

## Optimal Robust Linear Quadratic Regulator for Systems Subject to Uncertainties

Marco H. Terra, João P. Cerri, and João Y. Ishihara

**Abstract**—In this technical note, a robust recursive regulator for linear discrete-time systems, which are subject to parametric uncertainties, is proposed. The main feature of the optimal regulator developed is the absence of tuning parameters in online applications. To achieve this purpose, a quadratic cost function based on the combination of penalty function and robust weighted least-squares methods is formulated. The convergence and stability proofs for the stationary system and a numerical comparative study among the standard linear quadratic regulator, guaranteed cost and  $H_\infty$  controllers are provided.

**Index Terms**—Discrete-time systems, least squares, min-max problem, penalty function, Riccati Equation, robust regulator.

### I. INTRODUCTION

The regulation of linear discrete-time systems has been fundamental in a large number of applications such as aerospace, robotics, communication, finance, chemistry, and other fields. An important result related with optimal control and filtering was obtained by Rudolf E. Kalman in the early 1960's. At that time, Kalman was interested in providing an alternative solution to decrease the computational complexity of the optimal filter proposed by Norbert Wiener [1]. Kalman brought to the center of the filtering theory, arguments based on orthogonal projections which resulted in efficient recursive algorithms for online estimates. By duality, they were directly related with noise-free optimal regulator problem, as described in [2]. Recursiveness and easily attainable existence conditions are two great features of the standard regulator synthesized through Riccati equation, when the dynamic system is not subject to uncertainties. In the presence of uncertainties, the optimality of the standard regulator is not guaranteed. Consequently, its performance is compromised and even the stability is not assured. (See, e.g., in [3] and [4] some issues related with sensitivity of linear state feedback control systems and in [5] and [6] examples on the robustness properties of linear quadratic regulators.)

In the last four decades, several techniques have been proposed to solve robust control problems when external disturbances and internal uncertainties influence the performance and the stability of feedback systems, see for instance [7]–[23], and references therein. Recursive solutions have already been obtained for uncertain systems (see, e.g., [20] and [24]). Some approaches have used Riccati equations, rather than inequalities, to deal with robustness (see, e.g., [13], [15], [20], [25], and [26]). In all these references, it is always necessary to search the optimal robust performance through an auxiliary parameter whose adjustment depends on the solution of the Riccati equation. As a consequence, the parameter tuning remains as the main drawback to obtain optimal robust controllers directly in online applications. In order to overcome this problem, the question is how to obtain optimal

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robust regulators which resemble to the optimal recursive algorithms developed for standard regulators.

This note deals with the classical robust recursive control problem for time-varying linear systems subject to parametric uncertainties. A key issue in the developed approach is to transform a constrained control problem subject to uncertainties in an unconstrained one. Defining control inputs and states as variables to be minimized, it is possible to regularize the solution of this optimization problem. The new proposed quadratic cost function is optimized by the combination of a penalty function and a robust weighted least-squares method. An optimal equilibrium between the best performance and the maximum influence of uncertainties is found through the minimization of a continuous function which presents a unique global minimum. The optimal robust control law proposed recovers the simplicity of the standard optimal control in the sense that the performance tuning is no longer necessary.

This paper is organized as follows. In Section II, the robust control problem is introduced. In Section III, the recursive robust regulator with the respective Riccati equation is developed. In Section III-A, the convergence and stability of the robust regulator are proved. In Section IV, three examples to illustrate the performance of the robust optimal linear quadratic regulator are provided.

### II. PROBLEM FORMULATION

In this technical note, the problem of obtaining a robust linear quadratic regulator (RLQR) for the discrete-time linear system subject to parametric uncertainties

$$x_{i+1} = (F_i + \delta F_i)x_i + (G_i + \delta G_i)u_i; \quad i = 0, \dots, N \quad (1)$$

is dealt with. In (1),  $x_i \in \mathbb{R}^n$  is the state vector,  $u_i \in \mathbb{R}^m$  is the control input vector,  $F_i \in \mathbb{R}^{n \times n}$  and  $G_i \in \mathbb{R}^{n \times m}$  are nominal parameter matrices

$$[\delta F_i \ \delta G_i] = H_i \Delta_i [E_{F_i} \ E_{G_i}]; \quad i = 0, \dots, N \quad (2)$$

are uncertainty matrices in which  $H_i \in \mathbb{R}^{n \times p}$ ,  $E_{F_i} \in \mathbb{R}^{l \times n}$ ,  $E_{G_i} \in \mathbb{R}^{l \times m}$  are known matrices, and  $\Delta_i \in \mathbb{R}^{p \times l}$  is an arbitrary matrix such that  $\|\Delta_i\| \leq 1$ . It is supposed that  $H_i$  is a nonzero matrix for all  $i = 0, \dots, N$ . For this class of system, which has been studied since 1950's [27], for each  $u_i$  fixed we have a set of possible solutions  $x_{i+1}$ .

In order to obtain the RLQR, the following optimization problem is solved:

$$\min_{x_{i+1}, u_i} \max_{\delta F_i, \delta G_i} \{ \tilde{J}_i^\mu(x_{i+1}, u_i, \delta F_i, \delta G_i) \} \quad (3)$$

where the one-step quadratic cost function is given by (4) with  $Q_i \succ 0$ ,  $R_i \succ 0$ ,  $P_{i+1} \succ 0$ , and  $\mu > 0$

$$\begin{aligned} & \tilde{J}_i^\mu(x_{i+1}, u_i, \delta F_i, \delta G_i) \\ &= \begin{bmatrix} x_{i+1} \\ u_i \end{bmatrix}^T \begin{bmatrix} P_{i+1} & 0 \\ 0 & R_i \end{bmatrix} \begin{bmatrix} x_{i+1} \\ u_i \end{bmatrix} \\ &+ \left\{ \left( \begin{bmatrix} 0 & 0 \\ I & -G_i \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -\delta G_i \end{bmatrix} \right) \begin{bmatrix} x_{i+1} \\ u_i \end{bmatrix} \right. \\ &\quad \left. - \left( \begin{bmatrix} -I \\ F_i \end{bmatrix} x_i + \begin{bmatrix} 0 \\ \delta F_i \end{bmatrix} x_i \right) \right\}^T \begin{bmatrix} Q_i & 0 \\ 0 & \mu I \end{bmatrix} \{ \bullet \}. \quad (4) \end{aligned}$$

The matrices  $Q_i$ ,  $R_i$ , and  $P_{i+1}$  are weighting matrices and  $\mu$  is a penalty parameter. This parameter is responsible to guarantee that the

equality (1) holds and to assure the regularization of the RLQR. It also determines the robustness level of the RLQR. The optimization problem (3) takes two contradictory objectives into account defined to encompass the worst influence of uncertainties and the optimal control of the system.

The motivation to define the problem (3), (4) is inspired by the structural features of classical quadratic-cost designs, developed in the 1950's and 1960's, combined with penalty function, see details in [28]–[30], and [31]. Therefore, it is possible to extend the recursive Riccati equation-based control for the RLQR. In fact, if uncertainties are not considered, (3), (4) reduces to the following unconstrained minimization problem:

$$\min_{x_{i+1}, u_i} \{J_i^\mu(x_{i+1}, u_i)\} \quad (5)$$

where

$$J_i^\mu(x_{i+1}, u_i) = x_{i+1}^T P_{i+1} x_{i+1} + u_i^T R_i u_i + x_i^T Q_i x_i + \mu(x_{i+1} - F_i x_i - G_i u_i)^T (x_{i+1} - F_i x_i - G_i u_i), \quad (6)$$

which is equivalent, as  $\mu$  tends to infinity, to the constrained minimization problem

$$\begin{aligned} \min_{x_{i+1}, u_i} \{x_{i+1}^T P_{i+1} x_{i+1} + u_i^T R_i u_i + x_i^T Q_i x_i\} \\ \text{s.t. } x_{i+1} = F_i x_i + G_i u_i. \end{aligned} \quad (7)$$

Comparing (7) with the standard LQR problem ([32, p. 362]), the only difference is that the minimization in (7) depends on  $x_{i+1}$  and  $u_i$  rather than only on  $u_i$ . Independent of the minimization approach adopted, it is obtained the same solution for the standard LQR. This is due to the fact that for each  $u_i$ , there exists only one equivalent  $x_{i+1}$  when (1) is not subject to uncertainties. For the other hand, it is shown in Section III that the choice of variables to be optimized is fundamental to obtain a regularized RLQR.

### III. ROBUST LINEAR QUADRATIC REGULATOR

The optimization problem (3), (4) can be solved based on the solution of a general robust regularized least-squares problem:

$$\min_x \max_{\delta A, \delta b} \left\{ \|x\|_Q^2 + \|(A + \delta A)x - (b + \delta b)\|_W^2 \right\} \quad (8)$$

where  $A$  is a nominal matrix,  $b$  is a measurement vector,  $Q \succ 0$  and  $W \succ 0$  are weighting matrices,  $x$  is an unknown vector, and  $\{\delta A, \delta b\}$  are perturbations modeled by

$$[\delta A \ \delta b] = H \Delta [E_A \ E_b], \quad \|\Delta\| \leq 1 \quad (9)$$

see, e.g., [15], [33], and [34]. In these references it is shown that there exists a unique optimal solution for (8), (9) which depends on a continuous function. This unique solution can be obtained thanks to the transformation of (8), (9) in an equivalent min – min optimization problem, see [15]. An important consequence of this result is that saddle point analysis for the problem (8) as done, for instance in [35], can be replaced by a much easier convex analysis of an equivalent optimization problem.

Even with the great advantage to deal with continuous functions that present a global minimum, it is not an easy task to find this optimal solution in online applications. Mainly related to robust control problems, see an example of application provided in [15]. The novelty of the approach presented in this section is that with the penalty parameter  $\mu$  introduced in this note, the optimal operation point of

the RLQR comes up at each step of the algorithm. When appropriate identifications of (3), (4) with (8), (9) are performed, i.e.,

$$\begin{aligned} Q &\leftarrow \begin{bmatrix} P_{i+1} & 0 \\ 0 & R_i \end{bmatrix}, \quad x \leftarrow \begin{bmatrix} x_{i+1}(\mu) \\ u_i(\mu) \end{bmatrix}, \quad W \leftarrow \begin{bmatrix} Q_i & 0 \\ 0 & \mu I \end{bmatrix}, \\ A &\leftarrow \begin{bmatrix} 0 & 0 \\ I & -G_i \end{bmatrix}, \quad \delta A \leftarrow \begin{bmatrix} 0 & 0 \\ 0 & -\delta G_i \end{bmatrix}, \quad b \leftarrow \begin{bmatrix} -I \\ F_i \end{bmatrix} x_i, \\ \delta b &\leftarrow \begin{bmatrix} 0 \\ \delta F_i \end{bmatrix} x_i, \quad H \leftarrow \begin{bmatrix} 0 \\ H_i \end{bmatrix}, \\ \Delta &\leftarrow \Delta_i, \quad E_A \leftarrow [0 \ -E_{G_i}], \quad E_b \leftarrow E_{F_i} x_i \end{aligned}$$

the regularization of the robust regulator is obtained thanks to the minimization over both variables  $x_{i+1}(\mu)$  and  $u_i(\mu)$ . The advantage of the application of this property is that the existence condition of the RLQR is well defined, which is useful for online applications. In order to show this characteristic, the next lemma presents a framework given in terms of an array of matrices, according to [36], to compute the optimal state trajectory, input control, and cost function of the RLQR.

*Lemma 1:* Consider the optimization problem (3), (4). The optimal solution for each  $\mu > 0$  is given by

$$\begin{bmatrix} x_{i+1}^*(\mu) \\ u_i^*(\mu) \\ \tilde{J}_i^\mu(x_{i+1}^*(\mu), u_i^*(\mu)) \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & x_i^*(\mu)^T \end{bmatrix} \begin{bmatrix} L_{i,\mu} \\ K_{i,\mu} \\ P_{i,\mu} \end{bmatrix} x_i^*(\mu) \quad (10)$$

with

$$\begin{aligned} \begin{bmatrix} L_{i,\mu} \\ K_{i,\mu} \\ P_{i,\mu} \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & -I & \mathcal{F}_i^T & 0 & 0 \end{bmatrix} \\ &\times \begin{bmatrix} P_{i+1}^{-1} & 0 & 0 & 0 & I & 0 \\ 0 & R_i^{-1} & 0 & 0 & 0 & I \\ 0 & 0 & Q_i^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \Sigma_i(\mu, \lambda_i) & \mathcal{I} & -G_i \\ I & 0 & 0 & \mathcal{I}^T & 0 & 0 \\ 0 & I & 0 & -\mathcal{G}_i^T & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ -I \\ \mathcal{F}_i \\ 0 \\ 0 \end{bmatrix} \end{aligned} \quad (11)$$

where  $\Sigma_{i,\mu} := \Sigma_i(\mu, \lambda_i) = \begin{bmatrix} \mu^{-1}I - \lambda_i^{-1}H_i H_i^T & 0 \\ 0 & \lambda_i^{-1}I \end{bmatrix}$ ,

$\mathcal{I} = \begin{bmatrix} I \\ 0 \end{bmatrix}$ ,  $\mathcal{G}_i = \begin{bmatrix} G_i \\ E_{G_i} \end{bmatrix}$ ,  $\mathcal{F}_i = \begin{bmatrix} F_i \\ E_{F_i} \end{bmatrix}$ , and  $\lambda_i > \|\mu H_i^T H_i\|$ .

Furthermore, alternatively one has

$$\begin{aligned} P_{i,\mu} &= L_{i,\mu}^T P_{i+1} L_{i,\mu} + K_{i,\mu}^T R_i K_{i,\mu} + Q_i + \\ &(\mathcal{I} L_{i,\mu} - \mathcal{G}_i K_{i,\mu} - \mathcal{F}_i)^T \Sigma_{i,\mu}^{-1} (\mathcal{I} L_{i,\mu} - \mathcal{G}_i K_{i,\mu} - \mathcal{F}_i) \succ 0. \end{aligned} \quad (12)$$

*Proof:* Based on the results presented in [36]. ■

With Lemma 1, one can obtain the RLQR shown in Table I. Two important features of this regulator is that it does not depend on  $H_i$  and the resulting recursiveness can be performed without the need of tuning auxiliary parameters. In fact, it depends only on weighting and parameter matrices, which are known *a priori*. The penalty function method imposes that  $\mu \rightarrow +\infty$ , in consequence

$$\lim_{\mu \rightarrow +\infty} \Sigma_i(\mu, \lambda_i) = 0$$

and the quadratic term of (12) also vanishes,

$$(\mathcal{I} L_{i,\mu} - \mathcal{G}_i K_{i,\mu} - \mathcal{F}_i)^T \Sigma_{i,\mu}^{-1} (\mathcal{I} L_{i,\mu} - \mathcal{G}_i K_{i,\mu} - \mathcal{F}_i) \rightarrow 0.$$

TABLE I  
ROBUST LINEAR QUADRATIC REGULATOR

<p><b>Uncertain Model:</b> Consider (1), (2), and (4) with <math>F_i, G_i, E_{F_i}, E_{G_i}, Q_i \succ 0</math>, and <math>R_i \succ 0</math> known for all <math>i</math>.</p> <p><b>Initial Conditions:</b> Set <math>x_0</math> and <math>P_{N+1} \succ 0</math>.</p> <p><b>Step 1: (Backward)</b> Calculate for all <math>i = N, \dots, 0</math>:</p> $\begin{bmatrix} L_i \\ K_i \\ P_i \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -I \\ 0 & 0 & F_i \\ 0 & 0 & E_{F_i} \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix}^T \begin{bmatrix} P_{i+1}^{-1} & 0 & 0 & 0 & 0 & I & 0 \\ 0 & R_i^{-1} & 0 & 0 & 0 & 0 & I \\ 0 & 0 & Q_i^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & -G_i \\ 0 & 0 & 0 & 0 & 0 & 0 & -E_{G_i} \\ I & 0 & 0 & I & 0 & 0 & 0 \\ 0 & I & 0 & 0 & -G_i^T & -E_{G_i}^T & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ -I \\ F_i \\ E_{F_i} \\ 0 \\ 0 \end{bmatrix}.$ <p><b>Step 2: (Forward)</b> Obtain for each <math>i = 0, \dots, N</math>:</p> $\begin{bmatrix} x_{i+1}^* \\ u_i^* \end{bmatrix} = \begin{bmatrix} L_i \\ K_i \end{bmatrix} x_i^*,$ <p>with the total cost given by <math>J_r^* = x_0^T P_0 x_0</math>.</p>	
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For each robust iteration of (12), the matrix  $P_{i,\mu}$  is finite and  $\mathcal{L}L_{i,\mu} - \mathcal{G}_i K_{i,\mu} - F_i \rightarrow 0$  which leads to

$$\begin{cases} L_{i,\infty} = F_i + G_i K_{i,\infty} \\ E_{F_i} + E_{G_i} K_{i,\infty} = 0. \end{cases} \quad (13)$$

A sufficient condition to satisfy (13) is that  $\text{span}(E_{F_i}) \subset \text{span}(E_{G_i})$ , or equivalently

$$\text{rank}([E_{F_i} \ E_{G_i}]) = \text{rank}(E_{G_i}). \quad (14)$$

In this case, (12) reduces to

$$P_i = L_i^T P_{i+1} L_i + K_i^T R_i K_i + Q_i \quad (15)$$

where  $L_i := L_{i,\infty}$ ,  $K_i := K_{i,\infty}$ , and  $P_i := P_{i,\infty}$  are defined to simplify the notation. Notice that with (13) the closed-loop system of (1) becomes

$$\begin{aligned} x_{i+1}^* &= [(F_i + \delta F_i) + (G_i + \delta G_i)K_i] x_i^* \\ &= (F_i + G_i K_i) x_i^* = L_i x_i^*, \quad i = 0, \dots, N. \end{aligned} \quad (16)$$

In the robust regulator proposed by [15], the optimal parameter  $\lambda_i$  should be determined over an interval given by  $(a, +\infty)$ . The lower bound  $a$  depends on the solution of the Riccati equation  $P_{i+1}$  and *a posteriori* computations are necessary to check the stability of the closed-loop system. In the RLQR presented in Table I, the parameter  $\lambda_i$  vanished. The stability and the optimal robust performance are always guaranteed at each instant of time.

#### A. Convergence and Stability

Algebraic manipulations of expressions presented in Table I show that the array of matrices proposed reduces to the form of a standard Riccati equation. Hence, convergence and stability of the RLQR developed can be obtained performing some direct identifications with the standard optimal regulator problem, for systems not subject to uncertainties, see for instance [32]. Considering the recursive equation

$$P_i = \hat{F}_i^T \left( P_{i+1} - P_{i+1} \hat{G}_i (I + \hat{G}_i^T P_{i+1} \hat{G}_i)^{-1} \hat{G}_i^T P_{i+1} \right) \hat{F}_i + \hat{Q}_i \quad (17)$$

with

$$\begin{aligned} \hat{F}_i &= F_i - G_i R_i^{-1} E_{G_i}^T (E_{G_i} R_i^{-1} E_{G_i}^T)^{-1} E_{F_i}, \\ \hat{G}_i &= G_i \hat{R}_i^{-\frac{1}{2}}, \quad \hat{R}_i = R_i^{-1} - R_i^{-1} E_{G_i}^T (E_{G_i} R_i^{-1} E_{G_i}^T)^{-1} E_{G_i} R_i^{-1}, \\ \hat{Q}_i &= Q_i + E_{F_i}^T (E_{G_i} R_i^{-1} E_{G_i}^T)^{-1} E_{F_i} \end{aligned}$$

and bearing in mind that the standard Riccati equation is given by

$$P_i = F_i^T \left( P_{i+1} - P_{i+1} G_i (R_i + G_i^T P_{i+1} G_i)^{-1} G_i^T P_{i+1} \right) F_i + Q_i \quad (18)$$

the identifications follow directly:

$$F_i \leftarrow \hat{F}_i; \quad G_i \leftarrow \hat{G}_i; \quad R_i \leftarrow I; \quad Q_i \leftarrow \hat{Q}_i.$$

In addition, the optimal feedback gain  $\hat{K}_i$  and the closed-loop system matrix  $\hat{L}_i$ , for  $u_i^* = \hat{K}_i x_i^*$  and  $x_{i+1}^* = \hat{L}_i x_i^*$ , are given by

$$\hat{K}_i = - (I + \hat{G}_i^T P_{i+1} \hat{G}_i)^{-1} \hat{G}_i^T P_{i+1} \hat{F}_i, \quad (19)$$

$$\hat{L}_i = \hat{F}_i + \hat{G}_i \hat{K}_i. \quad (20)$$

With similar arguments used in robust filtering problems, see [33], [37], [38], convergence and stability are proved in a deterministic way. As aforementioned an interesting feature of this robust regulator is that it resembles to the standard LQR where the stability is related directly with the positiveness of the Riccati equation solution.

#### IV. NUMERICAL EXAMPLE

In this section, three examples are presented to illustrate the efficiency of the RLQR. In the first, the behavior of the robust regulator for a set of parameters  $\mu$ , including  $\mu \rightarrow +\infty$ , is shown. In the second, a comparative study among the approach proposed and three other approaches is presented where regularization and deregularization of the respective solutions are put in evidence. In the third, a fundamental limitation of the standard LQR and the  $H_\infty$  controller in online applications is shown.

*Example 1:* Consider the uncertain system (1), (2) with the following parameter matrices and initial conditions:

$$F = \begin{bmatrix} 1.1 & 0 & 0 \\ 0 & 0 & 1.2 \\ -1.0 & 1.0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 1.0 \\ 1.0 & 1.0 \\ -1.0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0.7 \\ 0.5 \\ -0.7 \end{bmatrix},$$

$$E_F = [0.4 \ 0.5 \ -0.6], \quad E_G = [0.4 \ -0.4], \quad -1 \leq \Delta_i \leq 1$$

and  $x_0 = [1 \ -1 \ 0.5]^T$ . With respect to the quadratic cost function (4), the weighting matrices are given by  $P_{N+1} = I_3$ ,  $Q = I_3$ , and  $R = I_2$ .

Some simulations are performed to illustrate the behavior of the robust RLQR proposed in Lemma 1, when the penalty parameter  $\mu$  tends to infinity. In Table II, it is presented the convergence of  $P_i$  and  $K_i$ , the total cost function, and the relation  $E_{F_i} + E_{G_i} K_i$  for different values of  $\mu$ .

TABLE II  
CONVERGENCE OF THE ROBUST LINEAR QUADRATIC REGULATOR WHEN  $\mu \rightarrow +\infty$

$\mu$	$P_i$			$K_i$			Cost	$E_{F_i} + E_{G_i} K_i$
$10^2$	25.4185 0.0837 -16.8601	0.0837 5.8550 -2.1563	-16.8601 -2.1563 14.2362	$\begin{bmatrix} -1.1581 & -0.2841 & 0.5248 \\ -0.3801 & 0.8652 & -0.7762 \end{bmatrix}$	19.9612	$[0.0888 \quad 0.0403 \quad -0.0796]$		
$10^5$	31.0526 0.4401 -21.0090	0.4401 6.1407 -2.5223	-21.0090 -2.5223 17.3339	$\begin{bmatrix} -1.3258 & -0.3345 & 0.6635 \\ -0.3261 & 0.9154 & -0.8363 \end{bmatrix}$	22.1599	$10^{-3} [0.1119 \quad 0.0440 \quad -0.0975]$		
$10^{10}$	31.0596 0.4406 -21.0142	0.4406 6.1411 -2.5228	-21.0142 -2.5228 17.3378	$\begin{bmatrix} -1.3260 & -0.3345 & 0.6636 \\ -0.3260 & 0.9155 & -0.8364 \end{bmatrix}$	22.1625	$10^{-8} [0.1120 \quad 0.0440 \quad -0.0976]$		
$\infty$	31.0596 0.4406 -21.0142	0.4406 6.1411 -2.5228	-21.0142 -2.5228 17.3378	$\begin{bmatrix} -1.3260 & -0.3345 & 0.6636 \\ -0.3260 & 0.9155 & -0.8364 \end{bmatrix}$	22.1625	$10^{-10} [0.1120 \quad 0.0440 \quad -0.0976]$		

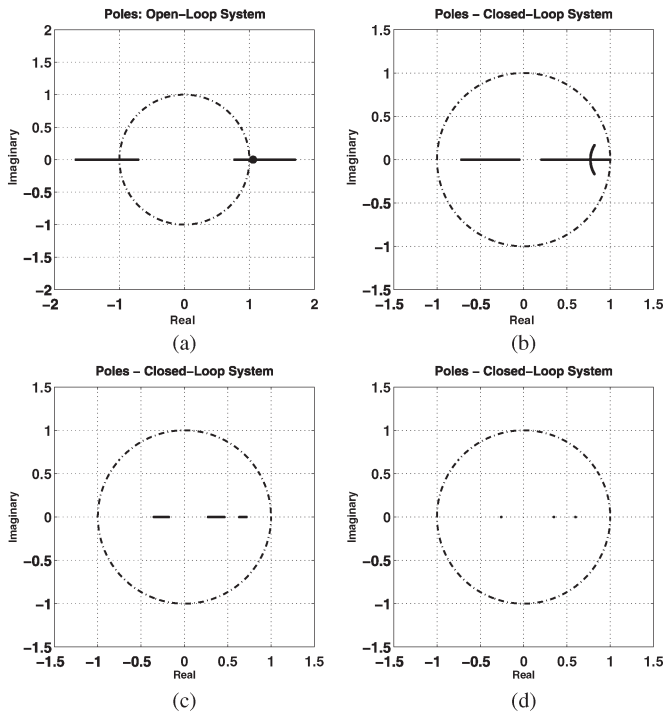


Fig. 1. Poles—open-loop and closed-loop system for  $(-1 \leq \Delta_i \leq 1)$ . (a); (b)  $\mu = 10$ ; (c)  $\mu = 10^2$ ; (d)  $\mu > 10^{10}$ .

Notice that for sufficiently large values of  $\mu$ , the sequence  $\{P_i\}_{i=0}^{+\infty}$  given in Table I converges to the solution  $P$  when  $i \rightarrow +\infty$ . In addition, the gain  $K$  is such that the resulting feedback System of (1), (2) is stable. One can see this feature through the open-loop and closed-loop poles depicted in Fig. 1, when  $\mu \rightarrow +\infty$ , for admissible uncertainties  $\{\delta F_i, \delta G_i\}$ .

It is worth emphasizing that the robust gains designed stabilize the system even for small values of  $\mu$ , as it can be seen in Figs. 1(b) and 1(c). The states and control inputs of the closed-loop system, where  $\Delta_i$  ( $-1 \leq \Delta_i \leq 1$ ) were selected randomly at each time instant, are shown in Fig. 2.

*Example 2:* It is presented in this example a comparative study on the performance of the RLQR and the standard LQR,  $H_\infty$ , and Guaranteed Cost controllers. It is based on the system of Example 1.

- 1) Standard LQR [39]: it is designed based on the nominal system to control the system subject to uncertainties. Fig. 3 illustrates the performance of the LQR.

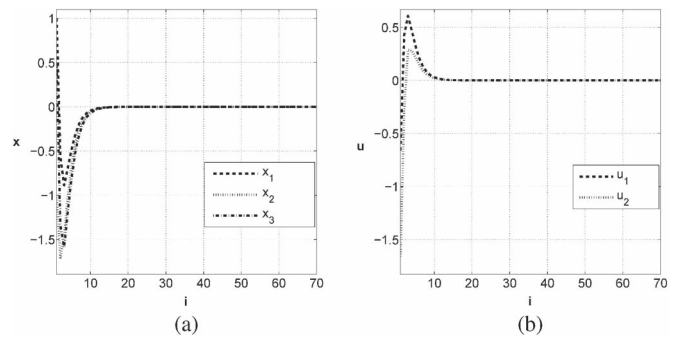


Fig. 2. Robust linear quadratic regulator—Table I. (a) States. (b) Control inputs.

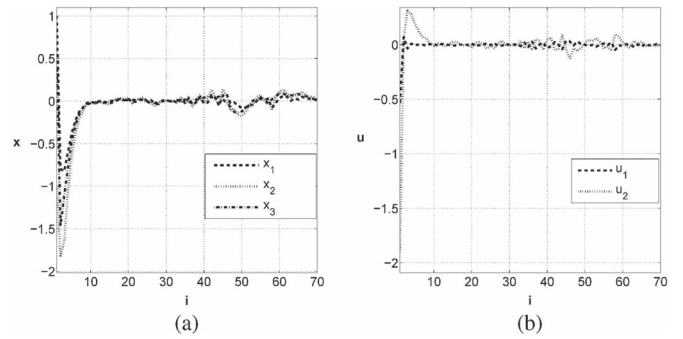


Fig. 3. Standard Linear Quadratic Regulator. (a) States. (b) Control inputs.

For each time instant  $i$ , Figs. 3(a) and 3(b) correspond to the mean of the states and control inputs, respectively, calculated over  $T = 3000$  experiments for a horizon  $N = 70$ . In Figs. 2(a) and 2(b) one can see that states and control inputs do not oscillate in steady state when the system is controlled through the RLQR.

- 2)  $H_\infty$  Control ([24] and [8]—Ch. 3): it is designed for the following linear system with disturbance

$$x_{i+1} = F_i x_i + G_{1,i} w_i + G_{2,i} u_i, \quad i = 0, \dots, N \quad (21)$$

where  $w_i$  is the disturbance. Figs. 4(a) and 4(b) show states and control inputs of the closed-loop system for  $\gamma$  tuned in 2.45 and  $x_0$  selected randomly. After 3000 experiments, it was obtained a mean total cost  $\bar{J} = 18.96$ .

- 3) Guaranteed Cost Control[20]: for this example, there exists a positive-definite solution  $P^*$  for this controller when

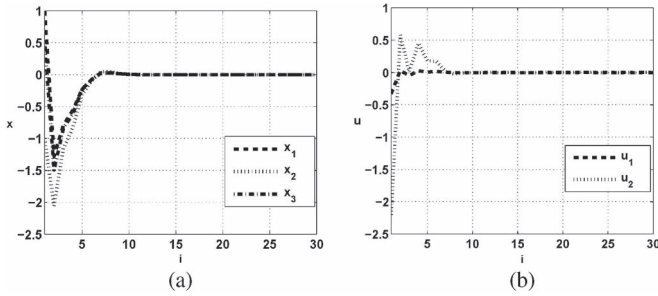


Fig. 4. Uncertain closed-loop system— $H_\infty$  Control: ( $\gamma = 2.45$ ). (a) States. (b) Control inputs.

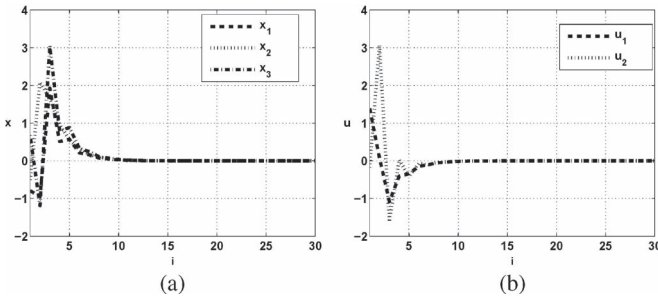


Fig. 5. Uncertain closed-loop system—Guaranteed Cost Control:  $\epsilon^* = 0.0182$ . (a) States. (b) Control inputs.

$\epsilon \in (0, 0.0220)$ . For  $\epsilon^* = 0.0182$ , it was obtained the optimal stabilizing solution  $P^*$  and feedback gain  $K^*$  as

$$P^* = \begin{bmatrix} 167.8105 & 11.0383 & -121.6068 \\ 11.0383 & 6.7848 & -10.2601 \\ -121.6068 & -10.2601 & 91.3164 \end{bmatrix}$$

and

$$K^* = \begin{bmatrix} -1.1788 & -0.2140 & 0.5062 \\ -0.7445 & 0.8380 & -0.5129 \end{bmatrix}$$

with an guaranteed cost upper bound of the uncertain closed-loop system computed as  $J^* = 53.7109$ . Figs. 5(a) and 5(b) show the behavior of the uncertain closed loop system, for initial states  $x_0$  also selected randomly.

The offline tuned parameters  $\gamma$  and  $\epsilon$  related with  $H_\infty$  and guaranteed cost controllers respectively, represent an important limitation for these methods mainly in cases where the uncertainties increase in online applications. Although the LQR does not depend on the tune of auxiliary parameters, its performance deteriorates when the system is subject to uncertainties. For the RLQR developed, the recursiveness can be performed without the need of tuning auxiliary parameters, it depends only on the parameters and weighting matrices which are known *a-priori*. In addition, the stability is guaranteed for all admissible  $\{\delta F_i, \delta G_i\}$ .

*Example 3:* Consider the same parameter matrices given in the

Example 1, except that  $F$  was replaced by  $F = \begin{bmatrix} 5.5 & 0 & 0 \\ 0 & 0 & 6.0 \\ -5.0 & 5.0 & 0 \end{bmatrix}$ .

For this case, all eigenvalues of  $(F + \delta F)$  are outside the open unit disc for all admissible uncertainties. In these circumstances, notice that the  $H_\infty$  control is not able to regulate the uncertain system, despite the tune of the parameter  $\gamma = 25.2$  and the existence conditions of the solution have been verified, see Fig. 6. The performance of the RLQR and the Guaranteed Cost Controller, which was tuned as  $\epsilon^* = 1 \times 10^{-7}$ , are equivalent.

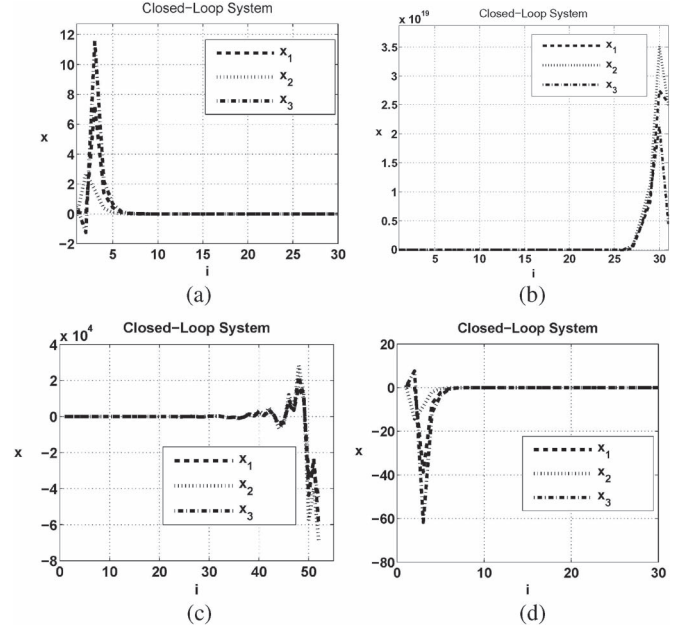


Fig. 6. Comparative study: closed-loop systems. (a) RLQR; (b) standard LQR; (c)  $H_\infty$  control; (d) guaranteed cost control.

V. CONCLUSION

In this technical note, it was proposed an optimal robust recursive regulator for uncertain discrete-time systems. The approach applied to design this class of regulator, based on penalty function and robust regularized least-squares method, provides an algorithm which does not depend on any parameter to be tuned. The proofs of convergence and stability presented, show that this robust regulator resembles to the standard regulator developed for systems not subject to uncertainties. The numerical examples showed the effectiveness of the RLQR proposed in comparison with other parameter-dependent classical robust controllers. This approach has been used to estimate and control uncertain systems subject to Markovian jumps [40] and [41]. The development of robust filters with the same features presented by this RLQR has been performed, preliminar results were presented in [42]. As future works, algebraic solutions for filtering and control problems of networked systems subject to uncertainties will be investigated.

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