Published in IET Control Theory and Applications
Received on 21st May 2009
Revised on 11th August 2009
doi: 10.1049/iet-cta.2009.0246

ISSN 1751-8644

Switched affine systems control design with application to DC–DC converters

G.S. Deaecto J.C. Geromel F.S. Garcia J.A. Pomilio
DSCE/School of Electrical and Computer Engineering, UNICAMP, CP 6101, 13081 – 970, Campinas, SP, Brazil
E-mail: grace@dsce.fee.unicamp.br

Abstract: This study presents a novel procedure for switched affine systems control design specially developed to deal with switched converters where the main goal is to attain a set of equilibrium points. The main contribution is on the determination of a switching function, which assures global stability and minimises a guaranteed quadratic cost. The implementation of the switching function taking into account only partial information is analysed and discussed with particular interest. The theoretical results are applied to buck, boost and buck–boost converters control design. Several simulations show the usefulness of the methodology and its favourable impact in a class of real-world control design problems.

1 Introduction

The last years have witnessed the crescent interest of the scientific community in the study of switched linear systems. They consist of a subclass of hybrid systems characterised by having a switching rule which selects, at each instant of time, a dynamic subsystem among a determined number of available ones. In general, the main goal is to design a switching strategy in order to guarantee closed-loop asymptotical stability with adequate guaranteed performance. The literature to date presents several important results in this research field, where the papers [1–3], and the books [4, 5] are surveys that treat these topics with deepness and particular attention to the most effective design techniques.

Roughly speaking, the techniques commonly used to study this kind of systems arise from the choice of distinct classes of Lyapunov functions, as for instance, quadratic [6–9], multiple [10–12], polyhedral [13] or piecewise quadratic ones [14–18], where the main difference among them is the level of conservativeness in the provided conditions. Some of the cited techniques seamed very effective to treat state feedback switched control design problems [7, 9, 11, 12, 14, 16, 17] and dynamic output feedback design problems [8, 9, 16], where some of these works consider robust control design as well. Despite the rapid progress made so far, many switched systems-related problems are still unexplored.

Motivated by the wide field of application in the real world, mainly in power electronics, we turned our attention to study switched affine systems with the following state-space realisation

$$\dot{x} = A_\sigma x + B_\sigma u, \quad x(0) = x_0 \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $\sigma(\cdot)$ is the switching strategy and $u(t) = u \in \mathbb{R}^m$ is an external input assumed to be constant for all $t \geq 0$. Notice that, when $u(t) \equiv 0$ the system, whenever globally asymptotically stable, presents an unique equilibrium point $x_e = 0$. In this case, as it can be seen by the references cited before, the literature is rich and provides results even for state and output feedback control design. It is interesting to notice that, depending on the switching strategy, when $u(t) \not\equiv 0$ the resulting affine system may have several equilibrium points composing a region in the state space. This aspect, important for practical applications, will be fully addressed in the sequel.

In this paper, our main goal is to calculate what we call the set of attainable equilibrium points defined as the state-space region composed by all points that can be reached from any initial condition by the proposed switched control technique. Our main goal is to calculate this set and design a switching rule in order to take any trajectory of the system to a desired point inside the mentioned set. In addition, this is done by minimising a quadratic in the state

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guaranteed cost to be defined later. Undoubtedly, this problem is more challenging than that defined with $u(t) \equiv 0$. Now, two points of concern are: (i) to find the set of attainable equilibrium points and (ii) to design a switching rule that guides the system trajectory to the desired equilibrium point. To the best of our knowledge, this problem is still unexplored, although it was cited before by [1] and [5], as motivation to study switched linear systems but, in the context presented here, has never been solved. However, it is worth mentioning the recent results of [19] dealing with convergence analysis of affine systems with bounded piecewise continuous inputs, that will be considered for comparison. The obtained theoretical results are based on a quadratic Lyapunov function and seemed very useful, in particular, for application in power electronics control design problems. In a first moment, the designer has to provide full information, that is the set of state variables at equilibrium. Afterwards, the possibility to implement the switching strategy with partial information, which represents a more realistic situation where only part of the equilibrium state variable is provided, will be treated and discussed. Three different and classical topologies of DC–DC converters, namely boost, buck and buck–boost, will be considered in detail. For these converters, our methodology has shown to be efficient in the sense that, under the full or partial information assumption, the switching strategy was able to make globally asymptotically stable all equilibrium points within a set that covers a wide range of the load voltage.

The paper is organised as follows. Section 2 presents the problem statement and some mathematical preliminaries needed for future developments. In Section 3, two switching strategies are designed. They differ from the fact that one of them is linear and easier to implement. Section 4 is devoted to design switching strategies for the three mentioned converters under full information. In Section 5, some relevant points on control design under partial information are addressed. We end the paper by a conclusion which puts in evidence the main contributions.

The notation used throughout is standard. The identity matrix of any dimension is denoted by I. For real matrices or vectors ($'$) indicates transpose. The convex combination of a set of matrices $\{A_1, \ldots, A_N\}$ is denoted by $A_\lambda = \sum_{i=1}^{N} \lambda_i A_i$, where λ belongs to the set Λ composed by all non-negative vectors such that $\sum_{i=1}^{N} \lambda_i = 1$. The set of square and Hurwitz real matrices, those with eigenvalues located in the left-hand side of the complex plane, is denoted by $\mathcal{H}$. Finally, the set composed by the first N positive integers, namely $\{1, \ldots, N\}$, is denoted by $\mathbb{K}$.

2 Problem statement and basic results

In this section, the problem to be dealt with is presented. The class of switched systems of interest is defined by the following state space realisation

$$\dot{x} = A_\sigma x + B_\sigma u, \quad x(0) = x_0 \tag{2}$$

$$z = C_\sigma x \tag{3}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) = u \in \mathbb{R}^m$ is the input supposed to be constant for all $t \geq 0$ and $z(t) \in \mathbb{R}^p$ is the controlled output. The switching rule $\sigma(t): t \geq 0 \to \mathbb{K}$ selects at each instant of time $t \geq 0$, a known subsystem among N available ones defined by

$$\mathcal{G}_i = \begin{bmatrix} A_i & B_i \\ C_i & 0 \end{bmatrix}, \quad i \in \mathbb{K} \tag{4}$$

where the matrices of each subsystem $\mathcal{G}_i$ have compatible dimensions. The control design problem to be addressed afterwards can be summarised as follows: Design a switching strategy $\sigma(x(t))$ for all $t \geq 0$ and determine the set of all equilibrium points $x_e \in \mathbb{R}^n$ denoted X_e that are attainable with such switching strategy, that is $x(t) \to x_e$ as $t \to \infty$. Of course, we would like to determine a switching strategy in order to be able to attain any point $x_e \in \mathbb{R}^n$, that is $X_e \equiv \mathbb{R}^n$. However, such a switching strategy often does not exist and, as a consequence, we have to determine the region $X_e \subset \mathbb{R}^n$ together with the associated switching strategy. In this framework, the practical usefulness of a proposed solution must be verified for each specific problem as it will be done in the sequel. Furthermore, ideally the final design requires the determination of $\sigma(\cdot)$ from the solution of the optimal control problem

$$\min_{\sigma} \int_0^{\infty} (z - C_\sigma x_e)'(z - C_\sigma x_e) \mathrm{d}t \tag{5}$$

for some given $x_e \in X_e$. Owing to the non-continuous nature of the switching function $\sigma(t)$, this problem is very hard to solve. Hence, we propose to replace it by a simpler problem which corresponds to minimise an upper bound of the objective function yielding a quadratic in the state guaranteed cost control problem. To ease the notation, throughout the paper we denote $Q_i = C_i' C_i \geq 0$ for all $i \in \mathbb{K}$. In the literature to date, several results dealing with the determination of a stabilising switching strategy are available, but in general, for the particular case of switched linear systems characterised by $u(t) \equiv 0$, which naturally imposes $X_e = \{0\}$. They differ from the use of different Lyapunov functions, as for instance multiple [1, 10], quadratic [6] and piecewise quadratic Lyapunov functions [15, 16], among others. An exception is [19] where the concept of quadratic convergence of affine systems is addressed by means of a quadratic Lyapunov function. In this paper, we restrict ourselves to use a quadratic Lyapunov function to establish the next theoretical results. This class of Lyapunov functions allows precise and simple comparisons with the already classical results on DC–DC converters design, which are based on some time approximation of the switching

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doi: 10.1049/iet-cta.2009.0246

strategy, known as the state variables averaging method, see [20]. The next lemma provides a well-known and important result on this subject.

Lemma 1: Consider the switched linear system (2) with $u(t) \equiv 0$ for all $t \geq 0$. If there exist $\lambda \in \Lambda$ and a symmetric positive-definite matrix $P \in \mathbb{R}^{n \times n}$ such that

$$A'_\lambda P + PA_\lambda < 0 \tag{6}$$

then the switching strategy $\sigma(x) = \arg\ \min_{i \in K} x' PA_i x$ makes the equilibrium point $x_e = 0$ globally asymptotically stable.

The proof of this result is simple. It suffices to consider the quadratic Lyapunov function $v(x) = x'Px$ whose time derivative with respect to an arbitrary trajectory of (2) with $u = 0$ provides

$$\begin{aligned} \dot{v}(x) &= x'(A'_\sigma P + PA_\sigma)x \\ &= \min_{i \in \mathbb{K}} x'(A'_i P + PA_i)x \\ &= \min_{\lambda \in \Lambda} x'(A'_\lambda P + PA_\lambda)x \\ &< 0, \quad \forall x \neq 0 \end{aligned} \tag{7}$$

where the last inequality follows from the existence of $\lambda \in \Lambda$ satisfying (6). A remarkable fact associated to this lemma is that the necessity also holds whenever $N = 2$, as has been proved in the interesting paper [6].

In the next section, we generalise this result in two directions. First, a constant input signal is taken into account in order to cope with affine models. As it will be clear afterwards, the behaviour of the closed-loop system with a state-dependent switching control strategy of the form $\sigma(x)$ is considerably richer in the sense that the set of attainable equilibrium points is not a single point anymore. Instead, it becomes curves or regions of the state-space $\mathbb{R}^n$. Second, a quadratic guaranteed cost associated to (5) is taken into account towards the switching strategy design.

3 State feedback switching control design

In this section, the state feedback switched control for system (2), (3) is designed. The main goal is to determine a set of equilibrium points $x_e \in \mathbb{R}^n$ denoted X_e such that the equality $\lim_{t \to \infty} x(t) = x_e$ holds for all initial condition $x_0 \in \mathbb{R}^n$ whenever the designed switching strategy $\sigma(x)$ is applied. The next two theorems provide the main results of this paper.

Theorem 1: Consider the switched affine system (2), (3) with constant input $u(t) = u$ for all $t \geq 0$ and let $x_e \in \mathbb{R}^n$ be given. If there exist $\lambda \in \Lambda$, and a symmetric positive-definite matrix $P \in \mathbb{R}^{n \times n}$ such that

$$A'_\lambda P + PA_\lambda + Q_\lambda < 0 \tag{8}$$

$$A_\lambda x_e + B_\lambda u = 0 \tag{9}$$

then the switching strategy

$$\sigma(x) = \arg \min_{i \in \mathbb{K}} \xi'(Q_i \xi + 2P(A_i x + B_i u)) \tag{10}$$

where $\xi = x - x_e$ makes the equilibrium point $x_e \in \mathbb{R}^n$ globally asymptotically stable and the guaranteed cost

$$\int_0^\infty (z - C_\sigma x_e)'(z - C_\sigma x_e)\, \mathrm{d}t < (x_0 - x_e)' P (x_0 - x_e) \tag{11}$$

holds.

Proof: Considering the switching strategy (10) and adopting the quadratic Lyapunov function $v(\xi) = \xi' P \xi$, its time derivative along an arbitrary trajectory of the switched system (2), (3) satisfies

$$\begin{aligned} \dot{v}(\xi) &= \dot{x}' P \xi + \xi' P \dot{x} \\ &= 2\xi' P(A_\sigma x + B_\sigma u) \\ &= \min_{i \in K} \xi'(Q_i \xi + 2P(A_i x + B_i u)) - \xi' Q_\sigma \xi \\ &= \min_{i \in K} (\xi'(A'_i P + PA_i + Q_i)\xi \\ &\quad + 2\xi' P(A_i x_e + B_i u)) - \xi' Q_\sigma \xi \\ &= \min_{\lambda \in \Lambda} (\xi'(A'_\lambda P + PA_\lambda + Q_\lambda)\xi \\ &\quad + 2\xi' P(A_\lambda x_e + B_\lambda u)) - \xi' Q_\sigma \xi \\ &< -\xi' Q_\sigma \xi \end{aligned} \tag{12}$$

where the fourth equality follows from the third one by substituting $x = \xi + x_e$ and the last inequality follows from the conditions (8), (9). Since $\dot{v}(\xi) < 0$ for all $\xi \neq 0 \in \mathbb{R}^n$, the conclusion is that x_e is a globally asymptotically stable equilibrium point. Moreover, integrating (12) from zero to infinity and taking into account that $v(\xi(\infty)) = 0$ we obtain (11) and the proof is concluded. □

It is interesting to observe that even in the particular case $Q_i = 0$ for all $i \in \mathbb{K}$ where only asymptotical stability is addressed, the switching function given in Theorem 1 is in general quadratic with respect to the state variable $x \in \mathbb{R}^n$. Moreover, whenever $Q_i = 0$ for all $i \in \mathbb{K}$ and $u = 0$ this theorem reduces to the result of Lemma 1. A simpler linear switching strategy will be given in the sequel. For the moment, the result of Theorem 1 admits the following considerations. The first one concerns the way we have to proceed in order to solve the design conditions (8), (9), which requires the determination of a specific vector $\lambda \in \Lambda$

doi: 10.1049/iet-cta.2009.0246
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associated to an equilibrium point x_e. Noticing that the inequality (8) imposes A_λ asymptotically stable, all x_e satisfying (9) constitute the set

$$X_e = \{-A_\lambda^{-1} B_\lambda u : A_\lambda \in \mathcal{H}, \forall \lambda \in \Lambda\} \tag{13}$$

which can be numerically determined. Only points $x_e \in X_e$ can be reached by the switching strategy provided in Theorem 1. Hence, for a given $x_e \in X_e$ the associated value $\lambda \in \Lambda$ is determined and, afterwards, a feasible solution for the Lyapunov inequality (8) is calculated. Since by construction $A_\lambda \in \mathcal{H}$, matrix $P > 0$ is readily obtained from

$$P = \int_0^\infty e^{A_\lambda' t}(Q_\lambda + S)e^{A_\lambda t}\, dt \tag{14}$$

where $S > 0$ is an arbitrary matrix with compatible dimensions. From this, all parameters required for the implementation of the switching strategy (10) follow.

Another relevant point is on the interpretation of Theorem 1 in the light of the well-known state variable averaging method reported in [20]. To this end, let us define the auxiliary dynamic system

$$\dot\eta = A_\lambda \eta + B_\lambda u, \quad x(0) = x_0 \tag{15}$$

$$\zeta = C_\lambda \eta \tag{16}$$

where we notice that $A_\lambda \in \mathcal{H}$ enforces $\eta(t) \to x_e \in X_e$ as $t \to \infty$. Moreover, simple calculations put in evidence that

$$\begin{aligned}\int_0^\infty (\zeta - C_\lambda x_e)'(\zeta - C_\lambda x_e)\, dt &= \int_0^\infty (x_0 - x_e)' e^{A_\lambda' t} C_\lambda' C_\lambda e^{A_\lambda t} \\ &\quad \times (x_0 - x_e)\, dt \\ &\le \int_0^\infty (x_0 - x_e)' e^{A_\lambda' t} Q_\lambda e^{A_\lambda t} \\ &\quad \times (x_0 - x_e)\, dt \\ &< (x_0 - x_e)' P (x_0 - x_e)\end{aligned} \tag{17}$$

where the first inequality follows from the fact that $C_\lambda' C_\lambda \le Q_\lambda$ for all $\lambda \in \Lambda$ and the second one is a direct consequence of $S > 0$ in (14). The important conclusion is that, with no approximation of any kind, the affine time invariant system (15), (16) admits the same steady-state solution $x_e \in X_e$ and the same guaranteed cost for the transition from $\eta(0) = x_0$ to $\eta(\infty) = x_e \in X_e$ than the switched affine system under consideration controlled with the switching strategy provided by Theorem 1. In our opinion, this gives a precise and favourable measure of the theoretical results obtained so far, as far as control design of power electronics converters is concerned. The next theorem provides a linear switching strategy, simpler to be implemented in practice.

Theorem 2: Consider the switched affine system (2), (3) with constant input $u(t) = u$ for all $t \ge 0$ and let $x_e \in \mathbb{R}^n$ be given. If there exist $\lambda \in \Lambda$, and a symmetric positive-definite matrix $P \in \mathbb{R}^{n\times n}$ such that

$$A_i' P + P A_i + Q_i < 0 \tag{18}$$

$$A_\lambda x_e + B_\lambda u = 0 \tag{19}$$

for all $i \in \mathbb{K}$, then the switching strategy

$$\sigma(x) = \arg\min_{i\in\mathbb{K}} \xi' P(A_i x_e + B_i u) \tag{20}$$

where $\xi = x - x_e$ makes the equilibrium point $x_e \in \mathbb{R}^n$ globally asymptotically stable and the guaranteed cost (11) holds.

Proof: Considering again the quadratic Lyapunov function $v(\xi) = \xi' P \xi$, its time derivative along an arbitrary trajectory of the switched system (2), (3) controlled with the switching rule (20), satisfies

$$\begin{aligned}\dot v(\xi) &= \dot x' P \xi + \xi' P \dot x \\ &= 2\xi' P(A_\sigma x + B_\sigma u) \\ &= 2\xi' P(A_\sigma x_e + B_\sigma u) + \xi'(A_\sigma' P + P A_\sigma)\xi \\ &= \min_{i\in\mathbb{K}} (2\xi' P(A_i x_e + B_i u)) + \xi'(A_\sigma' P + P A_\sigma)\xi \\ &< \min_{\lambda\in\Lambda} (2\xi' P(A_\lambda x_e + B_\lambda u)) - \xi' Q_\sigma \xi \\ &< -\xi' Q_\sigma \xi\end{aligned} \tag{21}$$

where the first inequality follows from the fact that $A_\sigma' P + P A_\sigma < -Q_\sigma$ for all $\sigma \in \mathbb{K}$ as an immediate consequence of (18) and the last inequality is due to (19). Finally, by simple integration, the inequality (11) is obtained and the proof is concluded. □

Notice that when $u = 0$, the global asymptotical stability is assured for an arbitrary switching rule. When compared to (10) the switching strategy provided by Theorem 2 is simpler and more amenable for practical purposes because the switching function is linear. However, the price to be payed is that the conditions (18) are much more stringent than (8). While (8) is a classical Lyapunov inequality the condition (18) represents a set of N LMIs with respect to a single matrix variable $P > 0$, being so more difficult to solve. Whenever Theorem 2 admits a solution, it is preferable to be used than Theorem 1. The main reason is that in the last, the matrix $P > 0$ does not depend on each $\lambda \in \Lambda$ associated to $x_e \in X_e$. Hence, for any equilibrium point $x_e \in X_e$ selected by the designer the same $P > 0$ and, consequently, the same switching strategy (20) is used.

An interesting property regarding piecewise affine systems is the quadratic convergence, fully addressed in [19]. For comparison purpose, restricting our attention to bimodal

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doi: 10.1049/iet-cta.2009.0246

switched systems characterised by $N = 2$ and $Q_1 = Q_2 = 0$, the linear switching strategy (20) allows us to rewrite the system (2) as

$$\dot{\xi} = \begin{cases} A_1\xi + b_1, & c_e'\xi \leq 0 \\ A_2\xi + b_2, & c_e'\xi > 0 \end{cases} \tag{22}$$

where $\xi = x - x_e$, $b_i = A_i x_e + B_i u$ for $i = 1, 2$ and $c_e = P(b_1 - b_2)$. Hence, applying Theorem 2 for any $x_e \in X_e$, the existence of a common positive-definite solution of the Lyapunov inequalities (18) is sufficient to conclude that the origin $\xi = 0$ is globally asymptotically stable. On the other hand, from the necessary and sufficient condition given in ([19], p. 1238), we cannot say that this affine system is quadratically convergent unless the supplementary condition $A_1 - A_2 = Gc_e'$ for some $G \in \mathbb{R}^n$ is also satisfied. As expected, the less restrictive result of Theorem 2, valid exclusively for affine switched systems with constant input, is not sufficient to assure quadratic convergence which, by definition, imposes an adequate behaviour of the system response for any bounded piecewise continuous input.

The next section is devoted to apply the theoretical results obtained so far to specific models of three classical DC–DC converters. Fortunately, as it will be seen, all of them satisfy the design conditions preconised by Theorem 2.

4 DC–DC converters design

In this section, the three DC–DC converters of interest are modelled as switched affine systems that consist of a group of $N = 2$ affine subsystems sharing the same state variables. At any instant of time only one of the subsystems determines the evolution of the states and this subsystem is said to be active. The decision of which subsystem is active is the control variable, resulting in a switching rule $\sigma(x(t)) \in \{1, 2\}$. This approach considers that the controller is able to determine the active subsystem, so converters that operate in discontinuous conduction mode are not considered here. In all converters under consideration, i_L denotes the inductor current, v_C denotes the capacitor voltage and they are the elements of the state variable $x = [i_L\ v_C]'$. The converters are modelled by two subsystems corresponding to each position of the switches that operate complementarily and depend on five parameters that, for simulation purpose, we have considered the following nominal values: $u = 100$ V, $R = 2\,\Omega$, $L = 500\,\mu$H, $C_o = 470\,\mu$F and $R_o = 50\,\Omega$.

In the present design framework, we propose to solve the guaranteed cost problem associated to

$$\min_{\sigma} \int_0^{\infty} R_o^{-1}(v_C - v_e)^2 + \rho R(i_L - i_e)^2\, dt \tag{23}$$

where $x_e = [i_e\ v_e]' \in X_e$ is an attainable equilibrium point and $\rho \geq 0$ is a parameter, both defined by the designer. The index appearing in (23) expresses the weighted sum of the energy of the error signal of each state variable, the error being taken relatively to the respective level defined by the chosen equilibrium point. It has been verified that the non-negative parameter $\rho \in \mathbb{R}$ plays a central role as far as the voltage transitory duration and the current peak value are concerned. Indeed, for $\rho \gg 1$ the current peak is reduced at the expense of a slower output voltage convergence to the equilibrium value. On the contrary, for $\rho \ll 1$, the voltage convergence is faster but a greater current peak generally occurs.

In this paper, we will consider the extreme case $\rho = 0$ which from (5) leads to

$$Q_1 = Q_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1/R_o \end{bmatrix} \tag{24}$$

The determination of the matrix $P > 0$, necessary to the implementation of the switching strategy given in Theorem 2, is done from the solution of the following convex programming problem

$$\inf_{P>0} \{\mathrm{Tr}(P) : A_i'P + PA_i + Q_i < 0, \forall i \in \mathbb{K}\} \tag{25}$$

which can be numerically handled by any LMI solver available in the literature, see [21]. The objective function of problem (25) corresponds to that of the right-hand side of (11) with the (unknown) vector $x_0 - x_e$ assumed to be uniformly distributed over the unit sphere. Furthermore, for the three DC–DC converters to be analysed in the sequel, problem (25) is always feasible because the energy Lyapunov matrix

$$P_o = \begin{bmatrix} L/2 & 0 \\ 0 & C_o/2 \end{bmatrix} \tag{26}$$

yields $P = \epsilon P_o$, which satisfies all constraints whenever $\epsilon > 1$. Unfortunately, it is important to mention that, in general, the matrix P_o may be far from the global optimal solution of the convex programming problem (25). Unless in the particular cases where $A_i = A\ \forall i \in \mathbb{K}$, the determination of the optimal solution is not trivial and, as already mentioned, requires the use of an appropriate numerical routine. Finally, we want to stress that for $N = 2$ the switching strategy (20) is particularly simple to be implemented. Actually, it can be written as

$$\sigma(x) = \begin{cases} 1 & \text{if } c_e'(x - x_e) \leq 0 \\ 2 & \text{if } c_e'(x - x_e) > 0 \end{cases} \tag{27}$$

where $c_e = P((A_1 - A_2)x_e + (B_1 - B_2)u)$. The actual value of $\sigma(x(t))$ is decided by verifying in which subspace defined by the hyperplane $c_e'(x - x_e) = 0$ passing through $x_e \in X_e$ the state vector $x(t)$ at time $t \geq 0$ belongs to. Needless to say that the vector c_e may depend on the equilibrium point x_e and must be recalculated whenever a different equilibrium point is selected. We are now in a position to analyse each DC–DC converter of interest. It is worth to be pointed out that no limit is imposed to the switching frequency; consequently, when the system trajectory evolves on a sliding mode, the

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switching signal changes very fast between its possible values. Therefore it is not possible to properly visualise the switching signal, which is omitted in the simulations.

4.1 Buck converter

The top of Fig. 1 shows the structure of a buck converter, which allows just output voltage magnitude lower than the input voltage. From the definition of the state variable, the switched system state-space model (2), (3) is defined by the following matrices

$$A_1 = A_2 = \begin{bmatrix} -R/L & -1/L \\ 1/C_o & -1/R_oC_o \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 1/L \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{28}$$

Before all, the set of all attainable equilibrium points is calculated as being

$$X_e = \{(i_e, v_e): v_e = R_oi_e, 0 \le i_e \le u/(R_o + R)\} \tag{29}$$

which is a line segment practically defined only by the load since $R \ll R_o$. The solution of problem (25) yields the vector

$$c_e = \begin{bmatrix} 0.0171 \\ 0.0128 \end{bmatrix} u \tag{30}$$

needed for the implementation of the switching strategy (27). It is interesting to notice that for this particular converter, the gradient of the switching surface does not depend on the equilibrium point $x_e \in X_e$. The bottom of Fig. 1 shows in solid lines some trajectories that start from zero initial condition and reach the equilibrium points with voltages $v_e = \{10, 20, \ldots, 90\}$ V belonging to the set X_e, viewed in dotted line. As it can be seen the proposed switching control is very effective. In all cases the transitory period was less than 5 ms but at the expense of current peaks near 40 A of magnitude.

4.2 Boost converter

The top of Fig. 2 shows a bidirectional boost converter feeding the resistive load R_o. The switched system state-space model (2), (3) is defined by the following matrices

$$A_1 = \begin{bmatrix} -R/L & 0 \\ 0 & -1/R_oC_o \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -R/L & -1/L \\ 1/C_o & -1/R_oC_o \end{bmatrix} \tag{31}$$

$$B_1 = B_2 = \begin{bmatrix} 1/L \\ 0 \end{bmatrix} \tag{32}$$

There is no difficulty to express the set of all attainable equilibrium points as

$$X_e = \{(i_e, v_e): u/(R + R_o) \le i_e \le u/R,$$
$$v_e^2 + (RR_o)i_e^2 - (R_ou)i_e = 0\} \tag{33}$$

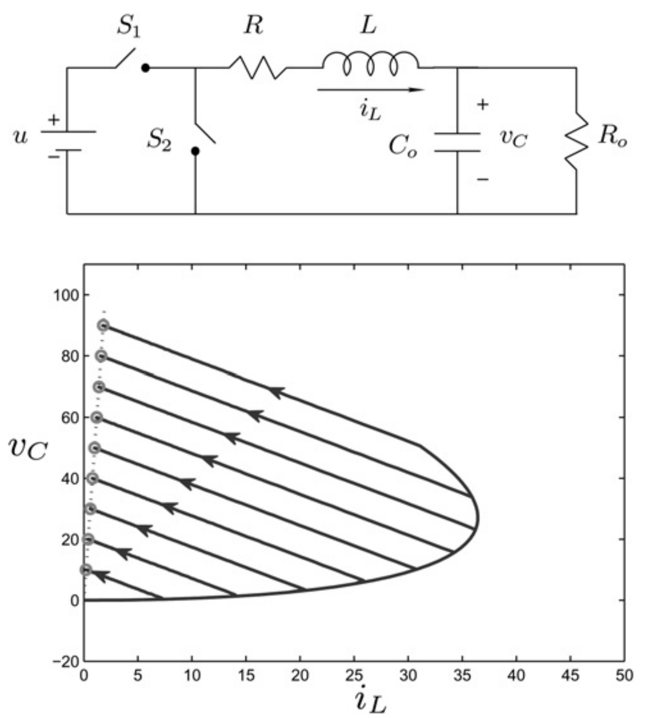

Figure 1 *Buck circuit and phase plane*

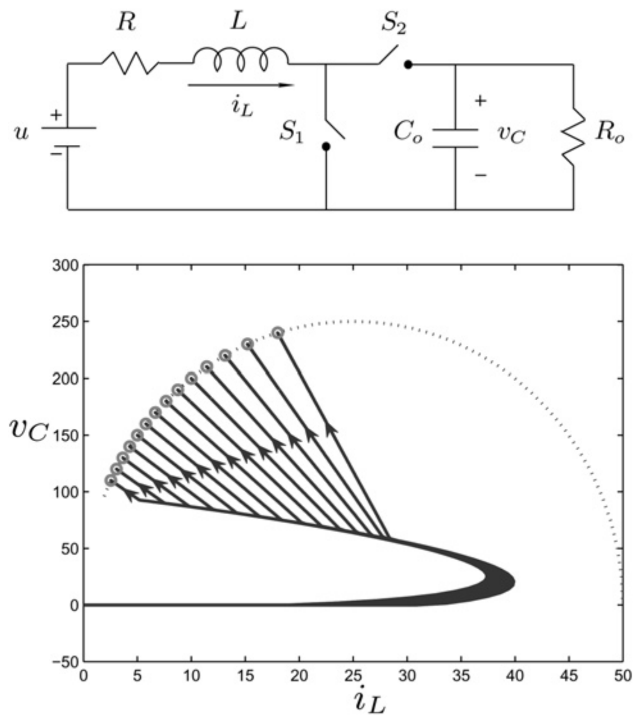

Figure 2 *Boost circuit and phase plane*

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IET Control Theory Appl., 2010, Vol. 4, Iss. 7, pp. 1201–1210
doi: 10.1049/iet-cta.2009.0246

which makes apparent that the equilibrium voltage attained by the proposed switched control law belongs to $0 \le v_e \le (\sqrt{R_o/4R})u$, which is an adequate voltage interval. Indeed, this is more than necessary because, in practice, the equilibrium current is limited to the region $u/(R+R_o) \le i_e \le u/2R$ enforcing $R_o u/(R+R_o) \le v_e \le (\sqrt{R_o/4R})u$. As already remarked we always have $R \ll R_o$ and consequently $v_e \ge u$ as a characteristic of this class of converters. From the optimal solution of problem (25), we obtain

$$c_e = \begin{bmatrix} -0.1182 & 0.5726 \\ -1.0506 & 0.1111 \end{bmatrix} x_e \quad (34)$$

used for the implementation of the switching strategy (27). The bottom of Fig. 2 shows the phase plane of the boost converter evolving from zero initial condition. In solid lines, we show the closed-loop system trajectories towards the equilibrium points corresponding to the following values of load voltage $v_e = \{110, 120, \ldots, 240\}$ V. These points of the set X_e given in (33) are shown in dotted line. In this case, the transitory period was less than 60 ms for the maximum current peak near 40 A of magnitude.

4.3 Buck–boost converter

The top of Fig. 3 shows a buck–boost converter feeding the resistive load R_o. The switched system state-space model (2),

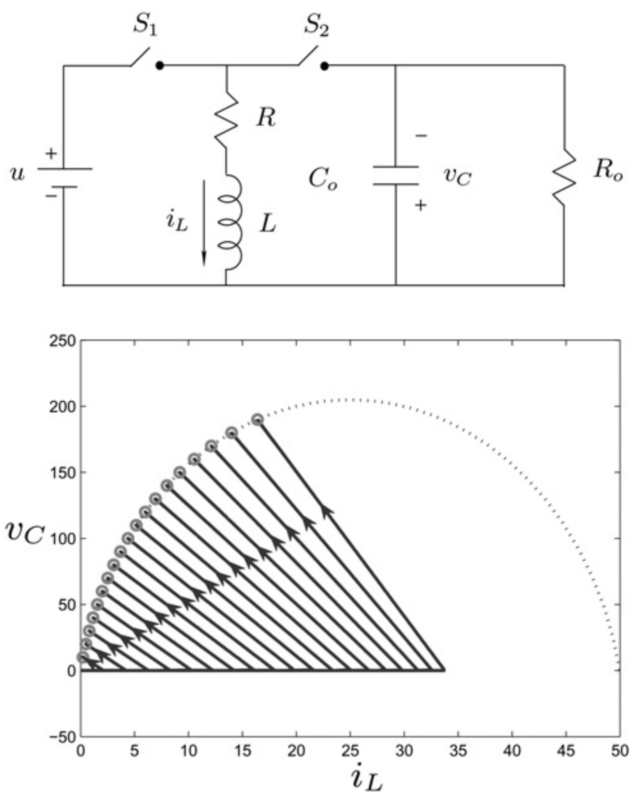

Figure 3 *Buck–boost circuit and phase plane*

(3) is defined by the following matrices

$$A_1 = \begin{bmatrix} -R/L & 0 \\ 0 & -1/R_oC_o \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -R/L & -1/L \\ 1/C_o & -1/R_oC_o \end{bmatrix} \quad (35)$$

$$B_1 = \begin{bmatrix} 1/L \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (36)$$

With a little more difficulty, the set of all attainable equilibrium points are calculated as being

$$X_e = \{(i_e, v_e) : 0 \le v_e \le R_o i_e,$$
$$v_e^2 + (RR_o)i_e^2 - (R_o u)i_e + uv_e = 0\} \quad (37)$$

from which it is verified that the attained voltage range is approximatively the same as in the previous case, that is $0 \le v_e \le (\sqrt{R_o/4R})u$, whenever the resistance of the load and the source satisfy $R \ll R_o$. Moreover, the optimal solution of problem (25) is the same as that of the boost converter, yielding

$$c_e = \begin{bmatrix} -0.1182 & 0.5726 \\ -1.0506 & 0.1111 \end{bmatrix} x_e + \begin{bmatrix} 0.5726 \\ 0.1111 \end{bmatrix} u \quad (38)$$

and by consequence the switching strategy (27). As before, the bottom part of Fig. 3 shows in solid lines the phase plane trajectories of the closed-loop system evolving from the origin. Notice in dotted line the equilibrium points of interest belonging to X_e and corresponding to $v_e = \{10, 20, \ldots, 190\}$ V. In this case, the converter has a transitory behaviour very close to that of the boost converter, namely, the transitory period was approximatively 60 ms for the maximum current peak near 35 A of magnitude.

The previous simulations show that the switching strategy is simple and is very effective to control three important classes of converters. It is linear and depends on the two coordinates of the equilibrium point $x_e = [i_e\, v_e]' \in X_e$ which must be provided by the designer. However, in practice, it would be important to work with partial information, a problem that will be stated and solved in the next section.

5 Partial information

A problem of great practical interest is the one raised from the possibility to implement the switching strategy based on partial information, see [22]. For instance, the equilibrium output voltage v_e of the converters already treated is precisely defined by the user who does not care about the corresponding current inductor i_e. In the framework of Theorem 1 applied to system (2), (3) it consists in making $\sigma(x)$ given in (10) independent of part of the equilibrium vector $x_e \in X_e$, that is, independent of the equilibrium

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current i_e. For simplicity of exposition we consider $Q_i = Q \geq 0$ for all $i \in \mathbb{K}$. The approach presented in [22] is adopted here. An idea is to introduce a low-pass filter with transfer function $F(s) = 1/(\tau s + 1)$ in order to estimate the steady-state value of the current $i_{\rm L}$. Choosing $\tau = R_o C_o$ the new state variable becomes $x = [i_{\rm L}\, v_{\rm C}\, \hat{i}_{\rm L}]'$ and the augmented switched system state-space model (2), (3) is defined by the following matrices

$$\mathcal{A}_1 = \begin{bmatrix} -R/L & 0 & 0 \\ 0 & -1/R_o C_o & 0 \\ 1/R_o C_o & 0 & -1/R_o C_o \end{bmatrix},$$

$$\mathcal{A}_2 = \begin{bmatrix} -R/L & -1/L & 0 \\ 1/C_o & -1/R_o C_o & 0 \\ 1/R_o C_o & 0 & -1/R_o C_o \end{bmatrix} \tag{39}$$

$$\mathcal{B}_1 = \begin{bmatrix} 1/L \\ 0 \\ 0 \end{bmatrix}, \quad \mathcal{B}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{40}$$

and $Q_1 = Q_2 = \text{diag}\{0, 1/R_o, 0\}$. The model of each converter is given by $(A_1, A_2, B_1, B_2) = (\mathcal{A}_2, \mathcal{A}_2, \mathcal{B}_1, \mathcal{B}_2)$ for buck, $(A_1, A_2, B_1, B_2) = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_1, \mathcal{B}_1)$ for boost and $(A_1, A_2, B_1, B_2) = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_1, \mathcal{B}_2)$ for buck–boost. The interesting issue in this case is that the set of equilibrium points X_e provided by Theorem 1 is composed by points of the form $x_e = [i_e\, v_e\, i_e]'$. Hence, setting the linear constraints

$$JP[A_i \quad B_i] = V, \quad \forall i \in \mathbb{K} \tag{41}$$

with respect to the Lyapunov matrix $P \in \mathbb{R}^{3\times 3}$ and the matrix variable $V \in \mathbb{R}^{1\times 4}$, where $J = [1\,0\,1]$, it is seen that the switching strategy $\sigma(x) = \arg\min_{i\in\mathbb{K}} (x - x_e)'\ P(A_i x + B_i u)$ does not depend on the equilibrium current i_e. Indeed, defining the vector $P(A_i x + B_i u) = [f_i(x)\, g_i(x)\, h_i(x)]'$, multiplying (41) to the right by $[x'\, u']'$ we conclude that $f_i(x) + h_i(x) = V[x'\, u']'$ for all $i \in \mathbb{K}$, that is the sum of these functions does not depend on the index $i \in \mathbb{K}$. Consequently, we obtain

$$\begin{aligned} \sigma(x) &= \arg\min_{i\in\mathbb{K}} (x - x_e)' P(A_i x + B_i u) \\ &= \arg\min_{i\in\mathbb{K}} (i_{\rm L} - i_e) f_i(x) + (v_{\rm C} - v_e) g_i(x) \\ &\quad + (\hat{i}_{\rm L} - i_e) h_i(x) \\ &= \arg\min_{i\in\mathbb{K}} (i_{\rm L} - \hat{i}_{\rm L}) f_i(x) + (v_{\rm C} - v_e) g_i(x) \end{aligned} \tag{42}$$

We notice that the functions $f_i(x)$ and $g_i(x)$ are linear with respect to $x \in \mathbb{R}^3$, which makes the switching surface quadratic. This switching strategy is globally asymptotically stable. For a given load voltage level v_e such that the corresponding equilibrium point x_e is attainable, Theorem 1 assures that for any initial condition $v_{\rm C}(t) \to v_e$ as $t \to \infty$. Moreover, due to the fact that only the error $v_{\rm C} - v_e$ is present in the objective function, this switching strategy generally imposes to the closed-loop system a small transitory in the load voltage at the expense of a greater one in the current error $i_{\rm L} - \hat{i}_{\rm L}$. This aspect will be illustrated afterwards. For the moment, we have to make explicit the determination of the switching strategy (42). Clearly, from Theorem 1 it follows that the solution of the convex

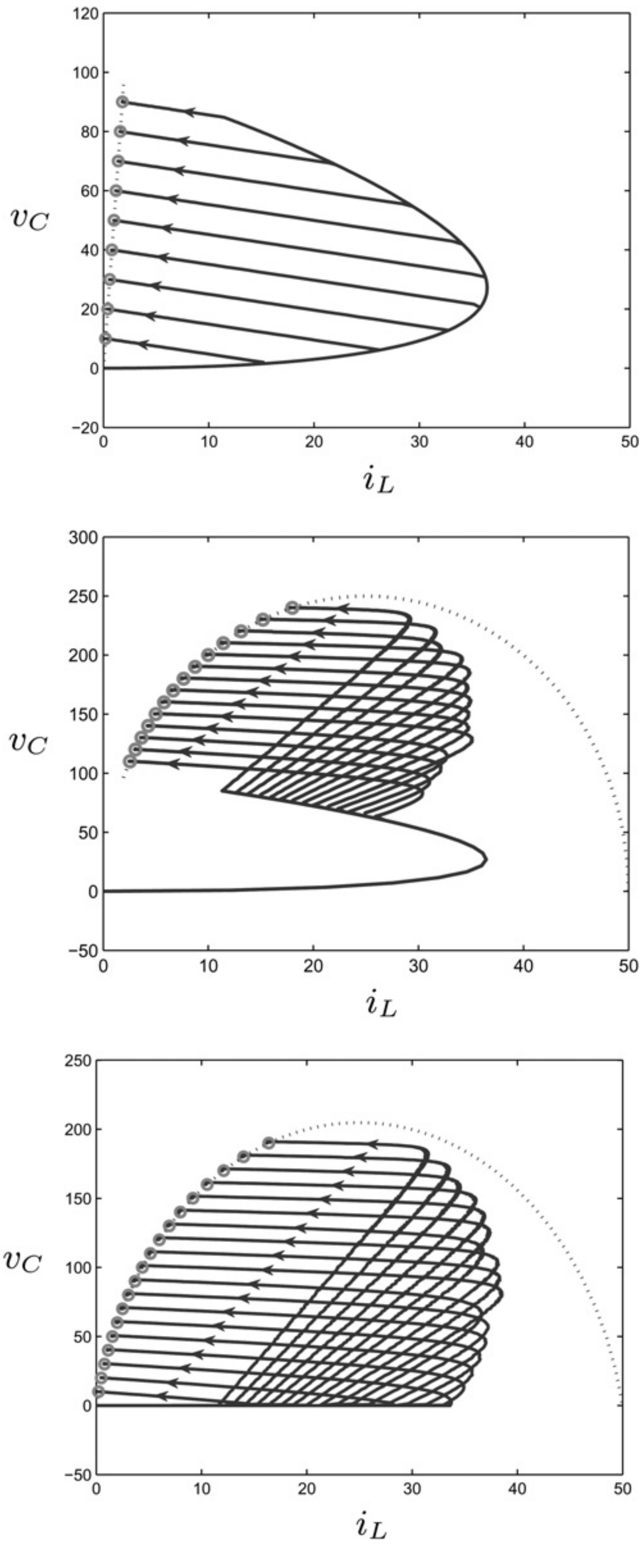

Figure 4 *Converters operating under partial information*

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IET Control Theory Appl., 2010, Vol. 4, Iss. 7, pp. 1201–1210
doi: 10.1049/iet-cta.2009.0246

programming problem

$$\inf_{(P,V)\in\Omega} \{\mathrm{Tr}(P) : A'_\lambda P + PA_\lambda + Q_\lambda \leq 0\} \tag{43}$$

where Ω is the set of all pairs (V, P) with $P > 0$ satisfying the equality constraints (41) and $\lambda \in \Lambda$ is associated to the desired equilibrium point $x_e \in X_e$. The inequality in (43) is not strict as indicated in (8) because the constraints (41) generally impose $JPA_\lambda J' = 0$. Indeed, it can be verified that this is precisely the case of the boost and buck–boost converters. As a consequence, to preserve feasibility of (43), we must have $JQ_iJ' = 0$ for all $i \in \mathbb{K}$, which is true for the matrices under consideration. Applying Theorem 1, these theoretical aspects led us to conclude that the time derivative of the quadratic Lyapunov function $v(\xi) = \xi' P\xi$ satisfies

$$\dot{v}(\xi) \leq -\xi' Q_\sigma \xi \tag{44}$$

and $\dot{v}(\xi) = 0$ for all $\xi \in \mathbb{R}^3$ of the form $\xi = J'\alpha$ for some $\alpha \in \mathbb{R}$, that is for all points of the state-space $\xi = [i_L - i_e \; v_C - v_e \; \hat{i}_L - i_e]'$ such that $i_L = \hat{i}_L = i_e + \alpha$ and $v_C = v_e$. On the other hand, since the filter output imposes $d\hat{i}_L/dt = 0$ whenever $i_L = \hat{i}_L$ and $v_C = v_e$ enforces $i_L = \hat{i}_L = i_e$, the only trajectory such that $\dot{v}(\xi) = 0$ is actually that defined by the equilibrium point $x_e \in X_e$ chosen by the designer.

Fig. 4 shows, from the top to the bottom, the phase plane of buck, boost and buck–boost converters operating under partial information and starting from zero initial conditions. In all cases, the time needed for the output voltage $v_C(t)$ be close enough to the equilibrium voltage v_e was approximatively the same than the one observed under full information. This fact, together with the phase plane trajectories that clearly indicate that all desired equilibrium points have been reached, put in evidence the usefulness and practical appealing of the proposed switching strategy design for switched affine systems.

6 Conclusion

In this paper, we have addressed the problem of designing a switching strategy for switched affine systems. Two different solutions have been proposed providing switching rules that are linear and quadratic with respect to the state-space vector supposed to be available for feedback. In both cases a quadratic guaranteed cost have been minimised. Moreover, problems with full information and partial information structures have been considered. These structures differ from one another by the important aspect that, in the second case, implementation constraints of the switching strategy are explicitly taken into account. Although it is possible to say that the theoretical results involving affine dynamic systems can be adopted in various real-world problems, due to its intrinsic practical importance, three classical DC–DC converters control design have been chosen for illustration. In this framework, it has been possible to give a new interpretation of the well-established state variable averaging method. Several simulations involving three types of converters namely buck, boost and buck–boost have shown the simplicity, quality and usefulness of the design methodology proposed.

7 Acknowledgment

This research was supported by grants from 'Conselho Nacional de Desenvolvimento Científico e Tecnológico - CNPq' and 'Fundação de Amparo à Pesquisa do Estado de São Paulo - FAPESP', Brazil.

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IET Control Theory Appl., 2010, Vol. 4, Iss. 7, pp. 1201–1210
doi: 10.1049/iet-cta.2009.0246