

# A Standardized State Space Approach: Cramer's rule, analysis, and applications

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With the advent of computer aided design, it is important to develop common mathematical frameworks within which Electrical, Mechanical, Chemical, Biological and other systems can be modelled, analysed and designed. One immediate use of this is that a common software platform can be used across various disciplines to compute solutions, as exemplified by Matlab.

A common problem that arises in many fields of engineering is that of modelling the behaviour of a complex interconnected system that is built up of simple elements. This is true in Electrical Engineering where Direct and Alternating Currents Circuits, Power Systems and Electronic Circuits are well known examples. Similar problems occur in Mechanical Engineering, Robotics, Hydraulic Networks, Truss Structures and Chemical Engineering. Complex interconnected feedback systems are created in Control Engineering from interconnections of elementary blocks. The use of Cramer's rule for solving systems of linear equations is of historical and theoretical importance. The Cramer's rule can be used in a pedagogical approach for calculating transfer functions from a polynomial description of the state space equations.

The main elements of the general unified model presented here is the definition of input vector  $\mathbf{u}$ , the output vector  $\mathbf{y}$ , the vector of internal state variables  $\mathbf{x}$ , the parameter vector  $\mathbf{p}$ , and the equations based on physical laws connecting them. If these equations are algebraic we call such a system as an algebraic system. If they involve time derivatives they are called dynamic. Even dynamic systems can be treated as algebraic ones by taking the Laplace transform. Because of that we refer to this model as a general unified model.

Many times when solving a linear system to obtain a variable of interest, it is common solve the whole state space equation or perform a matrix inversion to get just few variables of interest. In general, this is time consuming and can be avoided. Thus, in this lecture note we present in a pedagogical way an unified input-output-state model which is useful for describing direct current (DC) circuits, mechanical systems, block diagrams and other algebraic and dynamic systems.

The unified model can be explored to approach parameterized description of systems and various examples including circuits and hydraulic systems are given to illustrate the use of the well known Cramer's rule.

We revisit the use of the Cramer's rule and new applications are conceived by considering a system description with parameter dependence. We develop this general model and show how it can be used to obtain an input-output description of a parameterized system using a few measurements. The well known Thévenin's equivalent is also derived from this general model.

The examples presented show how to obtain currents or voltages of interest in DC circuits, transfer functions of complex dynamic systems and more importantly how the parameterized case can be handled and how outputs of interest can be obtained from a few measurements. In the parameterized case, the well known Thévenin's equivalent is treated as a special case.

The rest of the paper is organized as follows. First, the general unified model and the use of the Cramer's rule to obtain the system output and how the output of state space models can be obtained using the same framework are described. Subsequently, the parameterized version of the unified model is presented with the output solution expressed in terms of parameters of interest. Next, a measurement theorem is described to show that functions of the parameters of interest can be determined from a few measurements. Finally, some concluding remarks are made and future developments are described.

### Algebraic Models and Cramer's Rule

A general and unified approach to developing mathematical models for linear systems is described below. We regard a system as consisting of inputs  $u_1, u_2, \dots, u_r$ , outputs denoted  $y_1, y_2, \dots, y_m$ , which are external variables, and internal or state variables denoted by  $x_1, x_2, \dots, x_n$  as illustrated in Fig.1 Using vector notation

$$\mathbf{y} := \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}, \mathbf{u} := \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix}, \mathbf{x} := \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix},$$

denote the  $m$ -output vector,  $r$ -input vector and  $n$ -state or internal vector, respectively.

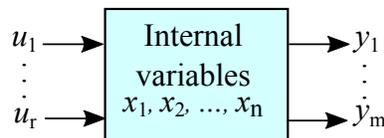


Figure 1. A state variable model relating input  $\mathbf{u}$  state  $\mathbf{x}$  and output  $\mathbf{y}$  vectors.

Often the only accessible or physically meaningful variables are  $\mathbf{y}$  and  $\mathbf{u}$ , and  $\mathbf{x}$  may not be accessible. When the algebraic equations describing the system are linear with constant coefficients we have a static linear time invariant system:

$$\begin{aligned} A\mathbf{x} + B\mathbf{u} &= 0 \\ C\mathbf{x} + D\mathbf{u} &= \mathbf{y} \end{aligned} \tag{1}$$

with output vector  $\mathbf{y}$  and input vector  $\mathbf{u}$  as already defined and  $(A, B, C, D)$  are matrices of size  $n \times n, n \times r, m \times n$  and  $m \times r$ , respectively. Let  $\mathbf{z} := [\mathbf{x} \ \mathbf{y}]^T$ . The system (1) may be thus rewritten as

$$\begin{bmatrix} A & 0 \\ -C & I \end{bmatrix} \mathbf{z} = \begin{bmatrix} -B \\ D \end{bmatrix} \mathbf{u}. \quad (2)$$

In (2), the mapping from input  $\mathbf{u}$  to output  $\mathbf{y}$  is made through the intermediate internal variables  $\mathbf{x}$ . A direct relationship between  $\mathbf{u}$  and  $\mathbf{y}$  can be obtained by eliminating  $\mathbf{x}$ , as shown below. This can be done systematically by using Cramer's rule. In [1], an approach for the presentation of linear systems theory in a first course showed that the Cramer's rule gives rise to a general method with pedagogical appeal for obtaining desired system transfer functions. However, it does not use the unified approach inspired in the input-output-state model, in which the solution is connected to the concept of output of linear dynamic systems. The approach presented here has been used in applications such as agricultural sprayers, power electronic converters, renewable sources and others as shown in some of our papers [2], [3], [4], [5], [6].

### Use of Cramer's rule

Let  $y_i$  denote the  $i$ -th output and  $u_j$  the  $j$ -th input in (1). The superposition principle is satisfied clearly, that is, output  $y_i$  is given as the sum of the outputs for each input  $u_j$ . If  $c_i$  denotes the  $i$ th row of  $C$ ,  $b_j$  the  $j$ -th column of  $B$  and  $d_{ij}$  the  $ij$  element of  $D$ , applying Cramer's rule to (2) it follows that

$$y_i = \frac{|T_{i1}|}{|T|} u_1 + \cdots + \frac{|T_{ir}|}{|T|} u_r, \quad i = 1, \dots, m \quad (3)$$

where

$$T_{ij} := \begin{bmatrix} A & -b_j \\ -c_i & d_{ij} \end{bmatrix}, \quad i = 1, \dots, m, \quad j = 1, \dots, r \quad (4)$$

and

$$T = \begin{bmatrix} A & 0 \\ -C & I \end{bmatrix}. \quad (5)$$

In the literature, several proofs of the Cramer's rule can be found [7]. A simple and direct proof is given here, at the end of the paper, for the sake of completeness.

**Example 1** (DC output voltages in terms of the determinants of the general form). *Let us illustrate the use of the general form to find the DC circuit outputs in Fig. 2. The input  $u = [V_1 \ V_2]'$  is given by voltage sources, the output  $y = [v_2 \ v_4]'$  is given by the voltages across the resistors  $R_2$  and  $R_4$  and the internal variables are the branch currents  $i_j$  and the branch voltages  $v_j$  across the resistors  $R_j, j = 1, \dots, 10$ :*

$$\mathbf{x} = \begin{bmatrix} i_1 & \cdots & i_{10} & v_1 & \cdots & v_{10} \end{bmatrix}^T.$$

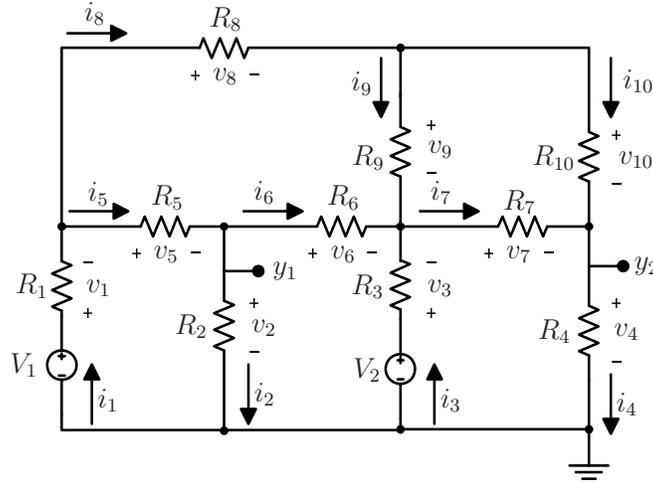


Figure 2. A resistive circuit.

Writing Kirchoff's equations for the circuit, we have the following node equations

$$\begin{aligned}
 i_1 - i_5 - i_8 &= 0, & i_5 - i_2 - i_6 &= 0 \\
 i_3 + i_6 + i_9 - i_7 &= 0, & i_7 + i_{10} - i_4 &= 0 \\
 i_8 - i_9 - i_{10} &= 0
 \end{aligned} \tag{6}$$

loop equations

$$\begin{aligned}
 V_1 - v_1 - v_5 - v_2 &= 0, & v_2 - v_6 + v_3 - V_2 &= 0 \\
 V_2 - v_3 - v_7 - v_4 &= 0, & v_8 + v_9 - v_6 - v_5 &= 0 \\
 v_9 - v_{10} + v_7 &= 0
 \end{aligned} \tag{7}$$

and Ohm's law relations

$$v_j = i_j R_j, \quad j = 1, \dots, 10. \tag{8}$$

The block diagram of a complex dynamic system can also be represented as the linear algebraic system (1) with the entries of the matrices being transfer functions. We illustrate with an example.

**Example 2** (Block diagram with transfer functions). Consider the block diagram of a 2-input, 2-output dynamic system shown in Fig.3. As usual in this kind of problem, to obtain the transfer functions of interest, it is necessary to apply some block diagram algebra simplification or even to replace several equations with consecutive substitutions, which is time consuming if the block diagram is as large as shown in Fig.3. However, it is possible to obtain the transfer functions of interest easily by using the present approach. The input and output of the block with transfer

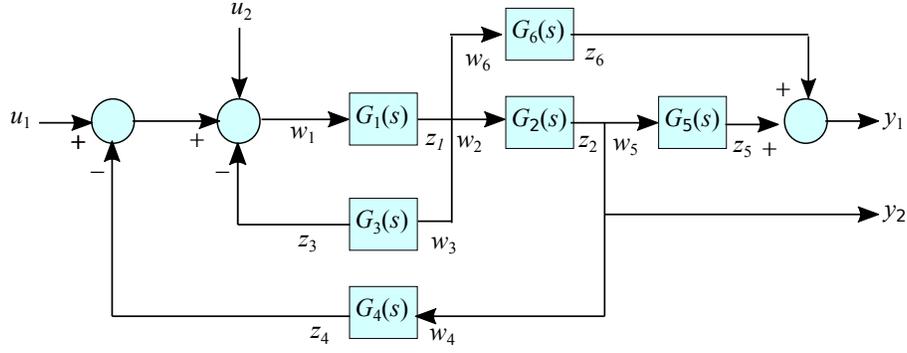


Figure 3. Block diagram of Example 2 showing input-output relationship.

functions  $G_k(s)$  are denoted  $w_k(s)$  and  $z_k(s)$ , respectively. The structural equations of the system are the six equations:

$$u_1 + u_2 - z_4 - z_3 = w_1; \quad z_1 = w_2 = w_6 = w_3; \quad z_2 = w_5 = w_4.$$

The output equations are

$$y_1 = z_5 + z_6; \quad y_2 = z_2$$

and the individual input/output equations are

$$z_k(s) = G_k(s)w_k(s), \quad k = 1, \dots, 6.$$

The above sets of equations displayed in matrix forms are

$$A \begin{bmatrix} z \\ w \end{bmatrix} + Bu = 0$$

$$C \begin{bmatrix} z \\ w \end{bmatrix} + Du = y$$

with

$$A = \left[ \begin{array}{cccccc|cccccc} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ \hline & & & & & & G_1 & & & & & \\ & & & & & & & & & & 0 & \\ & & & & & & & & & & \ddots & \\ & & & & & & & & & & 0 & G_6 \end{array} \right]$$

$$B = \begin{bmatrix} 1 & 1 \\ [0_{11 \times 2}] \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The transfer functions relating  $y_i$  to  $u_j$  can thus be obtained using Cramer's rule as shown in (3).

Table 1. Matlab code for generating the transfer functions of the block diagram in Example 2.

```

%% Clearing the variables

close all; clear all;clc
%% Defining the system
A11=[0 0 1 1 0 0
      1 0 0 0 0 0
      1 0 0 0 0 0
      1 0 0 0 0 0
      0 1 0 0 0 0
      0 1 0 0 0 0];
A12=[1 0 0 0 0 0
      0 0 -1 0 0 0
      0 -1 0 0 0 0
      0 0 0 0 0 -1
      0 0 0 0 -1 0
      0 0 0 -1 0 0];
A21=-eye(6);
G=[G1 G2 G3 G4 G5 G6];
A22=diag(G);

A=[A11 A12;A21 A22];
B=[1 1;zeros(11,2)];
D=[0 0;0 0];
C=[0 0 0 0 1 1 0 0 0 0 0 0
      0 1 0 0 0 0 0 0 0 0 0 0];
T=[A -B;0 eye(2)]
T=[A zeros(12,2);-C eye(2)];
T11=[A -B(:,1);-C(1,:) D(1,1)];
T12=[A -B(:,2);-C(1,:) D(1,2)];
T21=[A -B(:,1);-C(2,:) D(2,1)];
T22=[A -B(:,2);-C(2,:) D(2,2)];

```

$$y_1 = \det(T_{11}) * u_1 + \det(T_{12}) * u_2$$

## State Space Models

A special case of block diagrams is state space models described by the first order differential equation

$$\dot{x} = Ax(t) + Bu(t) \quad (9)$$

which obeys the superposition principle if and only if the initial condition  $x(0) = 0$  [8], [9]. The general description (1) can be extended to state space models as follows:

$$\begin{aligned} A(s)x(s) &= Bu(s) \\ y(s) &= Cx(s) + Du(s) \end{aligned} \quad (10)$$

with  $A(s) := sI - A$ . As before, we can write:

$$T(s) \mathbf{z} = \begin{bmatrix} -B \\ D \end{bmatrix} \mathbf{u}, \text{ with } T(s) := \begin{bmatrix} A(s) & 0 \\ -C & I \end{bmatrix}.$$

with  $z = [x(s) \ y(s)]^T$ . Let

$$T_{ij}(s) := \begin{bmatrix} A(s) & -b_j \\ -c_i & d_{ij} \end{bmatrix}, i = 1, \dots, m, j = 1, \dots, r \quad (11)$$

with  $c_i, i = 1, \dots, m$  being the  $i$ -th row of  $C$ ,  $b_j, j = 1, \dots, r$  the  $j$ -th column of  $B$ ,  $d_{ij}$  the  $ij$ -th element of  $D$ .

*Obtaining the input-output transfer functions*

Solving for  $y_i$ , we obtain

$$y_i = \frac{|T_{i1}(s)|}{|T(s)|} u_1 + \dots + \frac{|T_{ir}(s)|}{|T(s)|} u_r \quad (12)$$

where

$$T_{ij}(s) := \begin{bmatrix} A(s) & -b_j \\ -c_i & d_{ij} \end{bmatrix}, i = 1, \dots, m, j = 1, \dots, r \quad (13)$$

and

$$T(s) = \begin{bmatrix} A(s) & 0 \\ -C & I \end{bmatrix}. \quad (14)$$

The output  $y(s)$  can be written in terms of the transfer matrix entries denoted  $G_{ij}(s)$  as follows

$$y(s) = \begin{bmatrix} G_{11}(s) & \dots & G_{1r}(s) \\ \vdots & \dots & \vdots \\ G_{m1}(s) & \dots & G_{mr}(s) \end{bmatrix} u(s) \quad (15)$$

with

$$\begin{aligned}
 G_{ij}(s) &= \frac{|T_{ij}(s)|}{|T(s)|} \\
 &= \frac{\begin{bmatrix} A(s) & -b_j \\ -c_i & d_{ij} \end{bmatrix}}{|A(s)|}
 \end{aligned} \tag{16}$$

for  $i = 1, \dots, m$ ,  $j = 1, \dots, r$ .

**Example 3** (Inverted pendulum transfer functions). Consider the inverted pendulum shown in Fig. 4. The input is the force applied to the cart and the outputs are the position of the cart and the angular position of the pendulum. The dynamics of the pendulum on a cart are described by:

$$\begin{aligned}
 N\ddot{x}_a + b\dot{x}_a + m\ell\ddot{\theta}\cos\theta - m\ell\dot{\theta}^2\sin\theta &= k\dot{u} \\
 L\ddot{\theta} - mg\ell\sin\theta + m\ell\ddot{x}_a\cos\theta + d\dot{\theta} &= 0
 \end{aligned} \tag{17}$$

where  $N = m + M$ ,  $L = I + m\ell^2$  and  $F = k\dot{u}$  is the force acting on the cart with  $k$  a gain.

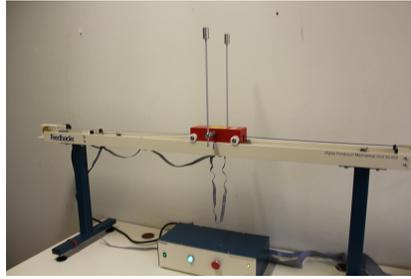


Figure 4. Feedback digital pendulum in the upright position.

Integrating both equations in (17), and defining  $x_1 = \theta$ ,  $x_2 = x_a$  and  $x_3 = \int \theta$  we obtain a space state description of the pendulum and cart. Linearising about  $\theta = 0$  we obtain the state space matrices as

$$\begin{aligned}
 A &= \begin{bmatrix} \frac{Nd}{\ell^2 m^2 - LN} & -\frac{b\ell m}{\ell^2 m^2 - LN} & -\frac{Ng\ell m}{\ell^2 m^2 - LN} \\ -\frac{d\ell m}{\ell^2 m^2 - LN} & \frac{Lb}{\ell^2 m^2 - LN} & \frac{g\ell^2 m^2}{\ell^2 m^2 - LN} \\ 1 & 0 & 0 \end{bmatrix} \\
 B &= \begin{bmatrix} \frac{k\ell m}{\ell^2 m^2 - LN} \\ -\frac{Lk}{\ell^2 m^2 - LN} \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}
 \end{aligned}$$

An experienced reader may now think of using the conversion state space to transfer the function, that is,  $C(sI - A)^{-1}B$  to find the outputs of the pendulum. However, calculating the

inverse matrix using the adjoint method needs calculating the determinant, the cofactor matrix, and at least three matrix multiplications, and if not all outputs are required then it is a really difficult way to get the result. Instead of this laborious work, the Cramer's rule in the presented approach gives an easier alternative to obtain the transfer function of interest by just calculating three determinants.

The outputs are given by:

$$y_1 = \frac{|T_{11}(s)|}{|T(s)|} u_1, \quad y_2 = \frac{|T_{21}(s)|}{|T(s)|} u_1$$

where

$$T(s) = \left[ \begin{array}{ccc|cc} s - \frac{Nd}{\ell^2 m^2 - LN} & \frac{b\ell m}{\ell^2 m^2 - LN} & \frac{Ng\ell m}{\ell^2 m^2 - LN} & 0 & 0 \\ \frac{d\ell m}{\ell^2 m^2 - LN} & s - \frac{Lb}{\ell^2 m^2 - LN} & -\frac{g\ell^2 m^2}{\ell^2 m^2 - LN} & 0 & 0 \\ -1 & 0 & s & 0 & 0 \\ \hline -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{array} \right]$$

$$T_{11}(s) = \left[ \begin{array}{ccc|c} s - \frac{Nd}{\ell^2 m^2 - LN} & \frac{b\ell m}{\ell^2 m^2 - LN} & \frac{Ng\ell m}{\ell^2 m^2 - LN} & \frac{k\ell m}{\ell^2 m^2 - LN} \\ \frac{d\ell m}{\ell^2 m^2 - LN} & s - \frac{Lb}{\ell^2 m^2 - LN} & -\frac{g\ell^2 m^2}{\ell^2 m^2 - LN} & -\frac{Lk}{\ell^2 m^2 - LN} \\ -1 & 0 & s & 0 \\ \hline -1 & 0 & 0 & 0 \end{array} \right]$$

$$T_{21}(s) = \left[ \begin{array}{ccc|c} s - \frac{Nd}{\ell^2 m^2 - LN} & \frac{b\ell m}{\ell^2 m^2 - LN} & \frac{Ng\ell m}{\ell^2 m^2 - LN} & \frac{k\ell m}{\ell^2 m^2 - LN} \\ \frac{d\ell m}{\ell^2 m^2 - LN} & s - \frac{Lb}{\ell^2 m^2 - LN} & -\frac{g\ell^2 m^2}{\ell^2 m^2 - LN} & -\frac{Lk}{\ell^2 m^2 - LN} \\ -1 & 0 & s & 0 \\ \hline 0 & -1 & 0 & 0 \end{array} \right]$$

Thus, we obtain the following transfer functions

$$G_{11}(s) = \frac{(-klm) s^2}{(\ell^2 m^2 - LN) s^3 + (-Lb - Nd) s^2 + (Ng\ell m - bd) s + bglm}$$

$$G_{21}(s) = \frac{(Lk) s^2 + (dk) s - gklm}{(\ell^2 m^2 - LN) s^3 + (-Lb - Nd) s^2 + (Ng\ell m - bd) s + bglm}$$

with

$$G_{11}(s) := \frac{|T_{11}(s)|}{|T(s)|}, \quad G_{21}(s) := \frac{|T_{21}(s)|}{|T(s)|}.$$

Table 2. Matlab code for generating the transfer functions of the pendulum in Example 3.

```

%% Clearing the variables

close all; clear all; clc

%% Defining the pendulum parameters and variables

syms N b m l k u g d L M I
syms x1 x2 x3 x1prime x2prime x3prime
syms s

%% Defining the system

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% x1 = teta
% x2 = xa
% x3 = integral theta
%
% thus
%
% x1prime = thetaprime
% x2prime = xaprime
% x3prime = teta = x1
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% diff equation system
eq1 = -k*u+N*x2prime+b*x2+m*l*x1prime;
eq2 = L*x1prime-m*g*l*x3+d*x1+m*l*x2prime;

% isol

x1prime = solve(eq1==0,x1prime);
eq2 = eval(eq2);
x2prime = solve(eq2==0,x2prime);
x1prime = eval(x1prime);
x3prime = x1;

x1prime;
x2prime;
x3prime;

%% Writing in state space form

x = [x1 x2 x3]; % define the state vector

```

```

[A, b] = equationsToMatrix([x1prime == 0, x2prime == 0, x3prime == 0 ], x);

A = simplify(A, 'Steps', 350)
[b, xxx] = equationsToMatrix(-b, u);
B = simplify(b, 'Steps', 350)

C = [1 0 0; 0 1 0]; % define C matrix

Ts = [s*eye(3)-A zeros(3,2); -C eye(2)]

T11 = [s*eye(3)-A -B; -C(1,:) 0]
T21 = [s*eye(3)-A -B; -C(2,:) 0]

G11 = collect(det(T11)/det(Ts), s)
G21 = collect(det(T21)/det(Ts), s)

```

### **Parameterized Input-Output Models**

In many design problems there is a set of parameters denoted by a vector  $\mathbf{p}$ , whose influence on the output is important to know. This can be incorporated into the present formulation as shown below. Let us thus consider the linear, parameterized, input-output model [10]:

$$\begin{aligned}
 A(\mathbf{p})\mathbf{x} &= B(\mathbf{p})\mathbf{u} \\
 \mathbf{y} &= C(\mathbf{p})\mathbf{x} + D(\mathbf{p})\mathbf{u}
 \end{aligned} \tag{18}$$

with output vector  $\mathbf{y}$ , input vector  $\mathbf{u}$  and parameter vector  $\mathbf{p}$ :

$$\mathbf{y} = [y_1 \ \cdots \ y_m]^T, \mathbf{u} = [u_1 \ \cdots \ u_r]^T, \mathbf{p} = [p_1 \ \cdots \ p_\ell]^T.$$

With  $\mathbf{z} := [\mathbf{x} \ \mathbf{y}]^T$ , (18) can be written as:

$$T(\mathbf{p}) \mathbf{z} = \begin{bmatrix} -B \\ D \end{bmatrix} \mathbf{u} \text{ with } T(\mathbf{p}) := \begin{bmatrix} A(\mathbf{p}) & 0 \\ -C(\mathbf{p}) & I \end{bmatrix}.$$

Let

$$T_{ij}(\mathbf{p}) := \begin{bmatrix} A(\mathbf{p}) & -b_j(\mathbf{p}) \\ -c_i(\mathbf{p}) & d_{ij}(\mathbf{p}) \end{bmatrix}, i = 1, \dots, m, j = 1, \dots, r \tag{19}$$

with  $c_i(\mathbf{p})$ ,  $i = 1, \dots, m$  being the  $i$ -th row of  $C(\mathbf{p})$ ,  $b_j(\mathbf{p})$ ,  $j = 1, \dots, r$  the  $j$ -th column of  $B$ ,  $d_{ij}(\mathbf{p})$  the  $ij$ -th element of  $D$ , and

$$\beta_{ij}(\mathbf{p}) := |T_{ij}(\mathbf{p})|, \alpha(\mathbf{p}) := |T(\mathbf{p})| = |A(\mathbf{p})|. \tag{20}$$

For the model (18), the outputs can be determined in terms of inputs and parameters. This is established below using the results given in [10], and omitting proofs.

**Assumption 1.** *The parameter  $\mathbf{p}$  appears affinely in  $A(\mathbf{p})$ ,  $B(\mathbf{p})$ ,  $C(\mathbf{p})$  and  $D(\mathbf{p})$ :*

$$\begin{aligned} A(\mathbf{p}) &= A_0 + p_1 A_1 + \cdots + p_\ell A_\ell \\ C(\mathbf{p}) &= C_0 + p_1 C_1 + \cdots + p_\ell C_\ell. \\ B(\mathbf{p}) &= B_0 + p_1 B_1 + \cdots + p_\ell P_\ell \\ D(\mathbf{p}) &= D_0 + p_1 D_1 + \cdots + p_\ell D_\ell. \end{aligned}$$

**Assumption 2.**

$$|T(\mathbf{p})| \neq 0. \quad (21)$$

**Theorem 1.** *For system (18), the output is given by*

$$y_i = \sum_{j=1}^r \frac{\beta_{ij}(\mathbf{p})}{\alpha(\mathbf{p})} u_j, \quad i = 1, 2, \dots, m \quad (22)$$

with  $\beta_{ij}(\mathbf{p})$  and  $\alpha(\mathbf{p})$  as already defined in (20). In matrix terms:

$$\mathbf{y} = \frac{1}{\alpha(\mathbf{p})} \begin{bmatrix} \beta_{11}(\mathbf{p}) & \cdots & \beta_{1r}(\mathbf{p}) \\ \vdots & & \vdots \\ \beta_{m1}(\mathbf{p}) & \cdots & \beta_{mr}(\mathbf{p}) \end{bmatrix} \mathbf{u}. \quad (23)$$

To describe the solution  $\mathbf{y}$  from (18), we give the form of the functions  $\alpha(\mathbf{p})$  and  $\beta_{ij}(\mathbf{p})$ .

**Lemma 1** ([11], [12]). *Let  $A(\mathbf{p}) = A_0 + p_1 A_1 + \cdots + p_\ell A_\ell$  with rank  $A_i = r_i, i = 1, 2, \dots, \ell$ . Then  $\alpha(\mathbf{p}) = |A(\mathbf{p})|$  is a multivariate polynomial in  $\mathbf{p}$ , of degree at most  $r_i$  in  $p_i, i = 1, 2, \dots, \ell$ .*

Then, it can be shown that the multivariate polynomials in  $\mathbf{p}$  can be written as:

$$|A(\mathbf{p})| = \sum_{i_\ell=0}^{r_\ell} \cdots \sum_{i_1=0}^{r_1} \alpha_{i_1 \dots i_\ell} p_1^{i_1} \cdots p_\ell^{i_\ell} \quad (24)$$

with rank  $(A_i) = r_i, i = 1, 2, \dots, \ell$ . In (24), the number of coefficients in  $|A(\mathbf{p})|$  is  $\mu := \sum_{i=1}^{\ell} (r_i + 1)$ .

**Example 4** (Determinant of parameterized matrices in terms of a multivariate polynomial). *Let*

$$A(\mathbf{p}) = \begin{bmatrix} 1 & 2p_1 & 0 \\ p_1 & p_2 & p_1 \\ 3 & p_1 & 3p_2 \end{bmatrix}.$$

As the parameter  $\mathbf{p}$  appears affinely in  $A(\mathbf{p})$ , following Assumption 1, we can write

$$A(\mathbf{p}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} p_1 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} p_2.$$

In this example,  $\text{rank}(A_1) = 2$ ,  $\text{rank}(A_2) = 2$ . Matrix  $A(\mathbf{p})$  is said to be rank 2 with respect to  $p_1$  and  $p_2$ , which yields  $r_1 = 2$  and  $r_2 = 2$ . Thus,

$$|A(\mathbf{p})| = \sum_{i_2=0}^2 \sum_{i_1=0}^2 \alpha_{i_1 i_2} p_1^{i_1} p_2^{i_2}$$

is a polynomial of degree at most 2 in both  $p_1$  and  $p_2$ . Calculating the determinant, it yields

$$|A(\mathbf{p})| = -6p_1^2 p_2 + 5p_1^2 + 3p_2^2.$$

Now consider (19) written in polynomial form:

$$T_{ij}(\mathbf{p}) = T_{ij0} + p_1 T_{ij1} + \cdots + p_\ell T_{ij\ell}.$$

Applying Lemma 1, we see that

$$|T_{ij}(\mathbf{p})| = \beta_{ij}(\mathbf{p})$$

is a multivariate polynomial in  $\mathbf{p}$  of degree at most  $t_{ijk}$  in  $p_k$  where

$$t_{ijk} = \text{rank}(T_{ijk}), i = 1, 2, \dots, m, j = 1, 2, \dots, r, k = 1, 2, \dots, \ell$$

and its determinant can be described in a similar manner as in (24).

**Example 5** (Parameterized output). Consider the gyrator circuit shown in Figure 5 taken from [13]. Using the gyrator resistance, denoted  $R_4$ , regulate the voltage across the resistor  $R_1$ , denoted  $y$ . Defining the internal variable vector as  $\mathbf{x} = [I_1 \ I_2 \ I_3]^T$  and using the Kirchhoff's current and voltage laws, we obtain:

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ R_1 & R_2 & -R_4 & 0 \\ K & -R_4 & R_3 & 0 \\ R_1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \end{bmatrix} u_1 + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_2 = 0.$$

In the general state space form, we have for output  $y$  and inputs  $u_1$  and  $u_2$ :

$$y = \frac{|T_1(\mathbf{p})|}{|A(\mathbf{p})|} u_1 + \frac{|T_2(\mathbf{p})|}{|A(\mathbf{p})|} u_2$$

$$\text{where } A = \begin{bmatrix} 1 & -1 & 0 \\ R_1 & R_2 & -R_4 \\ K & -R_4 & R_3 \end{bmatrix}, T_1 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ R_1 & R_2 & -R_4 & 1 \\ K & -R_4 & R_3 & 0 \\ R_1 & 0 & 0 & 1 \end{bmatrix}, T_2 = \begin{bmatrix} 1 & -1 & 0 & 1 \\ R_1 & R_2 & -R_4 & 0 \\ K & -R_4 & R_3 & 0 \\ R_1 & 0 & 0 & 0 \end{bmatrix}.$$

If we choose the design parameter as  $p = R_4$ , the matrices appearing in  $A(\mathbf{p})$  and  $T_1(\mathbf{p})$  and  $T_2(\mathbf{p})$  are rank 2 matrices in  $R_4$ . Thus,

$$y = \frac{\beta_{01} + \beta_{11}p + \beta_{21}p^2}{1 + \alpha_1 p + \alpha_2 p^2} u_1 + \frac{\beta_{02} + \beta_{12}p + \beta_{22}p^2}{1 + \alpha_1 p + \alpha_2 p^2} u_2.$$

This example illustrates the use of the approach described to analyse the output ranges of a variable of interest in terms of a parameter of the system.

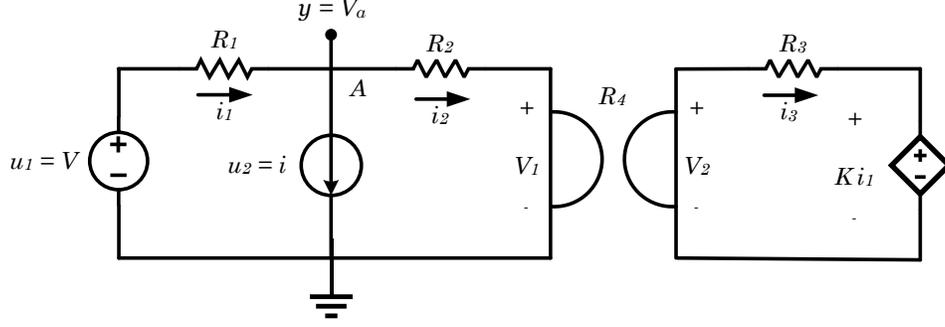


Figure 5.  $V$  and  $i$  are an ideal voltage and current source, respectively,  $R_1$ ,  $R_2$ ,  $R_3$  are linear resistors, and  $R_4$  is a gyrator resistance. The gyrator is a linear two port device where the instantaneous currents and the instantaneous voltages are related by  $V_2 = R_4 i_2$  and  $V_1 = -R_4 i_3$ ,  $K i_1$  describes a dependent voltage amplifier in which  $K$  is the gain.

### State Space Models with Delays

Consider the parameterized models described before for a state space description:

$$\begin{aligned} A(s, \mathbf{p})x(s) &= B(\mathbf{p})u(s) \\ y(s) &= C(\mathbf{p})x(s) + D(\mathbf{p})u(s) \end{aligned} \quad (25)$$

with  $A(s, \mathbf{p}) := sI - A(\mathbf{p})$ . As before, let us define:

$$T(s, \mathbf{p}) \mathbf{z} = \begin{bmatrix} B \\ -D \end{bmatrix} \mathbf{u}, \text{ with } T(s, \mathbf{p}) := \begin{bmatrix} A(s, \mathbf{p}) & 0 \\ -C(\mathbf{p}) & I \end{bmatrix}.$$

Also, let

$$T_{ij}(s) := \begin{bmatrix} A(s, \mathbf{p}) & -b_j(\mathbf{p}) \\ -c_i(\mathbf{p}) & d_{ij}(\mathbf{p}) \end{bmatrix}, i = 1, \dots, m, j = 1, \dots, r$$

with  $c_i(\mathbf{p})$ ,  $i = 1, \dots, m$  being the  $i$ -th row of  $C(\mathbf{p})$ ,  $b_j(\mathbf{p})$ ,  $j = 1, \dots, r$  the  $j$ -th column of  $B(\mathbf{p})$ ,  $d_{ij}(\mathbf{p})$  the  $ij$ -th element of  $D$ . We define the polynomials:

$$\beta_{i,j}(s, \mathbf{p}) := |T_{ij}(s, \mathbf{p})|, \alpha(s, \mathbf{p}) := |T(s, \mathbf{p})| = |A(s, \mathbf{p})|. \quad (26)$$

The determinants in (26) can be written as polynomials in  $\mathbf{p}$ :

$$\alpha(s, \mathbf{p}) = s^n + \alpha_1(\mathbf{p})s^{n-1} + \dots + \alpha_n(\mathbf{p}) \quad (27)$$

$$\beta_{i,j}(s, \mathbf{p}) = s^n + \beta_{i,j,1}(\mathbf{p})s^{n-1} + \dots + \beta_{i,j,n}(\mathbf{p}) \quad (28)$$

where the coefficients  $\alpha_v(\mathbf{p})$  and  $\beta_{i,j,v}(\mathbf{p})$ ,  $v = 1, \dots, n$  are multivariate polynomials in  $\mathbf{p}$  with degree at most  $r_k := \text{rank}(A_k)$  and  $t_{i,j,k} = \text{rank}(T_{ijk})$  in  $p_k$ , respectively. As before, using (16) we have

$$y_i(s, \mathbf{p}) = \sum_{j=1}^r \frac{\beta_{ij}(s, \mathbf{p})}{\alpha(s, \mathbf{p})} u_j, \quad i = 1, 2, \dots, m \quad (29)$$

A special case of (25) is the case of systems with delays which is illustrated next.

**Example 6** (Output of systems with delays in the state variables and output). Consider the space state system described by

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(t - L_1) \\ &+ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(t - L_2) + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u(t) \\ y &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(t - L_1) \end{aligned}$$

In this case, we define the parameter vector  $\mathbf{p}$  as the Laplace transform of the delays in the state. Let us thus take  $p_1 = e^{sL_1}$  and  $p_2 = e^{sL_2}$ , with  $L_1$  and  $L_2$  time delays. Thus, we have  $r_1 = 2$ ,  $r_2 = 1$ ,  $t_{1,1,1} = 2$ ,  $t_{1,1,2} = 1$ ,  $t_{2,1,1} = 3$ ,  $t_{2,1,2} = 1$  and it yields

$$y(s, \mathbf{p}) = \begin{bmatrix} \frac{-s^2 + (-p_1^2 - p_2 + p_1 + 2)s + p_1^2 - p_1^2 p_2 + p_1 p_2 + p_2 - p_1 - 1}{s^3 + (-p_1 - 3)s^2 + (-p_1^2 + 2p_1 + 3)s + p_1^2 - p_1 - 1} \\ \frac{-p_1}{s-1} \end{bmatrix} u(s)$$

Now, substituting for  $p_1$ ,  $p_2$  we obtain

$$y(s) = \begin{bmatrix} \frac{-s^2 + (-e^{s2L_1} - e^{sL_2} + e^{sL_1} + 2)s + e^{2sL_1} - e^{s(2L_1 + L_2)} + e^{s(L_1 + L_2)} + e^{sL_2} - e^{sL_1} - 1}{s^3 + (-e^{sL_1} - 3)s^2 + (-e^{2sL_1} + 2e^{sL_1} + 3)s + e^{2sL_1} - e^{sL_1} - 1} \\ \frac{-e^{sL_1}}{s-1} \end{bmatrix} u(s).$$

The solution given using the Cramer's rule displays the way the different delays appear in the output  $y(s)$ .

Table 3. Matlab code for generating the output of the system with delay in Example 6.

```

%% Clearing the variables

close all; clear all; clc

%% Defining the matrices and the parameters

syms s L1 L2

p1 = exp(s*L1);
p2 = exp(s*L2);

```

```

A0 = [1 0 0; 0 1 0; 0 0 1];
A1 = [0 1 0; 1 1 0; 0 0 0];
A2 = [0 0 1; 0 0 0; 0 0 0];

B = [1;0;1];

C0 = [1 0 0; 0 0 0];
C1 = [0 1 0; 0 0 1];

n = length(A0);

Ap = A0+A1*p1+A2*p2;
Cp = C0+C1*p1;
r1 = rank(A1);
r2 = rank(A2);

%% Calculate transfer functions

Asp = s*eye(n)-Ap;

Ts = Asp;
T11 = [Asp -B; -Cp(1,:) 0];
T21 = [Asp -B; -Cp(2,:) 0];

G11 = collect(det(T11)/det(Ts),s);
G21 = collect(det(T21)/det(Ts),s);

Gs = [G11;G21]

```

### ***System Behavior Estimation and Thévenin's Equivalent***

As showed before, a complex linear algebraic system described in the state space framework can have a simple input-output representation. *The solution (29) suggests that knowledge of the functions  $\alpha(\mathbf{p})$  and  $\beta_{ij}(\mathbf{p})$  are sufficient to determine the behavior of the outputs  $y_i$  as a function of  $\mathbf{p}$  and  $\mathbf{u}$  [12], [14]. The knowledge of  $\alpha(\mathbf{p})$  and  $\beta_{ij}(\mathbf{p})$  reduces to the knowledge of the coefficients of these polynomial functions. In an unknown system (black box, for instance) these coefficients are unknown a priori. However, if tests on the system can be performed by setting the design parameter  $\mathbf{p}$  and input  $\mathbf{u}$  to various values and measuring the corresponding output  $y_i$ , the polynomial function coefficients can be determined.*

**Theorem 2** (A Measurement Theorem). *Let  $y$  be as already defined in (23). The function  $y$  can be determined from  $\mu$  measurements and solution of a system of  $\mu$  linear equations in the*

unknown coefficient vectors  $\alpha(\mathbf{p})$  and  $\beta_{ij}(\mathbf{p})$ , called the measurement equations.

The above measurement result is illustrated in what follows with the special case of a single input single output  $y$  with  $\mathbf{p} = [p_1 \ p_2]'$ ,  $r_1 = 1$ ,  $r_2 = 2$ ,  $t_{111} = 1$ ,  $t_{112} = 2$ . Here,

$$y = \frac{\beta(\mathbf{p})}{\alpha(\mathbf{p})}u \quad (30)$$

with

$$\begin{aligned} \beta(\mathbf{p}) &= \beta_{00} + \beta_{10}p_1 + \beta_{01}p_2 + \beta_{11}p_1p_2 + \beta_{12}p_1p_2^2 \\ \alpha(\mathbf{p}) &= \alpha_{00} + \alpha_{10}p_1 + \alpha_{01}p_2 + \alpha_{11}p_1p_2 + \alpha_{12}p_1p_2^2. \end{aligned} \quad (31)$$

We may set  $\alpha_{12} = 1$  and conduct the following experiments. Measure  $y$  for seven different sets of values  $(p_1, p_2)$ . Let  $y(k)$ ,  $k = 1, \dots, 9$  denote the 9 different measurement values and  $\mathbf{p}(k) = [p_1(k) \ p_2(k)]$  the 9 sets of parameters. The 9 coefficients of  $\alpha(\mathbf{p})$  and  $\beta(\mathbf{p})$  in (31) can be determined from the measurement equation  $M\mathbf{f} = \mathbf{v}$  with

$$\mathbf{f} = \begin{bmatrix} \alpha_{00} \\ \alpha_{10} \\ \alpha_{01} \\ \alpha_{11} \\ \beta_{00} \\ \beta_{10} \\ \beta_{01} \\ \beta_{11} \\ \beta_{12} \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -p_1(1)p_2^2(1)y(1) \\ -p_1(2)p_2^2(2)y(2) \\ -p_1(3)p_2^2(3)y(3) \\ -p_1(4)p_2^2(4)y(4) \\ -p_1(5)p_2^2(5)y(5) \\ -p_1(6)p_2^2(6)y(6) \\ -p_1(7)p_2^2(7)y(7) \\ -p_1(8)p_2^2(8)y(8) \\ -p_1(9)p_2^2(9)y(9) \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{19} \\ m_{21} & m_{22} & \cdots & m_{29} \\ \vdots & \vdots & \vdots & \vdots \\ m_{91} & \cdots & \cdots & m_{99} \end{bmatrix}$$

where

$$\begin{aligned} m_{k1} &= y(k), \quad m_{k2} = p_1(k)y(k), \quad m_{k3} = p_2(k)y(k) \\ m_{k4} &= p_1(k)p_2(k)y(k), \quad m_{k5} = -u \\ m_{k6} &= -p_1(k)u, \quad m_{k7} = -p_2(k)u \\ m_{k8} &= -p_1(k)p_2(k)u, \quad m_{k9} = -p_1(k)p_2^2(k)u, \quad \text{for } k = \{1, 2, \dots, 9\}. \end{aligned}$$

Consider now a single input single output algebraic system with only one parameter denoted  $p$ . The input-output relationship can be written as

$$y = \frac{\beta_0 + \beta_1 p}{\alpha + p}u. \quad (32)$$

The description (32) holds when the coefficient matrix  $A_1$  of the parameter  $p$  as described before has unity rank. The formula (32) shows that if the constants  $(\beta_0, \beta_1, \alpha)$  are known the input output map is determined.

### Thevenin's Equivalent: Special Case of Circuits

In circuit theory, it is possible to obtain an equivalent circuit representation composed by a voltage source in series with a resistor using the Thévenin's equivalent. Consider an unknown DC circuit as shown in Fig. 6. The Thévenin's equivalent of circuit theory is summarized in the following theorem.

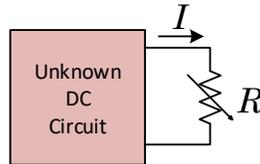


Figure 6. Thévenin's equivalent for a DC circuit illustrating its terminal connected to a load  $R$ .

**Theorem 3** (Thévenin's theorem),

$$V_{th} = V_{oc} \quad (33)$$

$$R_{th} = \frac{V_{oc}}{I_{sc}} \quad (34)$$

where  $I_{sc}$  is the short-circuit current and  $V_{oc}$  is the open-circuit voltage [15], [16].

From Theorem 3, it is necessary to open and then short circuit the terminals of the circuit of interest. However, it is known that many systems do not support short circuit conditions. Also, even in cases where this is possible, opening or short circuiting the terminals of a circuit may not be a trivial and a much less economically viable operation.

The single input-output description (32) represents a special case of Thévenin's theorem of circuit theory [15], [16] to arbitrary linear systems. To see this, consider again Example 5. To obtain the Thévenin's equivalent the inputs  $u_1$  and  $u_2$  are set as constants. Let us take the Thévenin's equivalent at the terminal defined by  $R$  and  $I$  in Figure 6 as the Thévenin's equivalent at the terminal defined by  $R_1$  and  $y = i_1$  in Fig. 5. Then, we can write

$$y = \frac{|T_1(\mathbf{p})|u_1 + |T_2(\mathbf{p})|u_2}{|A(\mathbf{p})|} \quad (35)$$

$$\text{where } A = \begin{bmatrix} 1 & -1 & 0 \\ p & R_2 & -R_4 \\ K & -R_4 & R_3 \end{bmatrix}, T_1 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ p & R_2 & -R_4 & 1 \\ K & -R_4 & R_3 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, T_2 = \begin{bmatrix} 1 & -1 & 0 & 1 \\ p & R_2 & -R_4 & 0 \\ K & -R_4 & R_3 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

As  $|T_1(\mathbf{p})|$  and  $|T_2(\mathbf{p})|$  are constants and the rank of matrix  $A$  with respect to parameter  $p$  is unity, we can write the solution

$$i_1 = \frac{\tilde{\beta}_0}{\tilde{\alpha} + R_1}. \quad (36)$$

In fact, the short circuit current denoted as  $I_{sc}$  is given by

$$I_{sc} = \frac{\tilde{\beta}_0}{\tilde{\alpha}} \quad (37)$$

obtained by setting  $R_1 = 0$ . Similarly, the open circuit voltage denoted as  $V_{oc}$  is given by multiplying (36) by  $R_1$  and taking the limit  $R_1 \rightarrow \infty$  which yields:

$$V_{oc} = \tilde{\beta}_0. \quad (38)$$

The Thévenin's equivalent is thus calculated as

$$R_{th} = \frac{V_{oc}}{I_{sc}} = \tilde{\alpha} \quad (39)$$

$$V_{th} = \tilde{\beta}_0. \quad (40)$$

We emphasize that calculating  $\tilde{\alpha}$ ,  $\tilde{\beta}_0$  does not require short-circuit or open-circuit experiments, only two arbitrary settings with corresponding measurements. Analytical models may be developed for mechanical, electrical, fluid, and thermal systems in a rather analogous manner. In view of the analogy, the Thevenin's equivalent can be generalized and the approach can also be applied to other dynamic systems.

In the following example, using electrical-hydraulic analogy, we consider the analysis of an hydraulic sprayer used in agriculture. The fluid pressure denoted  $\Delta P$ , the flow in a sprayer boom  $Q$  with the nozzle of interest and the fluidic resistance denoted  $R$ , are analogous to voltage, current, and electrical resistance, respectively [6].

**Example 7** (Electrical-hydraulic analogy and the Thévenin's equivalent). Consider now a sprayer system illustrated in Fig. 7 as in [6]. Let  $V = \sqrt{\Delta P}$ ,  $I = Q$ . The Thévenin's equivalent can be used to estimate the fluidic resistance of the boom.

With the presented Thévenin's equivalent method, it is no longer necessary to take the system to operate in two extreme situations, such as short-circuit and open-circuit. With two different safe operating points, it is possible to obtain the Thévenin's equivalent. Thus, let measurements of flow in the boom nozzle of interest be  $Q(1)$  and  $Q(2)$  and let the measurements of pressure in the corresponding boom be  $\Delta P(1)$  and  $\Delta P(2)$  thus:

$$\begin{pmatrix} Q(1) & -1 \\ Q(2) & -1 \end{pmatrix} \begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta}_0 \end{pmatrix} = \begin{pmatrix} -\sqrt{\Delta P(1)} \\ -\sqrt{\Delta P(2)} \end{pmatrix} \quad (41)$$

where  $\tilde{\alpha}$  e  $\tilde{\beta}_0$  are coefficients to be found.

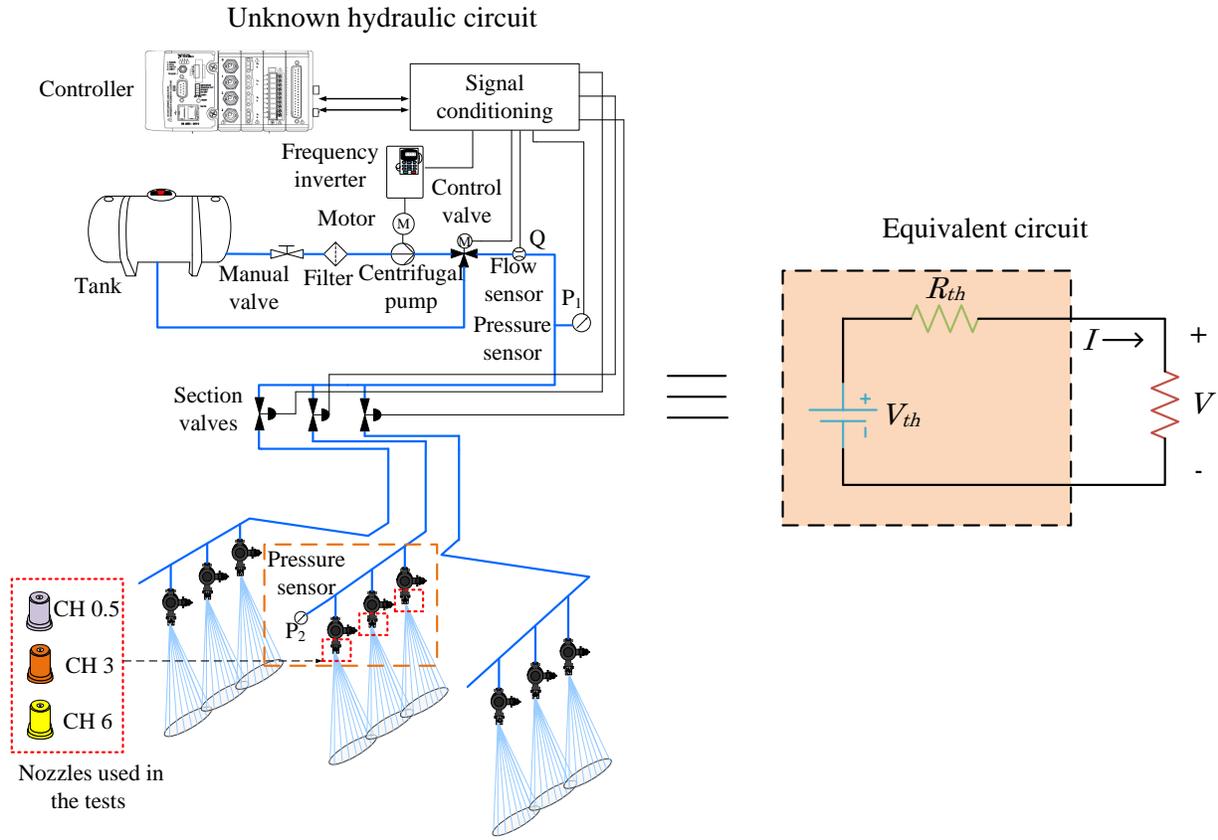


Figure 7. The sprayer boom used in the agricultural industry can be considered as an unknown hydraulic circuit, thus, the analogy between electrical and hydraulic circuits can be used to derive the pointwise hydraulic Thévenin's equivalent and thus obtain the fluidic resistance equivalent which is important because it directly affect the quality of, per example, pesticides application.

Therefore, the flow in the boom can be written in terms of coefficients of (41) as

$$Q = -\frac{\sqrt{\Delta P}}{\tilde{\alpha}} + \frac{\tilde{\beta}_0}{\tilde{\alpha}} \quad (42)$$

which yields

$$\begin{aligned} R_{th} &= \tilde{\alpha} \\ \Delta P_{th} &= \tilde{\beta}_0 \end{aligned}$$

where  $R_{th}$  and  $\Delta P_{th}$  are the internal loss and pressure equivalents, respectively. To extract the Thévenin's equivalent, only two different nozzles were required. To validate the Thévenin's equivalent obtained, a third different nozzle with an intermediate fluidic resistance value between

these two nozzles used for extraction of the equivalent was employed. The resulting data are shown in Table 4. Using (41), the following equivalent was obtained:

$$\Delta P_{th} = 1.85 \text{ [bar]}$$

$$R_{th} = 0.02 \text{ [bar min L}^{-1}\text{]}.$$

To validate the procedure, this equivalent was used to estimate the flow of arbitrary pressure values as shown in Fig. 8. The error of estimated flow using the Thévenin's equivalents was around 1.8%.

Table 4. Data obtained for different full cone nozzles

Nozzles	Pressure [bar]	Flow [L/min]
CH05	3.405	0.535
CH3	3.349	1.420
CH6	3.295	2.230

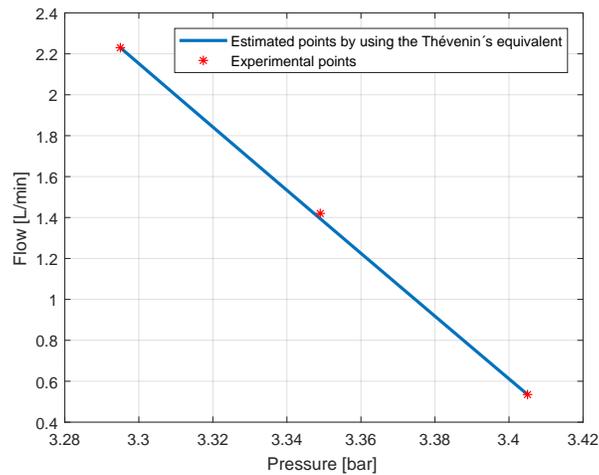


Figure 8. Validating the Thévenin's equivalent for full cone nozzles connected at the sprayer boom. The straight line is the estimate of the flow obtained via (42) and the dots are the experimental points. The pump pressure was set to 3.5 bar. It was necessary to guarantee that changing the fluidic resistance, a significant variation of pressure at the boom of interest was verified. A pressure variation of approximately 0.1 bar in relation to the pump pressure was considered significant taking into account the inherent noise of the spray sensors.

*The next example explores the popular DC-DC boost converter mainly used to step-up the voltage of photovoltaic module (usually in the range of 30V-40V to 60-cell crystalline modules)*

to provide the voltage required to the direct to alternating current (AC) converter stage to connect to the AC distribution power grid. As the photovoltaic module is a non-linear power source, the DC-DC boost converter is also used to find the maximum power point of this source [17]. This application is represented in Fig. 9.

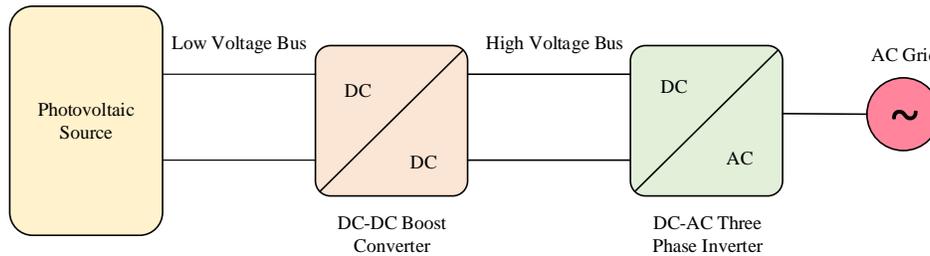


Figure 9. Typical application of a boost converter illustrating the connection of the direct current (DC) and alternating current (AC) devices and the AC distribution power grid.

**Example 8** (Parameterized functions for the boost DC-DC converter based on measurements). Consider the boost DC-DC converter circuit without loss shown in Fig. 10. The interest here is to analyse the circuit characteristic such as the boost converter gain and the input current denoted  $I_i$ , to help in selecting the converter parameters. The gain is related to the duty cycle, denoted  $D$ , of the pulse train which is called pulse width modulation (PWM). The input current denoted  $I_i$  is related to  $D$  and also to the load denoted  $R$ .

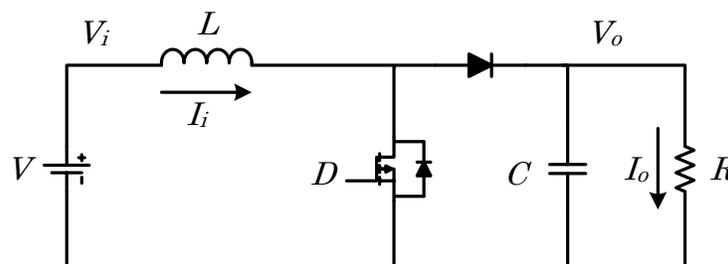


Figure 10. Model of an ideal boost DC-DC converter circuit showing the components. A boost DC-DC converter is a circuit which can deliver to the load a DC voltage equal or greater than the input voltage source.

For the purpose of comparison, we first obtain the converter circuit equations by the averaging method, a very common method used to analyse power electronics circuit. Defining

the following state, input and output as:

$$\begin{aligned}x &= [I_i \ V_o]^T \\u &= V_i \\y &= V_o.\end{aligned}$$

and following [18], the circuit from Fig. 10 can be modeled as a switched state space system which is defined by the following matrices:

$$\begin{aligned}A_1 &= \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{RC} \end{bmatrix} \\A_2 &= \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix} \\B_1 = B_2 &= \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} \\C_1 = C_2 &= [0 \ 1] \\D_1 = D_2 &= 0.\end{aligned}$$

Using the averaging method it can be shown that [19]:

$$\begin{aligned}\dot{x} &= [A_1d + A_2(1-d)]x + [B_1d + B_2(1-d)]u \\y &= [C_1d + C_2(1-d)]x + [D_1d + D_2(1-d)]u.\end{aligned}$$

Now, applying small signal perturbation denoted as  $\tilde{x}, \tilde{d}, \tilde{u}$ , it follows:

$$\begin{aligned}x &= X + \tilde{x} \\d &= D + \tilde{d} \\u &= U + \tilde{u}\end{aligned}$$

where the first term describes the mean value (DC) and the second the perturbation (AC) of the signal. Thus, we can write

$$\begin{aligned}\dot{\tilde{x}} &= AX + BU + A\tilde{x} + F\tilde{d} + B\tilde{u} \\y &= \bar{C}(X + \tilde{x})\end{aligned}\tag{43}$$

in which  $A$ ,  $B$ ,  $\bar{C}$  and  $F$  are given by:

$$\begin{aligned} A &= A_1 D + A_2(1 - D) \\ B &= B_1 D + B_2(1 - D) \\ \bar{C} &= C_1 D + C_2(1 - D) \\ F &= (A_1 - A_2)X + (B_1 - B_2)U. \end{aligned}$$

We thus obtain the steady state equations:

$$\begin{aligned} AX + BU &= 0 \\ y &= CX \end{aligned}$$

Finally, as in the boost circuit case  $U := V_i$  and  $y := V_o$ , using the Cramer's rule, we obtain the output as

$$V_o = \frac{|T_1|}{|A|} V_i$$

and the voltage gain is given by:

$$\begin{aligned} \frac{V_o}{V_i} &= \frac{|T_1|}{|A|} \\ &= \frac{1}{1 - D} \end{aligned}$$

with

$$\begin{aligned} A &= \begin{bmatrix} 0 & 0 \\ 0 & \frac{-1}{RC} \end{bmatrix} D + \begin{bmatrix} 0 & \frac{-1}{L} \\ \frac{1}{C} & \frac{-1}{RC} \end{bmatrix} (1 - D) \\ &= \begin{bmatrix} 0 & \frac{D-1}{L} \\ -\frac{D-1}{C} & \frac{D-1}{RC} - \frac{D}{RC} \end{bmatrix} \end{aligned}$$

and

$$T_1 = \begin{bmatrix} 0 & \frac{D-1}{L} & \frac{1}{L} \\ -\frac{D-1}{C} & \frac{D-1}{RC} - \frac{D}{RC} & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

**Table 5.** Matlab code to obtain the static voltage gain for the DC-DC converter in Example 8.

```
close all, clear all, clc

%% define the variables

syms R C L D
```

```

%% define the matrices

A1 = [0 0; 0 -1/(R*C)];
A2 = [0 -1/L; 1/C -1/(R*C)];
B1 = [1/L; 0];
B2 = B1;

%% calculate the gain

A = A1*D + A2*(1-D);
B = B1*D + B2*(1-D);

T1 = [A B; [0 1] 0];
Gain = simplifyFraction(det(T1)/det(A));

```

Now, without using the boost circuit model, we analyse the gain  $\frac{V_o}{V_i}$  in terms of the duty cycle defined as a parameter  $p = D$ , from measurements. The problem is thus reduced to the problem of finding the coefficients of the function:

$$V_o(D) = \frac{\beta_0 + \beta_1 D}{\alpha_0 + D} V_i.$$

The coefficients  $\alpha_0$ ,  $\alpha_1$  and  $\beta_0$  for  $V_i$  constant, can be found using three different measurements of the parameter  $p_1 = D$  denoted  $D(1)$ ,  $D(2)$  and  $D(3)$  and, corresponding outputs  $V_o$  denoted  $V_o(1)$ ,  $V_o(2)$  and  $V_o(3)$ . We can thus solve the following linear system of equations:

$$\begin{bmatrix} V_i & D(1)V_i & -V_o(1) \\ V_i & D(2)V_i & -V_o(2) \\ V_i & D(3)V_i & -V_o(3) \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \alpha_0 \end{bmatrix} = \begin{bmatrix} D(1)V_o(1) \\ D(2)V_o(2) \\ D(3)V_o(2) \end{bmatrix}.$$

Suppose the duty cycle range of interest is  $[0.5 \ 0.7]$  for  $V_i = 10V$ ,  $R = 100\Omega$ ,  $L = 1mH$  and  $C = 1000\mu F$ . Using PSIM software, we found the values shown in the Table 6. Solving the algebraic system above, as expected, we found  $\beta_0 = -1$ ,  $\beta_1 = 0$  and  $\alpha_0 = -1$ , which yields:

$$V_o(D) = \frac{-10}{-1 + D}.$$

and the boost converter gain:

$$\frac{V_o(D)}{V_i} = \frac{1}{1 - D}$$

which is identical to the theoretical gain analysed before. As for the Thévenin's equivalent the boost converter without loss reduces to a ideal voltage source as the output voltage of DC-DC boost converter does not depend on the current at the terminal.

Table 6. PSIM simulation data varying one parameter.

$D$	$V_o$ [V]
0.500	20.0
0.600	25.0
0.700	33.3

Table 7. PSIM simulation data varying two parameters

$D$	$R$ [ $\Omega$ ]	$I_i$ [A]
0.500	100	0.408
0.500	150	0.275
0.550	130	0.387
0.600	125	0.515
0.650	120	0.704
0.700	100	1.14
0.700	150	0.771

Finally, we analyse how the output  $I_i$  relates to the parameters  $p_1 = D$  and  $p_2 = R$ . For this case, we can write:

$$I_i(D, R) = \frac{\beta_0 + \beta_1 D + \beta_2 R + \beta_3 DR}{\alpha_0 + \alpha_1 D + \alpha_2 R + DR}.$$

Using seven different set of values of  $D$ ,  $R$  and corresponding measurements of  $V_o$ , we can find the constants  $\beta_0, \beta_1, \beta_2, \beta_3, \alpha_0, \alpha_1$  and  $\alpha_2$  by solving the following linear system of equations:

$$\begin{bmatrix} 1 & D(1) & R(1) & D(1)R(1) & -I_i(1) & -I_i(1)D(1) & -I_i(1)R(1) \\ 1 & D(2) & R(2) & D(2)R(2) & -I_i(2) & -I_i(2)D(2) & -I_i(2)R(2) \\ 1 & d(3) & R(3) & D(3)R(3) & -I_i(3) & -I_i(3)D(3) & -I_i(3)R(3) \\ 1 & D(4) & R(4) & D(4)R(4) & -I_i(4) & -I_i(4)D(4) & -I_i(4)R(4) \\ 1 & D(5) & R(5) & D(5)R(5) & -I_i(5) & -I_i(5)D(5) & -I_i(5)R(5) \\ 1 & D(6) & R(6) & D(6)R(6) & -I_i(6) & -I_i(6)D(6) & -I_i(6)R(6) \\ 1 & D(7) & R(7) & D(7)R(7) & -I_i(7) & -I_i(7)D(7) & -I_i(7)R(7) \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} D(1)I_i(1)R(1) \\ D(2)I_i(2)R(2) \\ D(3)I_i(3)R(3) \\ D(4)I_i(4)R(4) \\ D(5)I_i(5)R(5) \\ D(6)I_i(6)R(6) \\ D(7)I_i(7)R(7) \end{bmatrix}.$$

Suppose the duty cycle of interest lies in the interval  $[0.5 \ 0.7]$  and the load range is  $[100\Omega \ 150\Omega]$  for  $V_{th} = V_i = 10V$ ,  $L = 1mH$  and  $C = 1000\mu F$ . Using PSIM software, we found the values shown in Table 7. Solving the linear system equations, we found  $\beta_0 = 14.905, \beta_1 = -32.868, \beta_2 = -0.117, \beta_3 = -0.148, \alpha_0 = 50.104, \alpha_1 = -69.554$  and  $\alpha_2 = -0.797$ . Now, using (8), it is possible to draw a surface relating  $d$  and  $R$  with  $I_i$  as shown in Fig. 11. From this surface, we know the current limits corresponding to each load in the

range considered for the the same duty cycle. *This can be used to support the selection of the boost components.*

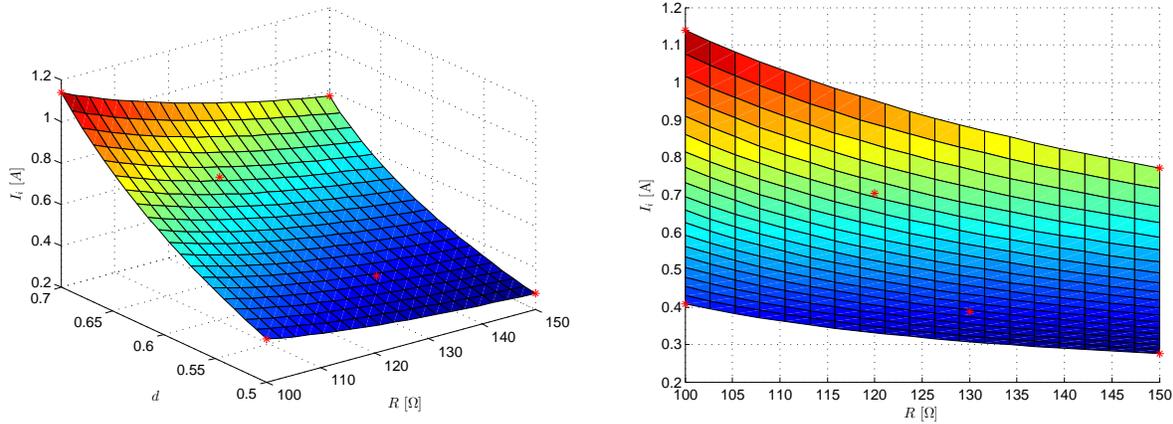


Figure 11. Relating the duty cycle parameter  $D$  and the load  $R$  of the circuit in Fig. 10 with current  $I_i$ . The red marked points are the measurements used. Note that extremal values of current  $I_i$  occur at the vertices.

*Real data are in general noisy with accuracy dependent on the tolerance of components and on the measurement device used. In [5] the analysis of the ill-posed problem arising in the measurement-based approach, with real experiments, was presented. The ill-posed problem should be handled with regularisation tools [20].*

### Conclusion

*In this tutorial we have shown that Cramer's rule applied to state space modeling of systems with parameters can lead to insightful results and computational tools. It also helps to unify apparently diverse types of models and thus can be a powerful educational tool. The use of Cramer's rule is simple and straightforward as it is based on the calculation of determinants. The students can understand the input-output-state description of systems using an attractive unified fashion. More importantly, the approach presented allows a practical use of input-output models via Cramer's rule as it can be used with measurements. Furthermore, a new way to calculate the Thévenin's equivalent without measuring the open circuit voltage and the short circuit current is given in detail and this opens up new insights for the design of data-based adaptive possibly control laws.*

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## A Simple Proof of Cramer's Rule

Consider the linear system equation

$$Az = b \tag{S1}$$

where  $A$  is  $n \times n$ ,  $z$  and  $b$  are  $n \times 1$ . Let  $z_k$  denote the  $k$ -th component of  $z$  and let  $A_k$  denote the  $n \times n$  matrix obtained by replacing the  $k$ -th column of  $A$  by  $b$ . Let  $|\cdot|$  denote determinant.

**Cramer's rule:** If  $|A| \neq 0$  then

$$z_k = \frac{|A_k|}{|A|}. \tag{S2}$$

*Proof.* Let  $e_i$  denote the  $i$ -th column of the identity matrix and

$$Z_k := [e_1 \ e_2 \ \cdots \ e_{k-1} \ z \ e_{k+1} \ \cdots \ e_n]. \tag{S3}$$

Then it is easy to see that (S1) is equivalent to

$$AZ_k = A_k. \tag{S4}$$

Taking the determinants of both sides of (S4) and using the fact that  $|Z_k| = z_k$ , we have

$$|A|z_k = |A_k| \tag{S5}$$

or

$$z_k = \frac{|A_k|}{|A|} \tag{S6}$$

which is (S2). □