Thévenin's Theorem, Cramer's Rule, and Parameterized Systems: Some Connections

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Engineering systems are complex systems comprising interconnections of simpler subsystems. Examples of this in electrical engineering are ac and dc circuits, motor drives, power systems, and electronic circuits. Similar examples occur in mechanical engineering, robotics, hydraulic networks, truss structures, and chemical reactors. In control engineering, complex feedback systems are created from interconnections of elementary blocks. In this article, we present a unified approach and framework applicable to the analysis of all such systems. The main unifying criterion is that the systems are described by linear equations. We show that the input–output maps of both algebraic and dynamic systems can be obtained by a common set of formulas developed by systematically applying Cramer's rule for solving linear equations.

Additionally, we consider systems containing parameters that may be uncertain or design parameters. For such parameterized systems, using Cramer's rule leads to the determination of a specific functional relationship between the parameter outputs. The unknown coefficients of this function can be found from a few strategic measurements of the system. This leads to a generalization and extension of Thévenin's theorem, rendering it applicable to arbitrary systems described by linear equations. This fundamental approach was developed in [1]. Here, we describe this approach and illustrate it with applications to electronic circuits and hydraulic systems.

The generalization proposed in this article makes the connection between Cramer's rule and Thévenin's theorem, thus extending the latter to a large class of systems described by linear equations. It will also be shown that this approach avoids short- or open-circuiting the system, as is required in applications of the classical Thévenin's theorem. In the presented approach, equivalent information can be obtained from a few measurements made using convenient system settings.

This article needs as background only basic algebraic, circuit theory, and dynamic physical system models concepts,

and thus, it should be accessible to students and professional engineers who work with system modeling. The main tools used in engineering dynamic systems, such as physical system modeling, are covered in several control textbooks. In some, linear graphs are also explored to obtain state-space and transfer function models (see [2] and [3]). In the former, the linear graph approach is used to obtain Thévenin and Norton equivalents.

As discussed in "Summary," this article presents a framework for addressing the input–output maps of systems, parameterized systems, and an extension of Thévenin's theorem. The parameterized state-space framework presented here was developed in [1] and used in applications such as agricultural sprayers [4], power electronic converters, and renewable sources [5], as shown in [6]–[8]. In the

Summary

The celebrated theorem by Thévenin is very useful in applications such as power system analysis because it states that it is possible to simplify a complex circuit to a much simpler equivalent circuit. Traditionally, it is accomplished by taking the power system to a short circuit that is a critical operating condition. This article, therefore, highlights the efficacy of using a proposed theorem that generalizes Thévenin's theorem and avoids submitting the system to critical operating conditions that differ from its conventional application. Additionally, this generalized Thévenin's theorem can be used in other fields, such as precision agriculture, to simplify complex interconnected systems. The connection between Thévenin's theorem and Cramer's rule in a parameterized state-space framework is also shown. The parameterized state-space solution is given in terms of parameters, which may include design parameters and unknown coefficients. Different problems in the agricultural, mechanical, and power electronic fields are used to illustrate practical applications of the theory presented.

following sections, the unified model and Cramer's rule are used to obtain the system output. The output of state-space models is obtained using the same framework. Subsequently, a parameterized version of the unified framework is presented, with the output solution expressed in terms of parameters of interest. Next, a measurement theorem is described to show that functions of the parameters of interest can be determined from a few measurements and obtain a generalization of the Thévenin theorem. Finally, concluding remarks are made, and future developments are described.

ALGEBRAIC SYSTEMS AND CRAMER'S RULE

Consider a system consisting of inputs $u_1, u_2, \ldots, u_r$, outputs denoted $y_1, y_2, \ldots, y_m$ (which are external variables), and internal or state variables denoted by $x_1, x_2, \ldots, x_n$, as illustrated in Figure 1. Using vector notation

$$\mathbf{y} := \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}, \mathbf{u} := \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix}, \mathbf{x} := \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix},$$

denote the m output vector, r input vector, and n state or internal vector, respectively.

When the system has no time derivative in the internal variable, it is an algebraic system. The algebraic equations describing the system are linear with constant coefficients. This results in a static, linear time-invariant system that is based on a state-space framework represented as

$$A\mathbf{x} + B\mathbf{u} = 0,$$
$$C\mathbf{x} + D\mathbf{u} = y, \tag{1}$$

with output vector $\mathbf{y}$, input vector $\mathbf{u}$ (as previously defined), and ($A, B, C,$ and D) matrices of size $n \times n$, $n \times r$, $m \times n$, and $m \times r$, respectively. Based on this state-space model, a general unified model is defined.

Definition 1: General Unified Model
Let $\mathbf{z} := [\mathbf{x}^T \ \mathbf{y}^T]^T$. Equation (1) may be rewritten as

$$\begin{bmatrix} A & 0 \\ -C & I \end{bmatrix} \mathbf{z} = \begin{bmatrix} -B \\ D \end{bmatrix} \mathbf{u}. \tag{2}$$

The system representation in (2), based on a state-space model, relates all of the inputs and internal and output variables to be solved in a single step. As a result, (2) is defined as a *general unified model*. However, in many cases, there is no interest in the solution for all of the internal variables in vector $\mathbf{x}$. The only physically meaningful variables are often $\mathbf{y}$ and $\mathbf{u}$ because $\mathbf{x}$ may not be accessible. To obtain a solution for the desired output, recall the well-known Cramer's rule.

Cramer's Rule: A Tool to Obtain a Solution of the General Unified Model
To teach students studying an introductory course on linear systems theory how to obtain transfer functions, [9] showed that Cramer's rule gives rise to a general method with pedagogical appeal. However, it does not use the general unified model inspired in the input–output-state model, in which the solution is connected to the output of linear dynamic systems. In this section, how best to obtain the solution for the general unified model is shown using Cramer's rule. There are several proofs of Cramer's rule in the literature, such as [10] and in "Simple Proof of Cramer's Rule."

Note that, in (2), the mapping from input $\mathbf{u}$ to output $\mathbf{y}$ is made through the intermediate internal variables $\mathbf{x}$. A direct relationship between $\mathbf{u}$ and $\mathbf{y}$ can be obtained by eliminating $\mathbf{x}$, as follows: let y_i denote the ith output and u_j the jth input in (1). Note that the superposition principle is satisfied, that is, output y_i is the sum of the outputs for each input u_j. Then, if c_i denotes the ith row of C, b_j the jth column of B, and d_{ij} the ij element of D, then applying Cramer's rule to (2) shows that

$$y_i = \frac{|T_{i1}|}{|T|} u_1 + \cdots + \frac{|T_{ir}|}{|T|} u_r, \quad i = 1, \ldots, m, \tag{3}$$

where

$$T_{ij} := \begin{bmatrix} A & -b_j \\ -c_i & d_{ij} \end{bmatrix}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, r, \tag{4}$$
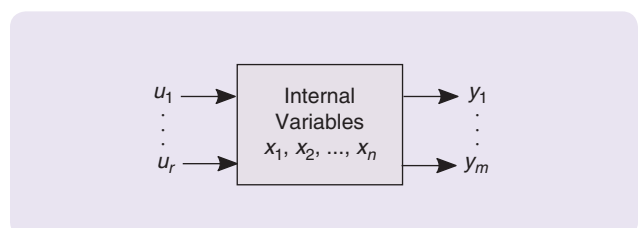

FIGURE 1 A state variable model relating input $\mathbf{u}$, state $\mathbf{x}$, and output $\mathbf{y}$ vectors.

Simple Proof of Cramer's Rule

Consider the linear system

$$Az = b, \qquad (S1)$$

where A is $n \times n$, z, and b are $n \times 1$. Let z_k denote the kth component of z and A_k denote the $n \times n$ matrix obtained by replacing the kth column of A by b. Let $|.|$ denote determinant.

CRAMER'S RULE
If $|A| \neq 0$, then

$$z_k = \frac{|A_k|}{|A|}. \qquad (S2)$$

PROOF
Let e_i denote the ith column of the identity matrix and

$$Z_k := [e_1 \, e_2 \cdots e_{k-1} \, z \, e_{k+1} \cdots e_n]. \qquad (S3)$$

It is easy to see that (S1) is equivalent to

$$AZ_k = A_k. \qquad (S4)$$

Taking the determinants of both sides of (S4) and the fact that $|Z_k| = z_k$,

$$|A| z_k = |A_k|, \qquad (S5)$$

or

$$z_k = \frac{|A_k|}{|A|}, \qquad (S6)$$

which is (S2).

and

$$T = \begin{bmatrix} A & 0 \\ -C & I \end{bmatrix}, \qquad (5)$$

which is a solution for a desired output of the general unified model. Thus far, only pure algebraic systems with no time derivatives were considered. Later in this article, dynamic systems are shown to be treated as algebraic systems using the Laplace transform (with zero initial conditions) and also recast as a unified model. Prior to this, the unified model and Cramer's rule were used in a purely algebraic example to obtain the desired outputs of a dc circuit.

Example 1: Output Voltages of dc Circuits in Terms of the Determinants

We illustrate the use of the general unified model to find the dc circuit outputs in Figure 2. The input $u = [V_1 \, V_2]'$ is given by voltage sources, the output $y = [v_2 \, v_4]'$ is given by the voltages across the resistors R_2 and R_4, the internal variables are the branch currents i_j, and the branch voltages v_j across the resistors R_j, $j = 1, \ldots, 10$ are represented as

$$\mathbf{x} = [i_1 \cdots i_{10} \quad v_1 \cdots v_{10}]^T.$$

Kirchhoff's point rule gives

$$i_1 - i_5 - i_8 = 0, \qquad (6)$$
$$i_5 - i_2 - i_6 = 0, \qquad (7)$$
$$i_3 + i_6 + i_9 - i_7 = 0, \qquad (8)$$
$$i_7 + i_{10} - i_4 = 0, \qquad (9)$$
$$i_8 - i_9 - i_{10} = 0, \qquad (10)$$

Kirchhoff's loop rule yields

$$V_1 - v_1 - v_5 - v_2 = 0, \qquad (11)$$
$$v_2 - v_6 + v_3 - V_2 = 0, \qquad (12)$$
$$V_2 - v_3 - v_7 - v_4 = 0, \qquad (13)$$
$$v_8 + v_9 - v_6 - v_5 = 0, \qquad (14)$$
$$v_9 - v_{10} + v_7 = 0, \qquad (15)$$

and Ohm's law gives the relations

$$v_j = i_j R_j, \quad j = 1, \ldots, 10. \qquad (16)$$

Without using Cramer's rule with the presented approach, it is necessary to solve for all of the circuit current variables to obtain the outputs of interest. However, as presented in this section, obtaining the desired outputs can be easier. Note that (6)–(15) are algebraic equations depending on the currents and voltage drops at the circuit nodes and loops, respectively, which contain no parameters. Only (16) contains the parameters R_j, $j = 1, \ldots, 10$. Rewriting (6)–(16) in the form of (2) gives

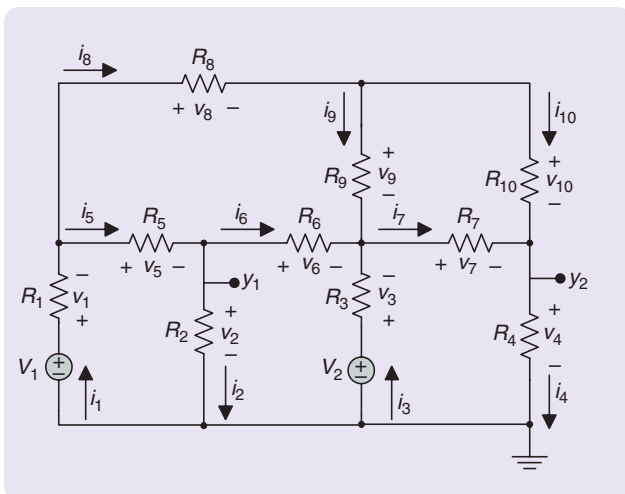$$A = \begin{bmatrix} A_{11} & A_{12} \\ -R & I \end{bmatrix}, \qquad (17)$$

where

$$A_{11} = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 \end{bmatrix},$$

$$R = \text{diag}[R_1, \dots, R_{10}], \quad I = I_{10},$$

with I_{10} being the identity matrix of order 10; $C := [c_{i,j}]$, $i = 1, 2$, $j = 1, \dots, 20$ with $c_{i,j} = 0$, except $c_{1,12} = 1$ and $c_{2,14} = 1$; $B := [b_{i,j}]$, $i = 1, \dots, 20$, $j = 1, 2$ with $b_{6,1} = 1$, $b_{7,2} = -1$, $b_{8,2} = 1$, $b_{14,2} = 1$; and $d_{ij} = 0$, $i = 1, 2$, $j = 1, 2$.

Using the general unified model (2),

$$\begin{bmatrix} A & 0 & 0 \\ -c_1 & 1 & 0 \\ -c_2 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -b_1 & -b_2 \\ d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \mathbf{u}.$$

From (5), $|T| = |A|$. Therefore, solving (18) for y_1 and y_2,

$$y_1 = \frac{\begin{vmatrix} A & -b_1 \\ -c_1 & d_{11} \end{vmatrix}}{|A|} u_1 + \frac{\begin{vmatrix} A & -b_2 \\ -c_1 & d_{12} \end{vmatrix}}{|A|} u_2, \tag{18}$$

$$y_2 = \frac{\begin{vmatrix} A & -b_1 \\ -c_2 & d_{21} \end{vmatrix}}{|A|} u_1 + \frac{\begin{vmatrix} A & -b_2 \\ -c_2 & d_{22} \end{vmatrix}}{|A|} u_2. \tag{19}$$

The next section addresses dynamic systems and how to represent them as an algebraic system using the general unified model.

DYNAMIC SYSTEMS MODEL
Consider the following state-space model

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t),$$

which obeys the superposition principle if and only if the initial condition $x(0) = 0$ [11], [12]. Applying the Laplace transform yields

$$A(s)x(s) = Bu(s),$$
$$y(s) = Cx(s) + Du(s), \tag{20}$$

with $A(s) := sI - A$. Defining $z = [x(s)^T \quad y(s)^T]^T$,

$$T(s)\mathbf{z} = \begin{bmatrix} -B \\ D \end{bmatrix} \mathbf{u}, \tag{21}$$

with

$$T(s) := \begin{bmatrix} A(s) & 0 \\ -C & I \end{bmatrix}, \tag{22}$$

which has exactly the same form of the general unified model in (2). Its solution for a prescribed output is obtained using Cramer's rule. To obtain the solution, let

$$T_{ij}(s) := \begin{bmatrix} A(s) & -b_j \\ -c_i & d_{ij} \end{bmatrix}, \quad i = 1, \dots, m, \quad j = 1, \dots, r, \tag{23}$$

with c_i, $i = 1, \dots, m$ being the ith row of C, b_j, $j = 1, \dots, r$ the jth column of B, and d_{ij} the ijth element of D. Solving for y_i gives

$$y_i = \frac{|T_{i1}(s)|}{|T(s)|} u_1 + \cdots + \frac{|T_{ir}(s)|}{|T(s)|} u_r, \tag{24}$$

where

$$T_{ij}(s) := \begin{bmatrix} A(s) & -b_j \\ -c_i & d_{ij} \end{bmatrix}, i = 1, \dots, m, j = 1, \dots, r \tag{25}$$

and $T(s)$ is defined in (22).

In dynamical systems, the transfer function plays an important role in both the analysis and design of controllers. An experienced reader may now consider using the state space to transfer function conversion, that is, finding the complete transfer function matrix using $C(sI - A)^{-1}B$. However, calculating the inverse matrix using the adjoint method requires the calculation of the determinant, the cofactor matrix, and at least three matrix multiplications. If not all outputs are required, then this step is not necessary to obtain the result. Instead of this laborious work, Cramer's rule gives an easier alternative to obtain the transfer function of interest by simply calculating three determinants. Next, we show how a transfer function of interest is found using the already-obtained solution for the general unified model in the state-space form.
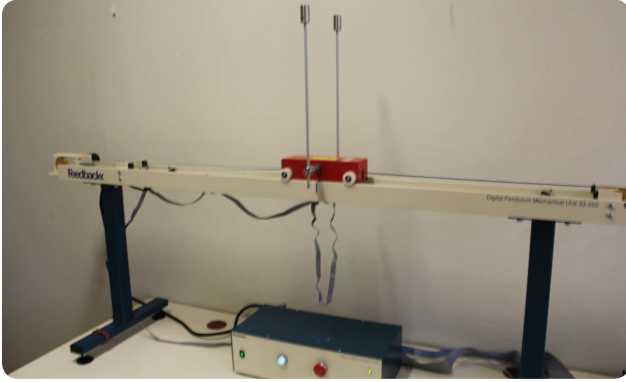
FIGURE 3 A feedback digital pendulum in the upright position.

Obtaining the Input–Output Transfer Functions Using the General Unified Model

The output $y(s)$ can be written in terms of the transfer matrix entries denoted $G_{ij}(s)$ as

$$y(s) = \begin{bmatrix} G_{11}(s) & \cdots & G_{1r}(s) \\ \vdots & \cdots & \vdots \\ G_{m1}(s) & \cdots & G_{mr}(s) \end{bmatrix} u(s), \tag{26}$$

with

$$G_{ij}(s) = \frac{|T_{ij}(s)|}{|T(s)|} = \frac{\begin{vmatrix} A(s) & -b_j \\ -c_i & d_{ij} \end{vmatrix}}{|A(s)|}, \tag{27}$$

for $i = 1, \ldots, m$, $j = 1, \ldots, r$. To illustrate the use of the framework to obtain some transfer functions of interest, consider the inverted pendulum example.

Example 2: Inverted Pendulum-Transfer Functions

Consider the inverted pendulum shown in Figure 3. The input is the force applied to the cart, and the outputs are the position of the cart and angular position of the pendulum. The dynamics of the pendulum on a cart are described with

$$N\ddot{x}_a + b\dot{x}_a + m\ell\ddot{\theta}\cos(\theta) - m\ell\dot{\theta}^2\sin(\theta) = k\dot{u}, \tag{28}$$

$$L\ddot{\theta} - mg\ell\sin(\theta) + m\ell\ddot{x}_a\cos(\theta) + d\dot{\theta} = 0, \tag{29}$$

where $N = m + M$ (with m the pole mass and M the cart mass), $L = I + m\ell^2$ (with I the moment of inertia of the arm), b and d the cart friction and pendulum-dumping coefficients, respectively, ℓ the arm length, g the gravity, and $F = k\dot{u}$ the force acting on the cart with gain k.

Integrating (28) and (29) and defining $x_1 = \theta$, $x_2 = x_a$, and $x_3 = \int\theta$, a state-space description of the pendulum and cart is obtained. Linearizing around $\theta = 0$, the state-space matrices are obtained as

$$A = \begin{bmatrix} \dfrac{Nd}{\ell^2 m^2 - LN} & -\dfrac{b\ell m}{\ell^2 m^2 - LN} & -\dfrac{Ng\ell m}{l^2 m^2 - LN} \\ -\dfrac{d\ell m}{\ell^2 m^2 - LN} & \dfrac{Lb}{\ell^2 m^2 - LN} & \dfrac{g\ell^2 m^2}{\ell^2 m^2 - LN} \\ 1 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} \dfrac{k\ell m}{\ell^2 m^2 - LN} \\ -\dfrac{Lk}{\ell^2 m^2 - LN} \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The outputs are

$$y_1 = \frac{|T_{11}(s)|}{|T(s)|} u_1, \quad y_2 = \frac{|T_{21}(s)|}{|T(s)|} u_1,$$

where

$$T(s) = \begin{vmatrix} s - \dfrac{Nd}{\ell^2 m^2 - LN} & \dfrac{b\ell m}{\ell^2 m^2 - LN} & \dfrac{Ng\ell m}{\ell^2 m^2 - LN} & 0 & 0 \\ \dfrac{d\ell m}{\ell^2 m^2 - LN} & s - \dfrac{Lb}{\ell^2 m^2 - LN} & -\dfrac{g\ell^2 m^2}{\ell^2 m^2 - LN} & 0 & 0 \\ -1 & 0 & s & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{vmatrix},$$

with

$T_{11}(s) =$

$$\begin{bmatrix} s - \dfrac{Nd}{\ell^2 m^2 - LN} & \dfrac{b\ell m}{\ell^2 m^2 - LN} & \dfrac{Ng\ell m}{\ell^2 m^2 - LN} & \dfrac{k\ell m}{\ell^2 m^2 - LN} \\ \dfrac{d\ell m}{\ell^2 m^2 - LN} & s - \dfrac{Lb}{\ell^2 m^2 - LN} & -\dfrac{g\ell^2 m^2}{\ell^2 m^2 - LN} & -\dfrac{Lk}{\ell^2 m^2 - LN} \\ -1 & 0 & s & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix},$$

$T_{21}(s) =$

$$\begin{bmatrix} s - \dfrac{Nd}{\ell^2 m^2 - LN} & \dfrac{b\ell m}{\ell^2 m^2 - LN} & \dfrac{Ng\ell m}{\ell^2 m^2 - LN} & \dfrac{k\ell m}{\ell^2 m^2 - LN} \\ \dfrac{d\ell m}{\ell^2 m^2 - LN} & s - \dfrac{Lb}{\ell^2 m^2 - LN} & -\dfrac{g\ell^2 m^2}{\ell^2 m^2 - LN} & -\dfrac{Lk}{\ell^2 m^2 - LN} \\ -1 & 0 & s & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

Thus, the transfer functions are obtained as

$$G_{11}(s) = \frac{(-k\ell m)s^2}{(\ell^2 m^2 - LN)s^3 + (-Lb - Nd)s^2 + (Ng\ell m - bd)s + bg\ell m},$$

$$G_{21}(s) = \frac{(Lk)s^2 + (dk)s - gk\ell m}{(\ell^2 m^2 - LN)s^3 + (-Lb - Nd)s^2 + (Ng\ell m - bd)s + bg\ell m},$$

with

$$G_{11}(s) = \frac{|T_{11}(s)|}{|T(s)|},$$

and

$$G_{21} = \frac{|T_{21}(s)|}{|T(s)|}.$$

The program code is given in "Matlab Code for Generating the Transfer Functions of the Pendulum in Example 2."

The block diagram of a complex dynamic system is represented as a general unified model (2) with the entries of the matrices being transfer functions. This case is directly illustrated in the following example.

Example 3: Block Diagram With Transfer Functions

Consider the block diagram of a two-input, two-output dynamic system, as shown in Figure 4. As is typical in this kind of problem, to obtain the transfer functions of interest, it is necessary to apply some block diagram algebraic simplification or replace several equations with consecutive substitutions, which is time-consuming if the block diagram is as large as the one shown in Figure 4. However, it is possible to obtain the transfer functions of interest easily using the presented approach.

The input and output of the block with transfer functions $G_k(s)$ are denoted by $w_k(s)$ and $z_k(s)$, respectively. The structural equations of the system are

$$u_1 + u_2 - z_4 - z_3 = w_1, \qquad (30)$$
$$z_1 = w_2 = w_6 = w_3, \qquad (31)$$
$$z_2 = w_5 = w_4. \qquad (32)$$

The outputs are

$$y_1 = z_5 + z_6, \qquad (33)$$
$$y_2 = z_2, \qquad (34)$$

and the individual input–output equations are

$$z_k(s) = G_k(s)\, w_k(s), \quad k = 1, \ldots, 6. \qquad (35)$$

Equations (30)–(35) are represented in matrix form as

$$A \begin{bmatrix} z \\ w \end{bmatrix} + Bu = 0, \quad C \begin{bmatrix} z \\ w \end{bmatrix} + Du = y,$$

with

$$A = \begin{bmatrix}
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
\hline
& & & & & & \multicolumn{6}{c}{G_1} \\
& & -I_6 & & & & & & & \ddots & & 0 \\
& & & & & & & & 0 & & & G_6
\end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 1 \\ [0_{11 \times 2}] \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$

Matlab Code for Generating the Transfer Functions of the Pendulum in Example 2

```
%% Clearing the variables
    close all; clear all; clc
%% Defining the pendulum parameters and variables
    syms N b m l k u g d L M I
    syms x1 x2 x3 x1prime x2prime x3prime
    syms s
%% Defining the system
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% x1 = teta
% x2 = xa
% x3 = integral theta
% thus
% x1prime = thetaprime
% x2prime = xaprime
% x3prime = teta = x1
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% diff equation system
    eq1 = −k*u + N*x2prime+b*x2+m*l*x1prime;
    eq2 = L*x1prime −m*g*l*x3+d*x1+m*l*x2prime;
% var. isol.
    x1prime = solve(eq1==0, x1prime);
    eq2 = eval(eq2);
    x2prime = solve(eq2==0, x2prime);
    x1prime = eval(x1prime);
    x3prime = x1;

    x1prime; x2prime; x3prime;

%% Writing in state-space form
    x = [x1 x2 x3]; % define the state vector
    [A, b] = equationsToMatrix([x1prime == 0, x2prime == 0,
        x3prime == 0], x);

    A = simplify(A, 'Steps', 350)
    [b, xxx] = equationsToMatrix(−b, u);
    B = simplify(b, 'Steps', 350)
    C = [1 0 0; 0 1 0]; % define Cmatrix

    Ts = [s*eye(3)−A zeros(3,2); −C eye(2)]
    T11 = [s*eye(3)−A −B; −C(1,:) 0]
    T21 = [s*eye(3)−A −B; −C(2,:) 0]

    G11 = collect(det(T11)/det(Ts), s)
    G21 = collect(det(T21)/det(Ts), s)
```

The transfer functions relating y_i to u_j can thus be obtained using Cramer's rule, as shown in (3). The program code is given in the "Matlab Code for Generating the Transfer Functions of the Block Diagram in Example 3."
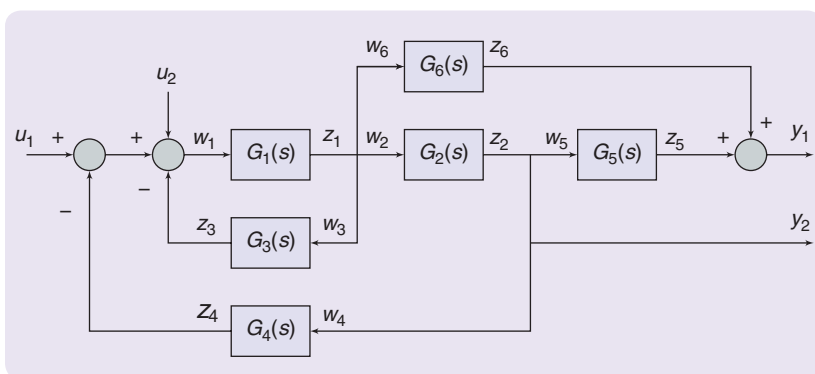
FIGURE 4 A block diagram of Example 3 showing the input–output relationship.

Matlab Code for Generating the Transfer Functions of the Block Diagram in Example 3

```
%% Clearing the variables
    close all; clear all;clc
%% Defining the system
    A11=[0 0 1 1 0 0
    1 0 0 0 0 0
    1 0 0 0 0 0
    1 0 0 0 0 0
    0 1 0 0 0 0
    0 1 0 0 0 0];
    A12=[1 0 0 0 0 0
    0 0 –1 0 0 0
    0 –1 0 0 0 0
    0 0 0 0 0 –1
    0 0 0 0 –1 0
    0 0 0 –1 0 0];
    A21=–eye(6);
    G=[G1 G2 G3 G4 G5 G6];
    A22 = diag(G);

    A=[A11 A12;A21 A22];
    B=[1 1;zeros(11,2)];
    D=[0 0;0 0];
    C=[0 0 0 0 1 1 0 0 0 0 0 0
    0 1 0 0 0 0 0 0 0 0 0 0];
    T=[A–B;0 eye(2)
    T=[A zeros(12,2);–C eye(2)];
    T11=[A –B(:,1);–C(1,:) D(1,1)];
    T12=[A –B(:,2);–C(1,:) D(1,2)];
    T21=[A –B(:,1);–C(2,:) D(2,1)];
    T22=[A –B(:,2);–C(2,:) D(2,2)];
    y1 = det(T11)*u1+det(T12)*u2
```

This section provided key details on the general unified model and its solution using Cramer's rule for both algebraic and dynamic systems. The next section presents the parameterized version of the general unified model. With the parameterized input-state-output model, the reader will become familiar with the concept of parameters in linear systems and some important results, such as the characterization of time-delay state-space systems into the form of the general unified model. This is also an important step in obtaining the Thévenin's equivalents explained later in this article.

PARAMETERIZED INPUT–OUTPUT MODELS

In many design problems, there is a set of parameters denoted by a vector $\mathbf{p}$, whose influence on the output is important. This can be incorporated into the general unified model as shown in (36). Consider the linear, parameterized, input–output model

$$A(\mathbf{p})\mathbf{x} = B(\mathbf{p})\mathbf{u},$$
$$\mathbf{y} = C(\mathbf{p})\mathbf{x} + D(\mathbf{p})\mathbf{u}, \tag{36}$$

with output vector $\mathbf{y}$, input vector $\mathbf{u}$, and parameter vector $\mathbf{p}$. Define

$$\mathbf{y} = [y_1 \cdots y_m]^T, \mathbf{u} = [u_1 \cdots u_r]^T, \mathbf{p} = [p_1 \cdots p_\ell]^T.$$

Using $\mathbf{z} := [\mathbf{x}^T \ \mathbf{y}^T]^T$, (36) can be written as

$$T(\mathbf{p}) \, \mathbf{z} = \begin{bmatrix} -B(\mathbf{p}) \\ D(\mathbf{p}) \end{bmatrix} \mathbf{u}, \tag{37}$$

with

$$T(\mathbf{p}) := \begin{bmatrix} A(\mathbf{p}) & 0 \\ -C(\mathbf{p}) & I \end{bmatrix}, \tag{38}$$

which is the general unified model in (2), except for the parameterized system matrices. Again, a solution for a desired output is obtained using Cramer's rule with important results. Let

$$T_{ij}(\mathbf{p}) := \begin{bmatrix} A(\mathbf{p}) & -b_j(\mathbf{p}) \\ -c_i(\mathbf{p}) & d_{ij}(\mathbf{p}) \end{bmatrix}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, r, \tag{39}$$

with $c_i(\mathbf{p}), i = 1, \ldots, m$ being the ith row of $C(\mathbf{p})$, $b_j(\mathbf{p}), j = 1, \ldots, r$ the jth column of B, $d_{ij}(\mathbf{p})$ the ijth element of D, and

$$\beta_{ij}(\mathbf{p}) := |T_{ij}(\mathbf{p})|, \tag{40}$$

$$\alpha(\mathbf{p}) := |T(\mathbf{p})| = |A(\mathbf{p})|. \tag{41}$$

For (36), the outputs can be determined in terms of inputs and parameters. This is established using the results given in [1] and [13], and omitting proofs.

Assumption 1

The parameter $\mathbf{p}$ appears affinely in $A(\mathbf{p}), B(\mathbf{p}), C(\mathbf{p})$, and $D(\mathbf{p})$ as

$$A(\mathbf{p}) = A_0 + p_1 A_1 + \cdots + p_\ell A_\ell,$$
$$C(\mathbf{p}) = C_0 + p_1 C_1 + \cdots + p_\ell C_\ell,$$
$$B(\mathbf{p}) = B_0 + p_1 B_1 + \cdots + p_\ell P_\ell,$$
$$D(\mathbf{p}) = D_0 + p_1 D_1 + \cdots + p_\ell D_\ell.$$

Assumption 2

$$|T(\mathbf{p})| \neq 0. \tag{42}$$

Theorem 1 [1], [13]

For (36), the output is

$$y_i = \sum_{j=1}^{r} \frac{\beta_{ij}(\mathbf{p})}{\alpha(\mathbf{p})} u_j, \quad i = 1, 2, \ldots, m, \tag{43}$$

with $\beta_{ij}(\mathbf{p})$ and $\alpha(\mathbf{p})$ defined in (40) and (41). In matrix terms,

$$\mathbf{y} = \frac{1}{\alpha(\mathbf{p})} \begin{bmatrix} \beta_{11}(\mathbf{p}) & \cdots & \beta_{1r}(\mathbf{p}) \\ \vdots & & \vdots \\ \beta_{m1}(\mathbf{p}) & \cdots & \beta_{mr}(\mathbf{p}) \end{bmatrix} \mathbf{u}. \tag{44}$$

To describe the solution $\mathbf{y}$ from (44), the form of the functions $\alpha(\mathbf{p})$ and $\beta_{ij}(\mathbf{p})$ are provided.

Lemma 1 [1], [14]

Let $A(\mathbf{p}) = A_0 + p_1 A_1 + \cdots + p_\ell A_\ell$ with rank $A_i = r_i, i = 1, 2, \ldots, \ell$. Then, $\alpha(\mathbf{p}) = |A(\mathbf{p})|$ is a multivariate polynomial in $\mathbf{p}$ of degree at most r_i in $p_i, i = 1, 2, \ldots, \ell$.

The multivariate polynomials in $\mathbf{p}$ can be written as

$$|A(\mathbf{p})| = \sum_{i_\ell=0}^{r_\ell} \cdots \sum_{i_1=0}^{r_1} \alpha_{i_1 \cdots i_\ell} p_1^{i_1} \ldots p_\ell^{i_\ell}, \tag{45}$$

with rank $(A_i) = r_i, i = 1, 2, \ldots, \ell$. In (45), the number of coefficients in $|A(\mathbf{p})|$ is $\mu := \prod_{i=1}^{\ell}(r_i + 1)$. Consider (39) written in polynomial form

$$T_{ij}(\mathbf{p}) = T_{ij0} + p_1 T_{ij1} + \cdots + p_\ell T_{ij\ell}.$$

Applying Lemma 1, we see that

$$|T_{ij}(\mathbf{p})| = \beta_{ij}(\mathbf{p})$$

is a multivariate polynomial in $\mathbf{p}$ of degree at most t_{ijk} in p_k, where

$$t_{ijk} = \text{rank}(T_{ijk}), \, i = 1, 2, \ldots, m, \, j = 1, 2, \ldots, r, \, k = 1, 2, \ldots, \ell,$$

and it can be described in a manner similar to that in (45). Note that the concepts of the rank and multivariate

polynomial may be new. To further explore these concepts, Example 4 shows how to use them.

Example 4: Determinant of Parameterized Matrices in Terms of a Multivariate Polynomial

Let

$$A(\mathbf{p}) = \begin{bmatrix} 1 & 2p_1 & 0 \\ p_1 & p_2 & p_1 \\ 3 & p_1 & 3p_2 \end{bmatrix}.$$

As the parameter $\mathbf{p}$ appears affinely in $A(\mathbf{p})$, then

$$A(\mathbf{p}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} p_1 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} p_2.$$

In this example, rank $(A_1) = 2$ and rank $(A_2) = 2$. Matrix $A(\mathbf{p})$ is said to be rank 2 with respect to p_1 and p_2, which yields $r_1 = 2$ and $r_2 = 2$. Thus,

$$|A(\mathbf{p})| = \sum_{i_2=0}^{2} \sum_{i_1=0}^{2} \alpha_{i_1 i_2} p_1^{i_1} p_2^{i_2}$$

is a polynomial of degree at most 2 in both p_1 and p_2. Calculating the determinant of $|A(\mathbf{p})|$ yields

$$|A(\mathbf{p})| = -6p_1^2 p_2 + 5p_1^2 + 3p_2^2.$$

Below is a class of time-delay systems that are shown to be a special case of the parameterized input–output model, where the parameters are the time delays of the system.

State-Space Models With Delays

First, consider a system with $\ell \in N$ time delays represented in state-space form

$$\dot{x}(t) = (A_0 x(t) + B_0 u(t)) + \left(\sum_{b=1}^{\ell} A_b x(t - L_b) + B_b u(t - L_b) \right), \tag{46}$$

$$y(t) = (C_0 x(t) + D_0 u(t)) + \left(\sum_{b=1}^{\ell} C_b x(t - L_b) + D_b u(t - L_b) \right). \tag{47}$$

Taking the Laplace transform of (46) and (47) with zero initial condition $T > 0$ and using $\mathcal{L}\{x(t - T)\} = e^{-sT} x(s)$,

$$sx(s) = \left(A_0 + \sum_{b=1}^{\ell} A_b e^{-sL_b} \right) x(s) + \left(B_0 + \sum_{b=1}^{\ell} B_b e^{-sL_b} \right) u(s), \tag{48}$$

$$y(s) = \left(C_0 + \sum_{b=1}^{\ell} C_b e^{-sL_b} \right) x(s) + \left(D_0 + \sum_{b=1}^{\ell} D_b e^{-sL_b} \right) u(s). \tag{49}$$

Define

$$\mathbf{p} = [p_1 \cdots p_\ell]^T = [e^{sL_1} \cdots e^{sL_\ell}]^T,$$

and

$$A(\mathbf{p}) = A_0 + p_1 A_1 + \cdots + p_\ell A_\ell,$$
$$B(\mathbf{p}) = B_0 + p_1 B_1 + \cdots + p_\ell P_\ell,$$
$$C(\mathbf{p}) = C_0 + p_1 C_1 + \cdots + p_\ell C_\ell,$$
$$D(\mathbf{p}) = D_0 + p_1 D_1 + \cdots + p_\ell D_\ell,$$

which has exactly the same representation as in Assumption 1. Finally, consider the parameterized models described previously for a state-space description,

$$A(s, \mathbf{p}) x(s) = B(\mathbf{p}) u(s),$$
$$y(s) = C(\mathbf{p}) x(s) + D(\mathbf{p}) u(s), \qquad (50)$$

with $A(s, \mathbf{p}) := sI - A(\mathbf{p})$. As previously mentioned, the general unified model is obtained by defining

$$T(s, \mathbf{p})\ \mathbf{z} = \begin{bmatrix} -B(\mathbf{p}) \\ D(\mathbf{p}) \end{bmatrix} \mathbf{u}, \qquad (51)$$

with

$$T(s, \mathbf{p}) := \begin{bmatrix} A(s, \mathbf{p}) & 0 \\ -C(\mathbf{p}) & I \end{bmatrix}.$$

To obtain the desired solution, use Cramer's rule with

$$T_{ij}(s) := \begin{bmatrix} A(s, \mathbf{p}) & -b_j(\mathbf{p}) \\ -c_i(\mathbf{p}) & d_{ij}(\mathbf{p}) \end{bmatrix}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, r,$$

where $c_i(\mathbf{p})$, $i = 1, \ldots, m$ is the ith row of $C(\mathbf{p})$, $b_j(\mathbf{p})$, $j = 1, \ldots, r$ is the jth column of $B(\mathbf{p})$, and $d_{ij}(\mathbf{p})$ is the ijth element of D. Define the polynomials

$$\beta_{i,j}(s, \mathbf{p}) := |T_{ij}(s, \mathbf{p})|, \qquad (52)$$

$$\alpha(s, \mathbf{p}) := |T(s, \mathbf{p})| = |A(s, \mathbf{p})|. \qquad (53)$$

The determinants in (52) and (53) can be written as polynomials in $\mathbf{p}$ as

$$\alpha(s, \mathbf{p}) = s^n + \alpha_1(\mathbf{p}) s^{n-1} + \cdots + \alpha_n(\mathbf{p}), \qquad (54)$$

$$\beta_{i,j}(s, \mathbf{p}) = s^n + \beta_{i,j,1}(\mathbf{p}) s^{n-1} + \cdots + \beta_{i,j,n}(\mathbf{p}), \qquad (55)$$

where the coefficients $\alpha_v(\mathbf{p})$ and $\beta_{i,j,v}(\mathbf{p})$, $v = 1, \ldots, n$ are multivariate polynomials in $\mathbf{p}$ with degree at most $r_k := \mathrm{rank}(A_k)$ and $t_{i,j,k} = \mathrm{rank}(T_{ijk})$ in p_k, respectively. As shown previously, Theorem 1 yields

$$y_i(s, \mathbf{p}) = \sum_{j=1}^{r} \frac{\beta_{ij}(s, \mathbf{p})}{\alpha(s, \mathbf{p})} u_j, \quad i = 1, 2, \ldots, m. \qquad (56)$$

The transfer functions can be obtained following the same procedure as in (24), but considering the parameterized state-space form. To conclude this section, two examples illustrate the parameterized form of the general unified model.

Example 5: Parameterized Output

Consider the gyrator circuit shown in Figure 5 from [15]. Using the gyrator resistance, denoted as R_4, regulate the voltage across the resistor R_1, denoted as y. Defining the internal variable vector as $\mathbf{x} = [I_1\ I_2\ I_3]^T$ and using the Kirchhoff's current and voltage laws

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ R_1 & R_2 & -R_4 & 0 \\ K & -R_4 & R_3 & 0 \\ R_1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \end{bmatrix} u_1 + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_2 = 0.$$

In the general unified model, for output y and inputs u_1 and u_2,

$$y = \frac{|T_1(\mathbf{p})|}{|A(\mathbf{p})|} u_1 + \frac{|T_2(\mathbf{p})|}{|A(\mathbf{p})|} u_2,$$

where

$$A = \begin{bmatrix} 1 & -1 & 0 \\ R_1 & R_2 & -R_4 \\ K & -R_4 & R_3 \end{bmatrix}, \quad T_1 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ R_1 & R_2 & -R_4 & 1 \\ K & -R_4 & R_3 & 0 \\ R_1 & 0 & 0 & 1 \end{bmatrix},$$

$$T_2 = \begin{bmatrix} 1 & -1 & 0 & 1 \\ R_1 & R_2 & -R_4 & 0 \\ K & -R_4 & R_3 & 0 \\ R_1 & 0 & 0 & 0 \end{bmatrix}.$$

Choosing the design parameter as $p = R_4$, the matrices appearing in $A(\mathbf{p})$ and $T_1(\mathbf{p})$ and $T_2(\mathbf{p})$ are rank 2 matrices in R_4. Thus,

$$y = \frac{\beta_{01} + \beta_{11} p + \beta_{21} p^2}{1 + \alpha_1 p + \alpha_2 p^2} u_1 + \frac{\beta_{02} + \beta_{12} p + \beta_{22} p^2}{1 + \alpha_1 p + \alpha_2 p^2} u_2.$$

This example illustrated how to analyze the output ranges of a variable of interest in terms of a parameter of the system using the unified model in its parameterized version. The second example illustrates how to obtain the parameterized output of a time-delay system.
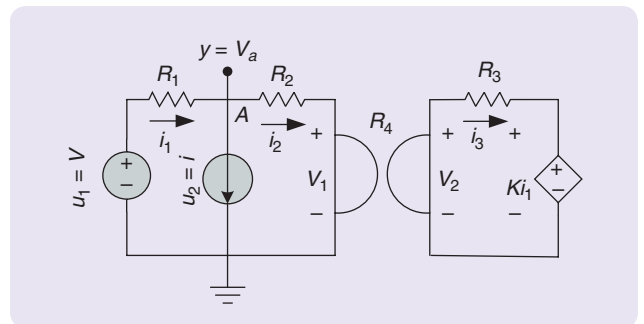

FIGURE 5 V and i are an ideal voltage and current source, respectively; R_1, R_2, and R_3 are linear resistors; and R_4 is a gyrator resistance. The gyrator is a linear two-port device, where the instantaneous currents and voltages are related by $V_2 = R_4 i_2$ and $V_1 = -R_4 i_3$, and Ki_1 describes a dependent voltage amplifier in which K is the gain.

Example 6: Output of Systems With Delays in the State Variables and Output

Consider the space-state system

$$\dot{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(t-L_1) + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(t-L_2) + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u(t),$$

$$y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(t-L_1).$$

Define the parameter vector $\mathbf{p}$ as the Laplace transform of the delays in the state. Let $p_1 = e^{sL_1}$ and $p_2 = e^{sL_2}$, with time delays L_1 and L_2. The parameter $\mathbf{p}$ depends on s. However, the state-space framework can still be used. Thus, the following representation is obtained:

$$A(\mathbf{p}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + p_1 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + p_2 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$C(\mathbf{p}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + p_1 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + p_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$B(\mathbf{p}) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + p_1 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + p_2 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$D(\mathbf{p}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + p_1 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + p_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

In this case, $r_1 = 2, r_2 = 1, t_{1,1,1} = 2, t_{1,1,2} = 1, t_{2,1,1} = 3$, and $t_{2,1,2} = 1$, which yields

$$y(s, \mathbf{p}) =$$
$$\begin{bmatrix} \dfrac{-s^2 + (-p_1^2 - p_2 + p_1 + 2)s + p_1^2 - p_1^2 p_2 + p_1 p_2 + p_2 - p_1 - 1}{s^3 + (-p_1 - 3)s^2 + (-p_1^2 + 2p_1 + 3)s + p_1^2 - p_1 - 1} \\ \dfrac{-p_1}{s-1} \end{bmatrix}$$
$$\times u(s).$$

Substituting for p_1 and p_2 provides the equation shown in the box at the bottom of the page.

The solution using Cramer's rule shows how the different delays appear in the output $y(s)$. The program code is given in "Matlab Code for Generating the Output of the System With Delay in Example 6."

The next section describes how to obtain the parameterized solution of the general unified model using measurements. A generalization of Thévenin's theorem developed

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Matlab Code for Generating the Output of the System With Delay in Example 6

```
%% Clearing the variables
    close all; clear all; clc
    %% Defining the matrices and the parameters
    syms s L1 L2

    p1 = exp(s*L1);
    p2 = exp(s*L2);

    A0 = [1 0 0; 0 1 0; 0 0 1];
    A1 = [0 1 0; 1 1 0; 0 0 0];
    A2 = [0 0 1; 0 0 0; 0 0 0];
    B = [1;0;1];
    C0 = [1 0 0; 0 0 0];
    C1 = [0 1 0; 0 0 1];
    n = length(A0);

    Ap = A0+A1*p1+A2*p2;
    Cp = C0+C1*p1;
    r1 = rank(A1);
    r2 = rank(A2);

    %% Calculate transfer functions
    Asp = s*eye(n)–Ap;

    Ts = Asp;
    T11 = [Asp –B; –Cp(1,:) 0];
    T21 = [Asp –B; –Cp(2,:) 0];

    G11 = collect(det(T11)/det(Ts),s);
    G21 = collect(det(T21)/det(Ts),s);

    Gs = [G11;G21]
```

in [1] is also presented to illustrate use of the results with practical applications.

SYSTEM PARAMETER ESTIMATION AND THÉVENIN'S EQUIVALENT

As shown previously, a complex linear algebraic system described in the state-space framework can have a simple input–output representation. Theorem 1's solution suggests that knowledge of the functions $\alpha(\mathbf{p})$ and $\beta_{ij}(\mathbf{p})$ are sufficient to determine the behavior of the outputs y_i as a function of $\mathbf{p}$ and $\mathbf{u}$ [1], [16]. The knowledge of $\alpha(\mathbf{p})$ and $\beta_{ij}(\mathbf{p})$ reduces to the knowledge of the coefficients of these polynomial functions. In

$$y(s) = \begin{bmatrix} \dfrac{-s^2 + (-e^{2sL_1} - e^{sL_2} + e^{sL_1} + 2)s + e^{2sL_1} - e^{s(2L_1 + L_2)} + e^{s(L_1 + L_2)} + e^{sL_2} - e^{sL_1} - 1}{s^3 + (-e^{sL_1} - 3)s^2 + (-e^{2sL_1} + 2e^{sL_1} + 3)s + e^{2sL_1} - e^{sL_1} - 1} \\ \dfrac{-e^{L_1 s}}{s-1} \end{bmatrix} u(s).$$

an unknown system (black box, for instance) these coefficients are unknown a priori. However, if tests on the system can be performed by setting the design parameter $\mathbf{p}$ and input $\mathbf{u}$ to various values and measuring the corresponding output y_i, then the polynomial function coefficients can be determined.

Theorem 2: A Measurement Theorem [1]

Let y be defined as in (44). The function y can be determined from μ measurements and solution of a system of μ linear equations in the unknown coefficient vectors $\alpha(\mathbf{p})$ and $\beta_{ij}(\mathbf{p})$, called the *measurement equations*.

Theorem 2 is a generalization of Thévenin's theorem of circuit theory, and the measurement result is illustrated in the special case of a single-input, single-output y with $\mathbf{p} = [p_1\ p_2]'$, $r_1 = 1$, $r_2 = 2$, $t_{111} = 1$, $t_{112} = 2$. Here,

$$y = \frac{\beta(\mathbf{p})}{\alpha(\mathbf{p})} u, \tag{57}$$

with

$$\beta(\mathbf{p}) = \beta_{00} + \beta_{10} p_1 + \beta_{01} p_2 + \beta_{11} p_1 p_2 + \beta_{12} p_1 p_2^2, \tag{58}$$

$$\alpha(\mathbf{p}) = \alpha_{00} + \alpha_{10} p_1 + \alpha_{01} p_2 + \alpha_{11} p_1 p_2 + \alpha_{12} p_1 p_2^2. \tag{59}$$

Set $\alpha_{12} = 1$ and conduct the following experiments: Measure y for nine different sets of values (p_1, p_2). Let $y(k)$, $k = 1, \ldots, 9$ denote the nine different measurement values and $\mathbf{p}(k) = [p_1(k)\ p_2(k)]$ the nine sets of parameters. The nine coefficients of $\alpha(\mathbf{p})$ and $\beta(\mathbf{p})$ in (58) and (59) can then be determined from the measurement equation $M\mathbf{f} = \mathbf{v}$ with

$$\mathbf{f} = \begin{bmatrix} \alpha_{00} \\ \alpha_{10} \\ \alpha_{01} \\ \alpha_{11} \\ \beta_{00} \\ \beta_{10} \\ \beta_{01} \\ \beta_{11} \\ \beta_{12} \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -p_1(1) p_2^2(1) y(1) \\ -p_1(2) p_2^2(2) y(2) \\ -p_1(3) p_2^2(3) y(3) \\ -p_1(4) p_2^2(4) y(4) \\ -p_1(5) p_2^2(5) y(5) \\ -p_1(6) p_2^2(6) y(6) \\ -p_1(7) p_2^2(7) y(7) \\ -p_1(8) p_2^2(8) y(8) \\ -p_1(9) p_2^2(9) y(9) \end{bmatrix} \text{ and } M = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{19} \\ m_{21} & m_{22} & \cdots & m_{29} \\ \vdots & \vdots & \vdots & \vdots \\ m_{91} & \cdots & \cdots & m_{99} \end{bmatrix},$$

where

$m_{k1} = y(k)$, $m_{k2} = p_1(k) y(k)$, $m_{k3} = p_2(k) y(k)$,
$m_{k4} = p_1(k) p_2(k) y(k)$, $m_{k5} = -u$,
$m_{k6} = -p_1(k) u$, $m_{k7} = -p_2(k) u$,
$m_{k8} = -p_1(k) p_2(k) u$, $m_{k9} = -p_1(k) p_2^2(k) u$, for $k = \{1, 2, \ldots, 9\}$.

Consider a single-input, single-output algebraic system with only one parameter denoted p. The input–output relationship can be written as

$$y = \frac{\beta_0 + \beta_1 p}{\alpha + p} u. \tag{60}$$

Equation (60) holds when the coefficient matrix A_1 of the parameter p as described in Assumption 1 has unity rank.

Equation (60) shows that, if the constants $(\beta_0, \beta_1, \alpha)$ are known, then the input–output map is determined.

Thévenin's Equivalent: Special Case of Circuits

In circuit theory, it is possible to obtain an equivalent circuit representation composed by a voltage source in series with an impedance using the Thévenin's equivalent. The Thévenin's equivalent is studied in several circuit theory textbooks such as in [16] and [17], Thévenin's theorem is extended to electromechanical systems. Consider an unknown dc circuit, as shown in Figure 6. The Thévenin's equivalent of circuit theory is summarized in the following theorem.

Theorem 3: Thévenin's Theorem

$$V_{\text{th}} = V_{\text{oc}}, \tag{61}$$

$$R_{\text{th}} = \frac{V_{\text{oc}}}{I_{\text{sc}}}, \tag{62}$$

where I_{sc} is the short circuit current and V_{oc} is the open circuit voltage [18], [19].

From Theorem 3, it is necessary to open- and short-circuit the terminals of the circuit of interest. However, many systems do not support short circuit conditions. Even in cases where this is possible, opening or short-circuiting the terminals of a circuit may not be a trivial or less economically feasible operation.

The single input–output description (60) represents a special case of Thévenin's theorem of circuit theory to arbitrary linear systems. To see this, consider again Example 5. To obtain the Thévenin's equivalent, the inputs u_1 and u_2 are set as constants. Take the Thévenin's equivalent at the terminal defined by R and I in Figure 6 as the Thévenin's equivalent at the terminal defined by R_1 and $y = i_1$ in Figure 5. In this case, the circuit in Figure 5 has a single parameter $\mathbf{p} = p = R_1$. Then

$$y = \frac{|T_1(p)| u_1 + |T_2(p)| u_2}{|A(p)|}, \tag{63}$$

where

$$A = \begin{bmatrix} 1 & -1 & 0 \\ p & R_2 & -R_4 \\ K & -R_4 & R_3 \end{bmatrix}, T_1 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ p & R_2 & -R_4 & 1 \\ K & -R_4 & R_3 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

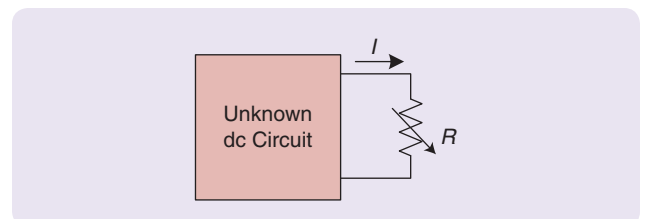$$T_2 = \begin{bmatrix} 1 & -1 & 0 & 1 \\ p & R_2 & -R_4 & 0 \\ K & -R_4 & R_3 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

FIGURE 6 Thévenin's equivalent for a dc circuit, illustrating its terminal connected to a load R.

As $|T_1(p)|$ and $|T_2(p)|$ are constants and the rank of matrix A with respect to parameter p is unity,

$$i_1 = \frac{\tilde{\beta}_0}{\tilde{\alpha} + R_1}. \tag{64}$$

The short circuit current I_{sc} is

$$I_{sc} = \frac{\tilde{\beta}_0}{\tilde{\alpha}}, \tag{65}$$

which is obtained by setting $R_1 = 0$. Similarly, the open circuit voltage denoted as V_{oc} is given by multiplying (64) by R_1 and taking the limit of $R_1 \to \infty$, which yields

$$V_{oc} = \tilde{\beta}_0. \tag{66}$$

The Thévenin's equivalent is thus calculated as

$$R_{th} = \frac{V_{oc}}{I_{sc}} = \tilde{\alpha}, \tag{67}$$

$$V_{th} = \tilde{\beta}_0, \tag{68}$$

where $\tilde{\alpha}, \tilde{\beta}_0$ are obtained by solving

$$\begin{bmatrix} i_1(1) & -1 \\ i_1(2) & -1 \end{bmatrix} \begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta}_0 \end{bmatrix} = \begin{bmatrix} -R_1(1)\,i_1(1) \\ -R_1(2)\,i_1(2) \end{bmatrix}.$$

Note that calculating $\tilde{\alpha}, \tilde{\beta}_0$ does not require short circuit or open circuit experiments, only two arbitrary settings with corresponding measurements. Analytical models may be developed for mechanical, electrical, fluid, and thermal systems in a rather analogous manner. In view of this analogy, the Thévenin's equivalent can be generalized, and the approach can also be applied to other dynamic systems.

In the following example, using the electrical-hydraulic analogy, consider the analysis of a hydraulic sprayer used in agriculture. The fluid pressure is ΔP, the flow in a sprayer boom is Q, and the nozzle of interest and fluidic resistance is R, which are analogous to voltage, current, and electrical resistance, respectively [4].

Example 7: Electrical-Hydraulic Analogy and the Thévenin's Equivalent

Consider a sprayer system illustrated in Figure 7, as in [4]. Let $V = \sqrt{\Delta P}$, and $I = Q$. The Thévenin's equivalent can be used to estimate the fluidic resistance of the boom.
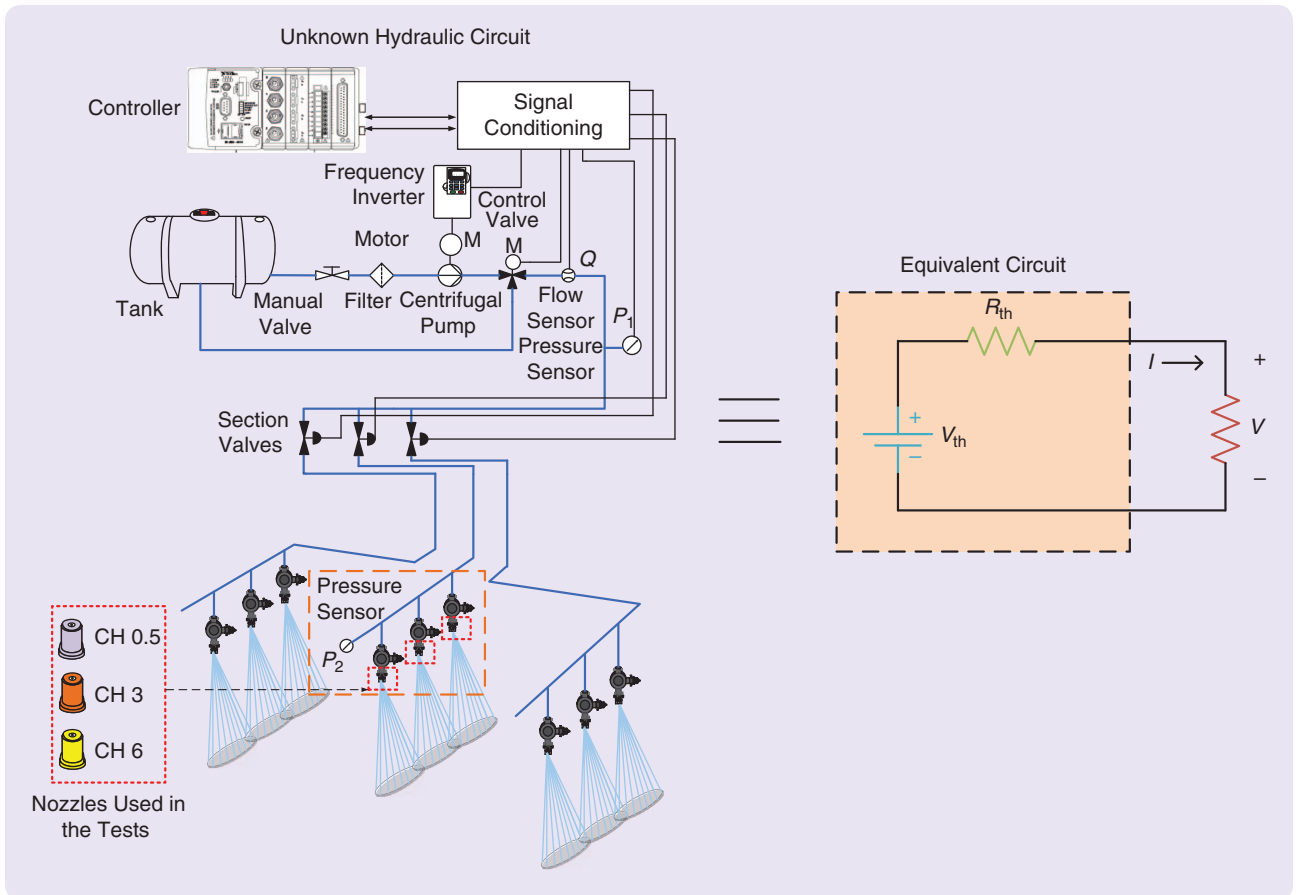

FIGURE 7 The sprayer boom used in the agricultural industry can be considered as an unknown hydraulic circuit. The analogy between electrical and hydraulic circuits can be used to derive the pointwise hydraulic Thévenin's equivalents, thus obtaining the fluidic resistance equivalent, which is important because it directly affects the quality of, for example, pesticides application.

Using the Thévenin's equivalent method presented, it is no longer necessary for the system to operate in two extreme situations, such as hydraulic short circuit and open circuit. Using two different safe operating points, it is possible to obtain the Thévenin's equivalent. Thus, let measurements of flow in the boom nozzle of interest be $Q(1)$ and $Q(2)$ and the measurements of pressure in the corresponding boom be $\Delta P(1)$ and $\Delta P(1)$. Thus,

$$
\begin{bmatrix} Q(1) & -1 \\ Q(2) & -1 \end{bmatrix} \begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta}_0 \end{bmatrix} = \begin{bmatrix} -\sqrt{\Delta P}(1) \\ -\sqrt{\Delta P}(2) \end{bmatrix}, \tag{69}
$$

where $\tilde{\alpha}$, e, and $\tilde{\beta}_0$ are coefficients to be determined. Therefore, the flow in the boom can be written in terms of the coefficients of (69) as

$$
Q = -\frac{\sqrt{\Delta P}}{\tilde{\alpha}} + \frac{\tilde{\beta}_0}{\tilde{\alpha}}, \tag{70}
$$

which yields

$$
R_{th} = \tilde{\alpha},
$$
$$
\Delta P_{th} = \tilde{\beta}_0,
$$

where R_{th} and ΔP_{th} are the internal loss and pressure equivalents, respectively. To extract the Thévenin's equivalent, only two different nozzles were required. To validate the obtained Thévenin's equivalent, a third nozzle with an intermediate fluidic resistance value between these two nozzles used for extraction of the equivalent was employed. The resulting data are shown in Table 1. Using (69),

$$
\Delta P_{th} = 1.85 \, [\text{bar}],
$$
$$
R_{th} = 0.02 \, [\text{bar min } L^{-1}].
$$

This equivalent estimated the flow of arbitrary pressure values, as shown in Figure 8. The error of the estimated flow using the Thévenin's equivalents was approximately 1.8%.

Example 8 explores the popular dc-dc boost converter used to step-up the voltage of a photovoltaic module to provide the voltage required by the dc-ac converter stage to connect to the ac distribution power grid [20]. This application is represented in Figure 9.

Example 8: Parameterized Functions for a dc-dc Boost Converter Based on Measurements

Consider the dc-dc boost converter circuit without loss in Figure 10. The goal is to analyze the circuit characteristics such as the boost converter gain and the input current (denoted I_i) to help select the converter parameters. The gain is related to the duty cycle, denoted D, of the pulse train, which is called *pulsewidth modulation*. The input current (denoted I_i) is related to D, and the load is denoted R.

For the purpose of comparison, we first obtain the converter circuit equations using the averaging method, a very common method used to analyze power electronics circuits. Let $x = [I_i \ V_o]^T$, $u = V_i$, and $y = V_o$ be the state, input, and output, respectively. Following [21], the circuit from Figure 10 can be modeled as a switched state-space system in which the matrices are defined as

TABLE 1 Data obtained for different full cone nozzles.

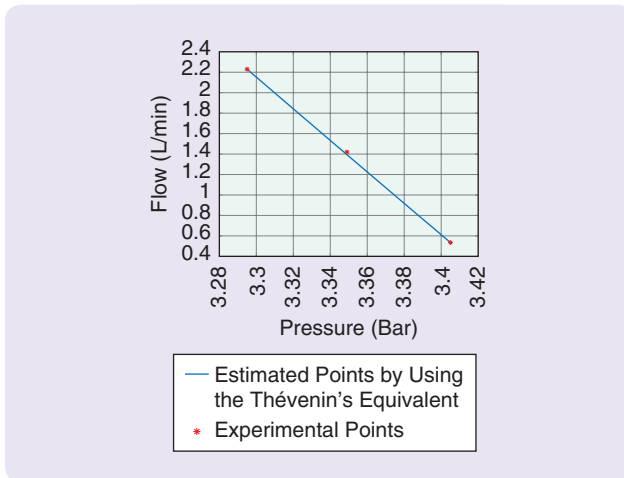| Nozzles	Pressure (bar)	Flow (L/min)
CH05	3.405	0.535
CH3	3.349	1.420
CH6	3.295	2.230

FIGURE 8 Validating the Thévenin's equivalent for full cone nozzles connected at the sprayer boom. The straight line is the estimate of the flow obtained via (70), and the dots are the experimental points. The pump pressure was set to 3.5 bar. It was necessary to ensure that changing the fluidic resistance (a significant variation of pressure at the boom of interest) was verified. A pressure variation of approximately 0.1 bar in relation to the pump pressure was considered significant due to the inherent noise of the spray sensors.
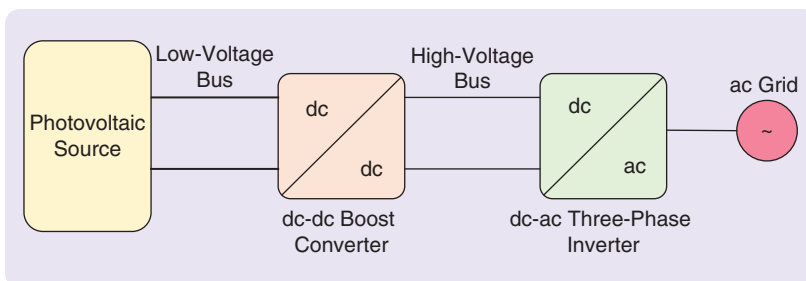
FIGURE 9 The typical application of a boost converter illustrating the connection of the dc and ac devices and the ac distribution power grid.

$$A_1 = \begin{bmatrix} 0 & 0 \\ 0 & -\dfrac{1}{RC} \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & -\dfrac{1}{L} \\ \dfrac{1}{C} & -\dfrac{1}{RC} \end{bmatrix}, \quad B_1 = B_2 = \begin{bmatrix} \dfrac{1}{L} \\ 0 \end{bmatrix},$$

$$C_1 = C_2 = [0 \ 1], \quad D_1 = D_2 = 0.$$

Using the averaging method as in [22],

$$\dot{x} = [A_1 d + A_2(1-d)]x + [B_1 d + B_2(1-d)]u,$$
$$y = [C_1 d + C_2(1-d)]x + [D_1 d + D_2(1-d)]u.$$

Applying small-signal perturbation denoted as $\tilde{x}$, $\tilde{d}$, and $\tilde{u}$,

$$x = X + \tilde{x},$$
$$d = D + \tilde{d},$$
$$u = U + \tilde{u},$$

where the first term describes the mean value and the second term is the perturbation of the signal. Thus,

$$\dot{\tilde{x}} = AX + BU + A\tilde{x} + F\tilde{d} + B\tilde{u}, \tag{71}$$
$$y = \bar{C}(X + \tilde{x}), \tag{72}$$

in which A, B, $\bar{C}$, and F are

$$A = A_1 D + A_2(1-D),$$
$$B = B_1 D + B_2(1-D),$$
$$\bar{C} = C_1 D + C_2(1-D),$$
$$F = (A_1 - A_2)X + (B_1 - B_2)U.$$

From (71) and (72), the steady-state condition is

$$AX + BU = 0,$$
$$y = \bar{C}X.$$

Finally, as in the boost circuit case, $u := V_i$ and $y := V_o$. Using Cramer's rule, the output is

$$V_o = \frac{|T_1|}{|A|} V_i,$$

and the voltage gain is

$$\frac{V_o}{V_i} = \frac{|T_1|}{|A|} = \frac{1}{1-D},$$

with

$$A = \begin{bmatrix} 0 & 0 \\ 0 & -\dfrac{1}{RC} \end{bmatrix} D + \begin{bmatrix} 0 & -\dfrac{1}{L} \\ \dfrac{1}{C} & -\dfrac{1}{RC} \end{bmatrix}(1-D) = \begin{bmatrix} 0 & \dfrac{D-1}{L} \\ -\dfrac{D-1}{C} & \dfrac{D-1}{RC} - \dfrac{D}{RC} \end{bmatrix},$$

and

$$T_1 = \begin{bmatrix} 0 & \dfrac{D-1}{L} & -\dfrac{1}{L} \\ -\dfrac{D-1}{C} & \dfrac{D-1}{RC} - \dfrac{D}{RC} & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$
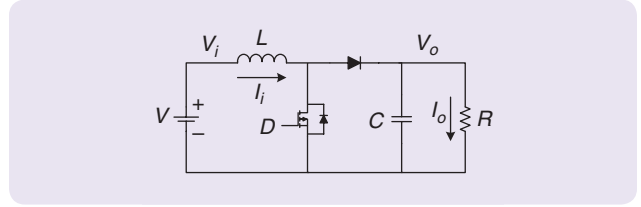

FIGURE 10 The model of an ideal boost dc-dc converter circuit showing its components. A boost dc-dc converter is a circuit that can deliver to the load a dc voltage equal to or greater than the input voltage source.

Matlab Code to Obtain the Static Voltage Gain for the dc-dc Converter in Example 8

```
close all, clear all, clc
    %% define the variables
    syms R C L D
    %% define the matrices
    A1 = [0 0; 0 –1/(R*C)];
    A2 = [0 –1/L; 1/C –1/(R*C)];
    B1 = [1/L;0];
    B2 = B1;

    %% calculate the gain
    A = A1*D + A2*(1–D);
    B = B1*D + B2*(1–D);

    T1 = [A –B; [0 –1] 0];
    Gain = simplifyFraction(det(T1)/det(A));
```

The program code is shown in "Matlab Code to Obtain the Static Voltage Gain for the dc-dc Converter in Example 8."

Without using the boost circuit model, the gain V_o/V_i is analyzed in terms of the duty cycle defined as a parameter $p = D$ from measurements. The problem is thus reduced to finding the coefficients of the function

$$V_o(D) = \frac{\beta_0 + \beta_1 D}{\alpha_0 + D} V_i.$$

The coefficients α_0, α_1, and β_0 for V_i are constant and can be found using three different measurements of the parameter $p_1 = D$, denoted $D(1)$, $D(2)$, and $D(3)$. The corresponding outputs V_o are denoted $V_o(1)$, $V_o(2)$, and $V_o(3)$. Solving the following linear system of equations,

$$\begin{bmatrix} V_i & D(1)V_i & -V_o(1) \\ V_i & D(2)V_i & -V_o(2) \\ V_i & D(3)V_i & -V_o(3) \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \alpha_0 \end{bmatrix} = \begin{bmatrix} D(1)V_o(1) \\ D(2)V_o(2) \\ D(3)V_o(2) \end{bmatrix}.$$

Suppose the duty cycle range of interest is [0.5 0.7] for $V_i = 10$ V, $R = 100 \ \Omega$, $L = 1$ mH, and $C = 1000 \ \mu$F. Using the PSIM software, the determined values are shown in Table 2.

Solving the algebraic system in β_0, β_1, and α_0, it is shown that $\beta_0 = -1$, $\beta_1 = 0$, and $\alpha_0 = -1$, which yields

$$V_o(D) = \frac{-10}{-1 + D},$$

and the boost converter gain

$$\frac{V_o(D)}{V_i} = \frac{1}{1 - D},$$

which is identical to the theoretical gain that was previously analyzed. For the Thévenin's equivalent, the boost converter without loss reduces to an ideal voltage source,

TABLE 2 PSIM simulation data based on varying one parameter.

D	V_o (V)
0.500	20
0.600	25
0.700	33.3

TABLE 3 PSIM simulation data based on varying two parameters.

D	$R(\Omega)$	I_i (A)
0.500	100	0.408
0.500	150	0.275
0.550	130	0.387
0.600	125	0.515
0.650	120	0.704
0.700	100	1.14
0.700	150	0.771

as the output voltage of the dc-dc boost converter does not depend on the current at the terminal.

Finally, the output I_i is analyzed as it relates to the parameters $p_1 = D$ and $p_2 = R$. For this case, we can write

$$I_i(D, R) = \frac{\beta_0 + \beta_1 D + \beta_2 R + \beta_3 DR}{\alpha_0 + \alpha_1 D + \alpha_2 R + DR}.$$

Using seven different sets of values of D, R, and corresponding measurements of I_i, the constants β_0, β_1, β_2, β_3, α_0, α_1, and α_2 are determined by solving the linear system of equations

$$\begin{bmatrix} 1 & D(1) & R(1) & D(1)R(1) & -I_i(1) & -I_i(1)D(1) & -I_i(1)R(1) \\ 1 & D(2) & R(2) & D(2)R(2) & -I_i(2) & -I_i(2)D(2) & -I_i(2)R(2) \\ 1 & D(3) & R(3) & D(3)R(3) & -I_i(3) & -I_i(3)D(3) & -I_i(3)R(3) \\ 1 & D(4) & R(4) & D(4)R(4) & -I_i(4) & -I_i(4)D(4) & -I_i(4)R(4) \\ 1 & D(5) & R(5) & D(5)R(5) & -I_i(5) & -I_i(5)D(5) & -I_i(5)R(5) \\ 1 & D(6) & R(6) & D(6)R(6) & -I_i(6) & -I_i(6)D(6) & -I_i(6)R(6) \\ 1 & D(7) & R(7) & D(7)R(7) & -I_i(7) & -I_i(7)D(7) & -I_i(7)R(7) \end{bmatrix}$$
$$\times \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} D(1)I_i(1)R(1) \\ D(2)I_i(2)R(2) \\ D(3)I_i(3)R(3) \\ D(4)I_i(4)R(4) \\ D(5)I_i(5)R(5) \\ D(6)I_i(6)R(6) \\ D(7)I_i(7)R(7) \end{bmatrix}.$$

Suppose the duty cycle of interest lies in the interval [0.5 0.7], and the load range is [100 Ω 150 Ω] for $V_i = 10$ V, $L = 1$ mH, and $C = 1000$ μF. Using PSIM software, the values shown in Table 3 were determined.

Solving the linear equations, $\beta_0 = 14.905$, $\beta_1 = -32.868$, $\beta_2 = -0.117$, $\beta_3 = -0.148$, $\alpha_0 = 50.104$, $\alpha_1 = -69.554$, and $\alpha_2 = -0.797$. Using $I_i(D, R)$, it is possible to draw a surface relating D and R with $I_i(D, R)$, as shown in Figure 11.
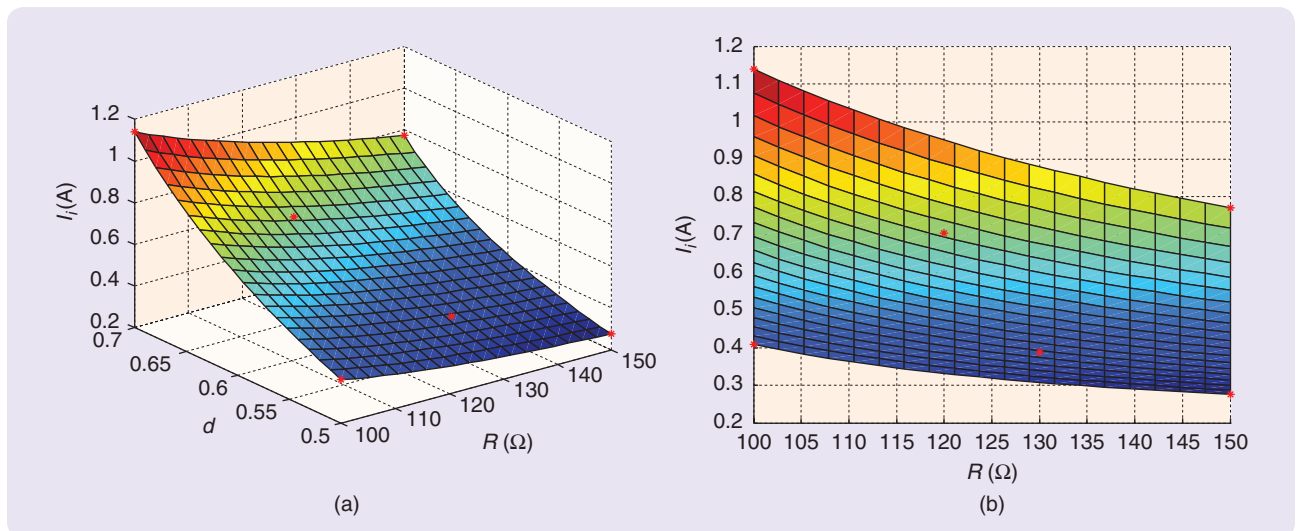

FIGURE 11 Relating the duty cycle parameter D and load R of the circuit in Figure 10 with current I_i. The red-marked points are the measurements used. Note that extremal values of current I_i occur at the vertices.

> **Real data are noisy, with accuracy dependent on the tolerance of components and the measurement device used.**

From this surface, the current limits corresponding to each load in the range considered for the same duty cycle are known. This can be used to support the selection of the boost components. For any fixed load R belonging to the interval $[100, 150]$ Ω, it is possible to know the dc current value at the inductor for any duty cycle in the considered range, which is essential information for selection [22].

Remark 1
Real data are noisy, with accuracy dependent on the tolerance of components and the measurement device used. In [8], the analysis of the ill-posed problem arising in the measurement-based approach, with real experiments, was presented. The ill-posed problem should be handled using regularization tools [23].

CONCLUSION
This article presented the general unified model: an alternative representation based on the state-space framework. Using the general unified model, it is possible to represent pure algebraic systems or dynamic systems and, using the well-known Cramer's rule, obtain a solution for a prescribed output of the system. Using these concepts, the parameterized state-space framework was introduced. It was shown how the parameters will appear in the solution of the system, considering the rank of matrices related to each parameter. The solution of the parameterized state-space model suggests the possibility of using measurement techniques to determine the unknown coefficients of the solution via a specific number of system experiments. Finally, from this solution, a generalization of Thévenin's theorem was shown, including its use in actual applications. Cramer's rule (applied to a general unified model based on the well-known state-space framework of systems with parameters) can lead to insightful results and computational tools. It also helps to unify apparently diverse types of models and, thus, can be a powerful educational tool. Furthermore, a new way to calculate the Thévenin's equivalent provides new insights for the design of data-based adaptive control laws.

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Cultural Lag

This occurred in a ninth-grade social studies class in the fall of 1944. Our teacher, Bradley Patterson, was discussing a concept that was familiar then in sociology: William F. Ogburn's notion of "cultural lag."

The idea was that the development of technology regularly moved much further and faster than other aspects of culture: our intuitions of government, values, habits, ethics, and understanding of society and ourselves. Indeed, the very notion of progress referred mainly to the technology. What lagged behind, what developed more slowly or not at all, was everything that bore on our ability to direct technology and to *control* it wisely, ethically, prudently.

—Daniel Ellsberg, *The Doomsday Machine*. Bloomsbury Publishing Plc, 2017, p. 25.