LMI. Mathworks offers a toolbox called LMILab on Matlab. However, the most common combination is the use of SeDuMi and YALMIP combination.

(SeDuMi) SeDuMi is a great piece of software for optimization over symmetric cones. It was developed by Jos F. Sturm, who passed away in 2003.



How to solve a LMI ?

(YALMIP) YALMIP is a toolbox for modeling and optimization in MATLAB created by Johan Löfberg.



How to solve a LMI ?

Johan Löfberg

FOLLOW

Automatic Control, Linköping University, Sweden
Verified email at liu.se - Homepage

Control Optimization Model Predictive Control Mathematical programming

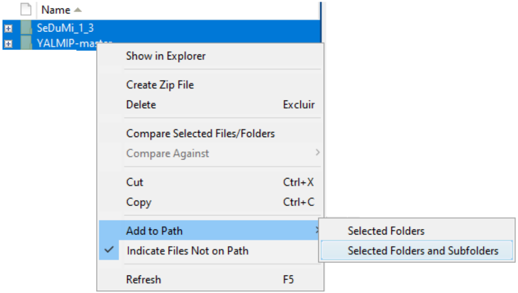
TITLE	CITED BY	YEAR
YALMIP: A toolbox for modeling and optimization in MATLAB J Löfberg Computer Aided Control Systems Design, 2004 IEEE International Symposium on ...	7799	2004

How to solve a LMI ?

You'll need,

1. Matlab installed;
2. Download SeDuMi package from `http://sedumi.ie.lehigh.edu/`
3. Download YALMIP from `https://yalmip.github.io/`

Then, extract those files from SuDuMi and YALMIP to a folder where you would like to work by adding the to path the folder and sub-folders as shown next.



How to solve a LMI ?

Hand's on! Write the following code

```
close all
clear
clc

A = [-1 2;-3 -4]; % define matrix A
P = sdpvar(2,2); % create symmetric P

% Set LMI
T = A'*P+P*A;
F = [P > 0; T < 0; trace(P)==1]; % trace(P)=1 is an additional restriction
% to ensure that we have a matrix P with values not so close to zero.

about = optimize(F) % solve LMI
P_feasible = double(P); % evaluate P
```

How to solve a LMI ?

The code should return something like

```
SeDuMi 1.3 by AdvOL, 2005-2008 and Jos F. Sturm, 1998-2003.
Alg = 2: xz-corrector, theta = 0.250, beta = 0.500
Put 1 free variables in a quadratic cone
eqs m = 3, order n = 7, dim = 11, blocks = 4
nnz(A) = 12 + 0, nnz(ADA) = 9, nnz(L) = 6
 it :     b*y       gap    delta  rate   t/tP*  t/tD*   feas cg cg  prec
  0 :            3.72E+01 0.000
  1 :   0.00E+00 3.25E+00 0.000 0.0874 0.9900 0.9900   1.07  1  1  2.6E+00
  2 :   0.00E+00 1.31E-01 0.000 0.0403 0.9900 0.9900   1.63  1  1  5.1E-02
  3 :   0.00E+00 4.43E-06 0.000 0.0000 1.0000 1.0000   1.00  1  1  2.8E-05
  4 :   0.00E+00 2.01E-12 0.000 0.0000 1.0000 1.0000   1.00  1  2  6.2E-12

iter seconds digits       c*x               b*y
  4      0.1   14.0   8.8470230081e-15  0.0000000000e+00
```

How to solve a LMI ?

```
|Ax-b| =   3.2e-13, [Ay-c]_+ =   3.6E-14, |x|=  9.2e-01, |y|=  7.5e-01

Detailed timing (sec)
   Pre             IPM            Post
1.500E-02    6.600E-02    4.003E-03
Max-norms: ||b||=0, ||c|| = 1,
Cholesky |add|=0, |skip| = 1, ||L.L|| = 1.

about =

  struct with fields:

    yalmiptime: 1.0610
    solvertime: 0.0870
          info: 'Successfully solved (SeDuMi-1.3)'
       problem: 0
```

How to solve a LMI ?

Focus your attention to "info: 'Successfully solved (SeDuMi-1.3)'" this allows you to move on and get P from "P_feasible" as

$$P = \begin{bmatrix} 0.678 & 0.058 \\ 0.058 & 0.322 \end{bmatrix} \tag{5}$$

using the command "eig(P_feasible)" we got the eigenvalues of P as 0.3130 and 0.6870, confirming that $P > 0$. Moreover, the solution satisfy $trace(P) = 1$. **When we deal with Lyapunov equation, P is unique, which does not follows for Lyapunov inequality, then if you got a different matrix P you're good.**

(On a side note) You can minimize the trace (or something else) of a variable X using "about = optimize(F,trace(X))". Minimizing the trace of a matrix is common for problems involving the $\mathcal{H}_2$ norm (;

On stability of switched linear systems

Consider the following general switched linear system with time-dependent switching function $\sigma(\cdot) : t \geq 0 \rightarrow \mathbb{K} = \{1, 2, \cdots, m\}$

$$\dot{x}(t) = A_{\sigma(t)}x(t), \quad x(0) = x_0, \quad x_0 \in \mathbb{R}^{n\times n}, \quad A_p \in \mathbb{R}^{n\times n} \quad \forall p \in \mathbb{K}. \tag{6}$$

A simple stability criterion is by assumption that all subsystems are Hurwitz, which means that any time a new subsystem is choose it will decrease to zero. Therefore, assume that $\{A_p : p \in \mathbb{K}\}$ is a compact set of Hurwitz matrix, then the following is true

Theorem 9

(Liberzon, 2003[a]) The system (6) *has global uniform exponential stability (GUES) if and only if it is locally attractive for every switching signal*

[a] Daniel Liberzon (2003). *Switching in systems and control.* Springer Science & Business Media.

On stability of switched linear systems

From that, it is natural to consider quadratic common Lyapunov functions as

$$V(x) = x'Px$$

which in view of the compactness assumption made earlier, it is suffice that

$$A_p'P + PA_p < 0, \quad \forall p \in \mathbb{K}. \tag{7}$$

Otherwise speaking, in that way the energy function is guaranteed to decrease for any subsystem. Moreover, in this framework, the switching function can be any.

On stability of switched linear systems

Example Let us consider the following subsystems

$$A_1 = \begin{bmatrix} -3 & 1 \\ 0 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & 1 \\ 0 & -5 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 5 \\ 10 \end{bmatrix} \tag{8}$$

Firstly, note that the eigenvalues of A_1 and A_2 are -3, -1 and -2, -5, respectively. Therefore, both A_1 and A_2 are **Hurwitz matrices**. Then, we can solve apply (7), resulting in

$$\begin{cases} A_1'P + PA_1 < 0 \\ A_2'P + PA_2 < 0 \end{cases}$$

Now, we need to find $P > 0$ which solves such problem, so let's go to Matlab.

On stability of switched linear systems

```
close all
clear
clc

A1 = [-3 1;0 -1];
A2 = [-2 1;0 -5]; % declare matrices
P = sdpvar(2,2);

% Set LMI problem
T1 = A1'*P+P*A1;
T2 = A2'*P+P*A2;
F = [P > 0; T1 < 0; T2<0];

about = optimize(F)
P_feasible = double(P);
```

On stability of switched linear systems

We got as solution,

$$P = \begin{bmatrix} 0.265 & 0.0267 \\ 0.0267 & 0.245 \end{bmatrix}$$

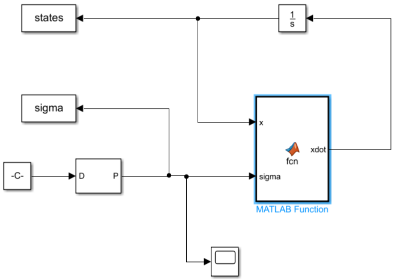
which has 0.2268 and 0.2838 as eigenvalues, therefore $P > 0$ and the switched system has GUES.

Well, theory is good, but how do we check this on practice ? We can simulate it and in order to simulate a switched system, we need

- Simulink;
- Matlab Function block.

On stability of switched linear systems

Simulation of a switched system



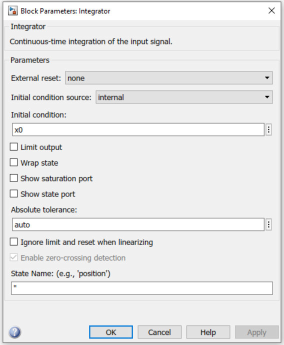


Figure 6: MATLAB/Simulink.

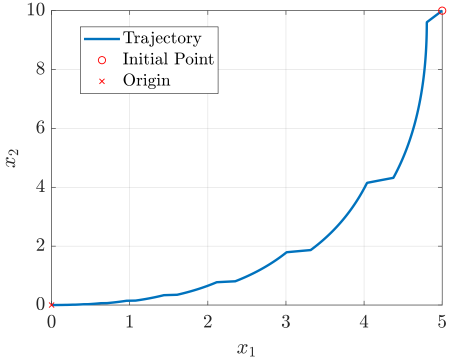
On stability of switched linear systems

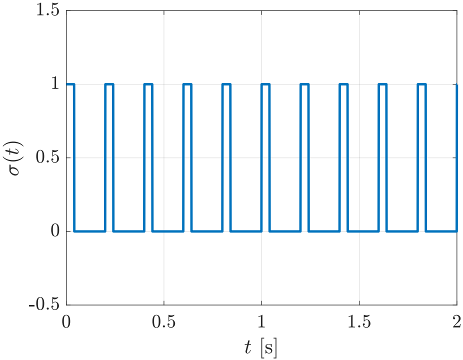
Inside of Matlab Function block use a code like as follows

```
function xdot = fcn(x,sigma,A1, A2)

if(sigma==1)
   xdot = A1*x;
else
   xdot = A2*x;
end

end
```

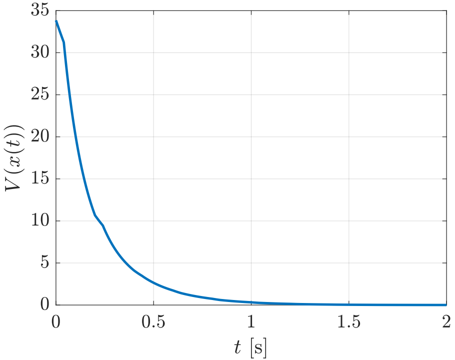
The idea is realize integration of $\dot{x}$ outside Matlab Function block to get x whereas $\sigma(t)$ selects the subsystem. To create $\sigma(t)$ you can use the block called "PWM Generator" or you can create with a random signal generator, be creative!.

On stability of switched linear systems





On stability of switched linear systems



On stability of switched linear systems

Although we got suffice condition of stability with

$$A_p'P + PA_p < 0, \quad \forall p \in \mathbb{K}. \tag{9}$$

such approach is very conservative! For instance, consider that

$$A_1 = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & -10 \\ 0.1 & -1 \end{bmatrix} \tag{10}$$

which are both Hurwitz. Now, without loss of generality, we can pick up

$$P = \begin{bmatrix} 1 & q \\ q & r \end{bmatrix}$$

On stability of switched linear systems

Now, $A_1'P + PA_1 < 0$ is true only if

$$q^2 + \frac{(r-3)^2}{8} < 1$$

and $A_2'P + PA_2 < 0$ is true only if

$$q^2 + \frac{(r-300)^2}{800} < 100$$

We can obtain the solution graphically as next.

On stability of switched linear systems

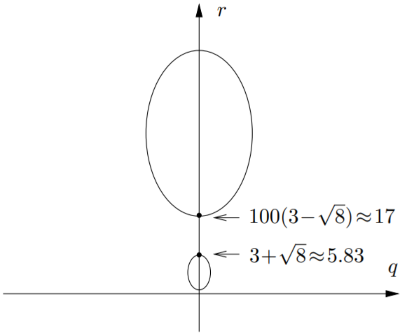


Figure 7: Ellipses showing that is not possible get a solution. Source: Liberzon, 2003[2]

[2] Daniel Liberzon (2003). *Switching in systems and control*. Springer Science & Business Media.

On stability of switched linear systems

Conclusion: Even the subsystems are Hurwitz we can not ensure that there exists a common matrix $P > 0$ for them. In other words, we can not say that the switched system is stable even with all subsystems stable. **Local stability does not implies in global stability**.

Solution: We can relax such condition by considering a matrix P for each subsystem.

The next theorem shows an alternative to obtain stability of switched linear system.

On stability of switched linear systems

Let $\{A_i : i \in \mathbb{K}\}$ be a compact set of Hurwitz matrix, t_k and t_{k+1} be successive switching times of $\sigma(t)$, consider that

$$\dot{x}(t) = A_{\sigma(t)}x(t), \quad x(0) = x_0, \quad x_0 \in \mathbb{R}^{n\times n}, \quad A_p \in \mathbb{R}^{n\times n} \forall p \in \mathbb{K}. \tag{11}$$

furthermore consider that

$$V(x) = x' P_{\sigma(t)} x$$

then the next theorem proposed by **Geromel and Colaneri, 2006**[3] uses the concept of **multiple Lyapunov function** with the innovation that the classical nonincreasing assumption at switching times is no longer needed.

[3] Jose C Geromel and Patrizio Colaneri (2006). "Stability and stabilization of continuous-time switched linear systems". In: *SIAM Journal on Control and Optimization* 45.5, pp. 1915–1930.

On stability of switched linear systems

Theorem 10

(Geromel and Colaneri, 2006[a]) Assume that for some $T > 0$, there exists a collection of positive definite matrices $\{P_1, \cdots, P_m\}$ of compatible dimensions such that

$$A_i' P_i + P_i A_i < 0, \quad \forall i \in \mathbb{K}$$

and

$$e^{A_i' T} P_j e^{A_i T} - P_i < 0, \quad \forall i \neq j \in \mathbb{K}.$$

The time-switching function $\sigma(t)$ with $t_{k+1} - t_k \geq T$ (Dwell-Time) makes the equilibrium solution $x = 0$ of (11) globally asymptotically stable (GAS).

[a] Jose C Geromel and Patrizio Colaneri (2006). "Stability and stabilization of continuous-time switched linear systems". In: *SIAM Journal on Control and Optimization* 45.5, pp. 1915–1930.

On stability of switched linear systems

Example Let us consider the following subsystems with **dwell-time** switching function with $T = 1/5$ as

$$A_1 = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & -10 \\ 0.1 & -1 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \tag{12}$$

From theorem 10 we have,

$$\begin{cases} A_1' P_1 + P_1 A_1 < 0 \\ A_2' P_2 + P_2 A_2 < 0 \\ e^{A_1' T} P_2 e^{A_1 T} - P_1 < 0 \\ e^{A_2' T} P_1 e^{A_2 T} - P_2 < 0. \end{cases}$$

We need to find $P_1 > 0$ and $P_2 > 0$, which can be done on Matlab with the next code.

On stability of switched linear systems

```
close all
clear
clc

% Parameters
A1 = [-3 1;0 -1];
A2 = [-2 1;0 -5];
x0 = [5;10];

T = 1/5;

% Declare SDP matrices
P1 = sdpvar(2,2);
P2 = sdpvar(2,2);
```

On stability of switched linear systems

```
% Set LMI
T1 = A1'*P1+P1*A1;
T2 = A2'*P2+P2*A2;
T3 = expm(A1'*T)*P2*expm(A1*T)-P1;
T4 = expm(A2'*T)*P1*expm(A2*T)-P2;
F = [P1>0; P2>0 ; T1<0; T2<0; T3<0; T4<0];

about = optimize(F)
P1_num = double(P1)
P2_num = double(P2)
```

On stability of switched linear systems

From that, we got

$$P_1 = \begin{bmatrix} 0.358 & 0.299 \\ 0.299 & 1.48 \end{bmatrix}$$

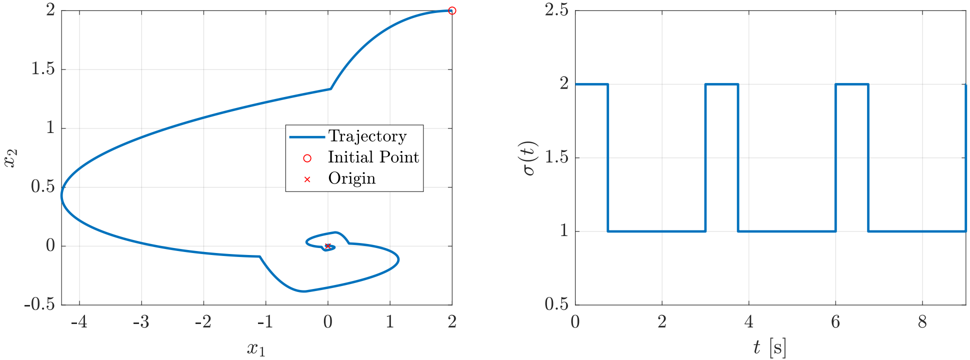
and

$$P_2 = \begin{bmatrix} 0.145 & -0.309 \\ -0.309 & 3.55 \end{bmatrix}$$

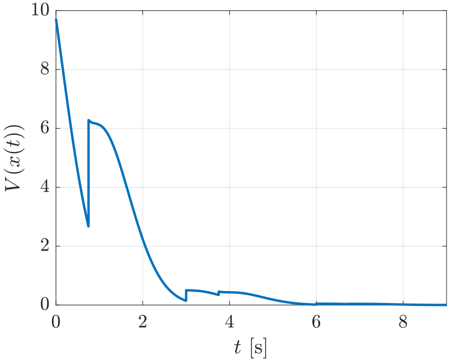
which are both symmetric positive definite matrices.

Therefore, from theorem 10, we can tell that system has GAS. As a matter of fact, next we showed the simulation upholding that.

On stability of switched linear systems



On stability of switched linear systems



Homework

Consider the following subsystems with **dwell-time** switching function with $T = 1/10$ as

$$A_1 = \begin{bmatrix} -2 & 0 \\ 1 & -3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0 \\ 6 & -4 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \tag{13}$$

1. Is the linear switched system stable with a common Lyapunov function ?
2. Is the linear switched system stable with a multiple Lyapunov function ?
3. If we change $x(0)$, it does affect the stability ?
4. Bring up a drawback in consider multiple Lyapunov function to state stability with switched linear systems.