4 Vector Geometry

SECTION 4.1 Vectors and Lines

In this chapter we study the geometry of 3-dimensional space. We view a point in 3-space as an arrow from the origin to that point. Doing so provides a "picture" of the point that is truly worth a thousand words. We used this idea earlier, in Section 2.6, to describe rotations, reflections, and projections of the plane $\mathbb{R}^2$. We now apply the same techniques to 3-space to examine similar transformations of $\mathbb{R}^3$. Moreover, the method enables us to completely describe all lines and planes in space.

Vectors in $\mathbb{R}^3$

Introduce a coordinate system in 3-dimensional space in the usual way. First choose a point O called the *origin*, then choose three mutually perpendicular lines through O, called the x, y, and z *axes*, and establish a number scale on each axis with zero at the origin. Given a point P in 3-space we associate three numbers x, y, and z with P, as described in Figure 1. These numbers are called the *coordinates* of P, and we denote the point as (x, y, z), or $P(x, y, z)$ to emphasize the label P. The result is called a *cartesian*[1] coordinate system for 3-space, and the resulting description of 3-space is called *cartesian geometry*.

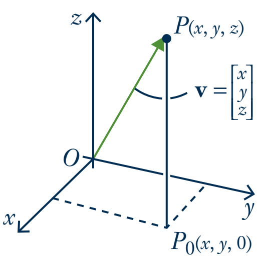


FIGURE 1

As in the plane, we introduce vectors by identifying each point $P(x, y, z)$ with the vector $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$, represented by the **arrow** from the origin to P as in Figure 1. Informally, we say that the point P *has vector* $\mathbf{v}$, and that vector $\mathbf{v}$ *has point* P. In this way 3-space is identified with $\mathbb{R}^3$, and this identification will be made throughout this chapter, often without comment. In particular, the terms "vector" and "point" are interchangeable.[2] The resulting description of 3-space is called **vector geometry**. Note that the origin is $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Length and Direction

We are going to discuss two fundamental geometric properties of vectors in $\mathbb{R}^3$: length and direction. First, if $\mathbf{v}$ is a vector with point P, the **length** $\|\mathbf{v}\|$ of vector

1 Named after René Descartes who introduced the idea in 1637.

2 Recall that we defined $\mathbb{R}^n$ as the set of all ordered n-tuples of real numbers, and reserved the right to denote them as rows or as columns.

$\mathbf{v}$ is defined to be the distance from the origin to P, that is the length of the arrow representing $\mathbf{v}$. The following properties of length will be used frequently.

Theorem 1

Let $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ *be a vector.*

(1) $\|\mathbf{v}\| = \sqrt{x^2 + y^2 + z^2}$.[3]

(2) $\mathbf{v} = \mathbf{0}$ *if and only if* $\|\mathbf{v}\| = 0$

(3) $\|a\mathbf{v}\| = |a|\,\|\mathbf{v}\|$ *for all scalars* a.[4]

PROOF

Let v have point $P = (x, y, z)$.

(1) In Figure 2, $\|\mathbf{v}\|$ is the hypotenuse of the right triangle OQP, and so $\|\mathbf{v}\|^2 = h^2 + z^2$ by Pythagoras' theorem.[5] But h is the hypotenuse of the right triangle ORQ, so $h^2 = x^2 + y^2$. Now (1) follows by eliminating h^2 and taking positive square roots.

(2) If $\|\mathbf{v}\| = 0$, then $x^2 + y^2 + z^2 = 0$ by (1). Because squares of real numbers are nonnegative, it follows that $x = y = z = 0$, and hence that $\mathbf{v} = \mathbf{0}$. The converse is because $\|\mathbf{0}\| = 0$.

(3) We have $a\mathbf{v} = (ax, ay, az)$ so (1) gives $\|a\mathbf{v}\|^2 = (ax)^2 + (ay)^2 + (az)^2 = a^2\|\mathbf{v}\|^2$. Hence $\|a\mathbf{v}\| = \sqrt{a^2}\,\|\mathbf{v}\|$, and we are done because $\sqrt{a^2} = |a|$ for any real number a.

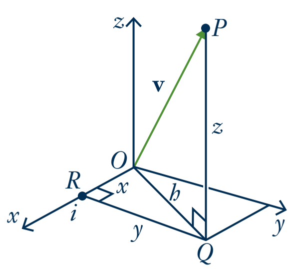


FIGURE 2

Of course the $\mathbb{R}^2$-version of Theorem 1 also holds.

EXAMPLE 1

If $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ then $\|\mathbf{v}\| = \sqrt{4 + 1 + 9} = \sqrt{14}$. Similarly if $\mathbf{v} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$ in 2-space then $\|\mathbf{v}\| = \sqrt{9 + 16} = 5$.

When we view two nonzero vectors as arrows emanating from the origin, it is clear geometrically what we mean by saying that they have the same or opposite **direction**. This leads to a fundamental new description of vectors.

3 When we write $\sqrt{p}$ we mean the positive square root of p.

4 Recall that the absolute value $|a|$ of a real number is defined by $|a| = \begin{cases} a \text{ if } a \geq 0 \\ -a \text{ if } a < 0 \end{cases}$.

5 Pythagoras' theorem states that if a and b are sides of right triangle with hypotenuse c, then $a^2 + b^2 = c^2$. A proof is given at the end of this section.

Theorem 2

Let $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{w} \neq \mathbf{0}$ *be vectors in* $\mathbb{R}^3$. *Then* $\mathbf{v} = \mathbf{w}$ *as matrices if and only if* $\mathbf{v}$ *and* $\mathbf{w}$ *have the same direction and the same length.*[6]

PROOF

If $\mathbf{v} = \mathbf{w}$, they clearly have the same direction and length. Conversely, let $\mathbf{v}$ and $\mathbf{w}$ be vectors with points $P(x, y, z)$ and $Q(x_1, y_1, z_1)$ respectively. If $\mathbf{v}$ and $\mathbf{w}$ have the same length and direction then, geometrically, P and Q must be the same point (see Figure 3). Hence $x = x_1$, $y = y_1$, and $z = z_1$, that is $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \mathbf{w}$.

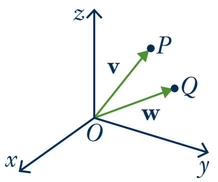


FIGURE 3

A characterization of a vector in terms of its length and direction only is called an **intrinsic** description of the vector. The point to note is that such a description does *not* depend on the choice of coordinate system in $\mathbb{R}^3$. Such descriptions are important in applications because physical laws are often stated in terms of vectors, and these laws cannot depend on the particular coordinate system used to describe the situation.

Geometric Vectors

If A and B are distinct points in space, the arrow from A to B has length and direction. Hence:

Definition 4.1

Suppose that A and B are any two points in $\mathbb{R}^3$. In Figure 4 the line segment from A to B is denoted $\overrightarrow{AB}$ and is called the **geometric vector** *from A to B. Point A is called the* **tail** *of $\overrightarrow{AB}$, B is called the* **tip** *of $\overrightarrow{AB}$, and the* **length** *of $\overrightarrow{AB}$ is denoted $\|\overrightarrow{AB}\|$.*

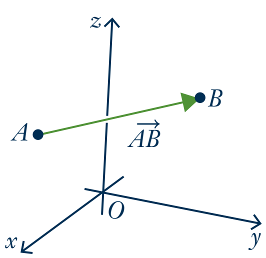


FIGURE 4

Note that if $\mathbf{v}$ is any vector in $\mathbb{R}^3$ with point P then $\mathbf{v} = \overrightarrow{OP}$ is itself a geometric vector where O is the origin. Referring to $\overrightarrow{AB}$ as a "vector" seems justified by Theorem 2 because it has a direction (from A to B) and a length $\|\overrightarrow{AB}\|$. However there appears to be a problem because two geometric vectors can have the same length and direction even if the tips and tails are different. For example $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ in Figure 5 have the same length $\sqrt{5}$ and the same direction (1 unit left and 2 units up) so, by Theorem 2, they are the same vector! The best way to understand this apparent paradox is to see $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ as different *representations* of the same underlying vector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.[7] Once it is clarified, this phenomenon is a great benefit because, thanks to Theorem 2, it means that the same geometric vector can be positioned anywhere in space; what is important is the length and direction, not the location of the tip and tail. This ability to move geometric vectors about is very useful as we shall soon see.

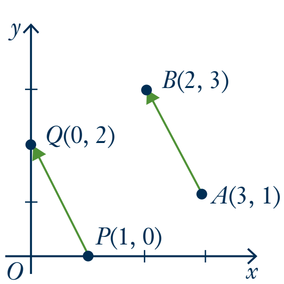


FIGURE 5

6 It is Theorem 2 that gives vectors their power in science and engineering because many physical quantities are determined by their length and magnitude (and are called **vector quantities**). For example, saying that an airplane is flying at 200 km/h does not describe where it is going; the direction must also be specified. The speed and direction comprise the **velocity** of the airplane, a vector quantity.

7 Fractions provide another example of quantities that can be the same but *look* different. For example $\frac{6}{9}$ and $\frac{14}{21}$ certainly appear different, but they are equal fractions—both equal $\frac{2}{3}$ in "lowest terms".

The Parallelogram Law

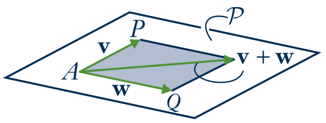


FIGURE 6

We now give an intrinsic description of the sum of two vectors **v** and **w** in $\mathbb{R}^3$, that is a description that depends only on the lengths and directions of **v** and **w** and not on the choice of coordinate system. Using Theorem 2 we can think of these vectors as having a common tail A. If their tips are P and Q respectively, then they both lie in a plane $\mathcal{P}$ containing A, P, and Q, as shown in Figure 6. The vectors **v** and **w** create a parallelogram[8] in $\mathcal{P}$, shaded in Figure 6, called the parallelogram **determined** by **v** and **w**.

If we now choose a coordinate system in the plane $\mathcal{P}$ with A as origin, then the parallelogram law in the plane (Section 2.6) shows that their sum **v** + **w** is the diagonal of the parallelogram they determine with tail A. This is an intrinsic description of the sum **v** + **w** because it makes no reference to coordinates. This discussion proves:

The Parallelogram Law

In the parallelogram determined by two vectors **v** *and* **w**, *the vector* **v** + **w** *is the diagonal with the same tail as* **v** *and* **w**.

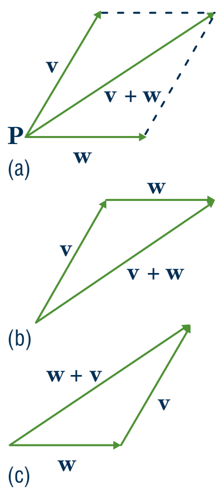


FIGURE 7

Because a vector can be positioned with its tail at any point, the parallelogram law leads to another way to view vector addition. In Figure 7(a) the sum **v** + **w** of two vectors **v** and **w** is shown as given by the parallelogram law. If **w** is moved so its tail coincides with the tip of **v** (Figure 7(b)) then the sum **v** + **w** is seen as "first **v** and then **w**. Similarly, moving the tail of **v** to the tip of **w** shows in Figure 7(c) that **v** + **w** is "first **w** and then **v**." This will be referred to as the **tip-to-tail rule**, and it gives a graphic illustration of why **v** + **w** = **w** + **v**.

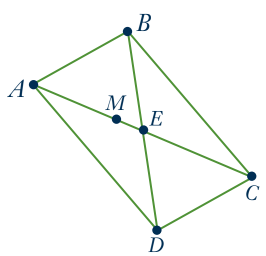
Since $\overrightarrow{AB}$ denotes the vector from a point A to a point B, the tip-to-tail rule takes the easily remembered form

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

for any points A, B, and C. The next example uses this to derive a theorem in geometry without using coordinates.

EXAMPLE 2

Show that the diagonals of a parallelogram bisect each other.



Solution ▶ Let the parallelogram have vertices A, B, C, and D, as shown; let E denote the intersection of the two diagonals; and let M denote the midpoint of diagonal AC. We must show that $M = E$ and that this is the midpoint of diagonal BD. This is accomplished by showing that $\overrightarrow{BM} = \overrightarrow{MD}$. (Then the fact that these vectors have the same direction means that $M = E$, and the fact that they have the same length means that $M = E$ is the midpoint of BD.) Now $\overrightarrow{AM} = \overrightarrow{MC}$ because M is the midpoint of AC, and $\overrightarrow{BA} = \overrightarrow{CD}$ because the figure is a parallelogram. Hence

$$\overrightarrow{BM} = \overrightarrow{BA} + \overrightarrow{AM} = \overrightarrow{CD} + \overrightarrow{MC} = \overrightarrow{MC} + \overrightarrow{CD} = \overrightarrow{MD}$$

where the first and last equalities use the tip-to-tail rule of vector addition.

8 Recall that a parallelogram is a four-sided figure whose opposite sides are parallel and of equal length.

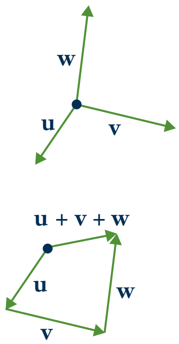


FIGURE 8

One reason for the importance of the tip-to-tail rule is that it means two or more vectors can be added by placing them tip-to-tail in sequence. This gives a useful "picture" of the sum of several vectors, and is illustrated for three vectors in Figure 8 where $\mathbf{u} + \mathbf{v} + \mathbf{w}$ is viewed as first $\mathbf{u}$, then $\mathbf{v}$, then $\mathbf{w}$.

There is a simple geometrical way to visualize the (matrix) **difference** $\mathbf{v} - \mathbf{w}$ of two vectors. If $\mathbf{v}$ and $\mathbf{w}$ are positioned so that they have a common tail A (see Figure 9), and if B and C are their respective tips, then the tip-to-tail rule gives $\mathbf{w} + \overrightarrow{CB} = \mathbf{v}$. Hence $\mathbf{v} - \mathbf{w} = \overrightarrow{CB}$ is the vector from the tip of $\mathbf{w}$ to the tip of $\mathbf{v}$. Thus both $\mathbf{v} - \mathbf{w}$ and $\mathbf{v} + \mathbf{w}$ appear as diagonals in the parallelogram determined by $\mathbf{v}$ and $\mathbf{w}$ (see Figure 9). We record this for reference.

Theorem 3

If $\mathbf{v}$ and $\mathbf{w}$ have a common tail, then $\mathbf{v} - \mathbf{w}$ is the vector from the tip of $\mathbf{w}$ to the tip of $\mathbf{v}$.

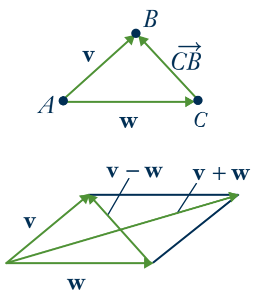


FIGURE 9

One of the most useful applications of vector subtraction is that it gives a simple formula for the vector from one point to another, and for the distance between the points.

Theorem 4

Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be two points. Then:

1. $\overrightarrow{P_1P_2} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix}$.
2. *The distance between P_1 and P_2 is* $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

PROOF

If O is the origin, write $\mathbf{v}_1 = \overrightarrow{OP_1} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\mathbf{v}_2 = \overrightarrow{OP_2} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ as in Figure 10.

Then Theorem 3 gives $\overrightarrow{P_1P_2} = \mathbf{v}_2 - \mathbf{v}_1$, and (1) follows. But the distance between P_1 and P_2 is $\|\overrightarrow{P_1P_2}\|$, so (2) follows from (1) and Theorem 1.

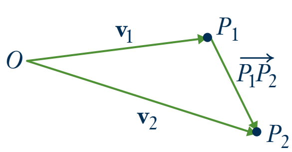


FIGURE 10

Of course the $\mathbb{R}^2$-version of Theorem 4 is also valid: If $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are points in $\mathbb{R}^2$, then $\overrightarrow{P_1P_2} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}$, and the distance between P_1 and P_2 is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

EXAMPLE 3

The distance between $P_1(2, -1, 3)$ and $P_2(1, 1, 4)$ is $\sqrt{(-1)^2 + (2)^2 + (1)^2} = \sqrt{6}$, and the vector from P_1 to P_2 is $\overrightarrow{P_1P_2} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$.

As for the parallelogram law, the intrinsic rule for finding the length and direction of a scalar multiple of a vector in $\mathbb{R}^3$ follows easily from the same situation in $\mathbb{R}^2$.

Scalar Multiplication

Scalar Multiple Law

If a is a real number and $\mathbf{v} \neq \mathbf{0}$ is a vector then:

(1) *The length of $a\mathbf{v}$ is $\|a\mathbf{v}\| = |a|\|\mathbf{v}\|$.*

(2) *If $a\mathbf{v} \neq \mathbf{0}$,[9] the direction of $a\mathbf{v}$ is* $\begin{cases} \text{the same as } \mathbf{v} \text{ if } a > 0, \\ \text{opposite to } \mathbf{v} \text{ if } a < 0. \end{cases}$

PROOF

(1) This part of Theorem 1.

(2) Let O denote the origin in $\mathbb{R}^3$, let $\mathbf{v}$ have point P, and choose any plane containing O and P. If we set up a coordinate system in this plane with O as origin, then $\mathbf{v} = \overrightarrow{OP}$ so the result in (2) follows from the scalar multiple law in the plane (Section 2.6).

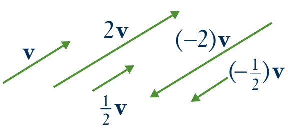


FIGURE 11

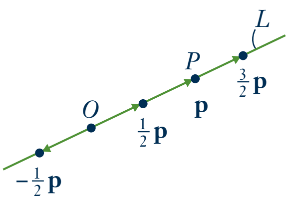


FIGURE 12

Figure 11 gives several examples of scalar multiples of a vector $\mathbf{v}$.

Consider a line L through the origin, let P be any point on L other than the origin O, and let $\mathbf{p} = \overrightarrow{OP}$. If $t \neq 0$, then $t\mathbf{p}$ is a point on L because it has direction the same or opposite as that of $\mathbf{p}$. Moreover $t > 0$ or $t < 0$ according as the point $t\mathbf{p}$ lies on the same or opposite side of the origin as P. This is illustrated in Figure 12.

A vector $\mathbf{u}$ is called a **unit vector** if $\|\mathbf{u}\| = 1$. Then $\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are unit vectors, called the **coordinate** vectors. We discuss them in more detail in Section 4.2.

EXAMPLE 4

If $\mathbf{v} \neq \mathbf{0}$ show that $\frac{1}{\|\mathbf{v}\|}\mathbf{v}$ is the unique unit vector in the same direction as $\mathbf{v}$.

Solution ▶ The vectors in the same direction as $\mathbf{v}$ are the scalar multiples $a\mathbf{v}$ where $a > 0$. But $\|a\mathbf{v}\| = |a|\|\mathbf{v}\| = a\|\mathbf{v}\|$ when $a > 0$, so $a\mathbf{v}$ is a unit vector if and only if $a = \frac{1}{\|\mathbf{v}\|}$.

The next example shows how to find the coordinates of a point on the line segment between two given points. The technique is important and will be used again below.

9 Since the zero vector has no direction, we deal only with the case $a\mathbf{v} \neq \mathbf{0}$.

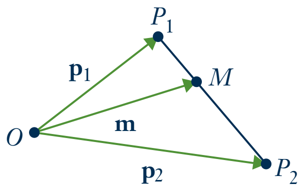
EXAMPLE 5

Let $\mathbf{p}_1$ and $\mathbf{p}_2$ be the vectors of two points P_1 and P_2. If M is the point one third the way from P_1 to P_2, show that the vector $\mathbf{m}$ of M is given by

$$\mathbf{m} = \tfrac{2}{3}\mathbf{p}_1 + \tfrac{1}{3}\mathbf{p}_2$$

Conclude that if $P_1 = P_1(x_1, y_1, z_1)$ and $P_2 = P_2(x_2, y_2, z_2)$, then M has coordinates

$$M = M(\tfrac{2}{3}x_1 + \tfrac{1}{3}x_2, \tfrac{2}{3}y_1 + \tfrac{1}{3}y_2, \tfrac{2}{3}z_1 + \tfrac{1}{3}z_2).$$



Solution ▶ The vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{m}$ are shown in the diagram. We have $\overrightarrow{P_1M} = \frac{1}{3}\overrightarrow{P_1P_2}$ because $\overrightarrow{P_1M}$ is in the same direction as $\overrightarrow{P_1P_2}$ and $\frac{1}{3}$ as long. By Theorem 3 we have $\overrightarrow{P_1P_2} = \mathbf{p}_2 - \mathbf{p}_1$, so tip-to-tail addition gives

$$\mathbf{m} = \mathbf{p}_1 + \overrightarrow{P_1M} = \mathbf{p}_1 + \tfrac{1}{3}(\mathbf{p}_2 - \mathbf{p}_1) = \tfrac{2}{3}\mathbf{p}_1 + \tfrac{1}{3}\mathbf{p}_2$$

as required. For the coordinates, we have $\mathbf{p}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\mathbf{p}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$, so

$$\mathbf{m} = \tfrac{2}{3}\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \tfrac{1}{3}\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{3}x_1 + \frac{1}{3}x_2 \\ \frac{2}{3}y_1 + \frac{1}{3}y_2 \\ \frac{2}{3}z_1 + \frac{1}{3}z_2 \end{bmatrix}$$

by matrix addition. The last statement follows.

Note that in Example 5 $\mathbf{m} = \frac{2}{3}\mathbf{p}_1 + \frac{1}{3}\mathbf{p}_2$ is a "weighted average" of $\mathbf{p}_1$ and $\mathbf{p}_2$ with more weight on $\mathbf{p}_1$ because $\mathbf{m}$ is closer to $\mathbf{p}_1$.

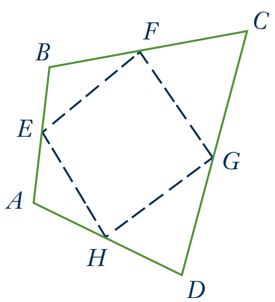
The point M halfway between points P_1 and P_2 is called the **midpoint** between these points. In the same way, the vector $\mathbf{m}$ of M is

$$\mathbf{m} = \tfrac{1}{2}\mathbf{p}_1 + \tfrac{1}{2}\mathbf{p}_2 = \tfrac{1}{2}(\mathbf{p}_1 + \mathbf{p}_2)$$

as the reader can verify, so $\mathbf{m}$ is the "average" of $\mathbf{p}_1$ and $\mathbf{p}_2$ in this case.

EXAMPLE 6

Show that the midpoints of the four sides of any quadrilateral are the vertices of a parallelogram. Here a quadrilateral is any figure with four vertices and straight sides.



Solution ▶ Suppose that the vertices of the quadrilateral are A, B, C, and D (in that order) and that E, F, G, and H are the midpoints of the sides as shown in the diagram. It suffices to show $\overrightarrow{EF} = \overrightarrow{HG}$ (because then sides EF and HG are parallel and of equal length). Now the fact that E is the midpoint of AB means that $\overrightarrow{EB} = \frac{1}{2}\overrightarrow{AB}$. Similarly, $\overrightarrow{BF} = \frac{1}{2}\overrightarrow{BC}$, so

$$\overrightarrow{EF} = \overrightarrow{EB} + \overrightarrow{BF} = \tfrac{1}{2}\overrightarrow{AB} + \tfrac{1}{2}\overrightarrow{BC} = \tfrac{1}{2}(\overrightarrow{AB} + \overrightarrow{BC}) = \tfrac{1}{2}\overrightarrow{AC}$$

A similar argument shows that $\overrightarrow{HG} = \frac{1}{2}\overrightarrow{AC}$ too, so $\overrightarrow{EF} = \overrightarrow{HG}$ as required.

Definition 4.2 *Two nonzero vectors are called* **parallel** *if they have the same or opposite direction.*

Many geometrical propositions involve this notion, so the following theorem will be referred to repeatedly.

Theorem 5

Two nonzero vectors $\mathbf{v}$ *and* $\mathbf{w}$ *are parallel if and only if one is a scalar multiple of the other.*

PROOF

If one of them is a scalar multiple of the other, they are parallel by the scalar multiple law.

Conversely, assume that $\mathbf{v}$ and $\mathbf{w}$ are parallel and write $d = \frac{\|\mathbf{v}\|}{\|\mathbf{w}\|}$ for convenience. Then $\mathbf{v}$ and $\mathbf{w}$ have the same or opposite direction. If they have the same direction we show that $\mathbf{v} = d\mathbf{w}$ by showing that $\mathbf{v}$ and $d\mathbf{w}$ have the same length and direction. In fact, $\|d\mathbf{w}\| = |d|\ \|\mathbf{w}\| = \|\mathbf{v}\|$ by Theorem 1; as to the direction, $d\mathbf{w}$ and $\mathbf{w}$ have the same direction because $d > 0$, and this is the direction of $\mathbf{v}$ by assumption. Hence $\mathbf{v} = d\mathbf{w}$ in this case by Theorem 2. In the other case, $\mathbf{v}$ and $\mathbf{w}$ have opposite direction and a similar argument shows that $\mathbf{v} = -d\mathbf{w}$. We leave the details to the reader.

EXAMPLE 7

Given points $P(2, -1, 4)$, $Q(3, -1, 3)$, $A(0, 2, 1)$, and $B(1, 3, 0)$, determine if $\overrightarrow{PQ}$ and $\overrightarrow{AB}$ are parallel.

Solution ▶ By Theorem 3, $\overrightarrow{PQ} = (1, 0, -1)$ and $\overrightarrow{AB} = (1, 1, -1)$. If $\overrightarrow{PQ} = t\overrightarrow{AB}$ then $(1, 0, -1) = (t, t, -t)$, so $1 = t$ and $0 = t$, which is impossible. Hence $\overrightarrow{PQ}$ is *not* a scalar multiple of $\overrightarrow{AB}$, so these vectors are not parallel by Theorem 5.

Lines in Space

These vector techniques can be used to give a very simple way of describing straight lines in space. In order to do this, we first need a way to specify the orientation of such a line, much as the slope does in the plane.

Definition 4.3 *With this in mind, we call a nonzero vector* $\mathbf{d} \neq \mathbf{0}$ *a* **direction vector** *for the line if it is parallel to* $\overrightarrow{AB}$ *for some pair of distinct points* A *and* B *on the line.*

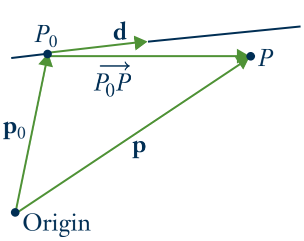


FIGURE 13

Of course it is then parallel to $\overrightarrow{CD}$ for *any* distinct points C and D on the line. In particular, any nonzero scalar multiple of $\mathbf{d}$ will also serve as a direction vector of the line.

We use the fact that there is exactly one line that passes through a particular point $P_0(x_0, y_0, z_0)$ and has a given direction vector $\mathbf{d} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. We want to describe this line by giving a condition on x, y, and z that the point $P(x, y, z)$ lies on this line. Let $\mathbf{p}_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$ and $\mathbf{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ denote the vectors of P_0 and P, respectively (see Figure 13). Then

$$\mathbf{p} = \mathbf{p}_0 + \overrightarrow{P_0P}$$

Hence P lies on the line if and only if $\overrightarrow{P_0P}$ is parallel to $\mathbf{d}$—that is, if and only if $\overrightarrow{P_0P} = t\mathbf{d}$ for some scalar t by Theorem 5. Thus $\mathbf{p}$ is the vector of a point on the line if and only if $\mathbf{p} = \mathbf{p}_0 + t\mathbf{d}$ for some scalar t. This discussion is summed up as follows.

Vector Equation of a Line

The line parallel to $\mathbf{d} \neq \mathbf{0}$ *through the point with vector* $\mathbf{p}_0$ *is given by*

$$\mathbf{p} = \mathbf{p}_0 + t\mathbf{d} \qquad t \text{ any scalar}$$

In other words, the point $\mathbf{p}$ *is on this line if and only if a real number* t *exists such that* $\mathbf{p} = \mathbf{p}_0 + t\mathbf{d}$.

In component form the vector equation becomes

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Equating components gives a different description of the line.

Parametric Equations of a Line

The line through $P_0(x_0, y_0, z_0)$ *with direction vector* $\mathbf{d} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \mathbf{0}$ *is given by*

$$\begin{aligned} x &= x_0 + ta \\ y &= y_0 + tb \qquad t \text{ any scalar} \\ z &= z_0 + tc \end{aligned}$$

In other words, the point $P(x, y, z)$ *is on this line if and only if a real number* t *exists such that* $x = x_0 + ta$, $y = y_0 + tb$, *and* $z = z_0 + tc$.

EXAMPLE 8

Find the equations of the line through the points $P_0(2, 0, 1)$ and $P_1(4, -1, 1)$.

Solution ▶ Let $\mathbf{d} = \overrightarrow{P_0P_1} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ denote the vector from P_0 to P_1. Then $\mathbf{d}$ is parallel to the line (P_0 and P_1 are *on* the line), so $\mathbf{d}$ serves as a direction vector for the line. Using P_0 as the point on the line leads to the parametric equations

$$\begin{aligned} x &= 2 + 2t \\ y &= -t \qquad t \text{ a parameter} \\ z &= 1 \end{aligned}$$

Note that if P_1 is used (rather than P_0), the equations are

$$\begin{aligned} x &= 4 + 2s \\ y &= -1 - s \qquad s \text{ a parameter} \\ z &= 1 \end{aligned}$$

These are different from the preceding equations, but this is merely the result of a change of parameter. In fact, $s = t - 1$.

EXAMPLE 9

Find the equations of the line through $P_0(3, -1, 2)$ parallel to the line with equations

$$\begin{aligned} x &= -1 + 2t \\ y &= 1 + t \\ z &= -3 + 4t \end{aligned}$$

Solution ▶ The coefficients of t give a direction vector $\mathbf{d} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$ of the given line. Because the line we seek is parallel to this line, $\mathbf{d}$ also serves as a direction vector for the new line. It passes through P_0, so the parametric equations are

$$\begin{aligned} x &= 3 + 2t \\ y &= -1 + t \\ z &= 2 + 4t \end{aligned}$$

EXAMPLE 10

Determine whether the following lines intersect and, if so, find the point of intersection.

$$\begin{aligned} x &= 1 - 3t \qquad & x &= -1 + s \\ y &= 2 + 5t \qquad & y &= 3 - 4s \\ z &= 1 + t \qquad & z &= 1 - s \end{aligned}$$

Solution ▶ Suppose $\mathbf{p} = P(x, y, z)$ lies on both lines. Then

$$\begin{bmatrix} 1 - 3t \\ 2 + 5t \\ 1 + t \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 + s \\ 3 - 4s \\ 1 - s \end{bmatrix} \text{ for some } t \text{ and } s,$$

where the first (second) equation is because P lies on the first (second) line. Hence the lines intersect if and only if the three equations

$$\begin{aligned} 1 - 3t &= -1 + s \\ 2 + 5t &= 3 - 4s \\ 1 + t &= 1 - s \end{aligned}$$

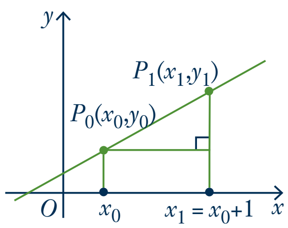
have a solution. In this case, $t = 1$ and $s = -1$ satisfy all three equations, so the lines *do* intersect and the point of intersection is

$$\mathbf{p} = \begin{bmatrix} 1 - 3t \\ 2 + 5t \\ 1 + t \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ 2 \end{bmatrix}$$

using $t = 1$. Of course, this point can also be found from $\mathbf{p} = \begin{bmatrix} -1+s \\ 3-4s \\ 1-s \end{bmatrix}$ using $s = -1$.

EXAMPLE 11

Show that the line through $P_0(x_0, y_0)$ with slope m has direction vector $\mathbf{d} = \begin{bmatrix} 1 \\ m \end{bmatrix}$ and equation $y - y_0 = m(x - x_0)$. This equation is called the *point-slope* formula.



Solution ▶ Let $P_1(x_1, y_1)$ be the point on the line one unit to the right of P_0 (see the diagram). Hence $x_1 = x_0 + 1$. Then $\mathbf{d} = \overrightarrow{P_0P_1}$ serves as direction vector of the line, and $\mathbf{d} = \begin{bmatrix} x_1 - x_0 \\ y_1 - y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ y_1 - y_0 \end{bmatrix}$. But the slope m can be computed as follows:

$$m = \frac{y_1 - y_0}{x_1 - x_0} = \frac{y_1 - y_0}{1} = y_1 - y_0$$

Hence $\mathbf{d} = \begin{bmatrix} 1 \\ m \end{bmatrix}$ and the parametric equations are $x = x_0 + t$, $y = y_0 + mt$. Eliminating t gives $y - y_0 = mt = m(x - x_0)$, as asserted.

Note that the vertical line through $P_0(x_0, y_0)$ has a direction vector $\mathbf{d} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ that is *not* of the form $\begin{bmatrix} 1 \\ m \end{bmatrix}$ for any m. This result confirms that the notion of slope makes no sense in this case. However, the vector method gives parametric equations for the line:

$$\begin{aligned} x &= x_0 \\ y &= y_0 + t \end{aligned}$$

Because y is arbitrary here (t is arbitrary), this is usually written simply as $x = x_0$.

Pythagoras' Theorem

The pythagorean theorem was known earlier, but Pythagoras (c. 550 B.C.) is credited with giving the first rigorous, logical, deductive proof of the result. The proof we give depends on a basic property of similar triangles: ratios of corresponding sides are equal.

Theorem 6

Pythagoras' Theorem
Given a right-angled triangle with hypotenuse c and sides a and b, then $a^2 + b^2 = c^2$.

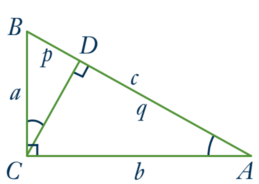


FIGURE 14

PROOF

Let A, B, and C be the vertices of the triangle as in Figure 14. Draw a perpendicular from C to the point D on the hypotenuse, and let p and q be the lengths of BD and DA respectively. Then DBC and CBA are similar triangles so $\frac{p}{a} = \frac{a}{c}$.

This means $a^2 = pc$. In the same way, the similarity of DCA and CBA gives $\frac{q}{b} = \frac{b}{c}$, whence $b^2 = qc$. But then

$$a^2 + b^2 = pc + qc = (p + q)c = c^2$$

because $p + q = c$. This proves Pythagoras' theorem.

EXERCISES 4.1

1. Compute $\|\mathbf{v}\|$ if $\mathbf{v}$ equals:

(a) $\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ ◆(b) $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ ◆(d) $\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

(e) $2\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ ◆(f) $-3\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

2. Find a unit vector in the direction of:

(a) $\begin{bmatrix} 7 \\ -1 \\ 5 \end{bmatrix}$ ◆(b) $\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$

3. (a) Find a unit vector in the direction from $\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$.

(b) If $\mathbf{u} \neq \mathbf{0}$, for which values of a is $a\mathbf{u}$ a unit vector?

4. Find the distance between the following pairs of points.

(a) $\begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ ◆(b) $\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$ ◆(d) $\begin{bmatrix} 4 \\ 0 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$

5. Use vectors to show that the line joining the midpoints of two sides of a triangle is parallel to the third side and half as long.

6. Let A, B, and C denote the three vertices of a triangle.

(a) If E is the midpoint of side BC, show that

$$\overrightarrow{AE} = \tfrac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC}).$$

◆(b) If F is the midpoint of side AC, show that

$$\overrightarrow{FE} = \tfrac{1}{2}\overrightarrow{AB}.$$

7. Determine whether $\mathbf{u}$ and $\mathbf{v}$ are parallel in each of the following cases.

(a) $\mathbf{u} = \begin{bmatrix} -3 \\ -6 \\ 3 \end{bmatrix}$; $\mathbf{v} = \begin{bmatrix} 5 \\ 10 \\ -5 \end{bmatrix}$

◆(b) $\mathbf{u} = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}$; $\mathbf{v} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$

(c) $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$; $\mathbf{v} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

◆(d) $\mathbf{u} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$; $\mathbf{v} = \begin{bmatrix} -8 \\ 0 \\ 4 \end{bmatrix}$

8. Let $\mathbf{p}$ and $\mathbf{q}$ be the vectors of points P and Q, respectively, and let R be the point whose vector is $\mathbf{p} + \mathbf{q}$. Express the following in terms of $\mathbf{p}$ and $\mathbf{q}$.

(a) $\overrightarrow{QP}$ ◆(b) $\overrightarrow{QR}$

(c) $\overrightarrow{RP}$ ◆(d) $\overrightarrow{RO}$ where O is the origin

9. In each case, find $\overrightarrow{PQ}$ and $\left\|\overrightarrow{PQ}\right\|$.

(a) $P(1, -1, 3)$, $Q(3, 1, 0)$

◆(b) $P(2, 0, 1)$, $Q(1, -1, 6)$

(c) $P(1, 0, 1)$, $Q(1, 0, -3)$

◆(d) $P(1, -1, 2)$, $Q(1, -1, 2)$

(e) $P(1, 0, -3)$, $Q(-1, 0, 3)$

◆(f) $P(3, -1, 6)$, $Q(1, 1, 4)$

10. In each case, find a point Q such that $\overrightarrow{PQ}$ has (i) the same direction as $\mathbf{v}$; (ii) the opposite direction to $\mathbf{v}$.

(a) $P(-1, 2, 2)$, $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$

◆(b) $P(3, 0, -1)$, $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

11. Let $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}$.
In each case, find $\mathbf{x}$ such that:
   (a) $3(2\mathbf{u} + \mathbf{x}) + \mathbf{w} = 2\mathbf{x} - \mathbf{v}$
   ◆(b) $2(3\mathbf{v} - \mathbf{x}) = 5\mathbf{w} + \mathbf{u} - 3\mathbf{x}$

12. Let $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. In each case, find numbers a, b, and c such that $\mathbf{x} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$.
   (a) $\mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$
   ◆(b) $\mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$

13. Let $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$, and $\mathbf{z} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. In each case, show that there are no numbers a, b, and c such that:
   (a) $a\mathbf{u} + b\mathbf{v} + c\mathbf{z} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$
   (b) $a\mathbf{u} + b\mathbf{v} + c\mathbf{z} = \begin{bmatrix} 5 \\ 6 \\ -1 \end{bmatrix}$

14. Let $P_1 = P_1(2, 1, -2)$ and $P_2 = P_2(1, -2, 0)$. Find the coordinates of the point P:
   (a) $\frac{1}{5}$ the way from P_1 to P_2
   ◆(b) $\frac{1}{4}$ the way from P_2 to P_1

15. Find the two points trisecting the segment between $P(2, 3, 5)$ and $Q(8, -6, 2)$.

16. Let $P_1 = P_1(x_1, y_1, z_1)$ and $P_2 = P_2(x_2, y_2, z_2)$ be two points with vectors $\mathbf{p}_1$ and $\mathbf{p}_2$, respectively. If r and s are positive integers, show that the point P lying $\frac{r}{r+s}$ the way from P_1 to P_2 has vector
$$\mathbf{p} = \left(\tfrac{s}{r+s}\right)\mathbf{p}_1 + \left(\tfrac{r}{r+s}\right)\mathbf{p}_2.$$

17. In each case, find the point Q:
   (a) $\overrightarrow{PQ} = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$ and $P = P(2, -3, 1)$
   ◆(b) $\overrightarrow{PQ} = \begin{bmatrix} -1 \\ 4 \\ 7 \end{bmatrix}$ and $P = P(1, 3, -4)$

18. Let $\mathbf{u} = \begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$. In each case find $\mathbf{x}$:
   (a) $2\mathbf{u} - \|\mathbf{v}\|\mathbf{v} = \frac{3}{2}(\mathbf{u} - 2\mathbf{x})$
   ◆(b) $3\mathbf{u} + 7\mathbf{v} = \|\mathbf{u}\|^2(2\mathbf{x} + \mathbf{v})$

19. Find all vectors $\mathbf{u}$ that are parallel to $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ and satisfy $\|\mathbf{u}\| = 3\|\mathbf{v}\|$.

20. Let P, Q, and R be the vertices of a parallelogram with adjacent sides PQ and PR. In each case, find the other vertex S.
   (a) $P(3, -1, -1)$, $Q(1, -2, 0)$, $R(1, -1, 2)$
   ◆(b) $P(2, 0, -1)$, $Q(-2, 4, 1)$, $R(3, -1, 0)$

21. In each case either prove the statement or give an example showing that it is false.
   (a) The zero vector $\mathbf{0}$ is the only vector of length 0.
   ◆(b) If $\|\mathbf{v} - \mathbf{w}\| = 0$, then $\mathbf{v} = \mathbf{w}$.
   (c) If $\mathbf{v} = -\mathbf{v}$, then $\mathbf{v} = \mathbf{0}$.
   ◆(d) If $\|\mathbf{v}\| = \|\mathbf{w}\|$, then $\mathbf{v} = \mathbf{w}$.
   (e) If $\|\mathbf{v}\| = \|\mathbf{w}\|$, then $\mathbf{v} = \pm\mathbf{w}$.
   ◆(f) If $\mathbf{v} = t\mathbf{w}$ for some scalar t, then $\mathbf{v}$ and $\mathbf{w}$ have the same direction.
   (g) If $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{v} + \mathbf{w}$ are nonzero, and $\mathbf{v}$ and $\mathbf{v} + \mathbf{w}$ parallel, then $\mathbf{v}$ and $\mathbf{w}$ are parallel.
   ◆(h) $\|-5\mathbf{v}\| = -5\|\mathbf{v}\|$, for all $\mathbf{v}$.
   (i) If $\|\mathbf{v}\| = \|2\mathbf{v}\|$, then $\mathbf{v} = \mathbf{0}$.
   ◆(j) $\|\mathbf{v} + \mathbf{w}\| = \|\mathbf{v}\| + \|\mathbf{w}\|$, for all $\mathbf{v}$ and $\mathbf{w}$.

22. Find the vector and parametric equations of the following lines.
   (a) The line parallel to $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and passing through $P(1, -1, 3)$.
   ◆(b) The line passing through $P(3, -1, 4)$ and $Q(1, 0, -1)$.
   (c) The line passing through $P(3, -1, 4)$ and $Q(3, -1, 5)$.
   ◆(d) The line parallel to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and passing through $P(1, 1, 1)$.

(e) The line passing through $P(1, 0, -3)$ and parallel to the line with parametric equations $x = -1 + 2t$, $y = 2 - t$, and $z = 3 + 3t$.

◆(f) The line passing through $P(2, -1, 1)$ and parallel to the line with parametric equations $x = 2 - t$, $y = 1$, and $z = t$.

(g) The lines through $P(1, 0, 1)$ that meet the line with vector equation $\mathbf{p} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ at points at distance 3 from $P_0(1, 2, 0)$.

23. In each case, verify that the points P and Q lie on the line.

(a) $\begin{aligned} x &= 3 - 4t \\ y &= 2 + t \\ z &= 1 - t \end{aligned}$ $\quad P(-1, 3, 0)$, $Q(11, 0, 3)$

◆(b) $\begin{aligned} x &= 4 - t \\ y &= 3 \\ z &= 1 - 2t \end{aligned}$ $\quad P(2, 3, -3)$, $Q(-1, 3, -9)$

24. Find the point of intersection (if any) of the following pairs of lines.

(a) $\begin{aligned} x &= 3 + t \\ y &= 1 - 2t \\ z &= 3 + 3t \end{aligned}$ $\quad \begin{aligned} x &= 4 + 2s \\ y &= 6 + 3s \\ z &= 1 + s \end{aligned}$

◆(b) $\begin{aligned} x &= 1 - t \\ y &= 2 + 2t \\ z &= -1 + 3t \end{aligned}$ $\quad \begin{aligned} x &= 2s \\ y &= 1 + s \\ z &= 3 \end{aligned}$

(c) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + s\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$

◆(d) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -7 \\ 12 \end{bmatrix} + s\begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}$

25. Show that if a line passes through the origin, the vectors of points on the line are all scalar multiples of some fixed nonzero vector.

26. Show that every line parallel to the z axis has parametric equations $x = x_0$, $y = y_0$, $z = t$ for some fixed numbers x_0 and y_0.

27. Let $\mathbf{d} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be a vector where a, b, and c are *all* nonzero. Show that the equations of the line through $P_0(x_0, y_0, z_0)$ with direction vector $\mathbf{d}$ can be written in the form

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

This is called the **symmetric form** of the equations.

28. A parallelogram has sides AB, BC, CD, and DA. Given $A(1, -1, 2)$, $C(2, 1, 0)$, and the midpoint $M(1, 0, -3)$ of AB, find $\overrightarrow{BD}$.

◆29. Find all points C on the line through $A(1, -1, 2)$ and $B = (2, 0, 1)$ such that $\|\overrightarrow{AC}\| = 2\|\overrightarrow{BC}\|$.

30. Let A, B, C, D, E, and F be the vertices of a regular hexagon, taken in order. Show that $\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{AF} = 3\overrightarrow{AD}$.

31. (a) Let P_1, P_2, P_3, P_4, P_5, and P_6 be six points equally spaced on a circle with centre C. Show that

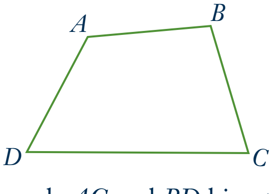
$$\overrightarrow{CP}_1 + \overrightarrow{CP}_2 + \overrightarrow{CP}_3 + \overrightarrow{CP}_4 + \overrightarrow{CP}_5 + \overrightarrow{CP}_6 = \mathbf{0}.$$

◆(b) Show that the conclusion in part (a) holds for any *even* set of points evenly spaced on the circle.

(c) Show that the conclusion in part (a) holds for *three* points.

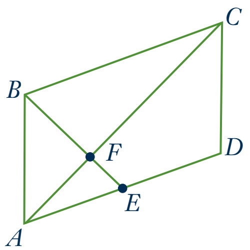
(d) Do you think it works for *any* finite set of points evenly spaced around the circle?

32. Consider a quadrilateral with vertices A, B, C, and D in order (as shown in the diagram).



If the diagonals AC and BD bisect each other, show that the quadrilateral is a parallelogram. (This is the converse of Example 2.) [*Hint*: Let E be the intersection of the diagonals. Show that $\overrightarrow{AB} = \overrightarrow{DC}$ by writing $\overrightarrow{AB} = \overrightarrow{AE} + \overrightarrow{EB}$.]

◆33. Consider the parallelogram $ABCD$ (see diagram), and let E be the midpoint of side AD.

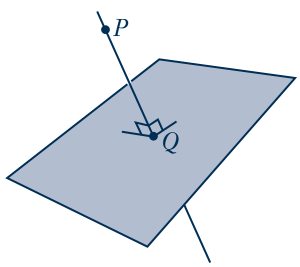


Show that BE and AC trisect each other; that is, show that the intersection point is one-third of the way from E to B and from A to C. [*Hint*: If F is one-third of the way from A to C, show that $2\overrightarrow{EF} = \overrightarrow{FB}$ and argue as in Example 2.]

34. The line from a vertex of a triangle to the midpoint of the opposite side is called a **median** of the triangle. If the vertices of a triangle have vectors **u**, **v**, and **w**, show that the point on each median that is $\frac{1}{3}$ the way from the midpoint to the vertex has vector $\frac{1}{3}(\mathbf{u} + \mathbf{v} + \mathbf{w})$. Conclude that the point C with vector $\frac{1}{3}(\mathbf{u} + \mathbf{v} + \mathbf{w})$ lies on all three medians. This point C is called the **centroid** of the triangle.

35. Given four noncoplanar points in space, the figure with these points as vertices is called a **tetrahedron**. The line from a vertex through the centroid (see previous exercise) of the triangle formed by the remaining vertices is called a **median** of the tetrahedron. If **u**, **v**, **w**, and **x** are the vectors of the four vertices, show that the point on a median one-fourth the way from the centroid to the vertex has vector $\frac{1}{4}(\mathbf{u} + \mathbf{v} + \mathbf{w} + \mathbf{x})$. Conclude that the four medians are concurrent.

SECTION 4.2 Projections and Planes



■ FIGURE 1

Any student of geometry soon realizes that the notion of perpendicular lines is fundamental. As an illustration, suppose a point P and a plane are given and it is desired to find the point Q that lies in the plane and is closest to P, as shown in Figure 1. Clearly, what is required is to find the line through P that is perpendicular to the plane and then to obtain Q as the point of intersection of this line with the plane. Finding the line *perpendicular* to the plane requires a way to determine when two vectors are perpendicular. This can be done using the idea of the dot product of two vectors.

The Dot Product and Angles

Definition 4.4 *Given vectors* $\mathbf{v} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ *and* $\mathbf{w} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$, *their* **dot product** $\mathbf{v} \cdot \mathbf{w}$ *is a number defined*

$$\mathbf{v} \cdot \mathbf{w} = x_1x_2 + y_1y_2 + z_1z_2 = \mathbf{v}^T\mathbf{w}$$

Because $\mathbf{v} \cdot \mathbf{w}$ is a number, it is sometimes called the **scalar product** of **v** and **w**.[10]

EXAMPLE 1

If $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$, then $\mathbf{v} \cdot \mathbf{w} = 2 \cdot 1 + (-1) \cdot 4 + 3 \cdot (-1) = -5$.

The next theorem lists several basic properties of the dot product.

10 Similarly, if $\mathbf{v} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ in $\mathbb{R}^2$, then $\mathbf{v} \cdot \mathbf{w} = x_1x_2 + y_1y_2$.

Theorem 1

Let $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *denote vectors in* $\mathbb{R}^3$ *(or* $\mathbb{R}^2$*).*

1. $\mathbf{v} \cdot \mathbf{w}$ *is a real number.*
2. $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$.
3. $\mathbf{v} \cdot \mathbf{0} = 0 = \mathbf{0} \cdot \mathbf{v}$.
4. $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$.
5. $(k\mathbf{v}) \cdot \mathbf{w} = k(\mathbf{w} \cdot \mathbf{v}) = \mathbf{v} \cdot (k\mathbf{w})$ *for all scalars* k.
6. $\mathbf{u} \cdot (\mathbf{v} \pm \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} \pm \mathbf{u} \cdot \mathbf{w}$

PROOF

(1), (2), and (3) are easily verified, and (4) comes from Theorem 1 Section 4.1. The rest are properties of matrix arithmetic (because $\mathbf{w} \cdot \mathbf{v} = \mathbf{v}^T\mathbf{w}$, and are left to the reader.

The properties in Theorem 1 enable us to do calculations like

$$3\mathbf{u} \cdot (2\mathbf{v} - 3\mathbf{w} + 4\mathbf{z}) = 6(\mathbf{u} \cdot \mathbf{v}) - 9(\mathbf{u} \cdot \mathbf{w}) + 12(\mathbf{u} \cdot \mathbf{z})$$

and such computations will be used without comment below. Here is an example.

EXAMPLE 2

Verify that $\|\mathbf{v} - 3\mathbf{w}\|^2 = 1$ when $\|\mathbf{v}\| = 2$, $\|\mathbf{w}\| = 1$, and $\mathbf{v} \cdot \mathbf{w} = 2$.

Solution ▶ We apply Theorem 1 several times:

$$\begin{aligned}\|\mathbf{v} - 3\mathbf{w}\|^2 &= (\mathbf{v} - 3\mathbf{w}) \cdot (\mathbf{v} - 3\mathbf{w}) \\ &= \mathbf{v} \cdot (\mathbf{v} - 3\mathbf{w}) - 3\mathbf{w} \cdot (\mathbf{v} - 3\mathbf{w}) \\ &= \mathbf{v} \cdot \mathbf{v} - 3(\mathbf{v} \cdot \mathbf{w}) - 3(\mathbf{w} \cdot \mathbf{v}) + 9(\mathbf{w} \cdot \mathbf{w}) \\ &= \|\mathbf{v}\|^2 - 6(\mathbf{v} \cdot \mathbf{w}) + 9\|\mathbf{v}\|^2 \\ &= 4 - 12 + 9 = 1.\end{aligned}$$

There is an intrinsic description of the dot product of two nonzero vectors in $\mathbb{R}^3$. To understand it we require the following result from trigonometry.

Law of Cosines

If a triangle has sides a, b, *and* c, *and if* θ *is the interior angle opposite* c *then*

$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$

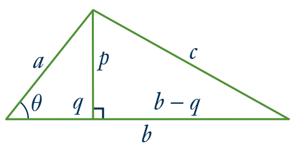


FIGURE 2

PROOF

We prove it when is θ acute, that is $0 \le \theta < \frac{\pi}{2}$; the obtuse case is similar. In Figure 2 we have $p = a \sin\theta$ and $q = a\cos\theta$. Hence Pythagoras' theorem gives

$$\begin{aligned} c^2 = p^2 + (b-q)^2 &= a^2 \sin^2\theta + (b - a\cos\theta)^2 \\ &= a^2(\sin^2\theta + \cos^2\theta) + b^2 - 2ab\cos\theta. \end{aligned}$$

The law of cosines follows because $\sin^2\theta + \cos^2\theta = 1$ for any angle θ.

Note that the law of cosines reduces to Pythagoras' theorem if θ is a right angle (because $\cos\frac{\pi}{2} = 0$).

Now let **v** and **w** be nonzero vectors positioned with a common tail as in Figure 3. Then they determine a unique angle θ in the range

$$0 \le \theta \le \pi$$

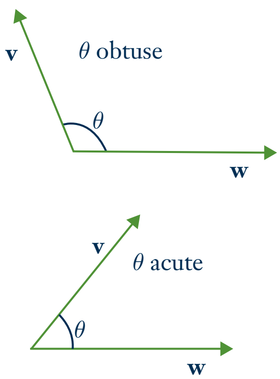


FIGURE 3

This angle θ will be called the **angle between v** and **w**. Figure 2 illustrates when θ is acute (less than $\frac{\pi}{2}$) and obtuse (greater than $\frac{\pi}{2}$). Clearly **v** and **w** are parallel if θ is either 0 or π. Note that we do not define the angle between **v** and **w** if one of these vectors is **0**.

The next result gives an easy way to compute the angle between two nonzero vectors using the dot product.

Theorem 2

Let **v** *and* **w** *be nonzero vectors. If θ is the angle between* **v** *and* **w**, *then*

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\|\cos\theta$$

PROOF

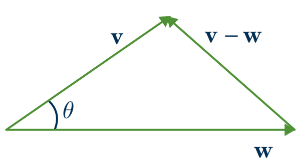


FIGURE 4

We calculate $\|\mathbf{v} - \mathbf{w}\|^2$ in two ways. First apply the law of cosines to the triangle in Figure 4 to obtain:

$$\|\mathbf{v} - \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\|\mathbf{v}\|\|\mathbf{w}\|\cos\theta$$

On the other hand, we use Theorem 1:

$$\begin{aligned} \|\mathbf{v} - \mathbf{w}\|^2 &= (\mathbf{v} - \mathbf{w}) \cdot (\mathbf{v} - \mathbf{w}) \\ &= \mathbf{v} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{w} - \mathbf{w} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{w} \\ &= \|\mathbf{v}\|^2 - 2(\mathbf{v} \cdot \mathbf{w}) + \|\mathbf{w}\|^2 \end{aligned}$$

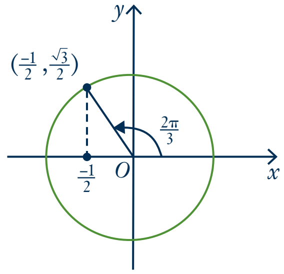
Comparing these we see that $-2\|\mathbf{v}\|\|\mathbf{w}\|\cos\theta = -2(\mathbf{v} \cdot \mathbf{w})$, and the result follows.

If **v** and **w** are nonzero vectors, Theorem 2 gives an intrinsic description of $\mathbf{v} \cdot \mathbf{w}$ because $\|\mathbf{v}\|$, $\|\mathbf{w}\|$, and the angle θ between **v** and **w** do not depend on the choice of coordinate system. Moreover, since $\|\mathbf{v}\|$ and $\|\mathbf{v}\|$ are nonzero (**v** and **w** are nonzero vectors), it gives a formula for the cosine of the angle θ:

$$\cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} \qquad (*)$$

Since $0 \le \theta \le \pi$, this can be used to find θ.

EXAMPLE 3



Compute the angle between $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$.

Solution ▶ Compute $\cos\theta = \dfrac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \dfrac{-2+1-2}{\sqrt{6}\sqrt{6}} = -\dfrac{1}{2}$. Now recall that $\cos\theta$ and $\sin\theta$ are defined so that $(\cos\theta, \sin\theta)$ is the point on the unit circle determined by the angle θ (drawn counterclockwise, starting from the positive x axis). In the present case, we know that $\cos\theta = -\frac{1}{2}$ and that $0 \le \theta \le \pi$. Because $\cos\frac{\pi}{3} = \frac{1}{2}$, it follows that $\theta = \frac{2\pi}{3}$ (see the diagram).

If $\mathbf{v}$ and $\mathbf{w}$ are nonzero, (∗) shows that $\cos\theta$ has the same sign as $\mathbf{v} \cdot \mathbf{w}$, so

$$\begin{array}{lll} \mathbf{v} \cdot \mathbf{w} > 0 & \text{if and only if} & \theta \text{ is acute } (0 \le \theta < \frac{\pi}{2}) \\ \mathbf{v} \cdot \mathbf{w} < 0 & \text{if and only if} & \theta \text{ is obtuse } (\frac{\pi}{2} < \theta \le 0) \\ \mathbf{v} \cdot \mathbf{w} = 0 & \text{if and only if} & \theta = \frac{\pi}{2} \end{array}$$

In this last case, the (nonzero) vectors are perpendicular. The following terminology is used in linear algebra:

Definition 4.5 *Two vectors* $\mathbf{v}$ *and* $\mathbf{w}$ *are said to be* **orthogonal** *if* $\mathbf{v} = \mathbf{0}$ *or* $\mathbf{w} = \mathbf{0}$ *or the angle between them is* $\frac{\pi}{2}$.

Since $\mathbf{v} \cdot \mathbf{w} = 0$ if either $\mathbf{v} = \mathbf{0}$ or $\mathbf{w} = \mathbf{0}$, we have the following theorem:

Theorem 3

Two vectors $\mathbf{v}$ *and* $\mathbf{w}$ *are orthogonal if and only if* $\mathbf{v} \cdot \mathbf{w} = 0$.

EXAMPLE 4

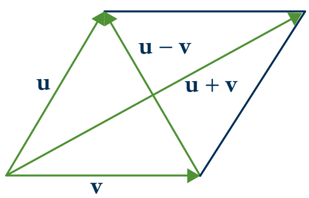
Show that the points $P(3, -1, 1)$, $Q(4, 1, 4)$, and $R(6, 0, 4)$ are the vertices of a right triangle.

Solution ▶ The vectors along the sides of the triangle are

$$\overrightarrow{PQ} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \overrightarrow{PR} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}, \quad \text{and} \quad \overrightarrow{QR} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

Evidently $\overrightarrow{PQ} \cdot \overrightarrow{QR} = 2 - 2 + 0 = 0$, so $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are orthogonal vectors. This means sides PQ and QR are perpendicular—that is, the angle at Q is a right angle.

Example 5 demonstrates how the dot product can be used to verify geometrical theorems involving perpendicular lines.



EXAMPLE 5

A parallelogram with sides of equal length is called a **rhombus**. Show that the diagonals of a rhombus are perpendicular.

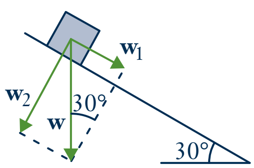
Solution ▶ Let $\mathbf{u}$ and $\mathbf{v}$ denote vectors along two adjacent sides of a rhombus, as shown in the diagram. Then the diagonals are $\mathbf{u} - \mathbf{v}$ and $\mathbf{u} + \mathbf{v}$, and we compute

$$\begin{aligned}(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) &= \mathbf{u} \cdot (\mathbf{u} + \mathbf{v}) - \mathbf{v} \cdot (\mathbf{u} + \mathbf{v}) \\ &= \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{v} \\ &= \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2 \\ &= 0\end{aligned}$$

because $\|\mathbf{u}\| = \|\mathbf{v}\|$ (it is a rhombus). Hence $\mathbf{u} - \mathbf{v}$ and $\mathbf{u} + \mathbf{v}$ are orthogonal.

Projections

In applications of vectors, it is frequently useful to write a vector as the sum of two orthogonal vectors. Here is an example.



EXAMPLE 6

Suppose a ten-kilogram block is placed on a flat surface inclined 30° to the horizontal as in the diagram. Neglecting friction, how much force is required to keep the block from sliding down the surface?

Solution ▶ Let $\mathbf{w}$ denote the weight (force due to gravity) exerted on the block. Then $\|\mathbf{w}\| = 10$ kilograms and the direction of $\mathbf{w}$ is vertically down as in the diagram. The idea is to write $\mathbf{w}$ as a sum $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$ where $\mathbf{w}_1$ is parallel to the inclined surface and $\mathbf{w}_2$ is perpendicular to the surface. Since there is no friction, the force required is $-\mathbf{w}_1$ because the force $\mathbf{w}_2$ has no effect parallel to the surface. As the angle between $\mathbf{w}$ and $\mathbf{w}_2$ is 30° in the diagram, we have $\frac{\|\mathbf{w}_1\|}{\|\mathbf{w}\|} = \sin 30° = \frac{1}{2}$. Hence $\|\mathbf{w}_1\| = \frac{1}{2}\|\mathbf{w}\| = \frac{1}{2}10 = 5$. Thus the required force has a magnitude of 5 kilograms weight directed up the surface.

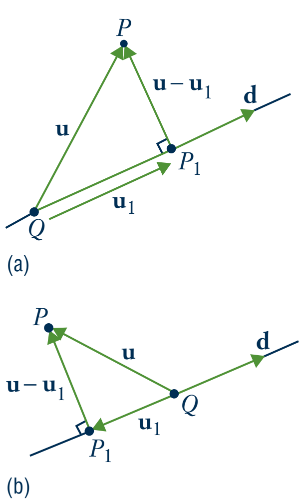


FIGURE 5

If a nonzero vector $\mathbf{d}$ is specified, the key idea in Example 6 is to be able to write an arbitrary vector $\mathbf{u}$ as a sum of two vectors,

$$\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$$

where $\mathbf{u}_1$ is parallel to $\mathbf{d}$ and $\mathbf{u}_2 = \mathbf{u} - \mathbf{u}_1$ is orthogonal to $\mathbf{d}$. Suppose that $\mathbf{u}$ and $\mathbf{d} \neq \mathbf{0}$ emanate from a common tail Q (see Figure 5). Let P be the tip of $\mathbf{u}$, and let P_1 denote the foot of the perpendicular from P to the line through Q parallel to $\mathbf{d}$. Then $\mathbf{u}_1 = \overrightarrow{QP_1}$ has the required properties:

1. *$\mathbf{u}_1$ is parallel to $\mathbf{d}$.*
2. *$\mathbf{u}_2 = \mathbf{u} - \mathbf{u}_1$ is orthogonal to $\mathbf{d}$.*
3. *$\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$.*

Definition 4.6

The vector $\mathbf{u}_1 = \overrightarrow{QP_1}$ in Figure 5 is called **the projection of u on d.** *It is denoted*

$$\mathbf{u}_1 = \operatorname{proj}_{\mathbf{d}} \mathbf{u}$$

In Figure 5(a) the vector $\mathbf{u}_1 = \text{proj}_{\mathbf{d}}\, \mathbf{u}$ has the same direction as $\mathbf{d}$; however, $\mathbf{u}_1$ and $\mathbf{d}$ have opposite directions if the angle between $\mathbf{u}$ and $\mathbf{d}$ is greater than $\frac{\pi}{2}$. (Figure 5(b)). Note that the projection $\mathbf{u}_1 = \text{proj}_{\mathbf{d}}\, \mathbf{u}$ is zero if and only if $\mathbf{u}$ and $\mathbf{d}$ are orthogonal.

Calculating the projection of $\mathbf{u}$ on $\mathbf{d} \neq \mathbf{0}$ is remarkably easy.

Theorem 4

Let $\mathbf{u}$ *and* $\mathbf{d} \neq \mathbf{0}$ *be vectors.*

1. *The projection of* $\mathbf{u}$ *on* $\mathbf{d}$ *is given by* $\text{proj}_{\mathbf{d}}\, \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{d}}{\|\mathbf{d}\|^2}\mathbf{d}$.
2. *The vector* $\mathbf{u} - \text{proj}_{\mathbf{d}}\, \mathbf{u}$ *is orthogonal to* $\mathbf{d}$.

PROOF

The vector $\mathbf{u}_1 = \text{proj}_{\mathbf{d}}\, \mathbf{u}$ is parallel to $\mathbf{d}$ and so has the form $\mathbf{u}_1 = t\mathbf{d}$ for some scalar t. The requirement that $\mathbf{u} - \mathbf{u}_1$ and $\mathbf{d}$ are orthogonal determines t. In fact, it means that $(\mathbf{u} - \mathbf{u}_1) \cdot \mathbf{d} = 0$ by Theorem 3. If $\mathbf{u}_1 = t\mathbf{d}$ is substituted here, the condition is

$$0 = (\mathbf{u} - t\mathbf{d}) \cdot \mathbf{d} = \mathbf{u} \cdot \mathbf{d} - t(\mathbf{d} \cdot \mathbf{d}) = \mathbf{u} \cdot \mathbf{d} - t\|\mathbf{d}\|^2$$

It follows that $t = \frac{\mathbf{u} \cdot \mathbf{d}}{\|\mathbf{d}\|^2}$, where the assumption that $\mathbf{d} \neq \mathbf{0}$ guarantees that $\|\mathbf{d}\|^2 \neq 0$.

EXAMPLE 7

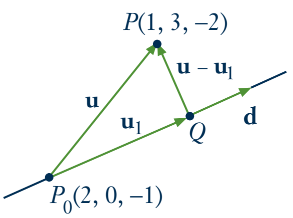
Find the projection of $\mathbf{u} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ on $\mathbf{d} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ and express $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$ where $\mathbf{u}_1$ is parallel to $\mathbf{d}$ and $\mathbf{u}_2$ is orthogonal to $\mathbf{d}$.

Solution ▶ The projection $\mathbf{u}_1$ of $\mathbf{u}$ on $\mathbf{d}$ is

$$\mathbf{u}_1 = \text{proj}_{\mathbf{d}}\, \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{d}}{\|\mathbf{d}\|^2}\mathbf{d} = \frac{2 + 3 + 3}{1^2 + (-1)^2 + 3^2}\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \frac{8}{11}\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

Hence $\mathbf{u}_2 = \mathbf{u} - \mathbf{u}_1 = \frac{1}{11}\begin{bmatrix} 14 \\ -25 \\ -13 \end{bmatrix}$, and this is orthogonal to $\mathbf{d}$ by Theorem 4 (alternatively, observe that $\mathbf{d} \cdot \mathbf{u}_2 = 0$). Since $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$, we are done.

EXAMPLE 8



Find the shortest distance (see diagram) from the point $P(1, 3, -2)$ to the line through $P_0(2, 0, -1)$ with direction vector $\mathbf{d} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. Also find the point Q that lies on the line and is closest to P.

Solution ▸ Let $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$ denote the vector from P_0 to P, and let $\mathbf{u}_1$ denote the projection of $\mathbf{u}$ on $\mathbf{d}$. Thus

$$\mathbf{u}_1 = \frac{\mathbf{u} \cdot \mathbf{d}}{\|\mathbf{d}\|^2}\mathbf{d} = \frac{-1 - 3 + 0}{1^2 + (-1)^2 + 0^2}\mathbf{d} = -2\mathbf{d} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$$

by Theorem 4. We see geometrically that the point Q on the line is closest to P, so the distance is

$$\left\|\overrightarrow{QP}\right\| = \|\mathbf{u} - \mathbf{u}_1\| = \left\|\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right\| = \sqrt{3}$$

To find the coordinates of Q, let $\mathbf{p}_0$ and $\mathbf{q}$ denote the vectors of P_0 and Q, respectively. Then $\mathbf{p}_0 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ and $\mathbf{q} = \mathbf{p}_0 + \mathbf{u}_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$.

Hence $Q(0, 2, -1)$ is the required point. It can be checked that the distance from Q to P is $\sqrt{3}$, as expected.

Planes

It is evident geometrically that among all planes that are perpendicular to a given straight line there is exactly one containing any given point. This fact can be used to give a very simple description of a plane. To do this, it is necessary to introduce the following notion:

Definition 4.7

A nonzero vector $\mathbf{n}$ *is called a* **normal** *for a plane if it is orthogonal to every vector in the plane.*

For example, the coordinate vector $\mathbf{k}$ is a normal for the x-y plane.

Given a point $P_0 = P_0(x_0, y_0, z_0)$ and a nonzero vector $\mathbf{n}$, there is a unique plane through P_0 with normal $\mathbf{n}$, shaded in Figure 6. A point $P = P(x, y, z)$ lies on this plane if and only if the vector $\overrightarrow{P_0P}$ is orthogonal to $\mathbf{n}$—that is, if and only if $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$. Because $\overrightarrow{P_0P} = \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix}$ this gives the following result:

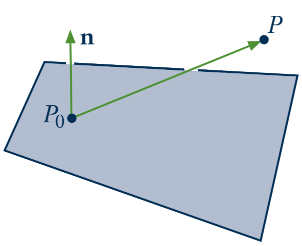


FIGURE 6

Scalar Equation of a Plane

The plane through $P_0(x_0, y_0, z_0)$ *with normal* $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \mathbf{0}$ *as a normal vector is given by*

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

In other words, a point $P(x, y, z)$ *is on this plane if and only if* x, y, *and* z *satisfy this equation.*

EXAMPLE 9

Find an equation of the plane through $P_0(1, -1, 3)$ with $\mathbf{n} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ as normal.

Solution ▶ Here the general scalar equation becomes

$$3(x - 1) - (y + 1) + 2(z - 3) = 0$$

This simplifies to $3x - y + 2z = 10$.

If we write $d = ax_0 + by_0 + cz_0$, the scalar equation shows that every plane with normal $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ has a linear equation of the form

$$ax + by + cz = d \tag{$*$}$$

for some constant d. Conversely, the graph of this equation is a plane with $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ as a normal vector (assuming that a, b, and c are not all zero).

EXAMPLE 10

Find an equation of the plane through $P_0(3, -1, 2)$ that is parallel to the plane with equation $2x - 3y = 6$.

Solution ▶ The plane with equation $2x - 3y = 6$ has normal $\mathbf{n} = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$. Because the two planes are parallel, $\mathbf{n}$ serves as a normal for the plane we seek, so the equation is $2x - 3y = d$ for some d by equation $(*)$. Insisting that $P_0(3, -1, 2)$ lies on the plane determines d; that is, $d = 2 \cdot 3 - 3(-1) = 9$. Hence, the equation is $2x - 3y = 9$.

Consider points $P_0(x_0, y_0, z_0)$ and $P(x, y, z)$ with vectors $\mathbf{p}_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$ and $\mathbf{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Given a nonzero vector $\mathbf{n}$, the scalar equation of the plane through $P_0(x_0, y_0, z_0)$ with normal $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ takes the vector form:

Vector Equation of a Plane

The plane with normal $\mathbf{n} \neq \mathbf{0}$ through the point with vector $\mathbf{p}_0$ is given by

$$\mathbf{n} \cdot (\mathbf{p} - \mathbf{p}_0) = 0$$

In other words, the point with vector $\mathbf{p}$ is on the plane if and only if $\mathbf{p}$ satisfies this condition.

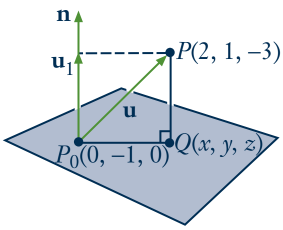
Moreover, equation $(*)$ translates as follows:

Every plane with normal $\mathbf{n}$ has vector equation $\mathbf{n} \cdot \mathbf{p} = d$ for some number d.

This is useful in the second solution of Example 11.

EXAMPLE 11

Find the shortest distance from the point $P(2, 1, -3)$ to the plane with equation $3x - y + 4z = 1$. Also find the point Q on this plane closest to P.



Solution 1 ▶ The plane in question has normal $\mathbf{n} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$. Choose any point P_0 on the plane—say $P_0(0, -1, 0)$—and let $Q(x, y, z)$ be the point on the plane closest to P (see the diagram). The vector from P_0 to P is $\mathbf{u} = \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix}$. Now erect $\mathbf{n}$ with its tail at P_0. Then $\overrightarrow{QP} = \mathbf{u}_1$ and $\mathbf{u}_1$ is the projection of $\mathbf{u}$ on $\mathbf{n}$:

$$\mathbf{u}_1 = \frac{\mathbf{n} \cdot \mathbf{u}}{\|\mathbf{n}\|^2}\mathbf{n} = \frac{-8}{26}\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} = \frac{-4}{13}\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$$

Hence the distance is $\left\|\overrightarrow{QP}\right\| = \|\mathbf{u}_1\| = \frac{4\sqrt{26}}{13}$. To calculate the point Q, let $\mathbf{q} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mathbf{p}_0 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$ be the vectors of Q and P_0. Then

$$\mathbf{q} = \mathbf{p}_0 + \mathbf{u} - \mathbf{u}_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix} + \frac{4}{13}\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{38}{13} \\ \frac{9}{13} \\ \frac{-23}{13} \end{bmatrix}$$

This gives the coordinates of $Q(\frac{38}{13}, \frac{9}{13}, \frac{-23}{13})$.

Solution 2 ▶ Let $\mathbf{q} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mathbf{p} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$ be the vectors of Q and P. Then Q is on the line through P with direction vector $\mathbf{n}$, so $\mathbf{q} = \mathbf{p} + t\mathbf{n}$ for some scalar t. In addition, Q lies on the plane, so $\mathbf{n} \cdot \mathbf{q} = 1$. This determines t:

$$1 = \mathbf{n} \cdot \mathbf{q} = \mathbf{n} \cdot (\mathbf{p} + t\mathbf{n}) = \mathbf{n} \cdot \mathbf{p} + t\|\mathbf{n}\|^2 = -7 + t(26)$$

This gives $t = \frac{8}{26} = \frac{4}{13}$, so

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{q} = \mathbf{p} + t\mathbf{n} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + \frac{4}{13}\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} 38 \\ 9 \\ -23 \end{bmatrix}$$

as before. This determines Q (in the diagram), and the reader can verify that the required distance is $\left\|\overrightarrow{QP}\right\| = \frac{4}{13}\sqrt{26}$, as before.

The Cross Product

If P, Q, and R are three distinct points in $\mathbb{R}^3$ that are not all on some line, it is clear geometrically that there is a unique plane containing all three. The vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ both lie in this plane, so finding a normal amounts to finding a nonzero vector orthogonal to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. The cross product provides a systematic way to do this.

Definition 4.8

Given vectors $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ *and* $\mathbf{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$, *define the* **cross product** $\mathbf{v}_1 \times \mathbf{v}_2$ *by*

$$\mathbf{v}_1 \times \mathbf{v}_2 = \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ -(x_1 z_2 - z_1 x_2) \\ x_1 y_2 - y_1 x_2 \end{bmatrix}.$$

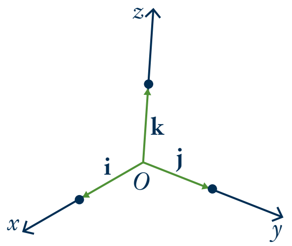


FIGURE 7

(Because it is a vector, $\mathbf{v}_1 \times \mathbf{v}_2$ is often called the **vector product**.) There is an easy way to remember this definition using the **coordinate vectors**:

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

They are vectors of length 1 pointing along the positive x, y, and z axes, respectively, as in Figure 7. The reason for the name is that any vector can be written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

With this, the cross product can be described as follows:

Determinant Form of the Cross Product

If $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ *and* $\mathbf{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ *are two vectors, then*

$$\mathbf{v}_1 \times \mathbf{v}_2 = \det \begin{bmatrix} \mathbf{i} & x_1 & x_2 \\ \mathbf{j} & y_1 & y_2 \\ \mathbf{k} & z_1 & z_2 \end{bmatrix} = \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} x_1 & x_2 \\ z_1 & z_2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \mathbf{k}$$

where the determinant is expanded along the first column.

EXAMPLE 12

If $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}$, then

$$\begin{aligned} \mathbf{v}_1 \times \mathbf{v}_2 = \det \begin{bmatrix} \mathbf{i} & 2 & 1 \\ \mathbf{j} & -1 & 3 \\ \mathbf{k} & 4 & 7 \end{bmatrix} &= \begin{vmatrix} -1 & 3 \\ 4 & 7 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ 4 & 7 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix} \mathbf{k} \\ &= -19\mathbf{i} - 10\mathbf{j} + 7\mathbf{k} \\ &= \begin{bmatrix} -19 \\ -10 \\ 7 \end{bmatrix} \end{aligned}$$

Observe that $\mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$ in Example 12. This holds in general as can be verified directly by computing $\mathbf{v} \cdot (\mathbf{v} \times \mathbf{w})$ and $\mathbf{w} \cdot (\mathbf{v} \times \mathbf{w})$, and is recorded as the first part of the following theorem. It will follow from a more general result which, together with the second part, will be proved in Section 4.3 where a more detailed study of the cross product will be undertaken.

Theorem 5

Let $\mathbf{v}$ *and* $\mathbf{w}$ *be vectors in* $\mathbb{R}^3$.

1. $\mathbf{v} \times \mathbf{w}$ *is a vector orthogonal to both* $\mathbf{v}$ *and* $\mathbf{w}$.
2. *If* $\mathbf{v}$ *and* $\mathbf{w}$ *are nonzero, then* $\mathbf{v} \times \mathbf{w} = \mathbf{0}$ *if and only if* $\mathbf{v}$ *and* $\mathbf{w}$ *are parallel.*

It is interesting to contrast Theorem 5(2) with the assertion (in Theorem 3) that

$$\mathbf{v} \cdot \mathbf{w} = 0 \quad \text{if and only if } \mathbf{v} \text{ and } \mathbf{w} \text{ are orthogonal.}$$

EXAMPLE 13

Find the equation of the plane through $P(1, 3, -2)$, $Q(1, 1, 5)$, and $R(2, -2, 3)$.

Solution ▶ The vectors $\overrightarrow{PQ} = \begin{bmatrix} 0 \\ -2 \\ 7 \end{bmatrix}$ and $\overrightarrow{PR} = \begin{bmatrix} 1 \\ -5 \\ 5 \end{bmatrix}$ lie in the plane, so

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \det \begin{bmatrix} \mathbf{i} & 0 & 1 \\ \mathbf{j} & -2 & -5 \\ \mathbf{k} & 7 & 5 \end{bmatrix} = 25\mathbf{i} + 7\mathbf{j} + 2\mathbf{k} = \begin{bmatrix} 25 \\ 7 \\ 2 \end{bmatrix}$$

is a normal for the plane (being orthogonal to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$). Hence the plane has equation

$$25x + 7y + 2z = d \quad \text{for some number } d.$$

Since $P(1, 3, -2)$ lies in the plane we have $25 \cdot 1 + 7 \cdot 3 + 2(-2) = d$. Hence $d = 42$ and the equation is $25x + 7y + 2z = 42$. Incidentally, the same equation is obtained (verify) if $\overrightarrow{QP}$ and $\overrightarrow{QR}$, or $\overrightarrow{RP}$ and $\overrightarrow{RQ}$, are used as the vectors in the plane.

EXAMPLE 14

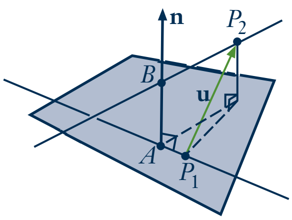
Find the shortest distance between the nonparallel lines

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Then find the points A and B on the lines that are closest together.

Solution ▶ Direction vectors for the two lines are $\mathbf{d}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{d}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, so

$$\mathbf{n} = \mathbf{d}_1 \times \mathbf{d}_2 = \det \begin{bmatrix} \mathbf{i} & 2 & 1 \\ \mathbf{j} & 0 & 1 \\ \mathbf{k} & 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$$



is perpendicular to both lines. Consider the plane shaded in the diagram containing the first line with $\mathbf{n}$ as normal. This plane contains $P_1(1, 0, -1)$ and is parallel to the second line. Because $P_2(3, 1, 0)$ is on the second line, the distance in question is just the shortest distance between $P_2(3, 1, 0)$ and this plane. The vector $\mathbf{u}$ from P_1 to P_2 is $\mathbf{u} = \overrightarrow{P_1P_2} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and so, as in Example 11, the distance is the length of the projection of $\mathbf{u}$ on $\mathbf{n}$.

$$\text{distance} = \left\| \frac{\mathbf{u} \cdot \mathbf{n}}{\|\mathbf{n}\|^2}\mathbf{n} \right\| = \frac{|\mathbf{u} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{3}{\sqrt{14}} = \frac{3\sqrt{14}}{14}$$

Note that it is necessary that $\mathbf{n} = \mathbf{d}_1 \times \mathbf{d}_2$ be nonzero for this calculation to be possible. As is shown later (Theorem 4 Section 4.3), this is guaranteed by the fact that $\mathbf{d}_1$ and $\mathbf{d}_2$ are *not* parallel.

The points A and B have coordinates $A(1 + 2t, 0, t - 1)$ and $B(3 + s, 1 + s, -s)$ for some s and t, so $\overrightarrow{AB} = \begin{bmatrix} 2 + s - 2t \\ 1 + s \\ 1 - s - t \end{bmatrix}$. This vector is orthogonal to both $\mathbf{d}_1$ and $\mathbf{d}_2$, and the conditions $\overrightarrow{AB} \cdot \mathbf{d}_1 = 0$ and $\overrightarrow{AB} \cdot \mathbf{d}_2 = 0$ give equations $5t - s = 5$ and $t - 3s = 2$. The solution is $s = \frac{-5}{14}$ and $t = \frac{13}{14}$, so the points are $A\left(\frac{40}{14}, 0, \frac{-1}{14}\right)$ and $B\left(\frac{37}{14}, \frac{9}{14}, \frac{5}{14}\right)$. We have $\|\overrightarrow{AB}\| = \frac{3\sqrt{14}}{14}$, as before.

EXERCISES 4.2

1. Compute $\mathbf{u} \cdot \mathbf{v}$ where:

   (a) $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

   ◆(b) $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \mathbf{v} = \mathbf{u}$

   (c) $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

   ◆(d) $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 6 \\ -7 \\ -5 \end{bmatrix}$

   (e) $\mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

   ◆(f) $\mathbf{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \mathbf{v} = \mathbf{0}$

2. Find the angle between the following pairs of vectors.

   (a) $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ ◆(b) $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -6 \\ 2 \\ 0 \end{bmatrix}$

   (c) $\mathbf{u} = \begin{bmatrix} 7 \\ -1 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$

   ◆(d) $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}$ (e) $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

   ◆(f) $\mathbf{u} = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 5\sqrt{2} \\ -7 \\ -1 \end{bmatrix}$

3. Find all real numbers x such that:

   (a) $\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} x \\ -2 \\ 1 \end{bmatrix}$ are orthogonal.

   ◆(b) $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ x \\ 2 \end{bmatrix}$ are at an angle of $\frac{\pi}{3}$.

4. Find all vectors $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ orthogonal to both:

(a) $\mathbf{u}_1 = \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

◆(b) $\mathbf{u}_1 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

(c) $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -4 \\ 0 \\ 2 \end{bmatrix}$

◆(d) $\mathbf{u}_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

5. Find two orthogonal vectors that are both orthogonal to $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

6. Consider the triangle with vertices $P(2, 0, -3)$, $Q(5, -2, 1)$, and $R(7, 5, 3)$.

(a) Show that it is a right-angled triangle.

◆(b) Find the lengths of the three sides and verify the Pythagorean theorem.

7. Show that the triangle with vertices $A(4, -7, 9)$, $B(6, 4, 4)$, and $C(7, 10, -6)$ is not a right-angled triangle.

8. Find the three internal angles of the triangle with vertices:

(a) $A(3, 1, -2)$, $B(3, 0, -1)$, and $C(5, 2, -1)$

◆(b) $A(3, 1, -2)$, $B(5, 2, -1)$, and $C(4, 3, -3)$

9. Show that the line through $P_0(3, 1, 4)$ and $P_1(2, 1, 3)$ is perpendicular to the line through $P_2(1, -1, 2)$ and $P_3(0, 5, 3)$.

10. In each case, compute the projection of $\mathbf{u}$ on $\mathbf{v}$.

(a) $\mathbf{u} = \begin{bmatrix} 5 \\ 7 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

◆(b) $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$

c) $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$

◆(d) $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -6 \\ 4 \\ 2 \end{bmatrix}$

11. In each case, write $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$, where $\mathbf{u}_1$ is parallel to $\mathbf{v}$ and $\mathbf{u}_2$ is orthogonal to $\mathbf{v}$.

(a) $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$

◆(b) $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$

(c) $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$

◆(d) $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -6 \\ 4 \\ -1 \end{bmatrix}$

12. Calculate the distance from the point P to the line in each case and find the point Q on the line closest to P.

(a) $P(3, 2, -1)$ line: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + t\begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$

◆(b) $P(1, -1, 3)$ line: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$

13. Compute $\mathbf{u} \times \mathbf{v}$ where:

(a) $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

◆(b) $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -6 \\ 2 \\ 0 \end{bmatrix}$

(c) $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

◆(d) $\mathbf{u} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$

14. Find an equation of each of the following planes.

(a) Passing through $A(2, 1, 3)$, $B(3, -1, 5)$, and $C(1, 2, -3)$.

◆(b) Passing through $A(1, -1, 6)$, $B(0, 0, 1)$, and $C(4, 7, -11)$.

(c) Passing through $P(2, -3, 5)$ and parallel to the plane with equation $3x - 2y - z = 0$.

◆(d) Passing through $P(3, 0, -1)$ and parallel to the plane with equation $2x - y + z = 3$.

(e) Containing $P(3, 0, -1)$ and the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

◆(f) Containing $P(2, 1, 0)$ and the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

(g) Containing the lines $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

◆(h) Containing the lines $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$.

(i) Each point of which is equidistant from $P(2, -1, 3)$ and $Q(1, 1, -1)$.

◆(j) Each point of which is equidistant from $P(0, 1, -1)$ and $Q(2, -1, -3)$.

15. In each case, find a vector equation of the line.

(a) Passing through $P(3, -1, 4)$ and perpendicular to the plane $3x - 2y - z = 0$.

◆(b) Passing through $P(2, -1, 3)$ and perpendicular to the plane $2x + y = 1$.

(c) Passing through $P(0, 0, 0)$ and perpendicular to the lines $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}$.

◆(d) Passing through $P(1, 1, -1)$, and perpendicular to the lines $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ -2 \end{bmatrix} + t\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$.

(e) Passing through $P(2, 1, -1)$, intersecting the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + t\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$, and perpendicular to that line.

◆(f) Passing through $P(1, 1, 2)$, intersecting the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and perpendicular to that line.

16. In each case, find the shortest distance from the point P to the plane and find the point Q on the plane closest to P.

(a) $P(2, 3, 0)$; plane with equation $5x + y + z = 1$.

◆(b) $P(3, 1, -1)$; plane with equation $2x + y - z = 6$.

17. (a) Does the line through $P(1, 2, -3)$ with direction vector $\mathbf{d} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ lie in the plane $2x - y - z = 3$? Explain.

◆(b) Does the plane through $P(4, 0, 5)$, $Q(2, 2, 1)$, and $R(1, -1, 2)$ pass through the origin? Explain.

18. Show that every plane containing $P(1, 2, -1)$ and $Q(2, 0, 1)$ must also contain $R(-1, 6, -5)$.

19. Find the equations of the line of intersection of the following planes.

(a) $2x - 3y + 2z = 5$ and $x + 2y - z = 4$.

◆(b) $3x + y - 2z = 1$ and $x + y + z = 5$.

20. In each case, find all points of intersection of the given plane and the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + t\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$.

(a) $x - 3y + 2z = 4$ ◆(b) $2x - y - z = 5$

(c) $3x - y + z = 8$ ◆(d) $-x - 4y - 3z = 6$

21. Find the equation of *all* planes:

(a) Perpendicular to the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$.

◆(b) Perpendicular to the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$.

(c) Containing the origin.

◆(d) Containing $P(3, 2, -4)$.

(e) Containing $P(1, 1, -1)$ and $Q(0, 1, 1)$.

◆(f) Containing $P(2, -1, 1)$ and $Q(1, 0, 0)$.

(g) Containing the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

◆(h) Containing the line $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} + t\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$.

22. If a plane contains two distinct points P_1 and P_2, show that it contains every point on the line through P_1 and P_2.

23. Find the shortest distance between the following pairs of parallel lines.

(a) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$; $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$

◆(b) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} + t\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$; $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} + t\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$

24. Find the shortest distance between the following pairs of nonparallel lines and find the points on the lines that are closest together.

(a) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$; $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

◆(b) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$; $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + t\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} + s\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$; $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

◆(d) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + s\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$; $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

25. Show that two lines in the plane with slopes m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$. [*Hint*: Example 11 Section 4.1.]

26. (a) Show that, of the four diagonals of a cube, no pair is perpendicular.

◆(b) Show that each diagonal is perpendicular to the face diagonals it does not meet.

27. Given a rectangular solid with sides of lengths 1, 1, and $\sqrt{2}$, find the angle between a diagonal and one of the longest sides.

◆28. Consider a rectangular solid with sides of lengths a, b, and c. Show that it has two orthogonal diagonals if and only if the sum of two of a^2, b^2, and c^2 equals the third.

29. Let A, B, and $C(2, -1, 1)$ be the vertices of a triangle where $\overrightarrow{AB}$ is parallel to $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\overrightarrow{AC}$ is parallel to $\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$, and angle $C = 90°$. Find the equation of the line through B and C.

30. If the diagonals of a parallelogram have equal length, show that the parallelogram is a rectangle.

31. Given $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in component form, show that the projections of $\mathbf{v}$ on $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are $x\mathbf{i}$, $y\mathbf{j}$, and $z\mathbf{k}$, respectively.

32. (a) Can $\mathbf{u} \cdot \mathbf{v} = -7$ if $\|\mathbf{u}\| = 3$ and $\|\mathbf{v}\| = 2$? Defend your answer.

(b) Find $\mathbf{u} \cdot \mathbf{v}$ if $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$, $\|\mathbf{v}\| = 6$, and the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\frac{2\pi}{3}$.

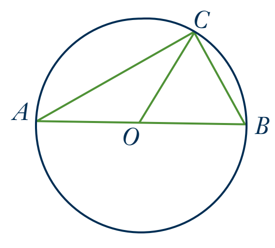
33. Show that $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2$ for any vectors $\mathbf{u}$ and $\mathbf{v}$.

34. (a) Show that $\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2)$ for any vectors $\mathbf{u}$ and $\mathbf{v}$.

◆(b) What does this say about parallelograms?

35. Show that if the diagonals of a parallelogram are perpendicular, it is necessarily a rhombus. [*Hint*: Example 5.]

36. Let A and B be the end points of a diameter of a circle (see the diagram). If C is any point on the circle, show that AC and BC are perpendicular. [*Hint*: Express $\overrightarrow{AC}$ and $\overrightarrow{BC}$ in terms of $\mathbf{u} = \overrightarrow{OA}$ and $\mathbf{v} = \overrightarrow{OC}$, where O is the centre.]



37. Show that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, if and only if $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

38. Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be pairwise orthogonal vectors.

(a) Show that $\|\mathbf{u} + \mathbf{v} + \mathbf{w}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2$.

◆(b) If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are all the same length, show that they all make the same angle with $\mathbf{u} + \mathbf{v} + \mathbf{w}$.

39. (a) Show that $\mathbf{n} = \begin{bmatrix} a \\ b \end{bmatrix}$ is orthogonal to every vector along the line $ax + by + c = 0$.

◆(b) Show that the shortest distance from $P_0(x_0, y_0)$ to the line is $\dfrac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$. [*Hint*: If P_1 is on the line, project $\mathbf{u} = \overrightarrow{P_1P_0}$ on $\mathbf{n}$.]

40. Assume $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors that are not parallel. Show that $\mathbf{w} = \|\mathbf{u}\|\mathbf{v} + \|\mathbf{v}\|\mathbf{u}$ is a nonzero vector that bisects the angle between $\mathbf{u}$ and $\mathbf{v}$.

41. Let α, β, and γ be the angles a vector $\mathbf{v} \neq \mathbf{0}$ makes with the positive x, y, and z axes, respectively. Then $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are called the **direction cosines** of the vector $\mathbf{v}$.

(a) If $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, show that $\cos\alpha = \dfrac{a}{\|\mathbf{v}\|}$, $\cos\beta = \dfrac{b}{\|\mathbf{v}\|}$, and $\cos\gamma = \dfrac{c}{\|\mathbf{v}\|}$.

◆(b) Show that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

42. Let $\mathbf{v} \neq \mathbf{0}$ be any nonzero vector and suppose that a vector $\mathbf{u}$ can be written as $\mathbf{u} = \mathbf{p} + \mathbf{q}$, where $\mathbf{p}$ is parallel to $\mathbf{v}$ and $\mathbf{q}$ is orthogonal to $\mathbf{v}$. Show that $\mathbf{p}$ must equal the projection of $\mathbf{u}$ on $\mathbf{v}$. [*Hint*: Argue as in the proof of Theorem 4.]

43. Let $\mathbf{v} \neq \mathbf{0}$ be a nonzero vector and let $a \neq 0$ be a scalar. If $\mathbf{u}$ is any vector, show that the projection of $\mathbf{u}$ on $\mathbf{v}$ equals the projection of $\mathbf{u}$ on $a\mathbf{v}$.

44. (a) Show that the **Cauchy-Schwarz inequality** $|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\|$ holds for all vectors $\mathbf{u}$ and $\mathbf{v}$. [*Hint*: $|\cos\theta| \leq 1$ for all angles θ.]

(b) Show that $|\mathbf{u} \cdot \mathbf{v}| = \|\mathbf{u}\|\|\mathbf{v}\|$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are parallel. [*Hint*: When is $\cos\theta = \pm 1$?]

(c) Show that $|x_1x_2 + y_1y_2 + z_1z_2| \leq \sqrt{x_1^2 + y_1^2 + z_1^2}\sqrt{x_2^2 + y_2^2 + z_2^2}$ holds for all numbers x_1, x_2, y_1, y_2, z_1, and z_2.

◆(d) Show that $|xy + yz + zx| \leq x^2 + y^2 + z^2$ for all x, y, and z.

(e) Show that $(x + y + z)^2 \leq 3(x^2 + y^2 + z^2)$ holds for all x, y, and z.

45. Prove that the **triangle inequality** $\|\mathbf{u} \cdot \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ holds for all vectors $\mathbf{u}$ and $\mathbf{v}$. [*Hint*: Consider the triangle with $\mathbf{u}$ and $\mathbf{v}$ as two sides.]

SECTION 4.3 More on the Cross Product

The cross product $\mathbf{v} \times \mathbf{w}$ of two $\mathbb{R}^3$-vectors $\mathbf{v} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ was defined in Section 4.2 where we observed that it can be best remembered using a determinant:

$$\mathbf{v} \times \mathbf{w} = \det\begin{bmatrix} \mathbf{i} & x_1 & x_2 \\ \mathbf{j} & y_1 & y_2 \\ \mathbf{k} & z_1 & z_2 \end{bmatrix} = \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} x_1 & x_2 \\ z_1 & z_2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}\mathbf{k} \qquad (*)$$

Here $\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{k} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ are the coordinate vectors, and the determinant is expanded along the first column. We observed (but did not prove) in Theorem 5 Section 4.2 that $\mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$. This follows easily from the next result.

Theorem 1

If $\mathbf{u} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$, *and* $\mathbf{w} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$, *then* $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \det \begin{bmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ z_0 & z_1 & z_2 \end{bmatrix}$.

PROOF

Recall that $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ is computed by multiplying corresponding components of $\mathbf{u}$ and $\mathbf{v} \times \mathbf{w}$ and then adding. Using (∗), the result is:

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = x_0\left(\begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix}\right) + y_0\left(-\begin{vmatrix} x_1 & x_2 \\ z_1 & z_2 \end{vmatrix}\right) + z_0\left(\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}\right) = \det \begin{bmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ z_0 & z_1 & z_2 \end{bmatrix}$$

where the last determinant is expanded along column 1.

The result in Theorem 1 can be succinctly stated as follows: If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are three vectors in $\mathbb{R}^3$, then

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \det[\mathbf{u}\ \mathbf{v}\ \mathbf{w}]$$

where $[\mathbf{u}\ \mathbf{v}\ \mathbf{w}]$ denotes the matrix with $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ as its columns. Now it is clear that $\mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$ because the determinant of a matrix is zero if two columns are identical.

Because of (∗) and Theorem 1, several of the following properties of the cross product follow from properties of determinants (they can also be verified directly).

Theorem 2

Let $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *denote arbitrary vectors in* $\mathbb{R}^3$.

1. $\mathbf{u} \times \mathbf{v}$ *is a vector.*
2. $\mathbf{u} \times \mathbf{v}$ *is orthogonal to both* $\mathbf{u}$ *and* $\mathbf{v}$.
3. $\mathbf{u} \times \mathbf{0} = \mathbf{0} = \mathbf{0} \times \mathbf{u}$.
4. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$.
5. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$.
6. $(k\mathbf{u}) \times \mathbf{v} = k(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times (k\mathbf{v})$ *for any scalar* k.
7. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$.
8. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = (\mathbf{v} \times \mathbf{u}) + (\mathbf{w} \times \mathbf{u})$.

PROOF

(1) is clear; (2) follows from Theorem 1; and (3) and (4) follow because the determinant of a matrix is zero if one column is zero or if two columns are identical. If two columns are interchanged, the determinant changes sign, and this proves (5). The proofs of (6), (7), and (8) are left as Exercise 15.

We now come to a fundamental relationship between the dot and cross products.

Joseph Louis Lagrange.
Photo © Corbis.

Theorem 3

Lagrange Identity[11]

If **u** *and* **v** *are any two vectors in* $\mathbb{R}^3$, *then*

$$\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$$

PROOF

Given **u** and **v**, introduce a coordinate system and write $\mathbf{u} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ in component form. Then all the terms in the identity can be computed in terms of the components. The detailed proof is left as Exercise 14.

An expression for the magnitude of the vector $\mathbf{u} \times \mathbf{v}$ can be easily obtained from the Lagrange identity. If θ is the angle between **u** and **v**, substituting $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta$ into the Lagrange identity gives

$$\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - \|\mathbf{u}\|^2\|\mathbf{v}\|^2 \cos^2\theta = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 \sin^2\theta$$

using the fact that $1 - \cos^2\theta = \sin^2\theta$. But $\sin\theta$ is nonnegative on the range $0 \le \theta \le \pi$, so taking the positive square root of both sides gives

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin\theta$$

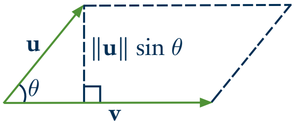


FIGURE 1

This expression for $\|\mathbf{u} \times \mathbf{v}\|$ makes no reference to a coordinate system and, moreover, it has a nice geometrical interpretation. The parallelogram determined by the vectors **u** and **v** has base length $\|\mathbf{v}\|$ and altitude $\|\mathbf{u}\| \sin\theta$ (see Figure 1). Hence the area of the parallelogram formed by **u** and **v** is

$$(\|\mathbf{u}\| \sin\theta)\, \|\mathbf{v}\| = \|\mathbf{u} \times \mathbf{v}\|$$

This proves the first part of Theorem 4.

Theorem 4

If **u** *and* **v** *are two nonzero vectors and* θ *is the angle between* **u** *and* **v**, *then*

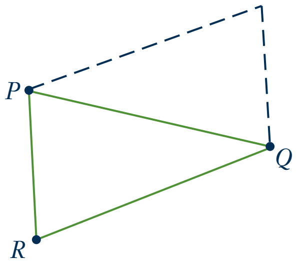
1. $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin\theta$ = *area of the parallelogram determined by* **u** *and* **v**.
2. **u** *and* **v** *are parallel if and only if* $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

11 Joseph Louis Lagrange (1736–1813) was born in Italy and spent his early years in Turin. At the age of 19 he solved a famous problem by inventing an entirely new method, known today as the calculus of variations, and went on to become one of the greatest mathematicians of all time. His work brought a new level of rigour to analysis and his *Mécanique Analytique* is a masterpiece in which he introduced methods still in use. In 1766 he was appointed to the Berlin Academy by Frederik the Great who asserted that the "greatest mathematician in Europe" should be at the court of the "greatest king in Europe." After the death of Frederick, Lagrange went to Paris at the invitation of Louis XVI. He remained there throughout the revolution and was made a count by Napoleon.

PROOF OF (2)

By (1), $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if the area of the parallelogram is zero. By Figure 1 the area vanishes if and only if $\mathbf{u}$ and $\mathbf{v}$ have the same or opposite direction—that is, if and only if they are parallel.

EXAMPLE 1



Find the area of the triangle with vertices $P(2, 1, 0)$, $Q(3, -1, 1)$, and $R(1, 0, 1)$.

Solution ▶ We have $\overrightarrow{RP} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $\overrightarrow{RQ} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$. The area of the triangle is half the area of the parallelogram (see the diagram), and so equals $\frac{1}{2}\|\overrightarrow{RP} \times \overrightarrow{RQ}\|$. We have

$$\overrightarrow{RP} \times \overrightarrow{RQ} = \det\begin{bmatrix} \mathbf{i} & 1 & 2 \\ \mathbf{j} & 1 & -1 \\ \mathbf{k} & -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix},$$

so the area of the triangle is $\frac{1}{2}\|\overrightarrow{RP} \times \overrightarrow{RQ}\| = \frac{1}{2}\sqrt{1+4+9} = \frac{1}{2}\sqrt{14}$.

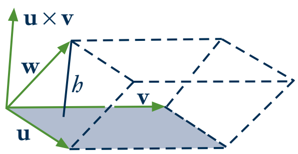


FIGURE 2

If three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are given, they determine a "squashed" rectangular solid called a **parallelepiped** (Figure 2), and it is often useful to be able to find the volume of such a solid. The base of the solid is the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$, so it has area $A = \|\mathbf{u} \times \mathbf{v}\|$ by Theorem 4. The height of the solid is the length h of the projection of $\mathbf{w}$ on $\mathbf{u} \times \mathbf{v}$. Hence

$$h = \left|\frac{\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})}{\|\mathbf{u} \times \mathbf{v}\|^2}\right| \|\mathbf{u} \times \mathbf{v}\| = \frac{|\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})|}{\|\mathbf{u} \times \mathbf{v}\|} = \frac{|\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})|}{A}$$

Thus the volume of the parallelepiped is $hA = |\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})|$. This proves

Theorem 5

The volume of the parallelepiped determined by three vectors $\mathbf{w}$, $\mathbf{u}$, and $\mathbf{v}$ (Figure 2) is given by $|\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})|$.

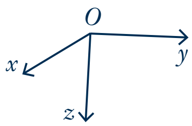
EXAMPLE 2

Find the volume of the parallelepiped determined by the vectors

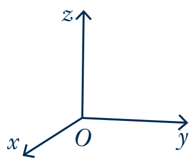
$$\mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \mathbf{v} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

Solution ▶ By Theorem 1, $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = \det\begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = -3$.

Hence the volume is $|\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})| = |-3| = 3$ by Theorem 5.



Left-hand system



Right-hand system

FIGURE 3

We can now give an intrinsic description of the cross product $\mathbf{u} \times \mathbf{v}$. Its magnitude $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin \theta$ is coordinate-free. If $\mathbf{u} \times \mathbf{v} \neq \mathbf{0}$, its direction is very nearly determined by the fact that it is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ and so points along the line normal to the plane determined by $\mathbf{u}$ and $\mathbf{v}$. It remains only to decide which of the two possible directions is correct.

Before this can be done, the basic issue of how coordinates are assigned must be clarified. When coordinate axes are chosen in space, the procedure is as follows: An origin is selected, two perpendicular lines (the x and y axes) are chosen through the origin, and a positive direction on each of these axes is selected quite arbitrarily. Then the line through the origin normal to this x-y plane is called the z axis, but there is a choice of which direction on this axis is the positive one. The two possibilities are shown in Figure 3, and it is a standard convention that cartesian coordinates are always **right-hand coordinate systems**. The reason for this terminology is that, in such a system, if the z axis is grasped in the right hand with the thumb pointing in the positive z direction, then the fingers curl around from the positive x axis to the positive y axis (through a right angle).

Suppose now that $\mathbf{u}$ and $\mathbf{v}$ are given and that θ is the angle between them (so $0 \leq \theta \leq \pi$). Then the direction of $\|\mathbf{u} \times \mathbf{v}\|$ is given by the right-hand rule.

Right-hand Rule

If the vector $\mathbf{u} \times \mathbf{v}$ is grasped in the right hand and the fingers curl around from $\mathbf{u}$ to $\mathbf{v}$ through the angle θ, the thumb points in the direction for $\mathbf{u} \times \mathbf{v}$.

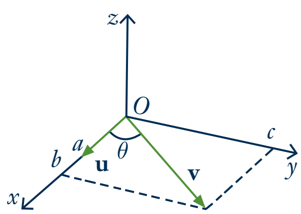


FIGURE 4

To indicate why this is true, introduce coordinates in $\mathbb{R}^3$ as follows: Let $\mathbf{u}$ and $\mathbf{v}$ have a common tail O, choose the origin at O, choose the x axis so that $\mathbf{u}$ points in the positive x direction, and then choose the y axis so that $\mathbf{v}$ is in the x-y plane and the positive y axis is on the same side of the x axis as $\mathbf{v}$. Then, in this system, $\mathbf{u}$ and $\mathbf{v}$ have component form $\mathbf{u} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} b \\ c \\ 0 \end{bmatrix}$ where $a > 0$ and $c > 0$. The situation is depicted in Figure 4. The right-hand rule asserts that $\mathbf{u} \times \mathbf{v}$ should point in the positive z direction. But our definition of $\mathbf{u} \times \mathbf{v}$ gives

$$\mathbf{u} \times \mathbf{v} = \det \begin{bmatrix} \mathbf{i} & a & b \\ \mathbf{j} & 0 & c \\ \mathbf{k} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ ac \end{bmatrix} = (ac)\mathbf{k}$$

and $(ac)\mathbf{k}$ has the positive z direction because $ac > 0$.

EXERCISES 4.3

1. If $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the coordinate vectors, verify that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$.

2. Show that $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ need not equal $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ by calculating both when
$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

3. Find two unit vectors orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ if:

(a) $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ ◆(b) $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

4. Find the area of the triangle with the following vertices.
   (a) $A(3, -1, 2)$, $B(1, 1, 0)$, and $C(1, 2, -1)$
   ◆(b) $A(3, 0, 1)$, $B(5, 1, 0)$, and $C(7, 2, -1)$
   (c) $A(1, 1, -1)$, $B(2, 0, 1)$, and $C(1, -1, 3)$
   ◆(d) $A(3, -1, 1)$, $B(4, 1, 0)$, and $C(2, -3, 0)$

5. Find the volume of the parallelepiped determined by $\mathbf{w}$, $\mathbf{u}$, and $\mathbf{v}$ when:
   (a) $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, and $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$
   ◆(b) $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$, and $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

6. Let P_0 be a point with vector $\mathbf{p}_0$, and let $ax + by + cz = d$ be the equation of a plane with normal $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.
   (a) Show that the point on the plane closest to P_0 has vector $\mathbf{p}$ given by
   $$\mathbf{p} = \mathbf{p}_0 + \frac{d - (\mathbf{p}_0 \cdot \mathbf{n})}{\|\mathbf{n}\|^2}\mathbf{n}.$$
   [*Hint*: $\mathbf{p} = \mathbf{p}_0 + t\mathbf{n}$ for some t, and $\mathbf{p} \cdot \mathbf{n} = \mathbf{d}$.]
   ◆(b) Show that the shortest distance from P_0 to the plane is $\dfrac{|d - (\mathbf{p}_0 \cdot \mathbf{n})|}{\|\mathbf{n}\|}$.
   (c) Let P_0' denote the reflection of P_0 in the plane—that is, the point on the opposite side of the plane such that the line through P_0 and P_0' is perpendicular to the plane. Show that $\mathbf{p}_0 + 2\dfrac{d - (\mathbf{p}_0 \cdot \mathbf{n})}{\|\mathbf{n}\|^2}\mathbf{n}$ is the vector of P_0'.

7. Simplify $(a\mathbf{u} + b\mathbf{v}) \times (c\mathbf{u} + d\mathbf{v})$.

8. Show that the shortest distance from a point P to the line through P_0 with direction vector $\mathbf{d}$ is $\dfrac{\|\overrightarrow{P_0P} \times \mathbf{d}\|}{\|\mathbf{d}\|}$.

9. Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero, nonorthogonal vectors. If θ is the angle between them, show that $\tan\theta = \dfrac{\|\mathbf{u} \times \mathbf{v}\|}{\mathbf{u} \cdot \mathbf{v}}$.

◆10. Show that points A, B, and C are all on one line if and only if $\overrightarrow{AB} \times \overrightarrow{AC} = \mathbf{0}$.

11. Show that points A, B, C, and D are all on one plane if and only if $\overrightarrow{AB} \cdot \left(\overrightarrow{AB} \times \overrightarrow{AC}\right) = 0$.

◆12. Use Theorem 5 to confirm that, if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are mutually perpendicular, the (rectangular) parallelepiped they determine has volume $\|\mathbf{u}\|\,\|\mathbf{v}\|\,\|\mathbf{w}\|$.

13. Show that the volume of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ is $\|\mathbf{u} \times \mathbf{v}\|^2$.

14. Complete the proof of Theorem 3.

15. Prove the following properties in Theorem 2.
    (a) Property 6 ◆(b) Property 7
    (c) Property 8

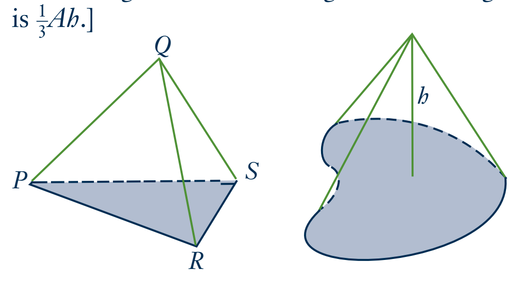
16. (a) Show that
    $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \times (\mathbf{w} \times \mathbf{u})$
    holds for all vectors $\mathbf{w}$, $\mathbf{u}$, and $\mathbf{v}$.
    ◆(b) Show that $\mathbf{v} - \mathbf{w}$ and $(\mathbf{u} \times \mathbf{v}) + (\mathbf{v} \times \mathbf{w}) + (\mathbf{w} \times \mathbf{u})$ are orthogonal.

17. Show that $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \times \mathbf{v})\mathbf{w}$. [*Hint*: First do it for $\mathbf{u} = \mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$; then write $\mathbf{u} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and use Theorem 2.]

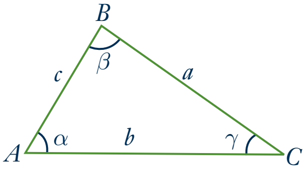
18. Prove the **Jacobi identity**:
    $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = \mathbf{0}$.
    [*Hint*: The preceding exercise.]

19. Show that
    $(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{w} \times \mathbf{z}) = \det\begin{bmatrix} \mathbf{u} \cdot \mathbf{w} & \mathbf{u} \cdot \mathbf{z} \\ \mathbf{v} \cdot \mathbf{w} & \mathbf{v} \cdot \mathbf{z} \end{bmatrix}$.
    [*Hint*: Exercises 16 and 17.]

20. Let P, Q, R, and S be four points, not all on one plane, as in the diagram. Show that the volume of the pyramid they determine is
    $\frac{1}{6}\left|\overrightarrow{PQ} \cdot \left(\overrightarrow{PR} \times \overrightarrow{PS}\right)\right|$.
    [*Hint*: The volume of a cone with base area A and height h as in the diagram below right is $\frac{1}{3}Ah$.]



21. Consider a triangle with vertices A, B, and C, as in the diagram below. Let α, β, and γ denote the angles at A, B, and C, respectively, and let a, b, and c denote the lengths of the sides opposite A, B, and C, respectively. Write $\mathbf{u} = \overrightarrow{AB}$, $\mathbf{v} = \overrightarrow{BC}$, and $\mathbf{w} = \overrightarrow{CA}$.



(a) Deduce that $\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}$.

(b) Show that $\mathbf{u} \times \mathbf{v} = \mathbf{w} \times \mathbf{u} = \mathbf{v} \times \mathbf{w}$. [*Hint*: Compute $\mathbf{u} \times (\mathbf{u} + \mathbf{v} + \mathbf{w})$ and $\mathbf{v} \times (\mathbf{u} + \mathbf{v} + \mathbf{w})$.]

(c) Deduce the **law of sines**:

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

◆22. Show that the (shortest) distance between two planes $\mathbf{n} \cdot \mathbf{p} = d_1$ and $\mathbf{n} \cdot \mathbf{p} = d_2$ with $\mathbf{n}$ as normal is $\dfrac{|d_2 - d_1|}{\|\mathbf{n}\|}$.

23. Let A and B be points other than the origin, and let $\mathbf{a}$ and $\mathbf{b}$ be their vectors. If $\mathbf{a}$ and $\mathbf{b}$ are not parallel, show that the plane through A, B, and the origin is given by

$$\{P(x, y, z) \mid \begin{bmatrix} x \\ y \\ z \end{bmatrix} = s\mathbf{a} + t\mathbf{b} \text{ for some } s \text{ and } t\}.$$

24. Let A be a 2×3 matrix of rank 2 with rows $\mathbf{r}_1$ and $\mathbf{r}_2$. Show that $P = \{XA \mid X = [x\ y];\ x, y \text{ arbitrary}\}$ is the plane through the origin with normal $\mathbf{r}_1 \times \mathbf{r}_2$.

25. Given the cube with vertices $P(x, y, z)$, where each of x, y, and z is either 0 or 2, consider the plane perpendicular to the diagonal through $P(0, 0, 0)$ and $P(2, 2, 2)$ and bisecting it.

(a) Show that the plane meets six of the edges of the cube and bisects them.

(b) Show that the six points in (a) are the vertices of a regular hexagon.

SECTION 4.4 Linear Operators on $\mathbb{R}^3$

Recall that a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called *linear* if $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ and $T(a\mathbf{x}) = aT(\mathbf{x})$ holds for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ and all scalars a. In this case we showed (in Theorem 2 Section 2.6) that there exists an $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$, and we say that T is the **matrix transformation induced** by A.

Definition 4.9

A linear transformation

$$T : \mathbb{R}^n \to \mathbb{R}^n$$

is called a **linear operator** *on* $\mathbb{R}^n$.

In Section 2.6 we investigated three important linear operators on $\mathbb{R}^2$: rotations about the origin, reflections in a line through the origin, and projections on this line.

In this section we investigate the analogous operators on $\mathbb{R}^3$: Rotations about a line through the origin, reflections in a plane through the origin, and projections onto a plane or line through the origin in $\mathbb{R}^3$. In every case we show that the operator is linear, and we find the matrices of all the reflections and projections.

To do this we must prove that these reflections, projections, and rotations are actually *linear* operators on $\mathbb{R}^3$. In the case of reflections and rotations, it is convenient to examine a more general situation. A transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is said to be **distance preserving** if the distance between $T(\mathbf{v})$ and $T(\mathbf{w})$ is the same as the distance between $\mathbf{v}$ and $\mathbf{w}$ for all $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$; that is,

$$\|T(\mathbf{v}) - T(\mathbf{w})\| = \|\mathbf{v} - \mathbf{w}\| \text{ for all } \mathbf{v} \text{ and } \mathbf{w} \text{ in } \mathbb{R}^3. \qquad (*)$$

Clearly reflections and rotations are distance preserving, and both carry $\mathbf{0}$ to $\mathbf{0}$, so the following theorem shows that they are both linear.

Theorem 1

If $T : \mathbb{R}^3 \to \mathbb{R}^3$ is distance preserving, and if $T(\mathbf{0}) = \mathbf{0}$, then T is linear.

PROOF

Since $T(\mathbf{0}) = \mathbf{0}$, taking $\mathbf{w} = \mathbf{0}$ in $(*)$ shows that $\|T(\mathbf{v})\| = \|\mathbf{v}\|$ for all $\mathbf{v}$ in $\mathbb{R}^3$, that is T preserves length. Also, $\|T(\mathbf{v}) - T(\mathbf{w})\|^2 = \|\mathbf{v} - \mathbf{w}\|^2$ by $(*)$. Since $\|\mathbf{v} - \mathbf{w}\|^2 = \|\mathbf{v}\|^2 - 2\mathbf{v} \cdot \mathbf{w} + \|\mathbf{w}\|^2$ always holds, it follows that $T(\mathbf{v}) \cdot T(\mathbf{w}) = \mathbf{v} \cdot \mathbf{w}$ for all $\mathbf{v}$ and $\mathbf{w}$. Hence (by Theorem 2 Section 4.2) the angle between $T(\mathbf{v})$ and $T(\mathbf{w})$ is the same as the angle between $\mathbf{v}$ and $\mathbf{w}$ for all (nonzero) vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$.

With this we can show that T is linear. Given nonzero vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$, the vector $\mathbf{v} + \mathbf{w}$ is the diagonal of the parallelogram determined by $\mathbf{v}$ and $\mathbf{w}$. By the preceding paragraph, the effect of T is to carry this *entire parallelogram* to the parallelogram determined by $T(\mathbf{v})$ and $T(\mathbf{w})$, with diagonal $T(\mathbf{v} + \mathbf{w})$. But this diagonal is $T(\mathbf{v}) + T(\mathbf{w})$ by the parallelogram law (see Figure 1).

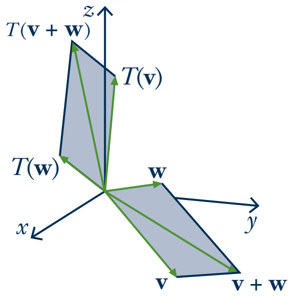


FIGURE 1

In other words, $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$. A similar argument shows that $T(a\mathbf{v}) = aT(\mathbf{v})$ for all scalars a, proving that T is indeed linear.

Distance-preserving linear operators are called **isometries**, and we return to them in Section 10.4.

Reflections and Projections

In Section 2.6 we studied the reflection $Q_m: \mathbb{R}^2 \to \mathbb{R}^2$ in the line $y = mx$ and projection $P_m: \mathbb{R}^2 \to \mathbb{R}^2$ on the same line. We found (in Theorems 5 and 6, Section 2.6) that they are both linear and

$$Q_m \text{ has matrix } \frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix} \quad \text{and} \quad P_m \text{ has matrix } \frac{1}{1+m^2}\begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix}.$$

We now look at the analogues in $\mathbb{R}^3$.

Let L denote a line through the origin in $\mathbb{R}^3$. Given a vector $\mathbf{v}$ in $\mathbb{R}^3$, the reflection $Q_L(\mathbf{v})$ of $\mathbf{v}$ in L and the projection $P_L(\mathbf{v})$ of $\mathbf{v}$ on L are defined in Figure 2. In the same figure, we see that

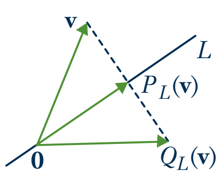


FIGURE 2

$$P_L(\mathbf{v}) = \mathbf{v} + \tfrac{1}{2}[Q_L(\mathbf{v}) - \mathbf{v}] = \tfrac{1}{2}[Q_L(\mathbf{v}) + \mathbf{v}] \qquad (**)$$

so the fact that Q_L is linear (by Theorem 1) shows that P_L is also linear.[12] However, Theorem 4 Section 4.2 gives us the matrix of P_L directly. In fact, if $\mathbf{d} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \mathbf{0}$ is a direction vector for L, and we write $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, then

$$P_L(\mathbf{v}) = \frac{\mathbf{v} \cdot \mathbf{d}}{\|\mathbf{d}\|^2}\mathbf{d} = \frac{ax + by + cz}{a^2 + b^2 + c^2}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{a^2 + b^2 + c^2}\begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

as the reader can verify. Note that this shows directly that P_L is a matrix transformation and so gives another proof that it is linear.

12 Note that Theorem 1 does *not* apply to P_L since it does not preserve distance.

Theorem 2

Let L denote the line through the origin in $\mathbb{R}^3$ with direction vector $\mathbf{d} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \mathbf{0}$. Then P_L and Q_L are both linear and

$$P_L \text{ has matrix } \frac{1}{a^2+b^2+c^2}\begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix},$$

$$Q_L \text{ has matrix } \frac{1}{a^2+b^2+c^2}\begin{bmatrix} a^2-b^2-c^2 & 2ab & 2ac \\ 2ab & b^2-a^2-c^2 & 2bc \\ 2ac & 2bc & c^2-a^2-b^2 \end{bmatrix}.$$

PROOF

It remains to find the matrix of Q_L. But (**) implies that $Q_L(\mathbf{v}) = 2P_L(\mathbf{v}) - \mathbf{v}$ for each $\mathbf{v}$ in $\mathbb{R}^3$, so if $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ we obtain (with some matrix arithmetic):

$$\begin{aligned} Q_L(\mathbf{v}) &= \left\{ \frac{2}{a^2+b^2+c^2}\begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ &= \frac{1}{a^2+b^2+c^2}\begin{bmatrix} a^2-b^2-c^2 & 2ab & 2ac \\ 2ab & b^2-a^2-c^2 & 2bc \\ 2ac & 2bc & c^2-a^2-b^2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \end{aligned}$$

as required.

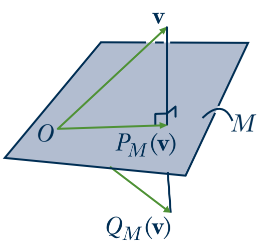


FIGURE 3

In $\mathbb{R}^3$ we can reflect in planes as well as lines. Let M denote a plane through the origin in $\mathbb{R}^3$. Given a vector $\mathbf{v}$ in $\mathbb{R}^3$, the reflection $Q_M(\mathbf{v})$ of $\mathbf{v}$ in M and the projection $P_M(\mathbf{v})$ of $\mathbf{v}$ on M are defined in Figure 3. As above, we have

$$P_M(\mathbf{v}) = \mathbf{v} + \tfrac{1}{2}[Q_M(\mathbf{v}) - \mathbf{v}] = \tfrac{1}{2}[Q_M(\mathbf{v}) + \mathbf{v}]$$

so the fact that Q_M is linear (again by Theorem 1) shows that P_M is also linear. Again we can obtain the matrix directly. If $\mathbf{n}$ is a normal for the plane M, then Figure 3 shows that

$$P_M(\mathbf{v}) = \mathbf{v} - \text{proj}_{\mathbf{n}}(\mathbf{v}) = \mathbf{v} - \frac{\mathbf{v} \cdot \mathbf{n}}{\|\mathbf{n}\|^2}\mathbf{n} \text{ for all vectors } \mathbf{v}.$$

If $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \mathbf{0}$ and $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, a computation like the above gives

$$P_M(\mathbf{v}) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \frac{ax+by+cz}{a^2+b^2+c^2}\begin{bmatrix} a \\ b \\ c \end{bmatrix} \right\} = \frac{1}{a^2+b^2+c^2}\begin{bmatrix} b^2+c^2 & -ab & -ac \\ -ab & a^2+c^2 & -bc \\ -ac & -bc & b^2+c^2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

This proves the first part of

Theorem 3

Let M denote the plane through the origin in $\mathbb{R}^3$ with normal $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \mathbf{0}$. Then P_M and Q_M are both linear and

$$P_M \text{ has matrix } \frac{1}{a^2+b^2+c^2}\begin{bmatrix} b^2+c^2 & -ab & -ac \\ -ab & a^2+c^2 & -bc \\ -ac & -bc & a^2+b^2 \end{bmatrix},$$

$$Q_M \text{ has matrix } \frac{1}{a^2+b^2+c^2}\begin{bmatrix} b^2+c^2-a^2 & -2ab & -2ac \\ -2ab & a^2+c^2-b^2 & -2bc \\ -2ac & -2bc & a^2+b^2-c^2 \end{bmatrix}.$$

PROOF

It remains to compute the matrix of Q_M. Since $Q_M(\mathbf{v}) = 2P_M(\mathbf{v}) - \mathbf{v}$ for each $\mathbf{v}$ in $\mathbb{R}^3$, the computation is similar to the above and is left as an exercise for the reader.

Rotations

In Section 2.6 we studied the rotation $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ counterclockwise about the origin through the angle θ. Moreover, we showed in Theorem 4 Section 2.6 that R_θ is linear and has matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. One extension of this is given in the following example.

EXAMPLE 1

Let $R_{z,\theta} : \mathbb{R}^3 \to \mathbb{R}^3$ denote rotation of $\mathbb{R}^3$ about the z axis through an angle θ from the positive x axis toward the positive y axis. Show that $R_{z,\theta}$ is linear and find its matrix.

Solution ▸ First R is distance preserving and so is linear by Theorem 1. Hence we apply Theorem 2 Section 2.6 to obtain the matrix of $R_{z,\theta}$.

Let $\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ denote the standard basis of $\mathbb{R}^3$; we must find $R_{z,\theta}(\mathbf{i})$, $R_{z,\theta}(\mathbf{j})$, and $R_{z,\theta}(\mathbf{k})$. Clearly $R_{z,\theta}(\mathbf{k}) = \mathbf{k}$. The effect of $R_{z,\theta}$ on the x-y plane is to rotate it counterclockwise through the angle θ. Hence Figure 4 gives

$$R_{z,\theta}(\mathbf{i}) = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}, \quad R_{z,\theta}(\mathbf{j}) = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix}$$

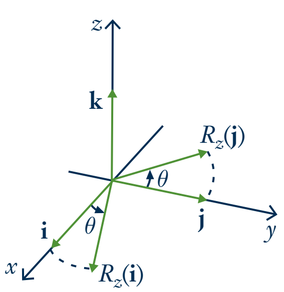


FIGURE 4

so, by Theorem 2 Section 2.6, $R_{z,\theta}$ has matrix

$$[R_{z,\theta}(\mathbf{i})\ \ R_{z,\theta}(\mathbf{j})\ \ R_{z,\theta}(\mathbf{k})] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Example 1 begs to be generalized. Given a line L through the origin in $\mathbb{R}^3$, every rotation about L through a fixed angle is clearly distance preserving, and so is a linear operator by Theorem 1. However, giving a precise description of the matrix of this rotation is not easy and will have to wait until more techniques are available.

Transformations of Areas and Volumes

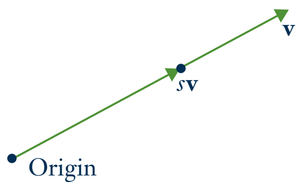


FIGURE 5

Let $\mathbf{v}$ be a nonzero vector in $\mathbb{R}^3$. Each vector in the same direction as $\mathbf{v}$ whose length is a fraction s of the length of $\mathbf{v}$ has the form $s\mathbf{v}$ (see Figure 5). With this, scrutiny of Figure 6 shows that a vector $\mathbf{u}$ is in the parallelogram determined by $\mathbf{v}$ and $\mathbf{w}$ if and only if it has the form $\mathbf{u} = s\mathbf{v} + t\mathbf{w}$ where $0 \le s \le 1$ and $0 \le t \le 1$. But then, if $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation, we have

$$T(s\mathbf{v} + t\mathbf{w}) = T(s\mathbf{v}) + T(t\mathbf{w}) = sT(\mathbf{v}) + tT(\mathbf{w}).$$

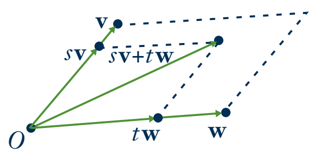


FIGURE 6

Hence $T(s\mathbf{v} + t\mathbf{w})$ is in the parallelogram determined by $T(\mathbf{v})$ and $T(\mathbf{w})$. Conversely, every vector in this parallelogram has the form $T(s\mathbf{v} + t\mathbf{w})$ where $s\mathbf{v} + t\mathbf{w}$ is in the parallelogram determined by $\mathbf{v}$ and $\mathbf{w}$. For this reason, the parallelogram determined by $T(\mathbf{v})$ and $T(\mathbf{w})$ is called the **image** of the parallelogram determined by $\mathbf{v}$ and $\mathbf{w}$. We record this discussion as:

Theorem 4

If $T : \mathbb{R}^3 \to \mathbb{R}^3$ (or $\mathbb{R}^2 \to \mathbb{R}^2$) is a linear operator, the image of the parallelogram determined by vectors $\mathbf{v}$ and $\mathbf{w}$ is the parallelogram determined by $T(\mathbf{v})$ and $T(\mathbf{w})$.

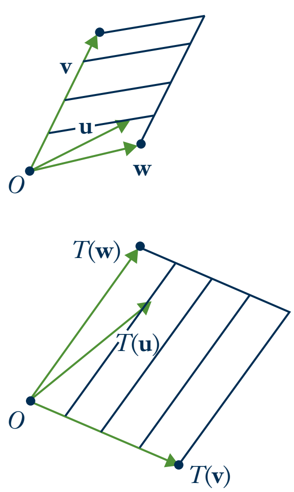


FIGURE 7

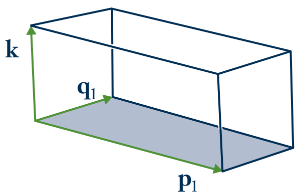
This result is illustrated in Figure 7, and was used in Examples 15 and 16 Section 2.2 to reveal the effect of expansion and shear transformations.

Now we are interested in the effect of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ on the parallelepiped determined by three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^3$ (see the discussion preceding Theorem 5 Section 4.3). If T has matrix A, Theorem 4 shows that this parallelepiped is carried to the parallelepiped determined by $T(\mathbf{u}) = A\mathbf{u}$, $T(\mathbf{v}) = A\mathbf{v}$, and $T(\mathbf{w}) = A\mathbf{w}$. In particular, we want to discover how the volume changes, and it turns out to be closely related to the determinant of the matrix A.

Theorem 5

Let $\text{vol}(\mathbf{u}, \mathbf{v}, \mathbf{w})$ *denote the volume of the parallelepiped determined by three vectors* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *in* $\mathbb{R}^3$, *and let* $\text{area}(\mathbf{p}, \mathbf{q})$ *denote the area of the parallelogram determined by two vectors* $\mathbf{p}$ *and* $\mathbf{q}$ *in* $\mathbb{R}^2$. *Then:*

1. *If A is a 3×3 matrix, then* $\text{vol}(A\mathbf{u}, A\mathbf{v}, A\mathbf{w}) = |\det(A)| \cdot \text{vol}(\mathbf{u}, \mathbf{v}, \mathbf{w})$.
2. *If A is a 2×2 matrix, then* $\text{area}(A\mathbf{p}, A\mathbf{q}) = |\det(A)| \cdot \text{area}(\mathbf{p}, \mathbf{q})$.



PROOF

1. Let $[\mathbf{u}\ \mathbf{v}\ \mathbf{w}]$ denote the 3×3 matrix with columns $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Then

$$\text{vol}(A\mathbf{u}, A\mathbf{v}, A\mathbf{w}) = |A\mathbf{u} \cdot (A\mathbf{v} \times A\mathbf{w})|$$

by Theorem 5 Section 4.3. Now apply Theorem 1 Section 4.3 twice to get

$$\begin{aligned} A\mathbf{u} \cdot (A\mathbf{v} \times A\mathbf{w}) = \det[A\mathbf{u}\ \ A\mathbf{v}\ \ A\mathbf{w}] &= \det\{A[\mathbf{u}\ \ \mathbf{v}\ \ \mathbf{w}]\} \\ &= \det(A)\det[\mathbf{u}\ \ \mathbf{v}\ \ \mathbf{w}] \\ &= \det(A)(\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})) \end{aligned}$$

where we used Definition 2.9 and the product theorem for determinants. Finally (1) follows from Theorem 5 Section 4.3 by taking absolute values.

2. Given $\mathbf{p} = \begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$, write $\mathbf{p}_1 = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$ in $\mathbb{R}^3$. By the diagram, $\text{area}(\mathbf{p}, \mathbf{q}) = \text{vol}(\mathbf{p}_1, \mathbf{q}_1, \mathbf{k})$ where $\mathbf{k}$ is the (length 1) coordinate vector along the z axis. If A is a 2×2 matrix, write $A_1 = \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$ in block form, and observe that $(A\mathbf{v})_1 = (A_1\mathbf{v}_1)$ for all $\mathbf{v}$ in $\mathbb{R}^2$ and $A_1\mathbf{k} = \mathbf{k}$. Hence part (1) if this theorem shows

$$\begin{aligned} \text{area}(A\mathbf{p}, A\mathbf{q}) &= \text{vol}(A_1\mathbf{p}_1, A_1\mathbf{q}_1, A_1\mathbf{k}) \\ &= |\det(A_1)|\,\text{vol}(\mathbf{p}_1, \mathbf{q}_1, \mathbf{k}) \\ &= |\det(A)|\,\text{area}(\mathbf{p}, \mathbf{q}) \end{aligned}$$

as required.

Define the **unit square** and **unit cube** to be the square and cube corresponding to the coordinate vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. Then Theorem 5 gives a geometrical meaning to the determinant of a matrix A:

- *If A is a 2×2 matrix, then $|\det(A)|$ is the area of the image of the unit square under multiplication by A;*
- *If A is a 3×3 matrix, then $|\det(A)|$ is the volume of the image of the unit cube under multiplication by A.*

These results, together with the importance of areas and volumes in geometry, were among the reasons for the initial development of determinants.

EXERCISES 4.4

1. In each case show that that T is either projection on a line, reflection in a line, or rotation through an angle, and find the line or angle.

(a) $T\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} x + 2y \\ 2x + 4y \end{bmatrix}$ ◆(b) $T\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} x - y \\ y - x \end{bmatrix}$

(c) $T\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -x - y \\ x - y \end{bmatrix}$ ◆(d) $T\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -3x + 4y \\ 4x + 3y \end{bmatrix}$

(e) $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ -x \end{bmatrix}$ ◆(f) $T\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} x - \sqrt{3}y \\ \sqrt{3}x + y \end{bmatrix}$

2. Determine the effect of the following transformations.

(a) Rotation through $\frac{\pi}{2}$, followed by projection on the y axis, followed by reflection in the line $y = x$.

◆(b) Projection on the line $y = x$ followed by projection on the line $y = -x$.

(c) Projection on the x axis followed by reflection in the line $y = x$.

3. In each case solve the problem by finding the matrix of the operator.

(a) Find the projection of $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ on the plane with equation $3x - 5y + 2z = 0$.

◆(b) Find the projection of $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$ on the plane with equation $2x - y + 4z = 0$.

(c) Find the reflection of $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ in the plane with equation $x - y + 3z = 0$.

◆(d) Find the reflection of $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$ in the plane with equation $2x + y - 5z = 0$.

(e) Find the reflection of $\mathbf{v} = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$ in the line with equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$.

◆(f) Find the projection of $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix}$ on the line with equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$.

(g) Find the projection of $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$ on the line with equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$.

◆(h) Find the reflection of $\mathbf{v} = \begin{bmatrix} 2 \\ -5 \\ 0 \end{bmatrix}$ in the line with equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$.

4. (a) Find the rotation of $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ about the z axis through $\theta = \frac{\pi}{4}$.

◆(b) Find the rotation of $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ about the z axis through $\theta = \frac{\pi}{6}$.

5. Find the matrix of the rotation in $\mathbb{R}^3$ about the x axis through the angle θ (from the positive y axis to the positive z axis).

◆6. Find the matrix of the rotation about the y axis through the angle θ (from the positive x axis to the positive z axis).

7. If A is 3×3, show that the image of the line in $\mathbb{R}^3$ through $\mathbf{p}_0$ with direction vector $\mathbf{d}$ is the line through $A\mathbf{p}_0$ with direction vector $A\mathbf{d}$, assuming that $A\mathbf{d} \neq \mathbf{0}$. What happens if $A\mathbf{d} = \mathbf{0}$?

8. If A is 3×3 and invertible, show that the image of the plane through the origin with normal $\mathbf{n}$ is the plane through the origin with normal $\mathbf{n}_1 = B\mathbf{n}$ where $B = (A^{-1})^T$. [*Hint*: Use the fact that $\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T\mathbf{w}$ to show that $\mathbf{n}_1 \cdot (A\mathbf{p}) = \mathbf{n} \cdot \mathbf{p}$ for each $\mathbf{p}$ in $\mathbb{R}^3$.]

9. Let L be the line through the origin in $\mathbb{R}^2$ with direction vector $\mathbf{d} = \begin{bmatrix} a \\ b \end{bmatrix} \neq 0$.

◆(a) If P_L denotes projection on L, show that P_L has matrix $\frac{1}{a^2 + b^2}\begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}$.

(b) If Q_L denotes reflection in L, show that Q_L has matrix $\frac{1}{a^2 + b^2}\begin{bmatrix} a^2 - b^2 & 2ab \\ 2ab & b^2 - a^2 \end{bmatrix}$.

10. Let $\mathbf{n}$ be a nonzero vector in $\mathbb{R}^3$, let L be the line through the origin with direction vector $\mathbf{n}$, and let M be the plane through the origin with normal $\mathbf{n}$. Show that $P_L(\mathbf{v}) = Q_L(\mathbf{v}) + P_M(\mathbf{v})$ for all $\mathbf{v}$ in $\mathbb{R}^3$. [In this case, we say that $P_L = Q_L + P_M$.]

11. If M is the plane through the origin in $\mathbb{R}^3$ with normal $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, show that Q_M has matrix

$$\frac{1}{a^2 + b^2 + c^2}\begin{bmatrix} b^2 + c^2 - a^2 & -2ab & -2ac \\ -2ab & a^2 + c^2 - b^2 & -2bc \\ -2ac & -2bc & a^2 + b^2 - c^2 \end{bmatrix}$$

SECTION 4.5 An Application to Computer Graphics

Computer graphics deals with images displayed on a computer screen, and so arises in a variety of applications, ranging from word processors, to *Star Wars* animations, to video games, to wire-frame images of an airplane. These images consist of a number of points on the screen, together with instructions on how to fill in areas bounded by lines and curves. Often curves are approximated by a set of short straight-line segments, so that the curve is specified by a series of points on the screen at the end of these segments. Matrix transformations are important here because matrix images of straight line segments are again line segments.[13] Note that a colour image requires that three images are sent, one to each of the red, green, and blue phosphorus dots on the screen, in varying intensities.

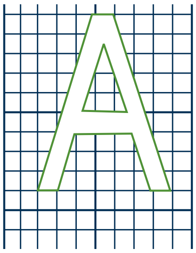


FIGURE 1

Consider displaying the letter A. In reality, it is depicted on the screen, as in Figure 1, by specifying the coordinates of the 11 corners and filling in the interior. For simplicity, we will disregard the thickness of the letter, so we require only five coordinates as in Figure 2. This simplified letter can then be stored as a data matrix

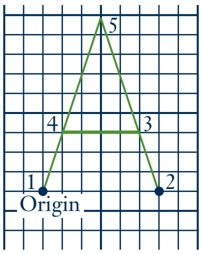


FIGURE 2

$$\begin{array}{c} \textit{Vertex} \\ D = \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 6 & 5 & 1 & 3 \\ 0 & 0 & 3 & 3 & 9 \end{bmatrix} \end{array}$$

where the columns are the coordinates of the vertices in order. Then if we want to transform the letter by a 2×2 matrix A, we left-multiply this data matrix by A (the effect is to multiply each column by A and so transform each vertex).

For example, we can slant the letter to the right by multiplying by an x-shear matrix $A = \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}$—see Section 2.2. The result is the letter with data matrix

$$AD = \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 6 & 5 & 1 & 3 \\ 0 & 0 & 3 & 3 & 9 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 5.6 & 1.6 & 4.8 \\ 0 & 0 & 3 & 3 & 9 \end{bmatrix}$$

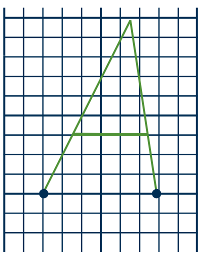


FIGURE 3

which is shown in Figure 3. If we want to make this slanted matrix narrower, we can now apply an x-scale matrix $B = \begin{bmatrix} 0.8 & 0 \\ 0 & 1 \end{bmatrix}$ that shrinks the x-coordinate by 0.8. The result is the composite transformation

$$BAD = \begin{bmatrix} 0.8 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 6 & 5 & 1 & 3 \\ 0 & 0 & 3 & 3 & 9 \end{bmatrix} = \begin{bmatrix} 0 & 4.8 & 4.48 & 1.28 & 3.84 \\ 0 & 0 & 3 & 3 & 9 \end{bmatrix}$$

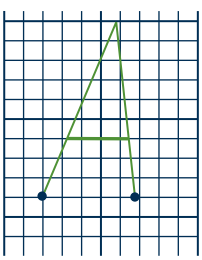


FIGURE 4

which is drawn in Figure 4.

On the other hand, we can rotate the letter about the origin through $\frac{\pi}{6}$ (or 30°) by multiplying by the matrix $R_{\frac{\pi}{2}} = \begin{bmatrix} \cos(\frac{\pi}{6}) & -\sin(\frac{\pi}{6}) \\ \sin(\frac{\pi}{6}) & \cos(\frac{\pi}{6}) \end{bmatrix} = \begin{bmatrix} 0.866 & -0.5 \\ 0.5 & 0.866 \end{bmatrix}$. This gives

$$R_{\frac{\pi}{2}}D = \begin{bmatrix} 0.866 & -0.5 \\ 0.5 & 0.866 \end{bmatrix}\begin{bmatrix} 0 & 6 & 5 & 1 & 3 \\ 0 & 0 & 3 & 3 & 9 \end{bmatrix} = \begin{bmatrix} 0 & 5.196 & 2.83 & -0.634 & -1.902 \\ 0 & 3 & 5.098 & 3.098 & 9.294 \end{bmatrix}$$

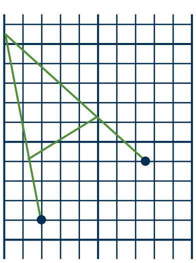


FIGURE 5

and is plotted in Figure 5.

This poses a problem: How do we rotate at a point other than the origin? It turns out that we can do this when we have solved another more basic problem. It is clearly important to be able to translate a screen image by a fixed vector $\mathbf{w}$, that is apply the transformation $T_{\mathbf{w}} : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T_{\mathbf{w}}(\mathbf{v}) = \mathbf{v} + \mathbf{w}$ for all $\mathbf{v}$ in $\mathbb{R}^2$. The problem is that these translations are not matrix transformations $\mathbb{R}^2 \to \mathbb{R}^2$ because they do not carry $\mathbf{0}$ to $\mathbf{0}$ (unless $\mathbf{w} = \mathbf{0}$). However, there is a clever way around this.

13 If $\mathbf{v}_0$ and $\mathbf{v}_1$ are vectors, the vector from $\mathbf{v}_0$ to $\mathbf{v}_1$ is $\mathbf{d} = \mathbf{v}_1 - \mathbf{v}_0$. So a vector $\mathbf{v}$ lies on the line segment between $\mathbf{v}_0$ and $\mathbf{v}_1$ if and only if $\mathbf{v} = \mathbf{v}_0 + t\mathbf{d}$ for some number t in the range $0 \leq t \leq 1$. Thus the image of this segment is the set of vectors $A\mathbf{v} = A\mathbf{v}_0 + tA\mathbf{d}$ with $0 \leq t \leq 1$, that is the image is the segment between $A\mathbf{v}_0$ and $A\mathbf{v}_1$.

The idea is to represent a point $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ as a 3×1 column $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$, called the **homogeneous coordinates** of $\mathbf{v}$. Then translation by $\mathbf{w} = \begin{bmatrix} p \\ q \end{bmatrix}$ can be achieved by multiplying by a 3×3 matrix:

$$\begin{bmatrix} 1 & 0 & p \\ 0 & 1 & q \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + p \\ y + q \\ 1 \end{bmatrix} = \begin{bmatrix} T_{\mathbf{w}}(\mathbf{v}) \\ 1 \end{bmatrix}$$

Thus, by using homogeneous coordinates we can implement the translation $T_{\mathbf{w}}$ in the top two coordinates. On the other hand, the matrix transformation induced by $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is also given by a 3×3 matrix:

$$\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \\ 1 \end{bmatrix} = \begin{bmatrix} A\mathbf{v} \\ 1 \end{bmatrix}$$

So everything can be accomplished at the expense of using 3×3 matrices and homogeneous coordinates.

EXAMPLE 1

Rotate the letter A in Figure 2 through $\frac{\pi}{6}$ about the point $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

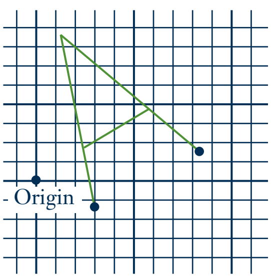


FIGURE 6

Solution ▶ Using homogenous coordinates for the vertices of the letter results in a data matrix with three rows:

$$K_d = \begin{bmatrix} 0 & 6 & 5 & 1 & 3 \\ 0 & 0 & 3 & 3 & 9 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

If we write $\mathbf{w} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, the idea is to use a composite of transformations: First translate the letter by $-\mathbf{w}$ so that the point $\mathbf{w}$ moves to the origin, then rotate this translated letter, and then translate it by $\mathbf{w}$ back to its original position. The matrix arithmetic is as follows (remember the order of composition!):

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 6 & 5 & 1 & 3 \\ 0 & 0 & 3 & 3 & 9 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3.036 & 8.232 & 5.866 & 2.402 & 1.134 \\ -1.33 & 1.67 & 3.768 & 1.768 & 7.964 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

This is plotted in Figure 6.

This discussion merely touches the surface of computer graphics, and the reader is referred to specialized books on the subject. Realistic graphic rendering requires an enormous number of matrix calculations. In fact, matrix multiplication algorithms are now embedded in microchip circuits, and can perform over 100 million matrix multiplications per second. This is particularly important in the field of three-dimensional graphics where the homogeneous coordinates have four components and 4×4 matrices are required.

EXERCISES 4.5

1. Consider the letter A described in Figure 2. Find the data matrix for the letter obtained by:
   (a) Rotating the letter through $\frac{\pi}{4}$ about the origin.
   ◆(b) Rotating the letter through $\frac{\pi}{4}$ about the point $\begin{bmatrix}1\\2\end{bmatrix}$.
2. Find the matrix for turning the letter A in Figure 2 upside-down in place.
3. Find the 3×3 matrix for reflecting in the line $y = mx + b$. Use $\begin{bmatrix}1\\m\end{bmatrix}$ as direction vector for the line.
4. Find the 3×3 matrix for rotating through the angle θ about the point $P(a, b)$.
5. Find the reflection of the point P in the line $y = 1 + 2x$ in $\mathbb{R}^2$ if:
   (a) $P = P(1, 1)$
   ◆(b) $P = P(1, 4)$
   (c) What about $P = P(1, 3)$? Explain. [*Hint*: Example 1 and Section 4.4.]

SUPPLEMENTARY EXERCISES FOR CHAPTER 4

1. Suppose that **u** and **v** are nonzero vectors. If **u** and **v** are not parallel, and $a\mathbf{u} + b\mathbf{v} = a_1\mathbf{u} + b_1\mathbf{v}$, show that $a = a_1$ and $b = b_1$.
2. Consider a triangle with vertices A, B, and C. Let E and F be the midpoints of sides AB and AC, respectively, and let the medians EC and FB meet at O. Write $\overrightarrow{EO} = s\overrightarrow{EC}$ and $\overrightarrow{FO} = t\overrightarrow{FB}$, where s and t are scalars. Show that $s = t = \frac{1}{3}$ by expressing $\overrightarrow{AO}$ two ways in the form $a\overrightarrow{EO} + b\overrightarrow{AC}$, and applying Exercise 1. Conclude that the medians of a triangle meet at the point on each that is one-third of the way from the midpoint to the vertex (and so are concurrent).
3. A river flows at 1 km/h and a swimmer moves at 2 km/h (relative to the water). At what angle must he swim to go straight across? What is his resulting speed?
◆4. A wind is blowing from the south at 75 knots, and an airplane flies heading east at 100 knots. Find the resulting velocity of the airplane.
5. An airplane pilot flies at 300 km/h in a direction 30° south of east. The wind is blowing from the south at 150 km/h.
   (a) Find the resulting direction and speed of the airplane.
   (b) Find the speed of the airplane if the wind is from the west (at 150 km/h).
◆6. A rescue boat has a top speed of 13 knots. The captain wants to go due east as fast as possible in water with a current of 5 knots due south. Find the velocity vector $\mathbf{v} = (x, y)$ that she must achieve, assuming the **x** and **y** axes point east and north, respectively, and find her resulting speed.
7. A boat goes 12 knots heading north. The current is 5 knots from the west. In what direction does the boat actually move and at what speed?
8. Show that the distance from a point A (with vector **a**) to the plane with vector equation $\mathbf{n} \cdot \mathbf{p} = d$ is $\frac{1}{\|\mathbf{n}\|}|\mathbf{n} \cdot \mathbf{a} - d|$.
9. If two distinct points lie in a plane, show that the line through these points is contained in the plane.
10. The line through a vertex of a triangle, perpendicular to the opposite side, is called an **altitude** of the triangle. Show that the three altitudes of any triangle are concurrent. (The intersection of the altitudes is called the **orthocentre** of the triangle.) [*Hint*: If P is the intersection of two of the altitudes, show that the line through P and the remaining vertex is perpendicular to the remaining side.]

5 The Vector Space $\mathbb{R}^n$

SECTION 5.1 Subspaces and Spanning

In Section 2.2 we introduced the set $\mathbb{R}^n$ of all n-tuples (called *vectors*), and began our investigation of the matrix transformations $\mathbb{R}^n \to \mathbb{R}^m$ given by matrix multiplication by an $m \times n$ matrix. Particular attention was paid to the euclidean plane $\mathbb{R}^2$ where certain simple geometric transformations were seen to be matrix transformations. Then in Section 2.6 we introduced linear transformations, showed that they are all matrix transformations, and found the matrices of rotations and reflections in $\mathbb{R}^2$. We returned to this in Section 4.4 where we showed that projections, reflections, and rotations of $\mathbb{R}^2$ and $\mathbb{R}^3$ were all linear, and where we related areas and volumes to determinants.

In this chapter we investigate $\mathbb{R}^n$ in full generality, and introduce some of the most important concepts and methods in linear algebra. The n-tuples in $\mathbb{R}^n$ will continue to be denoted $\mathbf{x}$, $\mathbf{y}$, and so on, and will be written as rows or columns depending on the context.

Subspaces of $\mathbb{R}^n$

Definition 5.1 *A set*[1] *U of vectors in $\mathbb{R}^n$ is called a* **subspace** *of $\mathbb{R}^n$ if it satisfies the following properties:*

S1. The zero vector $\mathbf{0}$ is in U.

S2. If $\mathbf{x}$ and $\mathbf{y}$ are in U, then $\mathbf{x} + \mathbf{y}$ is also in U.

S3. If $\mathbf{x}$ is in U, then $a\mathbf{x}$ is in U for every real number a.

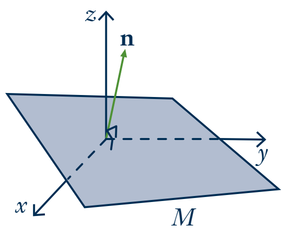
We say that the subset U is **closed under addition** if S2 holds, and that U is **closed under scalar multiplication** if S3 holds.

Clearly $\mathbb{R}^n$ is a subspace of itself. The set $U = \{\mathbf{0}\}$, consisting of only the zero vector, is also a subspace because $\mathbf{0} + \mathbf{0} = \mathbf{0}$ and $a\mathbf{0} = \mathbf{0}$ for each a in $\mathbb{R}$; it is called the **zero subspace**. Any subspace of $\mathbb{R}^n$ other than $\{\mathbf{0}\}$ or $\mathbb{R}^n$ is called a **proper** subspace.

1 We use the language of sets. Informally, a **set** X is a collection of objects, called the **elements** of the set. The fact that x is an element of X is denoted $x \in X$. Two sets X and Y are called equal (written $X = Y$) if they have the same elements. If every element of X is in the set Y, we say that X is a **subset** of Y, and write $X \subseteq Y$. Hence $X \subseteq Y$ and $Y \subseteq X$ both hold if and only if $X = Y$.

We saw in Section 4.2 that every plane M through the origin in $\mathbb{R}^3$ has equation $ax + by + cz = 0$ where a, b, and c are not all zero. Here $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is a normal for the plane and

$$M = \{\mathbf{v} \text{ in } \mathbb{R}^3 \mid \mathbf{n} \cdot \mathbf{v} = 0\}$$



where $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mathbf{n} \cdot \mathbf{v}$ denotes the dot product introduced in Section 2.2 (see the diagram).[2] Then M is a subspace of $\mathbb{R}^3$. Indeed we show that M satisfies S1, S2, and S3 as follows:

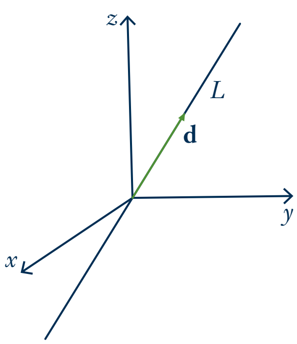
S1. $\mathbf{0}$ *is in* M *because* $\mathbf{n} \cdot \mathbf{0} = 0$;

S2. If $\mathbf{v}$ *and* $\mathbf{v}_1$ *are in* M, *then* $\mathbf{n} \cdot (\mathbf{v} + \mathbf{v}_1) = \mathbf{n} \cdot \mathbf{v} + \mathbf{n} \cdot \mathbf{v}_1 = 0 + 0 = 0$, *so* $\mathbf{v} + \mathbf{v}_1$ *is in* M;

S3. If $\mathbf{v}$ *is in* M, *then* $\mathbf{n} \cdot (a\mathbf{v}) = a(\mathbf{n} \cdot \mathbf{v}) = a(0) = 0$, *so* $a\mathbf{v}$ *is in* M.

This proves the first part of

EXAMPLE 1



Planes and lines through the origin in $\mathbb{R}^3$ are all subspaces of $\mathbb{R}^3$.

Solution ▶ We dealt with planes above. If L is a line through the origin with direction vector $\mathbf{d}$, then $L = \{t\mathbf{d} \mid t \text{ in } \mathbb{R}\}$ (see the diagram). We leave it as an exercise to verify that L satisfies S1, S2, and S3.

Example 1 shows that lines through the origin in $\mathbb{R}^2$ are subspaces; in fact, they are the *only* proper subspaces of $\mathbb{R}^2$ (Exercise 24). Indeed, we shall see in Example 14 Section 5.2 that lines and planes through the origin in $\mathbb{R}^3$ are the only proper subspaces of $\mathbb{R}^3$. Thus the geometry of lines and planes through the origin is captured by the subspace concept. (Note that *every* line or plane is just a translation of one of these.)

Subspaces can also be used to describe important features of an $m \times n$ matrix A. The **null space** of A, denoted null A, and the **image space** of A, denoted im A, are defined by

$$\text{null } A = \{\mathbf{x} \text{ in } \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\} \quad \text{and} \quad \text{im } A = \{A\mathbf{x} \mid \mathbf{x} \text{ in } \mathbb{R}^n\}$$

In the language of Chapter 2, null A consists of all solutions $\mathbf{x}$ in $\mathbb{R}^n$ of the homogeneous system $A\mathbf{x} = \mathbf{0}$, and im A is the set of all vectors $\mathbf{y}$ in $\mathbb{R}^m$ such that $A\mathbf{x} = \mathbf{y}$ *has* a solution $\mathbf{x}$. Note that $\mathbf{x}$ is in null A if it satisfies the *condition* $A\mathbf{x} = \mathbf{0}$, while im A consists of vectors of the *form* $A\mathbf{x}$ for some $\mathbf{x}$ in $\mathbb{R}^n$. These two ways to describe subsets occur frequently.

EXAMPLE 2

If A is an $m \times n$ matrix, then:

1. null A is a subspace of $\mathbb{R}^n$.
2. im A is a subspace of $\mathbb{R}^m$.

2 We are using set notation here. In general $\{q \mid p\}$ means the set of all objects q with property p.

Solution ▶

1. The zero vector $\mathbf{0}$ in $\mathbb{R}^n$ lies in null A because $A\mathbf{0} = \mathbf{0}$.[3] If $\mathbf{x}$ and $\mathbf{x}_1$ are in null A, then $\mathbf{x} + \mathbf{x}_1$ and $a\mathbf{x}$ are in null A because they satisfy the required condition:
$$A(\mathbf{x} + \mathbf{x}_1) = A\mathbf{x} + A\mathbf{x}_1 = \mathbf{0} + \mathbf{0} = \mathbf{0} \quad \text{and} \quad A(a\mathbf{x}) = a(A\mathbf{x}) = a\mathbf{0} = \mathbf{0}$$
Hence null A satisfies S1, S2, and S3, and so is a subspace of $\mathbb{R}^n$.
2. The zero vector $\mathbf{0}$ in $\mathbb{R}^m$ lies in im A because $\mathbf{0} = A\mathbf{0}$. Suppose that $\mathbf{y}$ and $\mathbf{y}_1$ are in im A, say $\mathbf{y} = A\mathbf{x}$ and $\mathbf{y}_1 = A\mathbf{x}_1$ where $\mathbf{x}$ and $\mathbf{x}_1$ are in $\mathbb{R}^n$. Then
$$\mathbf{y} + \mathbf{y}_1 = A\mathbf{x} + A\mathbf{x}_1 = A(\mathbf{x} + \mathbf{x}_1) \quad \text{and} \quad a\mathbf{y} = a(A\mathbf{x}) = A(a\mathbf{x})$$
show that $\mathbf{y} + \mathbf{y}_1$ and $a\mathbf{y}$ are both in im A (they have the required form). Hence im A is a subspace of $\mathbb{R}^m$.

There are other important subspaces associated with a matrix A that clarify basic properties of A. If A is an $n \times n$ matrix and λ is any number, let
$$E_\lambda(A) = \{\mathbf{x} \text{ in } \mathbb{R}^n \mid A\mathbf{x} = \lambda\mathbf{x}\}.$$
A vector $\mathbf{x}$ is in $E_\lambda(A)$ if and only if $(\lambda I - A)\mathbf{x} = \mathbf{0}$, so Example 2 gives:

EXAMPLE 3

$E_\lambda(A) = \text{null}(\lambda I - A)$ is a subspace of $\mathbb{R}^n$ for each $n \times n$ matrix A and number λ.

$E_\lambda(A)$ is called the **eigenspace** of A corresponding to λ. The reason for the name is that, in the terminology of Section 3.3, λ is an **eigenvalue** of A if $E_\lambda(A) \neq \{\mathbf{0}\}$. In this case the nonzero vectors in $E_\lambda(A)$ are called the **eigenvectors** of A corresponding to λ.

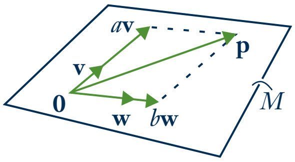
The reader should not get the impression that *every* subset of $\mathbb{R}^n$ is a subspace. For example:
$$U_1 = \left\{\begin{bmatrix} x \\ y \end{bmatrix} \mid x \geq 0\right\} \text{ satisfies S1 and S2, but not S3;}$$
$$U_2 = \left\{\begin{bmatrix} x \\ y \end{bmatrix} \mid x^2 = y^2\right\} \text{ satisfies S1 and S3, but not S2;}$$
Hence neither U_1 nor U_2 is a subspace of $\mathbb{R}^2$. (However, see Exercise 20.)

Spanning Sets

Let $\mathbf{v}$ and $\mathbf{w}$ be two nonzero, nonparallel vectors in $\mathbb{R}^3$ with their tails at the origin. The plane M through the origin containing these vectors is described in Section 4.2 by saying that $\mathbf{n} = \mathbf{v} \times \mathbf{w}$ is a *normal* for M, and that M consists of all vectors $\mathbf{p}$ such that $\mathbf{n} \cdot \mathbf{p} = 0$.[4] While this is a very useful way to look at planes, there is another approach that is at least as useful in $\mathbb{R}^3$ and, more importantly, works for all subspaces of $\mathbb{R}^n$ for any $n \geq 1$.

3 We are using $\mathbf{0}$ to represent the zero vector in both $\mathbb{R}^m$ and $\mathbb{R}^n$. This abuse of notation is common and causes no confusion once everybody knows what is going on.

4 The vector $\mathbf{n} = \mathbf{v} \times \mathbf{w}$ is nonzero because $\mathbf{v}$ and $\mathbf{w}$ are not parallel.



The idea is as follows: Observe that, by the diagram, a vector $\mathbf{p}$ is in M if and only if it has the form

$$\mathbf{p} = a\mathbf{v} + b\mathbf{w}$$

for certain real numbers a and b (we say that $\mathbf{p}$ is a *linear combination* of $\mathbf{v}$ and $\mathbf{w}$). Hence we can describe M as

$$M = \{a\mathbf{x} + b\mathbf{w} \mid a, b \text{ in } \mathbb{R}\}.^5$$

and we say that $\{\mathbf{v}, \mathbf{w}\}$ is a *spanning set* for M. It is this notion of a spanning set that provides a way to describe all subspaces of $\mathbb{R}^n$.

As in Section 1.3, given vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ in $\mathbb{R}^n$, a vector of the form

$$t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_k\mathbf{x}_k \quad \text{where the } t_i \text{ are scalars}$$

is called a **linear combination** of the $\mathbf{x}_i$, and t_i is called the **coefficient** of $\mathbf{x}_i$ in the linear combination.

Definition 5.2 *The set of all such linear combinations is called the* **span** *of the* $\mathbf{x}_i$ *and is denoted*

$$\text{span}\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\} = \{t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_k\mathbf{x}_k \mid t_i \text{ in } \mathbb{R}\}.$$

If $V = \text{span}\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$, *we say that* V *is* **spanned** *by the vectors* $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$, *and that the vectors* $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ **span** *the space* V.

Two examples:

$$\text{span}\{\mathbf{x}\} = \{t\mathbf{x} \mid t \text{ in } \mathbb{R}\},$$

which we write as $\text{span}\{\mathbf{x}\} = \mathbb{R}\mathbf{x}$ for simplicity.

$$\text{span}\{\mathbf{x}, \mathbf{y}\} = \{r\mathbf{x} + s\mathbf{y} \mid r, s \text{ in } \mathbb{R}\}.$$

In particular, the above discussion shows that, if $\mathbf{v}$ and $\mathbf{w}$ are two nonzero, nonparallel vectors in $\mathbb{R}^3$, then

$$M = \text{span}\{\mathbf{v}, \mathbf{w}\}$$

is the plane in $\mathbb{R}^3$ containing $\mathbf{v}$ and $\mathbf{w}$. Moreover, if $\mathbf{d}$ is any nonzero vector in $\mathbb{R}^3$ (or $\mathbb{R}^2$), then

$$L = \text{span}\{\mathbf{v}\} = \{t\mathbf{d} \mid t \text{ in } \mathbb{R}\} = \mathbb{R}\mathbf{d}$$

is the line with direction vector $\mathbf{d}$ (see also Lemma 1 Section 3.3). Hence lines and planes can both be described in terms of spanning sets.

EXAMPLE 4

Let $\mathbf{x} = (2, -1, 2, 1)$ and $\mathbf{y} = (3, 4, -1, 1)$ in $\mathbb{R}^4$. Determine whether $\mathbf{p} = (0, -11, 8, 1)$ or $\mathbf{q} = (2, 3, 1, 2)$ are in $U = \text{span}\{\mathbf{x}, \mathbf{y}\}$.

Solution ▶ The vector $\mathbf{p}$ is in U if and only if $\mathbf{p} = s\mathbf{x} + t\mathbf{y}$ for scalars s and t. Equating components gives equations

$$2s + 3t = 0, \quad -s + 4t = -11, \quad 2s - t = 8, \quad \text{and} \quad s + t = 1.$$

This linear system has solution $s = 3$ and $t = -2$, so $\mathbf{p}$ is in U. On the other hand, asking that $\mathbf{q} = s\mathbf{x} + t\mathbf{y}$ leads to equations

$$2s + 3t = 2, \quad -s + 4t = 3, \quad 2s - t = 1, \quad \text{and} \quad s + t = 2$$

and this system has *no* solution. So $\mathbf{q}$ does *not* lie in U.

5 In particular, this implies that any vector $\mathbf{p}$ orthogonal to $\mathbf{v} \times \mathbf{w}$ must be a linear combination $\mathbf{p} = a\mathbf{v} + b\mathbf{w}$ of $\mathbf{v}$ and $\mathbf{w}$ for some a and b. Can you prove this directly?

Theorem 1

Let $U = \text{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ in $\mathbb{R}^n$. Then:

1. *U is a subspace of $\mathbb{R}^n$ containing each X_i.*
2. *If W is a subspace of $\mathbb{R}^n$ and each X_i is in W, then $U \subseteq W$.*

PROOF

Write $U = \text{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ for convenience.

1. The zero vector $\mathbf{0}$ is in U because $\mathbf{0} = 0\mathbf{x}_1 + 0\mathbf{x}_2 + \cdots + 0\mathbf{x}_k$ is a linear combination of the $\mathbf{x}_i$. If $\mathbf{x} = t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_k\mathbf{x}_k$ and $\mathbf{y} = s_1\mathbf{x}_1 + s_2\mathbf{x}_2 + \cdots + s_k\mathbf{x}_k$ are in U, then $\mathbf{x} + \mathbf{y}$ and $a\mathbf{x}$ are in U because

$$\mathbf{x} + \mathbf{y} = (t_1 + s_1)\mathbf{x}_1 + (t_2 + s_2)\mathbf{x}_2 + \cdots + (t_k + s_k)\mathbf{x}_1, \quad \text{and}$$
$$a\mathbf{x} = (at_1)\mathbf{x}_1 + (at_2)\mathbf{x}_2 + \cdots + (at_k)\mathbf{x}_1.$$

   Hence S1, S2, and S3 are satisfied for U, proving (1).

2. Let $\mathbf{x} = t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_k\mathbf{x}_k$ where the t_i are scalars and each $\mathbf{x}_i$ is in W. Then each $t_i\mathbf{x}_i$ is in W because W satisfies S3. But then $\mathbf{x}$ is in W because W satisfies S2 (verify). This proves (2).

Condition (2) in Theorem 1 can be expressed by saying that $\text{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ is the *smallest* subspace of $\mathbb{R}^n$ that contains each $\mathbf{x}_i$. This is useful for showing that two subspaces U and W are equal, since this amounts to showing that both $U \subseteq W$ and $W \subseteq U$. Here is an example of how it is used.

EXAMPLE 5

If $\mathbf{x}$ and $\mathbf{y}$ are in $\mathbb{R}^n$, show that $\text{span}\{\mathbf{x}, \mathbf{y}\} = \text{span}\{\mathbf{x} + \mathbf{y}, \mathbf{x} - \mathbf{y}\}$.

Solution ▶ Since both $\mathbf{x} + \mathbf{y}$ and $\mathbf{x} - \mathbf{y}$ are in $\text{span}\{\mathbf{x}, \mathbf{y}\}$, Theorem 1 gives

$$\text{span}\{\mathbf{x} + \mathbf{y}, \mathbf{x} - \mathbf{y}\} \subseteq \text{span}\{\mathbf{x}, \mathbf{y}\}.$$

But $\mathbf{x} = \frac{1}{2}(\mathbf{x} + \mathbf{y}) + \frac{1}{2}(\mathbf{x} - \mathbf{y})$ and $\mathbf{y} = \frac{1}{2}(\mathbf{x} + \mathbf{y}) - \frac{1}{2}(\mathbf{x} - \mathbf{y})$ are both in $\text{span}\{\mathbf{x} + \mathbf{y}, \mathbf{x} - \mathbf{y}\}$, so

$$\text{span}\{\mathbf{x}, \mathbf{y}\} \subseteq \text{span}\{\mathbf{x} + \mathbf{y}, \mathbf{x} - \mathbf{y}\}$$

again by Theorem 1. Thus $\text{span}\{\mathbf{x}, \mathbf{y}\} = \text{span}\{\mathbf{x} + \mathbf{y}, \mathbf{x} - \mathbf{y}\}$, as desired.

It turns out that many important subspaces are best described by giving a spanning set. Here are three examples, beginning with an important spanning set for $\mathbb{R}^n$ itself. Column j of the $n \times n$ identity matrix I_n is denoted $\mathbf{e}_j$ and called the jth **coordinate vector** in $\mathbb{R}^n$, and the set $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is called the **standard basis** of $\mathbb{R}^n$. If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is any vector in $\mathbb{R}^n$, then $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$, as the reader can verify. This proves:

EXAMPLE 6

$\mathbb{R}^n = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ where $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ are the columns of I_n.

If A is an $m \times n$ matrix A, the next two examples show that it is a routine matter to find spanning sets for null A and im A.

EXAMPLE 7

Given an $m \times n$ matrix A, let $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ denote the basic solutions to the system $A\mathbf{x} = \mathbf{0}$ given by the gaussian algorithm. Then

$$\text{null } A = \text{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}.$$

Solution ▶ If $\mathbf{x}$ is in null A, then $A\mathbf{x} = \mathbf{0}$ so Theorem 2 Section 1.3 shows that $\mathbf{x}$ is a linear combination of the basic solutions; that is, null $A \subseteq \text{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$. On the other hand, if $\mathbf{x}$ is in $\text{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$, then $\mathbf{x} = t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_k\mathbf{x}_k$ for scalars t_i, so

$$A\mathbf{x} = t_1A\mathbf{x}_1 + t_2A\mathbf{x}_2 + \cdots + t_kA\mathbf{x}_k = t_1\mathbf{0} + t_2\mathbf{0} + \cdots + t_k\mathbf{0} = \mathbf{0}.$$

This shows that $\mathbf{x}$ is in null A, and hence that $\text{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\} \subseteq \text{null } A$. Thus we have equality.

EXAMPLE 8

Let $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n$ denote the columns of the $m \times n$ matrix A. Then

$$\text{im } A = \text{span}\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n\}.$$

Solution ▶ If $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is the standard basis of $\mathbb{R}^n$, observe that

$$[A\mathbf{e}_1 \;\; A\mathbf{e}_2 \;\; \cdots \;\; A\mathbf{e}_n] = A[\mathbf{e}_1 \;\; \mathbf{e}_2 \;\; \cdots \;\; \mathbf{e}_n] = AI_n = A = [\mathbf{c}_1 \;\; \mathbf{c}_2 \;\; \cdots \;\; \mathbf{c}_n].$$

Hence $\mathbf{c}_i = A\mathbf{e}_i$ is in im A for each i, so $\text{span}\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n\} \subseteq \text{im } A$.

Conversely, let $\mathbf{y}$ be in im A, say $\mathbf{y} = A\mathbf{x}$ for some $\mathbf{x}$ in $\mathbb{R}^n$. If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, then Definition 2.5 gives

$$\mathbf{y} = A\mathbf{x} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n \text{ is in span}\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n\}.$$

This shows that im $A \subseteq \text{span}\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n\}$, and the result follows.

EXERCISES 5.1

We often write vectors in $\mathbb{R}^n$ as rows.

1. In each case determine whether U is a subspace of $\mathbb{R}^3$. Support your answer.
   (a) $U = \{(1, s, t) \mid s \text{ and } t \text{ in } \mathbb{R}\}$.
   ◆(b) $U = \{(0, s, t) \mid s \text{ and } t \text{ in } \mathbb{R}\}$.
   (c) $U = \{(r, s, t) \mid r, s, \text{ and } t \text{ in } \mathbb{R}, -r + 3s + 2t = 0\}$.
   ◆(d) $U = \{(r, 3s, r - 2) \mid r \text{ and } s \text{ in } \mathbb{R}\}$.
   (e) $U = \{(r, 0, s) \mid r^2 + s^2 = 0, r \text{ and } s \text{ in } \mathbb{R}\}$.
   ◆(f) $U = \{(2r, -s^2, t) \mid r, s, \text{ and } t \text{ in } \mathbb{R}\}$.

2. In each case determine if $\mathbf{x}$ lies in $U = \text{span}\{\mathbf{y}, \mathbf{z}\}$. If $\mathbf{x}$ is in U, write it as a linear combination of $\mathbf{y}$ and $\mathbf{z}$; if $\mathbf{x}$ is not in U, show why not.
   (a) $\mathbf{x} = (2, -1, 0, 1)$, $\mathbf{y} = (1, 0, 0, 1)$, and $\mathbf{z} = (0, 1, 0, 1)$.
   ◆(b) $\mathbf{x} = (1, 2, 15, 11)$, $\mathbf{y} = (2, -1, 0, 2)$, and $\mathbf{z} = (1, -1, -3, 1)$.
   (c) $\mathbf{x} = (8, 3, -13, 20)$, $\mathbf{y} = (2, 1, -3, 5)$, and $\mathbf{z} = (-1, 0, 2, -3)$.
   ◆(d) $\mathbf{x} = (2, 5, 8, 3)$, $\mathbf{y} = (2, -1, 0, 5)$, and $\mathbf{z} = (-1, 2, 2, -3)$.
3. In each case determine if the given vectors span $\mathbb{R}^4$. Support your answer.
   (a) $\{(1, 1, 1, 1), (0, 1, 1, 1), (0, 0, 1, 1), (0, 0, 0, 1)\}$.
   ◆(b) $\{(1, 3, -5, 0), (-2, 1, 0, 0), (0, 2, 1, -1), (1, -4, 5, 0)\}$.
4. Is it possible that $\{(1, 2, 0), (2, 0, 3)\}$ can span the subspace $U = \{(r, s, 0) \mid r \text{ and } s \text{ in } \mathbb{R}\}$? Defend your answer.
5. Give a spanning set for the zero subspace $\{\mathbf{0}\}$ of $\mathbb{R}^n$.
6. Is $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$? Defend your answer.
7. If $U = \text{span}\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ in $\mathbb{R}^n$, show that $U = \text{span}\{\mathbf{x} + t\mathbf{z}, \mathbf{y}, \mathbf{z}\}$ for every t in $\mathbb{R}$.
8. If $U = \text{span}\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ in $\mathbb{R}^n$, show that $U = \text{span}\{\mathbf{x} + \mathbf{y}, \mathbf{y} + \mathbf{z}, \mathbf{z} + \mathbf{x}\}$.
9. If $a \neq 0$ is a scalar, show that $\text{span}\{a\mathbf{x}\} = \text{span}\{\mathbf{x}\}$ for every vector $\mathbf{x}$ in $\mathbb{R}^n$.

◆10. If $a_1, a_2, \dots, a_k$ are nonzero scalars, show that $\text{span}\{a_1\mathbf{x}_1, a_2\mathbf{x}_2, \dots, a_k\mathbf{x}_k\} = \text{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ for any vectors $\mathbf{x}_i$ in $\mathbb{R}^n$.

11. If $\mathbf{x} \neq \mathbf{0}$ in $\mathbb{R}^n$, determine all subspaces of $\text{span}\{\mathbf{x}\}$.

◆12. Suppose that $U = \text{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ where each $\mathbf{x}_i$ is in $\mathbb{R}^n$. If A is an $m \times n$ matrix and $A\mathbf{x}_i = \mathbf{0}$ for each i, show that $A\mathbf{y} = \mathbf{0}$ for every vector $\mathbf{y}$ in U.

13. If A is an $m \times n$ matrix, show that, for each invertible $m \times m$ matrix U, $\text{null}(A) = \text{null}(UA)$.
14. If A is an $m \times n$ matrix, show that, for each invertible $n \times n$ matrix V, $\text{im}(A) = \text{im}(AV)$.
15. Let U be a subspace of $\mathbb{R}^n$, and let $\mathbf{x}$ be a vector in $\mathbb{R}^n$.
   (a) If $a\mathbf{x}$ is in U where $a \neq 0$ is a number, show that $\mathbf{x}$ is in U.
   ◆(b) If $\mathbf{y}$ and $\mathbf{x} + \mathbf{y}$ are in U where $\mathbf{y}$ is a vector in $\mathbb{R}^n$, show that $\mathbf{x}$ is in U.
16. In each case either show that the statement is true or give an example showing that it is false.
   (a) If $U \neq \mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ and $\mathbf{x} + \mathbf{y}$ is in U, then $\mathbf{x}$ and $\mathbf{y}$ are both in U.
   ◆(b) If U is a subspace of $\mathbb{R}^n$ and $r\mathbf{x}$ is in U for all r in $\mathbb{R}$, then $\mathbf{x}$ is in U.
   (c) If U is a subspace of $\mathbb{R}^n$ and $\mathbf{x}$ is in U, then $-\mathbf{x}$ is also in U.
   ◆(d) If $\mathbf{x}$ is in U and $U = \text{span}\{\mathbf{y}, \mathbf{z}\}$, then $U = \text{span}\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$.
   (e) The empty set of vectors in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.
   ◆(f) $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is in $\text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}\right\}$.
17. (a) If A and B are $m \times n$ matrices, show that $U = \{\mathbf{x} \text{ in } \mathbb{R}^n \mid A\mathbf{x} = B\mathbf{x}\}$ is a subspace of $\mathbb{R}^n$.
   (b) What if A is $m \times n$, B is $k \times n$, and $m \neq k$?
18. Suppose that $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ are vectors in $\mathbb{R}^n$. If $\mathbf{y} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_k\mathbf{x}_k$ where $a_1 \neq 0$, show that $\text{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\} = \text{span}\{\mathbf{y}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$.
19. If $U \neq \{\mathbf{0}\}$ is a subspace of $\mathbb{R}$, show that $U = \mathbb{R}$.

◆20. Let U be a nonempty subset of $\mathbb{R}^n$. Show that U is a subspace if and only if S2 and S3 hold.

21. If S and T are nonempty sets of vectors in $\mathbb{R}^n$, and if $S \subseteq T$, show that $\text{span}\{S\} \subseteq \text{span}\{T\}$.
22. Let U and W be subspaces of $\mathbb{R}^n$. Define their **intersection** $U \cap W$ and their **sum** $U + W$ as follows:

   $U \cap W = \{\mathbf{x} \text{ in } \mathbb{R}^n \mid \mathbf{x} \text{ belongs to both } U \text{ and } W\}$.

   $U + W = \{\mathbf{x} \text{ in } \mathbb{R}^n \mid \mathbf{x} \text{ is a sum of a vector in } U \text{ and a vector in } W\}$.

   (a) Show that $U \cap W$ is a subspace of $\mathbb{R}^n$.
   ◆(b) Show that $U + W$ is a subspace of $\mathbb{R}^n$.
23. Let P denote an invertible $n \times n$ matrix. If λ is a number, show that $E_\lambda(PAP^{-1}) = \{P\mathbf{x} \mid \mathbf{x} \text{ is in } E_\lambda(A)\}$ for each $n \times n$ matrix A.

24. Show that every proper subspace U of $\mathbb{R}^2$ is a line through the origin. [*Hint*: If $\mathbf{d}$ is a nonzero vector in U, let $L = \mathbb{R}\mathbf{d} = \{r\mathbf{d} \mid r \text{ in } \mathbb{R}\}$ denote the line with direction vector $\mathbf{d}$. If $\mathbf{u}$ is in U but not in L, argue geometrically that every vector $\mathbf{v}$ in $\mathbb{R}^2$ is a linear combination of $\mathbf{u}$ and $\mathbf{d}$.]

SECTION 5.2 Independence and Dimension

Some spanning sets are better than others. If $U = \text{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ is a subspace of $\mathbb{R}^n$, then every vector in U can be written as a linear combination of the $\mathbf{x}_i$ in at least one way. Our interest here is in spanning sets where each vector in U has a *exactly one* representation as a linear combination of these vectors.

Linear Independence

Given $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ in $\mathbb{R}^n$, suppose that two linear combinations are equal:

$$r_1\mathbf{x}_1 + r_2\mathbf{x}_2 + \cdots + r_k\mathbf{x}_k = s_1\mathbf{x}_1 + s_2\mathbf{x}_2 + \cdots + s_k\mathbf{x}_k.$$

We are looking for a condition on the set $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ of vectors that guarantees that this representation is *unique*; that is, $r_i = s_i$ for each i. Taking all terms to the left side gives

$$(r_1 - s_1)\mathbf{x}_1 + (r_2 - s_2)\mathbf{x}_2 + \cdots + (r_k - s_k)\mathbf{x}_k = \mathbf{0}.$$

so the required condition is that this equation forces all the coefficients $r_i - s_i$ to be zero.

Definition 5.3 *With this in mind, we call a set* $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ *of vectors* **linearly independent** *(or simply* **independent***) if it satisfies the following condition:*

$$\text{If} \quad t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_k\mathbf{x}_k = \mathbf{0} \quad \text{then} \quad t_1 = t_2 = \cdots = t_k = 0.$$

We record the result of the above discussion for reference.

Theorem 1

If $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ *is an independent set of vectors in* $\mathbb{R}^n$*, then every vector in* $\text{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ *has a* **unique** *representation as a linear combination of the* $\mathbf{x}_i$.

It is useful to state the definition of independence in different language. Let us say that a linear combination **vanishes** if it equals the zero vector, and call a linear combination **trivial** if every coefficient is zero. Then the definition of independence can be compactly stated as follows:

> A set of vectors is independent if and only if the only linear combination that vanishes is the trivial one.

Hence we have a procedure for checking that a set of vectors is independent:

Independence Test

To verify that a set $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ *of vectors in* $\mathbb{R}^n$ *is independent, proceed as follows:*

1. *Set a linear combination equal to zero:* $t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_k\mathbf{x}_k = \mathbf{0}$.
2. *Show that* $t_i = 0$ *for each* i *(that is, the linear combination is trivial).*

Of course, if some nontrivial linear combination vanishes, the vectors are not *independent.*

EXAMPLE 1

Determine whether $\{(1, 0, -2, 5), (2, 1, 0, -1), (1, 1, 2, 1)\}$ is independent in $\mathbb{R}^4$.

Solution ▶ Suppose a linear combination vanishes:

$$r(1, 0, -2, 5) + s(2, 1, 0, -1) + t(1, 1, 2, 1) = (0, 0, 0, 0).$$

Equating corresponding entries gives a system of four equations:

$$r + 2s + t = 0, \quad s + t = 0, \quad -2r + 2t = 0, \quad \text{and} \quad 5r - s + t = 0.$$

The only solution is the trivial one $r = s = t = 0$ (verify), so these vectors are independent by the independence test.

EXAMPLE 2

Show that the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_k\}$ of $\mathbb{R}^n$ is independent.

Solution ▶ The components of $t_1\mathbf{e}_1 + t_2\mathbf{e}_2 + \cdots + t_n\mathbf{e}_n$ are $t_1, t_2, \dots, t_n$ (see the discussion preceding Example 6 Section 5.1) So the linear combination vanishes if and only if each $t_i = 0$. Hence the independence test applies.

EXAMPLE 3

If $\{\mathbf{x}, \mathbf{y}\}$ is independent, show that $\{2\mathbf{x} + 3\mathbf{y}, \mathbf{x} - 5\mathbf{y}\}$ is also independent.

Solution ▶ If $s(2\mathbf{x} + 3\mathbf{y}) + t(\mathbf{x} - 5\mathbf{y}) = \mathbf{0}$, collect terms to get $(2s + t)\mathbf{x} + (3s - 5t)\mathbf{y} = \mathbf{0}$. Since $\{\mathbf{x}, \mathbf{y}\}$ is independent this combination must be trivial; that is, $2s + t = 0$ and $3s - 5t = 0$. These equations have only the trivial solution $s = t = 0$, as required.

EXAMPLE 4

Show that the zero vector in $\mathbb{R}^n$ does not belong to any independent set.

Solution ▶ No set $\{\mathbf{0}, \mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ of vectors is independent because we have a vanishing, nontrivial linear combination $1 \cdot \mathbf{0} + 0\mathbf{x}_1 + 0\mathbf{x}_2 + \cdots + 0\mathbf{x}_k = \mathbf{0}$.

EXAMPLE 5

Given $\mathbf{x}$ in $\mathbb{R}^n$, show that $\{\mathbf{x}\}$ is independent if and only if $\mathbf{x} \neq \mathbf{0}$.

Solution ▶ A vanishing linear combination from $\{\mathbf{x}\}$ takes the form $t\mathbf{x} = \mathbf{0}$, t in $\mathbb{R}$. This implies that $t = 0$ because $\mathbf{x} \neq \mathbf{0}$.

The next example will be needed later.

EXAMPLE 6

Show that the nonzero rows of a row-echelon matrix R are independent.

Solution ▶ We illustrate the case with 3 leading 1s; the general case is analogous. Suppose R has the form $R = \begin{bmatrix} 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ where $*$ indicates a nonspecified number. Let R_1, R_2, and R_3 denote the nonzero rows of R. If $t_1R_1 + t_2R_2 + t_3R_3 = 0$ we show that $t_1 = 0$, then $t_2 = 0$, and finally $t_3 = 0$. The condition $t_1R_1 + t_2R_2 + t_3R_3 = 0$ becomes

$$(0, t_1, *, *, *, *) + (0, 0, 0, t_2, *, *) + (0, 0, 0, 0, t_3, *) = (0, 0, 0, 0, 0, 0).$$

Equating second entries show that $t_1 = 0$, so the condition becomes $t_2R_2 + t_3R_3 = 0$. Now the same argument shows that $t_2 = 0$. Finally, this gives $t_3R_3 = 0$ and we obtain $t_3 = 0$.

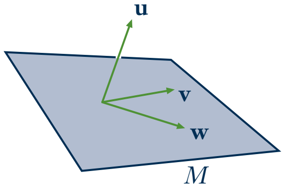
A set of vectors in $\mathbb{R}^n$ is called **linearly dependent** (or simply **dependent**) if it is *not* linearly independent, equivalently if some nontrivial linear combination vanishes.

EXAMPLE 7

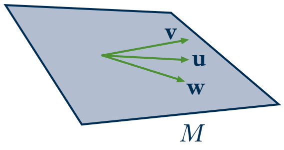
If $\mathbf{v}$ and $\mathbf{w}$ are nonzero vectors in $\mathbb{R}^3$, show that $\{\mathbf{v}, \mathbf{w}\}$ is dependent if and only if $\mathbf{v}$ and $\mathbf{w}$ are parallel.

Solution ▶ If $\mathbf{v}$ and $\mathbf{w}$ are parallel, then one is a scalar multiple of the other (Theorem 4 Section 4.1), say $\mathbf{v} = a\mathbf{w}$ for some scalar a. Then the nontrivial linear combination $\mathbf{v} - a\mathbf{w} = \mathbf{0}$ vanishes, so $\{\mathbf{v}, \mathbf{w}\}$ is dependent.

Conversely, if $\{\mathbf{v}, \mathbf{w}\}$ is dependent, let $s\mathbf{v} + t\mathbf{w} = \mathbf{0}$ be nontrivial, say $s \neq 0$. Then $\mathbf{v} = -\frac{t}{s}\mathbf{w}$, so $\mathbf{v}$ and $\mathbf{w}$ are parallel (by Theorem 4 Section 4.1). A similar argument works if $t \neq 0$.



$\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ independent



$\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ not independent

With this we can give a geometric description of what it means for a set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ in $\mathbb{R}^3$ to be independent. Note that this requirement means that $\{\mathbf{v}, \mathbf{w}\}$ is also independent ($a\mathbf{v} + b\mathbf{w} = \mathbf{0}$ means that $0\mathbf{u} + a\mathbf{v} + b\mathbf{w} = \mathbf{0}$), so $M = \text{span}\{\mathbf{v}, \mathbf{w}\}$ is the plane containing $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{0}$ (see the discussion preceding Example 4 Section 5.1). So we assume that $\{\mathbf{v}, \mathbf{w}\}$ is independent in the following example.

EXAMPLE 8

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be nonzero vectors in $\mathbb{R}^3$ where $\{\mathbf{v}, \mathbf{w}\}$ independent. Show that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is independent if and only if $\mathbf{u}$ is not in the plane $M = \text{span}\{\mathbf{v}, \mathbf{w}\}$. This is illustrated in the diagrams.

Solution ▶ If $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is independent, suppose $\mathbf{u}$ is in the plane $M = \text{span}\{\mathbf{v}, \mathbf{w}\}$, say $\mathbf{u} = a\mathbf{v} + b\mathbf{w}$, where a and b are in $\mathbb{R}$. Then $1\mathbf{u} - a\mathbf{v} - b\mathbf{w} = \mathbf{0}$, contradicting the independence of $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.

On the other hand, suppose that $\mathbf{u}$ is not in M; we must show that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is independent. If $r\mathbf{u} + s\mathbf{v} + t\mathbf{w} = \mathbf{0}$ where r, s, and t are in $\mathbb{R}^3$, then $r = 0$ since otherwise $\mathbf{u} = \frac{-s}{r}\mathbf{v} + \frac{-t}{r}\mathbf{w}$ is in M. But then $s\mathbf{v} + t\mathbf{w} = \mathbf{0}$, so $s = t = 0$ by our assumption. This shows that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is independent, as required.

By Theorem 5 Section 2.4, the following conditions are equivalent for an $n \times n$ matrix A:

1. *A is invertible.*
2. *If $A\mathbf{x} = \mathbf{0}$ where $\mathbf{x}$ is in $\mathbb{R}^n$, then $\mathbf{x} = \mathbf{0}$.*
3. *$A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$ for every vector $\mathbf{b}$ in $\mathbb{R}^n$.*

While condition 1 makes no sense if A is not square, conditions 2 and 3 are meaningful for any matrix A and, in fact, are related to independence and spanning. Indeed, if $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n$ are the columns of A, and if we write $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, then

$$A\mathbf{x} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n$$

by Definition 2.5. Hence the definitions of independence and spanning show, respectively, that condition 2 is equivalent to the independence of $\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n\}$ and condition 3 is equivalent to the requirement that $\text{span}\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n\} = \mathbb{R}^m$. This discussion is summarized in the following theorem:

Theorem 2

If A is an $m \times n$ matrix, let $\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n\}$ denote the columns of A.

1. *$\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n\}$ is independent in $\mathbb{R}^m$ if and only if $A\mathbf{x} = \mathbf{0}$, $\mathbf{x}$ in $\mathbb{R}^n$, implies $\mathbf{x} = \mathbf{0}$.*
2. *$\mathbb{R}^m = \text{span}\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n\}$ if and only if $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$ for every vector $\mathbf{b}$ in $\mathbb{R}^m$.*

For a *square* matrix A, Theorem 2 characterizes the invertibility of A in terms of the spanning and independence of its columns (see the discussion preceding Theorem 2). It is important to be able to discuss these notions for *rows*. If $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ are $1 \times n$ rows, we define $\text{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ to be the set of all linear combinations of the $\mathbf{x}_i$ (as matrices), and we say that $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ is linearly independent if the only vanishing linear combination is the trivial one (that is, if $\{\mathbf{x}_1^T, \mathbf{x}_2^T, \dots, \mathbf{x}_k^T\}$ is independent in $\mathbb{R}^n$, as the reader can verify).[6]

Theorem 3

The following are equivalent for an $n \times n$ matrix A:

1. *A is invertible.*

6 It is best to view columns and rows as just two different *notations* for ordered n-tuples. This discussion will become redundant in Chapter 6 where we define the general notion of a vector space.

2. *The columns of A are linearly independent.*
3. *The columns of A span $\mathbb{R}^n$.*
4. *The rows of A are linearly independent.*
5. *The rows of A span the set of all $1 \times n$ rows.*

PROOF

Let $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n$ denote the columns of A.

(1) $\Leftrightarrow$ (2). By Theorem 5 Section 2.4, A is invertible if and only if $A\mathbf{x} = \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}$; this holds if and only if $\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n\}$ is independent by Theorem 2.

(1) $\Leftrightarrow$ (3). Again by Theorem 5 Section 2.4, A is invertible if and only if $A\mathbf{x} = \mathbf{b}$ has a solution for every column B in $\mathbb{R}^n$; this holds if and only if $\text{span}\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n\} = \mathbb{R}^n$ by Theorem 2.

(1) $\Leftrightarrow$ (4). The matrix A is invertible if and only if A^T is invertible (by the Corollary to Theorem 4 Section 2.4); this in turn holds if and only if A^T has independent columns (by (1) $\Leftrightarrow$ (2)); finally, this last statement holds if and only if A has independent rows (because the rows of A are the transposes of the columns of A^T).

(1) $\Leftrightarrow$ (5). The proof is similar to (1) $\Leftrightarrow$ (4).

EXAMPLE 9

Show that $S = \{(2, -2, 5), (-3, 1, 1), (2, 7, -4)\}$ is independent in $\mathbb{R}^3$.

Solution ▸ Consider the matrix $A = \begin{bmatrix} 2 & -2 & 5 \\ -3 & 1 & 1 \\ 2 & 7 & -4 \end{bmatrix}$ with the vectors in S as its rows. A routine computation shows that $\det A = -117 \neq 0$, so A is invertible. Hence S is independent by Theorem 3. Note that Theorem 3 also shows that $\mathbb{R}^3 = \text{span S}$.

Dimension

It is common geometrical language to say that $\mathbb{R}^3$ is 3-dimensional, that planes are 2-dimensional and that lines are 1-dimensional. The next theorem is a basic tool for clarifying this idea of "dimension". Its importance is difficult to exaggerate.

Theorem 4

Fundamental Theorem
Let U be a subspace of $\mathbb{R}^n$. If U is spanned by m vectors, and if U contains k linearly independent vectors, then $k \leq m$.

This proof is given in Theorem 2 Section 6.3 in much greater generality.

Definition 5.4

If U is a subspace of $\mathbb{R}^n$, a set $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ of vectors in U is called a **basis** *of U if it satisfies the following two conditions:*

1. $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ *is linearly independent.*
2. $U = \operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$.

The most remarkable result about bases[7] is:

Theorem 5

Invariance Theorem
If $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ and $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_k\}$ are bases of a subspace U of $\mathbb{R}^n$, then $m = k$.

PROOF

We have $k \le m$ by the fundamental theorem because $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ spans U, and $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_k\}$ is independent. Similarly, by interchanging **x**s and **y**s we get $m \le k$. Hence $m = k$.

The invariance theorem guarantees that there is no ambiguity in the following definition:

Definition 5.5

If U is a subspace of $\mathbb{R}^n$ and $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ is any basis of U, the number, m, of vectors in the basis is called the **dimension** *of U, denoted*

$$\dim U = m.$$

The importance of the invariance theorem is that the dimension of U can be determined by counting the number of vectors in *any* basis.[8]

Let $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ denote the standard basis of $\mathbb{R}^n$, that is the set of columns of the identity matrix. Then $\mathbb{R}^n = \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ by Example 6 Section 5.1, and $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is independent by Example 2. Hence it is indeed a basis of $\mathbb{R}^n$ in the present terminology, and we have

EXAMPLE 10

$\dim(\mathbb{R}^n) = n$ and $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is a basis.

This agrees with our geometric sense that $\mathbb{R}^2$ is two-dimensional and $\mathbb{R}^3$ is three-dimensional. It also says that $\mathbb{R}^1 = \mathbb{R}$ is one-dimensional, and $\{1\}$ is a basis. Returning to subspaces of $\mathbb{R}^n$, we define

$$\dim \{\mathbf{0}\} = 0.$$

This amounts to saying $\{\mathbf{0}\}$ has a basis containing *no* vectors. This makes sense because $\mathbf{0}$ cannot belong to *any* independent set (Example 4).

7 The plural of "basis" is "bases".

8 We will show in Theorem 6 that every subspace of $\mathbb{R}^n$ does indeed *have* a basis.

EXAMPLE 11

Let $U = \left\{\begin{bmatrix} r \\ s \\ r \end{bmatrix} \mid r, s \text{ in } \mathbb{R}\right\}$. Show that U is a subspace of $\mathbb{R}^3$, find a basis, and calculate dim U.

Solution ▶ Clearly, $\begin{bmatrix} r \\ s \\ r \end{bmatrix} = r\mathbf{u} + s\mathbf{v}$ where $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. It follows that $U = \text{span}\{\mathbf{u}, \mathbf{v}\}$, and hence that U is a subspace of $\mathbb{R}^3$. Moreover, if $r\mathbf{u} + s\mathbf{v} = \mathbf{0}$, then $\begin{bmatrix} r \\ s \\ r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ so $r = s = 0$. Hence $\{\mathbf{u}, \mathbf{v}\}$ is independent, and so a **basis** of U. This means dim $U = 2$.

EXAMPLE 12

Let $B = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ be a basis of $\mathbb{R}^n$. If A is an invertible $n \times n$ matrix, then $D = \{A\mathbf{x}_1, A\mathbf{x}_2, \dots, A\mathbf{x}_n\}$ is also a basis of $\mathbb{R}^n$.

Solution ▶ Let $\mathbf{x}$ be a vector in $\mathbb{R}^n$. Then $A^{-1}\mathbf{x}$ is in $\mathbb{R}^n$ so, since B is a basis, we have $A^{-1}\mathbf{x} = t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_n\mathbf{x}_n$ for t_i in $\mathbb{R}$. Left multiplication by A gives $\mathbf{x} = t_1(A\mathbf{x}_1) + t_2(A\mathbf{x}_2) + \cdots + t_n(A\mathbf{x}_n)$, and it follows that D spans $\mathbb{R}^n$. To show independence, let $s_1(A\mathbf{x}_1) + s_2(A\mathbf{x}_2) + \cdots + s_n(A\mathbf{x}_n) = \mathbf{0}$, where the s_i are in $\mathbb{R}$. Then $A(s_1\mathbf{x}_1 + s_2\mathbf{x}_2 + \cdots + s_n\mathbf{x}_n) = \mathbf{0}$ so left multiplication by A^{-1} gives $s_1\mathbf{x}_1 + s_2\mathbf{x}_2 + \cdots + s_n\mathbf{x}_n = \mathbf{0}$. Now the independence of B shows that each $s_i = 0$, and so proves the independence of D. Hence D is a basis of $\mathbb{R}^n$.

While we have found bases in many subspaces of $\mathbb{R}^n$, we have not yet shown that *every* subspace *has* a basis. This is part of the next theorem, the proof of which is deferred to Section 6.4 where it will be proved in more generality.

Theorem 6

Let $U \neq \{\mathbf{0}\}$ be a subspace of $\mathbb{R}^n$. Then:

1. *U has a basis and dim $U \leq n$.*
2. *Any independent set in U can be enlarged (by adding vectors from the standard basis) to a basis of U.*
3. *Any spanning set for U can be cut down (by deleting vectors) to a basis of U.*

EXAMPLE 13

Find a basis of $\mathbb{R}^4$ containing $S = \{\mathbf{u}, \mathbf{v}\}$ where $\mathbf{u} = (0, 1, 2, 3)$ and $\mathbf{v} = (2, -1, 0, 1)$.

Solution ▶ By Theorem 6 we can find such a basis by adding vectors from the standard basis of $\mathbb{R}^4$ to S. If we try $\mathbf{e}_1 = (1, 0, 0, 0)$, we find easily that $\{\mathbf{e}_1, \mathbf{u}, \mathbf{v}\}$ is independent. Now add another vector from the standard basis, say $\mathbf{e}_2$.

Again we find that $B = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{u}, \mathbf{v}\}$ is independent. Since B has $4 = \dim \mathbb{R}^4$ vectors, then B must span $\mathbb{R}^4$ by Theorem 7 below (or simply verify it directly). Hence B is a basis of $\mathbb{R}^4$.

Theorem 6 has a number of useful consequences. Here is the first.

Theorem 7

Let U be a subspace of $\mathbb{R}^n$ where $\dim U = m$ and let $B = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ be a set of m vectors in U. Then B is independent if and only if B spans U.

PROOF

Suppose B is independent. If B does not span U then, by Theorem 6, B can be enlarged to a basis of U containing more than m vectors. This contradicts the invariance theorem because $\dim U = m$, so B spans U. Conversely, if B spans U but is not independent, then B can be cut down to a basis of U containing fewer than m vectors, again a contradiction. So B is independent, as required.

As we saw in Example 13, Theorem 7 is a "labour-saving" result. It asserts that, given a subspace U of dimension m and a set B of exactly m vectors in U, to prove that B is a basis of U it suffices to show either that B spans U or that B is independent. It is not necessary to verify both properties.

Theorem 8

Let $U \subseteq W$ be subspaces of $\mathbb{R}^n$. Then:

1. $\dim U \leq \dim W$.
2. *If* $\dim U = \dim W$, *then* $U = W$.

PROOF

Write $\dim W = k$, and let B be a basis of U.

1. If $\dim U > k$, then B is an independent set in W containing more than k vectors, contradicting the fundamental theorem. So $\dim U \leq k = \dim W$.
2. If $\dim U = k$, then B is an independent set in W containing $k = \dim W$ vectors, so B spans W by Theorem 7. Hence $W = \operatorname{span} B = U$, proving (2).

It follows from Theorem 8 that if U is a subspace of $\mathbb{R}^n$, then $\dim U$ is one of the integers $0, 1, 2, \ldots, n$, and that:

$$\begin{aligned} \dim U = 0 &\quad \text{if and only if } U = \{\mathbf{0}\}, \\ \dim U = n &\quad \text{if and only if } U = \mathbb{R}^n \end{aligned}$$

The other subspaces are called **proper**. The following example uses Theorem 8 to show that the proper subspaces of $\mathbb{R}^2$ are the lines through the origin, while the proper subspaces of $\mathbb{R}^3$ are the lines and planes through the origin.

EXAMPLE 14

1. If U is a subspace of $\mathbb{R}^2$ or $\mathbb{R}^3$, then $\dim U = 1$ if and only if U is a line through the origin.
2. If U is a subspace of $\mathbb{R}^3$, then $\dim U = 2$ if and only if U is a plane through the origin.

PROOF

1. Since $\dim U = 1$, let $\{\mathbf{u}\}$ be a basis of U. Then $U = \text{span}\{\mathbf{u}\} = \{t\mathbf{u} \mid t \text{ in } \mathbb{R}\}$, so U is the line through the origin with direction vector $\mathbf{u}$. Conversely each line L with direction vector $\mathbf{d} \neq \mathbf{0}$ has the form $L = \{t\mathbf{d} \mid t \text{ in } \mathbb{R}\}$. Hence $\{\mathbf{d}\}$ is a basis of U, so U has dimension 1.
2. If $U \subseteq \mathbb{R}^3$ has dimension 2, let $\{\mathbf{v}, \mathbf{w}\}$ be a basis of U. Then $\mathbf{v}$ and $\mathbf{w}$ are not parallel (by Example 7) so $\mathbf{n} = \mathbf{v} \times \mathbf{w} \neq \mathbf{0}$. Let $P = \{\mathbf{x} \text{ in } \mathbb{R}^3 \mid \mathbf{n} \cdot \mathbf{x} = 0\}$ denote the plane through the origin with normal $\mathbf{n}$. Then P is a subspace of $\mathbb{R}^3$ (Example 1 Section 5.1) and both $\mathbf{v}$ and $\mathbf{w}$ lie in P (they are orthogonal to $\mathbf{n}$), so $U = \text{span}\{\mathbf{v}, \mathbf{w}\} \subseteq P$ by Theorem 1 Section 5.1. Hence
$$U \subseteq P \subseteq \mathbb{R}^3.$$
Since $\dim U = 2$ and $\dim(\mathbb{R}^3) = 3$, it follows from Theorem 8 that $\dim P = 2$ or 3, whence $P = U$ or $\mathbb{R}^3$. But $P \neq \mathbb{R}^3$ (for example, $\mathbf{n}$ is not in P) and so $U = P$ is a plane through the origin.
Conversely, if U is a plane through the origin, then $\dim U = 0$, 1, 2, or 3 by Theorem 8. But $\dim U \neq 0$ or 3 because $U \neq \{\mathbf{0}\}$ and $U \neq \mathbb{R}^3$, and $\dim U \neq 1$ by (1). So $\dim U = 2$.

Note that this proof shows that if $\mathbf{v}$ and $\mathbf{w}$ are nonzero, nonparallel vectors in $\mathbb{R}^3$, then $\text{span}\{\mathbf{v}, \mathbf{w}\}$ is the plane with normal $\mathbf{n} = \mathbf{v} \times \mathbf{w}$. We gave a geometrical verification of this fact in Section 5.1.

EXERCISES 5.2

In Exercises 1–6 we write vectors $\mathbb{R}^n$ as rows.

1. Which of the following subsets are independent? Support your answer.
   (a) $\{(1, -1, 0), (3, 2, -1), (3, 5, -2)\}$ in $\mathbb{R}^3$.
   ◆(b) $\{(1, 1, 1), (1, -1, 1), (0, 0, 1)\}$ in $\mathbb{R}^3$.
   (c) $\{(1, -1, 1, -1), (2, 0, 1, 0), (0, -2, 1, -2)\}$ in $\mathbb{R}^4$.
   ◆(d) $\{(1, 1, 0, 0), (1, 0, 1, 0), (0, 0, 1, 1), (0, 1, 0, 1)\}$ in $\mathbb{R}^4$.
2. Let $\{\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}\}$ be an independent set in $\mathbb{R}^n$. Which of the following sets is independent? Support your answer.
   (a) $\{\mathbf{x} - \mathbf{y}, \mathbf{y} - \mathbf{z}, \mathbf{z} - \mathbf{x}\}$
   ◆(b) $\{\mathbf{x} + \mathbf{y}, \mathbf{y} + \mathbf{z}, \mathbf{z} + \mathbf{x}\}$
   (c) $\{\mathbf{x} - \mathbf{y}, \mathbf{y} - \mathbf{z}, \mathbf{z} - \mathbf{w}, \mathbf{w} - \mathbf{x}\}$
   ◆(d) $\{\mathbf{x} + \mathbf{y}, \mathbf{y} + \mathbf{z}, \mathbf{z} + \mathbf{w}, \mathbf{w} + \mathbf{x}\}$
3. Find a basis and calculate the dimension of the following subspaces of $\mathbb{R}^4$.
   (a) $\text{span}\{(1, -1, 2, 0), (2, 3, 0, 3), (1, 9, -6, 6)\}$.
   ◆(b) $\text{span}\{(2, 1, 0, -1), (-1, 1, 1, 1), (2, 7, 4, 1)\}$.
   (c) $\text{span}\{(-1, 2, 1, 0), (2, 0, 3, -1), (4, 4, 11, -3), (3, -2, 2, -1)\}$.
   ◆(d) $\text{span}\{(-2, 0, 3, 1), (1, 2, -1, 0), (-2, 8, 5, 3), (-1, 2, 2, 1)\}$.

4. Find a basis and calculate the dimension of the following subspaces of $\mathbb{R}^4$.

(a) $U = \left\{ \begin{bmatrix} a \\ a+b \\ a-b \\ b \end{bmatrix} \mid a \text{ and } b \text{ in } \mathbb{R} \right\}$.

◆(b) $U = \left\{ \begin{bmatrix} a+b \\ a-b \\ b \\ a \end{bmatrix} \mid a \text{ and } b \text{ in } \mathbb{R} \right\}$.

(c) $U = \left\{ \begin{bmatrix} a \\ b \\ c+a \\ c \end{bmatrix} \mid a, b, \text{ and } c \text{ in } \mathbb{R} \right\}$.

◆(d) $U = \left\{ \begin{bmatrix} a-b \\ b+c \\ a \\ b+c \end{bmatrix} \mid a, b, \text{ and } c \text{ in } \mathbb{R} \right\}$.

(e) $U = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \mid a+b-c+d = 0 \text{ in } \mathbb{R} \right\}$.

◆(f) $U = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \mid a+b = c+d \text{ in } \mathbb{R} \right\}$.

5. Suppose that $\{\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}\}$ is a basis of $\mathbb{R}^4$. Show that:

(a) $\{\mathbf{x} + a\mathbf{w}, \mathbf{y}, \mathbf{z}, \mathbf{w}\}$ is also a basis of $\mathbb{R}^4$ for any choice of the scalar a.

◆(b) $\{\mathbf{x} + \mathbf{w}, \mathbf{y} + \mathbf{w}, \mathbf{z} + \mathbf{w}, \mathbf{w}\}$ is also a basis of $\mathbb{R}^4$.

(c) $\{\mathbf{x}, \mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y} + \mathbf{z}, \mathbf{x} + \mathbf{y} + \mathbf{z} + \mathbf{w}\}$ is also a basis of $\mathbb{R}^4$.

6. Use Theorem 3 to determine if the following sets of vectors are a basis of the indicated space.

(a) $\{(3, -1), (2, 2)\}$ in $\mathbb{R}^2$.

◆(b) $\{(1, 1, -1), (1, -1, 1), (0, 0, 1)\}$ in $\mathbb{R}^3$.

(c) $\{(-1, 1, -1), (1, -1, 2), (0, 0, 1)\}$ in $\mathbb{R}^3$.

◆(d) $\{(5, 2, -1), (1, 0, 1), (3, -1, 0)\}$ in $\mathbb{R}^3$.

(e) $\{(2, 1, -1, 3), (1, 1, 0, 2), (0, 1, 0, -3), (-1, 2, 3, 1)\}$ in $\mathbb{R}^4$.

◆(f) $\{(1, 0, -2, 5), (4, 4, -3, 2), (0, 1, 0, -3), (1, 3, 3, -10)\}$ in $\mathbb{R}^4$.

7. In each case show that the statement is true or give an example showing that it is false.

(a) If $\{\mathbf{x}, \mathbf{y}\}$ is independent, then $\{\mathbf{x}, \mathbf{y}, \mathbf{x} + \mathbf{y}\}$ is independent.

◆(b) If $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is independent, then $\{\mathbf{y}, \mathbf{z}\}$ is independent.

(c) If $\{\mathbf{y}, \mathbf{z}\}$ is dependent, then $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is dependent for any $\mathbf{x}$.

◆(d) If all of $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ are nonzero, then $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ is independent.

(e) If one of $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ is zero, then $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ is dependent.

◆(f) If $a\mathbf{x} + b\mathbf{y} + c\mathbf{z} = \mathbf{0}$, then $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is independent.

(g) If $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is independent, then $a\mathbf{x} + b\mathbf{y} + c\mathbf{z} = \mathbf{0}$ for some a, b, and c in $\mathbb{R}$.

◆(h) If $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ is dependent, then $t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_k\mathbf{x}_k = \mathbf{0}$ for some numbers t_i in $\mathbb{R}$ not all zero.

(i) If $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ is independent, then $t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_k\mathbf{x}_k = \mathbf{0}$ for some t_i in $\mathbb{R}$.

8. If A is an $n \times n$ matrix, show that $\det A = 0$ if and only if some column of A is a linear combination of the other columns.

9. Let $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ be a linearly independent set in $\mathbb{R}^4$. Show that $\{\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{e}_k\}$ is a basis of $\mathbb{R}^4$ for some $\mathbf{e}_k$ in the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$.

◆10. If $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5, \mathbf{x}_6\}$ is an independent set of vectors, show that the subset $\{\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_5\}$ is also independent.

11. Let A be any $m \times n$ matrix, and let $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \dots, \mathbf{b}_k$ be columns in $\mathbb{R}^m$ such that the system $A\mathbf{x} = \mathbf{b}_i$ has a solution $\mathbf{x}_i$ for each i. If $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \dots, \mathbf{b}_k\}$ is independent in $\mathbb{R}^m$, show that $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_k\}$ is independent in $\mathbb{R}^n$.

◆12. If $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_k\}$ is independent, show that $\{\mathbf{x}_1, \mathbf{x}_1 + \mathbf{x}_2, \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3, \dots, \mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_k\}$ is also independent.

13. If $\{\mathbf{y}, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_k\}$ is independent, show that $\{\mathbf{y} + \mathbf{x}_1, \mathbf{y} + \mathbf{x}_2, \mathbf{y} + \mathbf{x}_3, \dots, \mathbf{y} + \mathbf{x}_k\}$ is also independent.

14. If $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ is independent in $\mathbb{R}^n$, and if $\mathbf{y}$ is not in $\operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$, show that $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k, \mathbf{y}\}$ is independent.

15. If A and B are matrices and the columns of AB are independent, show that the columns of B are independent.

16. Suppose that $\{\mathbf{x}, \mathbf{y}\}$ is a basis of $\mathbb{R}^2$, and let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
    (a) If A is invertible, show that $\{a\mathbf{x} + b\mathbf{y}, c\mathbf{x} + d\mathbf{y}\}$ is a basis of $\mathbb{R}^2$.
    ◆(b) If $\{a\mathbf{x} + b\mathbf{y}, c\mathbf{x} + d\mathbf{y}\}$ is a basis of $\mathbb{R}^2$, show that A is invertible.

17. Let A denote an $m \times n$ matrix.
    (a) Show that null A = null(UA) for every invertible $m \times m$ matrix U.
    ◆(b) Show that dim(null A) = dim(null(AV)) for every invertible $n \times n$ matrix V. [*Hint*: If $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ is a basis of null A, show that $\{V^{-1}\mathbf{x}_1, V^{-1}\mathbf{x}_2, \ldots, V^{-1}\mathbf{x}_k\}$ is a basis of null(AV).]

18. Let A denote an $m \times n$ matrix.
    (a) Show that im A = im(AV) for every invertible $n \times n$ matrix V.
    (b) Show that dim(im A) = dim(im(UA)) for every invertible $m \times m$ matrix U. [*Hint*: If $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k\}$ is a basis of im(UA), show that $\{U^{-1}\mathbf{y}_1, U^{-1}\mathbf{y}_2, \ldots, U^{-1}\mathbf{y}_k\}$ is a basis of im A.]

19. Let U and W denote subspaces of $\mathbb{R}^n$, and assume that $U \subseteq W$. If dim $U = n - 1$, show that either $W = U$ or $W = \mathbb{R}^n$.

◆20. Let U and W denote subspaces of $\mathbb{R}^n$, and assume that $U \subseteq W$. If dim $W = 1$, show that either $U = \{\mathbf{0}\}$ or $U = W$.

SECTION 5.3 Orthogonality

Length and orthogonality are basic concepts in geometry and, in $\mathbb{R}^2$ and $\mathbb{R}^3$, they both can be defined using the dot product. In this section we extend the dot product to vectors in $\mathbb{R}^n$, and so endow $\mathbb{R}^n$ with euclidean geometry. We then introduce the idea of an orthogonal basis—one of the most useful concepts in linear algebra, and begin exploring some of its applications.

Dot Product, Length, and Distance

If $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ are two n-tuples in $\mathbb{R}^n$, recall that their **dot product** was defined in Section 2.2 as follows:

$$\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

Observe that if $\mathbf{x}$ and $\mathbf{y}$ are written as columns then $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T\mathbf{y}$ is a matrix product (and $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}\mathbf{y}^T$ if they are written as rows). Here $\mathbf{x} \cdot \mathbf{y}$ is a 1×1 matrix, which we take to be a number.

Definition 5.6

As in $\mathbb{R}^3$, the **length** $\|\mathbf{x}\|$ *of the vector is defined by*

$$\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

Where $\sqrt{(\ )}$ indicates the positive square root.

A vector $\mathbf{x}$ of length 1 is called a **unit vector**. If $\mathbf{x} \neq \mathbf{0}$, then $\|\mathbf{x}\| \neq 0$ and it follows easily that $\frac{1}{\|\mathbf{x}\|}\mathbf{x}$ is a unit vector (see Theorem 6 below), a fact that we shall use later.

EXAMPLE 1

If $\mathbf{x} = (1, -1, -3, 1)$ and $\mathbf{y} = (2, 1, 1, 0)$ in $\mathbb{R}^4$, then $\mathbf{x} \cdot \mathbf{y} = 2 - 1 - 3 + 0 = -2$ and $\|\mathbf{x}\| = \sqrt{1 + 1 + 9 + 1} = \sqrt{12} = 2\sqrt{3}$. Hence $\frac{1}{2\sqrt{3}}\mathbf{x}$ is a unit vector; similarly $\frac{1}{\sqrt{6}}\mathbf{y}$ is a unit vector.

These definitions agree with those in $\mathbb{R}^2$ and $\mathbb{R}^3$, and many properties carry over to $\mathbb{R}^n$:

Theorem 1

Let $\mathbf{x}$, $\mathbf{y}$, *and* $\mathbf{z}$ *denote vectors in* $\mathbb{R}^n$. *Then:*

1. $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$.
2. $\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}$.
3. $(a\mathbf{x}) \cdot \mathbf{y} = a(\mathbf{x} \cdot \mathbf{y}) = \mathbf{x} \cdot (a\mathbf{y})$ *for all scalars* a.
4. $\|\mathbf{x}\|^2 = \mathbf{x} \cdot \mathbf{x}$.
5. $\|\mathbf{x}\| \geq 0$, *and* $\|\mathbf{x}\| = 0$ *if and only if* $\mathbf{x} = \mathbf{0}$.
6. $\|a\mathbf{x}\| = |a| \, \|\mathbf{x}\|$ *for all scalars* a.

PROOF

(1), (2), and (3) follow from matrix arithmetic because $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T\mathbf{y}$; (4) is clear from the definition; and (6) is a routine verification since $|a| = \sqrt{a^2}$. If $\mathbf{x} = (x_1, x_2, \dots, x_n)$, then $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$, so $\|\mathbf{x}\| = 0$ if and only if $x_1^2 + x_2^2 + \cdots + x_n^2 = 0$. Since each x_i is a real number this happens if and only if $x_i = 0$ for each i; that is, if and only if $\mathbf{x} = \mathbf{0}$. This proves (5).

Because of Theorem 1, computations with dot products in $\mathbb{R}^n$ are similar to those in $\mathbb{R}^3$. In particular, the dot product

$$(\mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_m) \cdot (\mathbf{y}_1 + \mathbf{y}_2 + \cdots + \mathbf{y}_k)$$

equals the sum of mk terms, $\mathbf{x}_i \cdot \mathbf{y}_j$, one for each choice of i and j. For example:

$$\begin{aligned}(3\mathbf{x} - 4\mathbf{y}) \cdot (7\mathbf{x} + 2\mathbf{y}) &= 21(\mathbf{x} \cdot \mathbf{x}) + 6(\mathbf{x} \cdot \mathbf{y}) - 28(\mathbf{y} \cdot \mathbf{x}) - 8(\mathbf{y} \cdot \mathbf{y}) \\ &= 21\|\mathbf{x}\|^2 - 22(\mathbf{x} \cdot \mathbf{y}) - 8\|\mathbf{y}\|^2\end{aligned}$$

holds for all vectors $\mathbf{x}$ and $\mathbf{y}$.

EXAMPLE 2

Show that $\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + 2(\mathbf{x} \cdot \mathbf{y}) + \|\mathbf{y}\|^2$ for any $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$.

Solution ▶ Using Theorem 1 several times:

$$\begin{aligned}\|\mathbf{x} + \mathbf{y}\|^2 &= (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) = \mathbf{x} \cdot \mathbf{x} + \mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{x} + \mathbf{y} \cdot \mathbf{y} \\ &= \|\mathbf{x}\|^2 + 2(\mathbf{x} \cdot \mathbf{y}) + \|\mathbf{y}\|^2\end{aligned}$$

EXAMPLE 3

Suppose that $\mathbb{R}^n = \operatorname{span}\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_k\}$ for some vectors $\mathbf{f}_i$. If $\mathbf{x} \cdot \mathbf{f}_i = 0$ for each i where $\mathbf{x}$ is in $\mathbb{R}^n$, show that $\mathbf{x} = \mathbf{0}$.

Solution ▶ We show $\mathbf{x} = \mathbf{0}$ by showing that $\|\mathbf{x}\| = 0$ and using (5) of Theorem 1. Since the $\mathbf{f}_i$ span $\mathbb{R}^n$, write $\mathbf{x} = t_1\mathbf{f}_1 + t_2\mathbf{f}_2 + \cdots + t_k\mathbf{f}_k$ where the t_i are in $\mathbb{R}$. Then

$$\begin{aligned}\|\mathbf{x}\|^2 &= \mathbf{x}\cdot\mathbf{x} = \mathbf{x}\cdot(t_1\mathbf{f}_1 + t_2\mathbf{f}_2 + \cdots + t_k\mathbf{f}_k)\\ &= t_1(\mathbf{x}\cdot\mathbf{f}_1) + t_2(\mathbf{x}\cdot\mathbf{f}_2) + \cdots + t_k(\mathbf{x}\cdot\mathbf{f}_k)\\ &= t_1(0) + t_2(0) + \cdots + t_k(0)\\ &= 0.\end{aligned}$$

We saw in Section 4.2 that if $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in $\mathbb{R}^3$, then $\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \cos\theta$ where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. Since $|\cos\theta| \le 1$ for any angle θ, this shows that $|\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\|\mathbf{v}\|$. In this form the result holds in $\mathbb{R}^n$.


Augustin Louis Cauchy.
Photo © Corbis.

Theorem 2

Cauchy Inequality[9]

If $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbb{R}^n$, then

$$|\mathbf{x}\cdot\mathbf{y}| \le \|\mathbf{x}\|\|\mathbf{y}\|.$$

Moreover $|\mathbf{x}\cdot\mathbf{y}| = \|\mathbf{x}\|\|\mathbf{y}\|$ if and only if one of $\mathbf{x}$ and $\mathbf{y}$ is a multiple of the other.

PROOF

The inequality holds if $\mathbf{x} = \mathbf{0}$ or $\mathbf{y} = \mathbf{0}$ (in fact it is equality). Otherwise, write $\|\mathbf{x}\| = a > 0$ and $\|\mathbf{y}\| = b > 0$ for convenience. A computation like that preceding Example 2 gives

$$\|b\mathbf{x} - a\mathbf{y}\|^2 = 2ab(ab - \mathbf{x}\cdot\mathbf{y}) \quad\text{and}\quad \|b\mathbf{x} - a\mathbf{y}\|^2 = 2ab(ab + \mathbf{x}\cdot\mathbf{y}). \qquad (*)$$

It follows that $ab - \mathbf{x}\cdot\mathbf{y} \ge 0$ and $ab + \mathbf{x}\cdot\mathbf{y} \ge 0$, and hence that $-ab \le \mathbf{x}\cdot\mathbf{y} \le ab$. Hence $|\mathbf{x}\cdot\mathbf{y}| \le ab = \|\mathbf{x}\|\|\mathbf{y}\|$, proving the Cauchy inequality.

If equality holds, then $|\mathbf{x}\cdot\mathbf{y}| = ab$, so $\mathbf{x}\cdot\mathbf{y} = ab$ or $\mathbf{x}\cdot\mathbf{y} = -ab$. Hence $(*)$ shows that $b\mathbf{x} - a\mathbf{y} = 0$ or $b\mathbf{x} + a\mathbf{y} = 0$, so one of $\mathbf{x}$ and $\mathbf{y}$ is a multiple of the other (even if $a = 0$ or $b = 0$).

The Cauchy inequality is equivalent to $(\mathbf{x}\cdot\mathbf{y})^2 \le \|\mathbf{x}\|^2\|\mathbf{y}\|^2$. In $\mathbb{R}^5$ this becomes

$$\begin{aligned}&(x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 + x_5y_5)^2\\ &\qquad \le \left(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2\right)\left(y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2\right)\end{aligned}$$

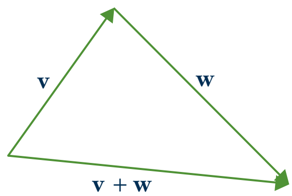
for all x_i and y_i in $\mathbb{R}$.

There is an important consequence of the Cauchy inequality. Given $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$, use Example 2 and the fact that $\mathbf{x}\cdot\mathbf{y} \le \|\mathbf{x}\|\|\mathbf{y}\|$ to compute

$$\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + 2(\mathbf{x}\cdot\mathbf{y}) + \|\mathbf{y}\|^2 \le \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\| + \|\mathbf{y}\|^2 = (\|\mathbf{x}+\mathbf{y}\|)^2.$$

Taking positive square roots gives:

9 Augustin Louis Cauchy (1789–1857) was born in Paris and became a professor at the École Polytechnique at the age of 26. He was one of the great mathematicians, producing more than 700 papers, and is best remembered for his work in analysis in which he established new standards of rigour and founded the theory of functions of a complex variable. He was a devout Catholic with a long-term interest in charitable work, and he was a royalist, following King Charles X into exile in Prague after he was deposed in 1830. Theorem 2 first appeared in his 1812 memoir on determinants.



Corollary 1

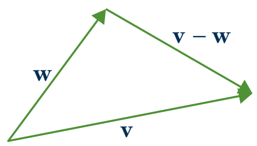
Triangle Inequality
If $\mathbf{x}$ *and* $\mathbf{y}$ *are vectors in* $\mathbb{R}^n$, *then* $\|\mathbf{x}+\mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$.

The reason for the name comes from the observation that in $\mathbb{R}^3$ the inequality asserts that the sum of the lengths of two sides of a triangle is not less than the length of the third side. This is illustrated in the first diagram.

Definition 5.7

If $\mathbf{x}$ *and* $\mathbf{y}$ *are two vectors in* $\mathbb{R}^n$, *we define the* **distance** $d(\mathbf{x}, \mathbf{y})$ *between* $\mathbf{x}$ *and* $\mathbf{y}$ *by*

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$$



The motivation again comes from $\mathbb{R}^3$ as is clear in the second diagram. This distance function has all the intuitive properties of distance in $\mathbb{R}^3$, including another version of the triangle inequality.

Theorem 3

If $\mathbf{x}$, $\mathbf{y}$, *and* $\mathbf{z}$ *are three vectors in* $\mathbb{R}^n$ *we have:*

1. $d(\mathbf{x}, \mathbf{y}) \ge 0$ *for all* $\mathbf{x}$ *and* $\mathbf{y}$.
2. $d(\mathbf{x}, \mathbf{y}) = 0$ *if and only if* $\mathbf{x} = \mathbf{y}$.
3. $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$.
4. $d(\mathbf{x}, \mathbf{z}) \le d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$. *Triangle inequality.*

PROOF

(1) and (2) restate part (5) of Theorem 1 because $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$, and (3) follows because $\|\mathbf{u}\| = \|-\mathbf{u}\|$ for every vector $\mathbf{u}$ in $\mathbb{R}^n$. To prove (4) use the Corollary to Theorem 2:

$$\begin{aligned} d(\mathbf{x}, \mathbf{z}) = \|\mathbf{x} - \mathbf{z}\| &= \|(\mathbf{x} - \mathbf{y}) + (\mathbf{y} - \mathbf{z})\| \\ &\le \|(\mathbf{x} - \mathbf{y})\| + \|(\mathbf{y} - \mathbf{z})\| = d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}) \end{aligned}$$

Orthogonal Sets and the Expansion Theorem

Definition 5.8

We say that two vectors $\mathbf{x}$ *and* $\mathbf{y}$ *in* $\mathbb{R}^n$ *are* **orthogonal** *if* $\mathbf{x} \cdot \mathbf{y} = 0$, *extending the terminology in* $\mathbb{R}^3$ *(See Theorem 3 Section 4.2). More generally, a set* $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ *of vectors in* $\mathbb{R}^n$ *is called an* **orthogonal set** *if*

$$\mathbf{x}_i \cdot \mathbf{x}_j = 0 \text{ for all } i \ne j \quad \text{and} \quad \mathbf{x}_i \ne \mathbf{0} \text{ for all } i.$$ [10]

Note that $\{\mathbf{x}\}$ *is an orthogonal set if* $\mathbf{x} \ne \mathbf{0}$. *A set* $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ *of vectors in* $\mathbb{R}^n$ *is called* **orthonormal** *if it is orthogonal and, in addition, each* $\mathbf{x}_i$ *is a unit vector:*

$$\|\mathbf{x}_i\| = 1 \text{ for each } i.$$

10 The reason for insisting that orthogonal sets consist of *nonzero* vectors is that we will be primarily concerned with orthogonal bases.

EXAMPLE 4

The standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is an orthonormal set in $\mathbb{R}^n$.

The routine verification is left to the reader, as is the proof of:

EXAMPLE 5

If $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ is orthogonal, so also is $\{a_1\mathbf{x}_1, a_2\mathbf{x}_2, \dots, a_k\mathbf{x}_k\}$ for any nonzero scalars a_i.

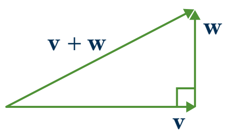
If $\mathbf{x} \neq \mathbf{0}$, it follows from item (6) of Theorem 1 that $\frac{1}{\|\mathbf{x}\|}\mathbf{x}$ is a unit vector, that is it has length 1.

Definition 5.9

Hence if $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ is an orthogonal set, then $\left\{\frac{1}{\|\mathbf{x}_1\|}\mathbf{x}_1, \frac{1}{\|\mathbf{x}_2\|}\mathbf{x}_2, \dots, \frac{1}{\|\mathbf{x}_k\|}\mathbf{x}_k\right\}$ is an orthonormal set, and we say that it is the result of **normalizing** *the orthogonal set $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$.*

EXAMPLE 6

If $\mathbf{f}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{f}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{f}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{f}_4 = \begin{bmatrix} -1 \\ 3 \\ -1 \\ 1 \end{bmatrix}$ then $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \mathbf{f}_4\}$ is an orthogonal set in $\mathbb{R}^4$ as is easily verified. After normalizing, the corresponding orthonormal set is $\left\{\frac{1}{2}\mathbf{f}_1, \frac{1}{\sqrt{6}}\mathbf{f}_2, \frac{1}{\sqrt{2}}\mathbf{f}_3, \frac{1}{2\sqrt{3}}\mathbf{f}_4\right\}$.



The most important result about orthogonality is Pythagoras' theorem. Given orthogonal vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$, it asserts that $\|\mathbf{v} + \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2$ as in the diagram. In this form the result holds for any orthogonal set in $\mathbb{R}^n$.

Theorem 4

Pythagoras' Theorem

If $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ is a orthogonal set in $\mathbb{R}^n$, then

$$\|\mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_k\|^2 = \|\mathbf{x}_1\|^2 + \|\mathbf{x}_2\|^2 + \cdots + \|\mathbf{x}_k\|^2.$$

PROOF

The fact that $\mathbf{x}_i \cdot \mathbf{x}_j = 0$ whenever $i \neq j$ gives

$$\begin{aligned} \|\mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_k\|^2 &= (\mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_k) \cdot (\mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_k) \\ &= (\mathbf{x}_1 \cdot \mathbf{x}_1 + \mathbf{x}_2 \cdot \mathbf{x}_2 + \cdots + \mathbf{x}_k \cdot \mathbf{x}_k) + \sum_{i \neq j} \mathbf{x}_i \cdot \mathbf{x}_j \\ &= \|\mathbf{x}_1\|^2 + \|\mathbf{x}_2\|^2 + \cdots + \|\mathbf{x}_k\|^2 + 0. \end{aligned}$$

This is what we wanted.

If $\mathbf{v}$ and $\mathbf{w}$ are orthogonal, nonzero vectors in $\mathbb{R}^3$, then they are certainly not parallel, and so are linearly independent by Example 7 Section 5.2. The next theorem gives a far-reaching extension of this observation.

Theorem 5

Every orthogonal set in $\mathbb{R}^n$ is linearly independent.

PROOF

Let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ be an orthogonal set in $\mathbb{R}^n$ and suppose a linear combination vanishes: $t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_k\mathbf{x}_k = \mathbf{0}$. Then

$$\begin{aligned} 0 = \mathbf{x}_1 \cdot \mathbf{0} &= \mathbf{x}_1 \cdot (t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_k\mathbf{x}_k) \\ &= t_1(\mathbf{x}_1 \cdot \mathbf{x}_1) + t_2(\mathbf{x}_1 \cdot \mathbf{x}_2) + \cdots + t_k(\mathbf{x}_1 \cdot \mathbf{x}_k) \\ &= t_1\|\mathbf{x}_1\|^2 + t_2(0) + \cdots + t_k(0) \\ &= t_1\|\mathbf{x}_1\|^2 \end{aligned}$$

Since $\|\mathbf{x}_1\|^2 \neq 0$, this implies that $t_1 = 0$. Similarly $t_i = 0$ for each i.

Theorem 5 suggests considering orthogonal bases for $\mathbb{R}^n$, that is orthogonal sets that span $\mathbb{R}^n$. These turn out to be the best bases in the sense that, when expanding a vector as a linear combination of the basis vectors, there are explicit formulas for the coefficients.

Theorem 6

Expansion Theorem

Let $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_m\}$ be an orthogonal basis of a subspace U of $\mathbb{R}^n$. If $\mathbf{x}$ is any vector in U, we have

$$\mathbf{x} = \left(\frac{\mathbf{x} \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2}\right)\mathbf{f}_1 + \left(\frac{\mathbf{x} \cdot \mathbf{f}_2}{\|\mathbf{f}_2\|^2}\right)\mathbf{f}_2 + \cdots + \left(\frac{\mathbf{x} \cdot \mathbf{f}_m}{\|\mathbf{f}_m\|^2}\right)\mathbf{f}_m.$$

PROOF

Since $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_m\}$ spans U, we have $\mathbf{x} = t_1\mathbf{f}_1 + t_2\mathbf{f}_2 + \cdots + t_m\mathbf{f}_m$ where the t_i are scalars. To find t_1 we take the dot product of both sides with $\mathbf{f}_1$:

$$\begin{aligned} \mathbf{x} \cdot \mathbf{f}_1 &= (t_1\mathbf{f}_1 + t_2\mathbf{f}_2 + \cdots + t_m\mathbf{f}_m) \cdot \mathbf{f}_1 \\ &= t_1(\mathbf{f}_1 \cdot \mathbf{f}_1) + t_2(\mathbf{f}_2 \cdot \mathbf{f}_1) + \cdots + t_m(\mathbf{f}_m \cdot \mathbf{f}_1) \\ &= t_1\|\mathbf{f}_1\|^2 + t_2(0) + \cdots + t_m(0) \\ &= t_1\|\mathbf{f}_1\|^2 \end{aligned}$$

Since $\mathbf{f}_1 \neq \mathbf{0}$, this gives $t_1 = \dfrac{\mathbf{x} \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2}$. Similarly, $t_i = \dfrac{\mathbf{x} \cdot \mathbf{f}_i}{\|\mathbf{f}_i\|^2}$ for each i.

The expansion in Theorem 6 of $\mathbf{x}$ as a linear combination of the orthogonal basis $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_m\}$ is called the **Fourier expansion** of $\mathbf{x}$, and the coefficients $t_1 = \dfrac{\mathbf{x} \cdot \mathbf{f}_i}{\|\mathbf{f}_i\|^2}$ are called the **Fourier coefficients**. Note that if $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_m\}$ is actually orthonormal, then $t_i = \mathbf{x} \cdot \mathbf{f}_i$ for each i. We will have a great deal more to say about this in Section 10.5.

EXAMPLE 7

Expand $\mathbf{x} = (a, b, c, d)$ as a linear combination of the orthogonal basis $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \mathbf{f}_4\}$ of $\mathbb{R}^4$ given in Example 6.

Solution ▶ We have $\mathbf{f}_1 = (1, 1, 1, -1)$, $\mathbf{f}_2 = (1, 0, 1, 2)$, $\mathbf{f}_3 = (-1, 0, 1, 0)$, and $\mathbf{f}_4 = (-1, 3, -1, 1)$ so the Fourier coefficients are

$$t_1 = \frac{\mathbf{x} \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2} = \tfrac{1}{4}(a + b + c + d) \qquad t_3 = \frac{\mathbf{x} \cdot \mathbf{f}_3}{\|\mathbf{f}_3\|^2} = \tfrac{1}{2}(-a + c)$$

$$t_2 = \frac{\mathbf{x} \cdot \mathbf{f}_2}{\|\mathbf{f}_2\|^2} = \tfrac{1}{6}(a + c + 2d) \qquad t_4 = \frac{\mathbf{x} \cdot \mathbf{f}_4}{\|\mathbf{f}_4\|^2} = \tfrac{1}{12}(-a + 3b - c + d)$$

The reader can verify that indeed $\mathbf{x} = t_1\mathbf{f}_1 + t_2\mathbf{f}_2 + t_3\mathbf{f}_3 + t_4\mathbf{f}_4$.

A natural question arises here: Does every subspace U of $\mathbb{R}^n$ *have* an orthogonal basis? The answer is "yes"; in fact, there is a systematic procedure, called the Gram-Schmidt algorithm, for turning any basis of U into an orthogonal one. This leads to a definition of the projection onto a subspace U that generalizes the projection along a vector used in $\mathbb{R}^2$ and $\mathbb{R}^3$. All this is discussed in Section 8.1.

EXERCISES 5.3

We often write vectors in $\mathbb{R}^n$ as row n-tuples.

1. Obtain orthonormal bases of $\mathbb{R}^3$ by normalizing the following.

   (a) $\{(1, -1, 2), (0, 2, 1), (5, 1, -2)\}$

   ◆(b) $\{(1, 1, 1), (4, 1, -5), (2, -3, 1)\}$

2. In each case, show that the set of vectors is orthogonal in $\mathbb{R}^4$.

   (a) $\{(1, -1, 2, 5), (4, 1, 1, -1), (-7, 28, 5, 5)\}$

   (b) $\{(2, -1, 4, 5), (0, -1, 1, -1), (0, 3, 2, -1)\}$

3. In each case, show that B is an orthogonal basis of $\mathbb{R}^3$ and use Theorem 6 to expand $\mathbf{x} = (a, b, c)$ as a linear combination of the basis vectors.

   (a) $B = \{(1, -1, 3), (-2, 1, 1), (4, 7, 1)\}$

   ◆(b) $B = \{(1, 0, -1), (1, 4, 1), (2, -1, 2)\}$

   (c) $B = \{(1, 2, 3), (-1, -1, 1), (5, -4, 1)\}$

   ◆(d) $B = \{(1, 1, 1), (1, -1, 0), (1, 1, -2)\}$

4. In each case, write $\mathbf{x}$ as a linear combination of the orthogonal basis of the subspace U.

   (a) $\mathbf{x} = (13, -20, 15)$;
   $U = \text{span}\{(1, -2, 3), (-1, 1, 1)\}$

   ◆(b) $\mathbf{x} = (14, 1, -8, 5)$;
   $U = \text{span}\{(2, -1, 0, 3), (2, 1, -2, -1)\}$

5. In each case, find all (a, b, c, d) in $\mathbb{R}^4$ such that the given set is orthogonal.

   (a) $\{(1, 2, 1, 0), (1, -1, 1, 3), (2, -1, 0, -1), (a, b, c, d)\}$

   ◆(b) $\{(1, 0, -1, 1), (2, 1, 1, -1), (1, -3, 1, 0), (a, b, c, d)\}$

6. If $\|\mathbf{x}\| = 3$, $\|\mathbf{y}\| = 1$, and $\mathbf{x} \cdot \mathbf{y} = -2$, compute:

   (a) $\|3\mathbf{x} - 5\mathbf{y}\|$ ◆(b) $\|2\mathbf{x} + 7\mathbf{y}\|$

   (c) $(3\mathbf{x} - \mathbf{y}) \cdot (2\mathbf{y} - \mathbf{x})$

   ◆(d) $(\mathbf{x} - 2\mathbf{y}) \cdot (3\mathbf{x} + 5\mathbf{y})$

7. In each case either show that the statement is true or give an example showing that it is false.

   (a) Every independent set in $\mathbb{R}^n$ is orthogonal.

   ◆(b) If $\{\mathbf{x}, \mathbf{y}\}$ is an orthogonal set in $\mathbb{R}^n$, then $\{\mathbf{x}, \mathbf{x} + \mathbf{y}\}$ is also orthogonal.

(c) If $\{\mathbf{x}, \mathbf{y}\}$ and $\{\mathbf{z}, \mathbf{w}\}$ are both orthogonal in $\mathbb{R}^n$, then $\{\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}\}$ is also orthogonal.

◆(d) If $\{\mathbf{x}_1, \mathbf{x}_2\}$ and $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ are both orthogonal and $\mathbf{x}_i \cdot \mathbf{y}_j = 0$ for all i and j, then $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ is orthogonal.

(e) If $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ is orthogonal in $\mathbb{R}^n$, then $\mathbb{R}^n = \operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$.

◆(f) If $\mathbf{x} \neq \mathbf{0}$ in $\mathbb{R}^n$, then $\{\mathbf{x}\}$ is an orthogonal set.

8. Let $\mathbf{v}$ denote a nonzero vector in $\mathbb{R}^n$.

(a) Show that $P = \{\mathbf{x} \text{ in } \mathbb{R}^n \mid \mathbf{x} \cdot \mathbf{v} = 0\}$ is a subspace of $\mathbb{R}^n$.

(b) Show that $\mathbb{R}\mathbf{v} = \{t\mathbf{v} \mid t \text{ in } \mathbb{R}\}$ is a subspace of $\mathbb{R}^n$.

(c) Describe P and $\mathbb{R}\mathbf{v}$ geometrically when $n = 3$.

◆9. If A is an $m \times n$ matrix with orthonormal columns, show that $A^TA = I_n$. [*Hint*: If $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n$ are the columns of A, show that column j of A^TA has entries $\mathbf{c}_1 \cdot \mathbf{c}_j, \mathbf{c}_2 \cdot \mathbf{c}_j, \dots, \mathbf{c}_n \cdot \mathbf{c}_j$].

10. Use the Cauchy inequality to show that $\sqrt{xy} \le \frac{1}{2}(x + y)$ for all $x \ge 0$ and $y \ge 0$. Here $\sqrt{xy}$ and $\frac{1}{2}(x + y)$ are called, respectively, the *geometric mean* and *arithmetic mean* of x and y.

[*Hint*: Use $\mathbf{x} = \begin{bmatrix} \sqrt{x} \\ \sqrt{y} \end{bmatrix}$ *and* $\mathbf{y} = \begin{bmatrix} \sqrt{y} \\ \sqrt{x} \end{bmatrix}$.]

11. Use the Cauchy inequality to prove that:

(a) $(r_1 + r_2 + \cdots + r_n)^2 \le n(r_1^2 + r_2^2 + \cdots + r_n^2)$ for all r_i in $\mathbb{R}$ and all $n \ge 1$.

◆(b) $r_1r_2 + r_1r_3 + r_2r_3 \le r_1^2 + r_2^2 + r_3^2$ for all r_1, r_2, and r_3 in $\mathbb{R}$. [*Hint*: See part (a).]

12. (a) Show that $\mathbf{x}$ and $\mathbf{y}$ are orthogonal in $\mathbb{R}^n$ if and only if $\|\mathbf{x} + \mathbf{y}\| = \|\mathbf{x} - \mathbf{y}\|$.

◆(b) Show that $\mathbf{x} + \mathbf{y}$ and $\mathbf{x} - \mathbf{y}$ are orthogonal in $\mathbb{R}^n$ if and only if $\|\mathbf{x}\| = \|\mathbf{y}\|$.

13. (a) Show that $\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$ if and only if $\mathbf{x}$ is orthogonal to $\mathbf{y}$.

(b) If $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{z} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, show that $\|\mathbf{x} + \mathbf{y} + \mathbf{z}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + \|\mathbf{z}\|^2$ but $\mathbf{x} \cdot \mathbf{y} \neq 0$, $\mathbf{x} \cdot \mathbf{z} \neq 0$, and $\mathbf{y} \cdot \mathbf{z} \neq 0$.

14. (a) Show that $\mathbf{x} \cdot \mathbf{y} = \frac{1}{4}[\|\mathbf{x} + \mathbf{y}\|^2 - \|\mathbf{x} - \mathbf{y}\|^2]$ for all $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$.

(b) Show that $\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 = \frac{1}{2}\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2$ for all $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$.

◆15. If A is $n \times n$, show that every eigenvalue of A^TA is nonnegative. [*Hint*: Compute $\|A\mathbf{x}\|^2$ where $\mathbf{x}$ is an eigenvector.]

16. If $\mathbb{R}^n = \operatorname{span}\{\mathbf{x}_1, \dots, \mathbf{x}_m\}$ and $\mathbf{x} \cdot \mathbf{x}_i = 0$ for all i, show that $\mathbf{x} = 0$. [*Hint*: Show $\|\mathbf{x}\| = 0$.]

17. If $\mathbb{R}^n = \operatorname{span}\{\mathbf{x}_1, \dots, \mathbf{x}_m\}$ and $\mathbf{x} \cdot \mathbf{x}_i = \mathbf{y} \cdot \mathbf{x}_i$ for all i, show that $\mathbf{x} = \mathbf{y}$. [*Hint*: Preceding Exercise.]

18. Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be an orthogonal basis of $\mathbb{R}^n$. Given $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$, show that

$$\mathbf{x} \cdot \mathbf{y} = \frac{(\mathbf{x} \cdot \mathbf{e}_1)(\mathbf{y} \cdot \mathbf{e}_1)}{\|\mathbf{e}_1\|^2} + \cdots + \frac{(\mathbf{x} \cdot \mathbf{e}_n)(\mathbf{y} \cdot \mathbf{e}_n)}{\|\mathbf{e}_n\|^2}.$$

SECTION 5.4 Rank of a Matrix

In this section we use the concept of dimension to clarify the definition of the rank of a matrix given in Section 1.2, and to study its properties. This requires that we deal with rows and columns in the same way. While it has been our custom to write the n-tuples in $\mathbb{R}^n$ as columns, in this section we will frequently write them as rows. Subspaces, independence, spanning, and dimension are defined for rows using matrix operations, just as for columns. If A is an $m \times n$ matrix, we define:

Definition 5.10

The **column space**, *col A, of A is the subspace of $\mathbb{R}^m$ spanned by the columns of A. The* **row space**, *row A, of A is the subspace of $\mathbb{R}^n$ spanned by the rows of A.*

Much of what we do in this section involves these subspaces. We begin with:

Lemma 1

Let A and B denote $m \times n$ matrices.

1. *If $A \to B$ by elementary row operations, then* row A = row B.
2. *If $A \to B$ by elementary column operations, then* col A = col B.

PROOF

We prove (1); the proof of (2) is analogous. It is enough to do it in the case when $A \to B$ by a single row operation. Let $R_1, R_2, \dots, R_m$ denote the rows of A. The row operation $A \to B$ either interchanges two rows, multiplies a row by a nonzero constant, or adds a multiple of a row to a different row. We leave the first two cases to the reader. In the last case, suppose that a times row p is added to row q where $p < q$. Then the rows of B are $R_1, \dots, R_p, \dots, R_q + aR_p, \dots, R_m$, and Theorem 1 Section 5.1 shows that

$$\operatorname{span}\{R_1, \dots, R_p, \dots, R_q, \dots, R_m\} = \operatorname{span}\{R_1, \dots, R_p, \dots, R_q + aR_p, \dots, R_m\}.$$

That is, row A = row B.

If A is any matrix, we can carry $A \to R$ by elementary row operations where R is a row-echelon matrix. Hence row A = row R by Lemma 1; so the first part of the following result is of interest.

Lemma 2

If R is a row-echelon matrix, then

1. *The nonzero rows of R are a basis of* row R.
2. *The columns of R containing leading ones are a basis of* col R.

PROOF

The rows of R are independent by Example 6 Section 5.2, and they span row R by definition. This proves *1*.

Let $\mathbf{c}j_1, \mathbf{c}j_2, \dots, \mathbf{c}j_r$ denote the columns of R containing leading 1s. Then $\{\mathbf{c}j_1, \mathbf{c}j_2, \dots, \mathbf{c}j_r\}$ is independent because the leading 1s are in different rows (and have zeros below and to the left of them). Let U denote the subspace of all columns in $\mathbb{R}^m$ in which the last $m - r$ entries are zero. Then dim $U = r$ (it is just $\mathbb{R}^r$ with extra zeros). Hence the independent set $\{\mathbf{c}j_1, \mathbf{c}j_2, \dots, \mathbf{c}j_r\}$ is a basis of U by Theorem 7 Section 5.2. Since each $\mathbf{c}_{j_i}$ is in col R, it follows that col $R = U$, proving (2).

With Lemma 2 we can fill a gap in the definition of the rank of a matrix given in Chapter 1. Let A be any matrix and suppose A is carried to some row-echelon matrix R by row operations. Note that R is not unique. In Section 1.2 we defined the **rank** of A, denoted rank A, to be the number of leading 1s in R, that is the number of nonzero rows of R. The fact that this number does not depend on the choice of R was not proved in Section 1.2. However part 1 of Lemma 2 shows that

$$\text{rank } A = \dim(\text{row } A)$$

and hence that rank A is independent of R.

Lemma 2 can be used to find bases of subspaces of $\mathbb{R}^n$ (written as rows). Here is an example.

EXAMPLE 1

Find a basis of $U = \text{span}\{(1, 1, 2, 3), (2, 4, 1, 0), (1, 5, -4, -9)\}$.

Solution ▶ U is the row space of $\begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 4 & 1 & 0 \\ 1 & 5 & -4 & -9 \end{bmatrix}$. This matrix has row-echelon form $\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & -\frac{3}{2} & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, so $\{(1, 1, 2, 3), (0, 1, -\frac{3}{2}, -3)\}$ is basis of U by Lemma 2.

Note that $\{(1, 1, 2, 3), (0, 2, -3, -6)\}$ is another basis that avoids fractions.

Lemmas 1 and 2 are enough to prove the following fundamental theorem.

Theorem 1

Let A denote any $m \times n$ matrix of rank r. *Then*

$$\dim(\text{col } A) = \dim(\text{row } A) = r.$$

Moreover, if A is carried to a row-echelon matrix R by row operations, then

1. *The r nonzero rows of R are a basis of* row A.
2. *If the leading 1s lie in columns $j_1, j_2, \dots, j_r$ of R, then columns $j_1, j_2, \dots, j_r$ of A are a basis of* col A.

PROOF

We have row A = row R by Lemma 1, so (1) follows from Lemma 2. Moreover, $R = UA$ for some invertible matrix U by Theorem 1 Section 2.5. Now write $A = [\mathbf{c}_1 \ \mathbf{c}_2 \ \cdots \ \mathbf{c}_n]$ where $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n$ are the columns of A. Then

$$R = UA = U[\mathbf{c}_1 \ \mathbf{c}_2 \ \cdots \ \mathbf{c}_n] = [U\mathbf{c}_1 \ U\mathbf{c}_2 \ \cdots \ U\mathbf{c}_n].$$

Thus, in the notation of (2), the set $B = \{U\mathbf{c}_{j_1}, U\mathbf{c}_{j_2}, \dots, U\mathbf{c}_{j_r}\}$ is a basis of col R by Lemma 2. So, to prove (2) and the fact that $\dim(\text{col } A) = r$, it is enough to show that $D = \{\mathbf{c}_{j_1}, \mathbf{c}_{j_2}, \dots, \mathbf{c}_{j_r}\}$ is a basis of col A. First, D is linearly independent because U is invertible (verify), so we show that, for each j, column $\mathbf{c}_j$ is a linear combination of the $\mathbf{c}_{j_i}$. But $U\mathbf{c}_j$ is column j of R, and so is a linear combination of the $U\mathbf{c}_{j_i}$, say $U\mathbf{c}_j = a_1 U\mathbf{c}_{j_1} + a_2 U\mathbf{c}_{j_2} + \cdots + a_r U\mathbf{c}_{j_r}$ where each a_i is a real number.

Since U is invertible, it follows that $\mathbf{c}_j = a_1\mathbf{c}_{j_1} + a_2\mathbf{c}_{j_2} + \cdots + a_r\mathbf{c}_{j_r}$ and the proof is complete.

EXAMPLE 2

Compute the rank of $A = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 3 & 6 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{bmatrix}$ and find bases for row A and col A.

Solution ▶ The reduction of A to row-echelon form is as follows:

$$\begin{bmatrix} 1 & 2 & 2 & -1 \\ 3 & 6 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & -1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & -1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & -1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence rank $A = 2$, and $\{[1\ \ 2\ \ 2\ \ -1], [0\ \ 0\ \ 1\ \ -3]\}$ is a basis of row A by Lemma 2. Since the leading 1s are in columns 1 and 3 of the row-echelon matrix, Theorem 1 shows that columns 1 and 3 of A are a basis $\left\{\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}\right\}$ of col A.

Theorem 1 has several important consequences. The first, Corollary 1 below, follows because the rows of A are independent (respectively span row A) if and only if their transposes are independent (respectively span col A).

Corollary 1

If A is any matrix, then rank A = rank(A^T).

If A is an $m \times n$ matrix, we have col $A \subseteq \mathbb{R}^m$ and row $A \subseteq \mathbb{R}^n$. Hence Theorem 8 Section 5.2 shows that $\dim(\text{col } A) \le \dim(\mathbb{R}^m) = m$ and $\dim(\text{row } A) \le \dim(\mathbb{R}^n) = n$. Thus Theorem 1 gives:

Corollary 2

If A is an $m \times n$ matrix, then rank $A \le m$ *and* rank $A \le n$.

Corollary 3

Rank A = rank(UA) = rank(AV) *whenever U and V are invertible.*

PROOF

Lemma 1 gives rank A = rank(UA). Using this and Corollary 1 we get

$$\text{rank}(AV) = \text{rank}(AV)^T = \text{rank}(V^TA^T) = \text{rank}(A^T) = \text{rank } A.$$

The next corollary requires a preliminary lemma.

Lemma 3

Let A, U, and V be matrices of sizes $m \times n$, $p \times m$, and $n \times q$ respectively.

(1) $\text{col}(AV) \subseteq \text{col } A$, *with equality if V is (square and) invertible.*

(2) $\text{row}(UA) \subseteq \text{row } A$, *with equality if U is (square and) invertible.*

PROOF

For (1), write $V = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_q]$ where $\mathbf{v}_j$ is column j of V. Then we have $AV = [A\mathbf{v}_1, A\mathbf{v}_2, \dots, A\mathbf{v}_q]$, and each $A\mathbf{v}_j$ is in col A by Definition 1 Section 2.2. It follows that $\text{col}(AV) \subseteq \text{col } A$. If V is invertible, we obtain $\text{col } A = \text{col}[(AV)V^{-1}] \subseteq \text{col}(AV)$ in the same way. This proves (1).

As to (2), we have $\text{col}[(UA)^T] = \text{col}(A^T U^T) \subseteq \text{col}(A^T)$ by (1), from which $\text{row}(UA) \subseteq \text{row } A$. If U is invertible, this is equality as in the proof of (1).

Corollary 4

If A is $m \times n$ and B is $n \times m$, then $\text{rank } AB \leq \text{rank } A$ *and* $\text{rank } AB \leq \text{rank } B$.

PROOF

By Lemma 3, $\text{col}(AB) \subseteq \text{col } A$ and $\text{row}(BA) \subseteq \text{row } A$, so Theorem 1 applies.

In Section 5.1 we discussed two other subspaces associated with an $m \times n$ matrix A: the null space $\text{null}(A)$ and the image space $\text{im}(A)$

$$\text{null}(A) = \{\mathbf{x} \text{ in } \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\} \quad \text{and} \quad \text{im}(A) = \{A\mathbf{x} \mid \mathbf{x} \text{ in } \mathbb{R}^n\}.$$

Using rank, there are simple ways to find bases of these spaces. If A has rank r, we have $\text{im}(A) = \text{col}(A)$ by Example 8 Section 5.1, so $\dim[\text{im}(A)] = \dim[\text{col}(A)] = r$. Hence Theorem 1 provides a method of finding a basis of $\text{im}(A)$. This is recorded as part (2) of the following theorem.

Theorem 2

Let A denote an $m \times n$ matrix of rank r. *Then*

(1) *The $n - r$ basic solutions to the system $A\mathbf{x} = \mathbf{0}$ provided by the gaussian algorithm are a basis of* $\text{null}(A)$, *so* $\dim[\text{null}(A)] = n - r$.

(2) *Theorem 1 provides a basis of* $\text{im}(A) = \text{col}(A)$, *and* $\dim[\text{im}(A)] = r$.

PROOF

It remains to prove (1). We already know (Theorem 1 Section 2.2) that $\text{null}(A)$ is spanned by the $n - r$ basic solutions of $A\mathbf{x} = \mathbf{0}$. Hence using Theorem 7 Section 5.2, it suffices to show that $\dim[\text{null}(A)] = n - r$. So let $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ be a basis of $\text{null}(A)$, and extend it to a basis $\{\mathbf{x}_1, \dots, \mathbf{x}_k, \mathbf{x}_{k+1}, \dots, \mathbf{x}_n\}$ of $\mathbb{R}^n$ (by Theorem 6 Section 5.2). It is enough to show that $\{A\mathbf{x}_{k+1}, \dots, A\mathbf{x}_n\}$ is a basis of $\text{im}(A)$; then $n - k = r$ by the above and so $k = n - r$ as required.

Spanning. Choose $A\mathbf{x}$ in $\text{im}(A)$, $\mathbf{x}$ in $\mathbb{R}^n$, and write $\mathbf{x} = a_1\mathbf{x}_1 + \cdots + a_k\mathbf{x}_k + a_{k+1}\mathbf{x}_{k+1} + \cdots + a_n\mathbf{x}_n$ where the a_i are in $\mathbb{R}$. Then $A\mathbf{x} = a_{k+1}A\mathbf{x}_{k+1} + \cdots + a_nA\mathbf{x}_n$ because $\{\mathbf{x}_1, \dots, \mathbf{x}_k\} \subseteq \text{null}(A)$.

Independence. Let $t_{k+1}A\mathbf{x}_{k+1} + \cdots + t_nA\mathbf{x}_n = \mathbf{0}$, t_i in $\mathbb{R}$. Then $t_{k+1}\mathbf{x}_{k+1} + \cdots + t_n\mathbf{x}_n$ is in null A, so $t_{k+1}\mathbf{x}_{k+1} + \cdots + t_n\mathbf{x}_n = t_1\mathbf{x}_1 + \cdots + t_k\mathbf{x}_k$ for some $t_1, \dots, t_k$ in $\mathbb{R}$. But then the independence of the $\mathbf{x}_i$ shows that $t_i = 0$ for every i.

EXAMPLE 3

If $A = \begin{bmatrix} 1 & -2 & 1 & 1 \\ -1 & 2 & 0 & 1 \\ 2 & -4 & 1 & 0 \end{bmatrix}$, find bases of $\text{null}(A)$ and $\text{im}(A)$, and so find their dimensions.

Solution ▶ If $\mathbf{x}$ is in $\text{null}(A)$, then $A\mathbf{x} = \mathbf{0}$, so $\mathbf{x}$ is given by solving the system $A\mathbf{x} = \mathbf{0}$. The reduction of the augmented matrix to reduced form is

$$\left[\begin{array}{rrrr|r} 1 & -2 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 & 0 \\ 2 & -4 & 1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{rrrr|r} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Hence $r = \text{rank}(A) = 2$. Here, $\text{im}(A) = \text{col}(A)$ has basis $\left\{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$ by Theorem 1 because the leading 1s are in columns 1 and 3. In particular, $\dim[\text{im}(A)] = 2 = r$ as in Theorem 2.

Turning to $\text{null}(A)$, we use gaussian elimination. The leading variables are x_1 and x_3, so the nonleading variables become parameters: $x_2 = s$ and $x_4 = t$. It follows from the reduced matrix that $x_1 = 2s + t$ and $x_3 = -2t$, so the general solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2s + t \\ s \\ -2t \\ t \end{bmatrix} = s\mathbf{x}_1 + t\mathbf{x}_2 \text{ where } \mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \text{ and } \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

Hence $\text{null}(A)$. But $\mathbf{x}_1$ and $\mathbf{x}_2$ *are* solutions (basic), so

$$\text{null}(A) = \text{span}\{\mathbf{x}_1, \mathbf{x}_2\}$$

However Theorem 2 asserts that $\{\mathbf{x}_1, \mathbf{x}_2\}$ is a *basis* of $\text{null}(A)$. (In fact it is easy to verify directly that $\{\mathbf{x}_1, \mathbf{x}_2\}$ is independent in this case.) In particular, $\dim[\text{null}(A)] = 2 = n - r$, as Theorem 2 asserts.

Let A be an $m \times n$ matrix. Corollary 2 of the Theorem 1 asserts that rank $A \leq m$ and rank $A \leq n$, and it is natural to ask when these extreme cases arise. If $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n$ are the columns of A, Theorem 2 Section 5.2 shows that $\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n\}$ spans $\mathbb{R}^m$ if and only if the system $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbb{R}^m$, and that $\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n\}$ is independent if and only if $A\mathbf{x} = \mathbf{0}$, $\mathbf{x}$ in $\mathbb{R}^n$, implies $\mathbf{x} = \mathbf{0}$. The next two useful theorems improve on both these results, and relate them to when the rank of A is n or m.

Theorem 3

The following are equivalent for an $m \times n$ matrix A:

1. rank $A = n$.
2. *The rows of A span $\mathbb{R}^n$.*
3. *The columns of A are linearly independent in $\mathbb{R}^m$.*
4. *The $n \times n$ matrix A^TA is invertible.*
5. *$CA = I_n$ for some $n \times m$ matrix C.*
6. *If $A\mathbf{x} = \mathbf{0}$, $\mathbf{x}$ in $\mathbb{R}^n$, then $\mathbf{x} = \mathbf{0}$.*

PROOF

(1) $\Rightarrow$ (2). We have row $A \subseteq \mathbb{R}^n$, and dim(row A) $= n$ by (1), so row $A = \mathbb{R}^n$ by Theorem 8 Section 5.2. This is (2).

(2) $\Rightarrow$ (3). By (2), row $A = \mathbb{R}^n$, so rank $A = n$. This means dim(col A) $= n$. Since the n columns of A span col A, they are independent by Theorem 7 Section 5.2.

(3) $\Rightarrow$ (4). If $(A^TA)\mathbf{x} = \mathbf{0}$, $\mathbf{x}$ in $\mathbb{R}^n$, we show that $\mathbf{x} = \mathbf{0}$ (Theorem 5 Section 2.4). We have

$$\|A\mathbf{x}\|^2 = (A\mathbf{x})^TA\mathbf{x} = \mathbf{x}^TA^TA\mathbf{x} = \mathbf{x}^T\mathbf{0} = 0.$$

Hence $A\mathbf{x} = \mathbf{0}$, so $\mathbf{x} = \mathbf{0}$ by (3) and Theorem 2 Section 5.2.

(4) $\Rightarrow$ (5). Given (4), take $C = (A^TA)^{-1}A^T$.

(5) $\Rightarrow$ (6). If $A\mathbf{x} = \mathbf{0}$, then left multiplication by C (from (5)) gives $\mathbf{x} = \mathbf{0}$.

(6) $\Rightarrow$ (1). Given (6), the columns of A are independent by Theorem 2 Section 5.2. Hence dim(col A) $= n$, and (1) follows.

Theorem 4

The following are equivalent for an $m \times n$ matrix A:

1. rank $A = m$.
2. *The columns of A span $\mathbb{R}^m$.*
3. *The rows of A are linearly independent in $\mathbb{R}^n$.*
4. *The $m \times m$ matrix AA^T is invertible.*
5. *$AC = I_m$ for some $n \times m$ matrix C.*
6. *The system $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbb{R}^m$.*

PROOF

(1) $\Rightarrow$ (2). By (1), dim(col A) $= m$, so col $A = \mathbb{R}^m$ by Theorem 8 Section 5.2.

(2) $\Rightarrow$ (3). By (2), col $A = \mathbb{R}^m$, so rank $A = m$. This means dim(row A) $= m$. Since the m rows of A span row A, they are independent by Theorem 7 Section 5.2.

(3) ⇒ (4). We have rank $A = m$ by (3), so the $n \times m$ matrix A^T has rank m. Hence applying Theorem 3 to A^T in place of A shows that $(A^T)^T A^T$ is invertible, proving (4).

(4) ⇒ (5). Given (4), take $C = A^T(AA^T)^{-1}$ in (5).

(5) ⇒ (6). Comparing columns in $AC = I_m$ gives $A\mathbf{c}_j = \mathbf{e}_j$ for each j, where $\mathbf{c}_j$ and $\mathbf{e}_j$ denote column j of C and I_m respectively. Given $\mathbf{b}$ in $\mathbb{R}^m$, write $\mathbf{b} = \sum_{j=1}^{m} r_j\mathbf{e}_j$, r_j in $\mathbb{R}$. Then $A\mathbf{x} = \mathbf{b}$ holds with $\mathbf{x} = \sum_{j=1}^{m} r_j\mathbf{c}_j$, as the reader can verify.

(6) ⇒ (1). Given (6), the columns of A span $\mathbb{R}^m$ by Theorem 2 Section 5.2. Thus col $A = \mathbb{R}^m$ and (1) follows.

EXAMPLE 4

Show that $\begin{bmatrix} 3 & x+y+z \\ x+y+z & x^2+y^2+z^2 \end{bmatrix}$ is invertible if x, y, and z are not all equal.

Solution ▶ The given matrix has the form A^TA where $a = \begin{bmatrix} 1 & x \\ 1 & y \\ 1 & z \end{bmatrix}$ has independent columns because x, y, and z are not all equal (verify). Hence Theorem 3 applies.

Theorems 4 and 5 relate several important properties of an $m \times n$ matrix A to the invertibility of the square, symmetric matrices A^TA and AA^T. In fact, even if the columns of A are not independent or do not span $\mathbb{R}^m$, the matrices A^TA and AA^T are both symmetric and, as such, have real eigenvalues as we shall see. We return to this in Chapter 7.

EXERCISES 5.4

1. In each case find bases for the row and column spaces of A and determine the rank of A.

(a) $A = \begin{bmatrix} 2 & -4 & 6 & 8 \\ 2 & -1 & 3 & 2 \\ 4 & -5 & 9 & 10 \\ 0 & -1 & 1 & 2 \end{bmatrix}$ ♦(b) $A = \begin{bmatrix} 2 & -1 & 1 \\ -2 & 1 & 1 \\ 4 & -2 & 3 \\ -6 & 3 & 0 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & -1 & 5 & -2 & 2 \\ 2 & -2 & -2 & 5 & 1 \\ 0 & 0 & -12 & 9 & -3 \\ -1 & 1 & 7 & -7 & 1 \end{bmatrix}$

♦(d) $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ -3 & -6 & 3 & -2 \end{bmatrix}$

2. In each case find a basis of the subspace U.

(a) $U = \text{span}\{(1, -1, 0, 3), (2, 1, 5, 1), (4, -2, 5, 7)\}$

♦(b) $U = \text{span}\{(1, -1, 2, 5, 1), (3, 1, 4, 2, 7), (1, 1, 0, 0, 0), (5, 1, 6, 7, 8)\}$

(c) $U = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}\right\}$

♦(d) $U = \text{span}\left\{\begin{bmatrix} 1 \\ 5 \\ -6 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ -8 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ -10 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \\ 12 \end{bmatrix}\right\}$

3. (a) Can a 3×4 matrix have independent columns? Independent rows? Explain.

♦(b) If A is 4×3 and rank $A = 2$, can A have independent columns? Independent rows? Explain.

(c) If A is an $m \times n$ matrix and rank $A = m$, show that $m \le n$.

◆(d) Can a nonsquare matrix have its rows independent and its columns independent? Explain.

(e) Can the null space of a 3×6 matrix have dimension 2? Explain.

◆(f) Suppose that A is 5×4 and $\text{null}(A) = \mathbb{R}\mathbf{x}$ for some column $\mathbf{x} \ne \mathbf{0}$. Can $\dim(\text{im } A) = 2$?

◆4. If A is $m \times n$ show that $\text{col}(A) = \{A\mathbf{x} \mid \mathbf{x} \text{ in } \mathbb{R}^n\}$.

5. If A is $m \times n$ and B is $n \times m$, show that $AB = 0$ if and only if col $B \subseteq$ null A.

6. Show that the rank does not change when an elementary row or column operation is performed on a matrix.

7. In each case find a basis of the null space of A. Then compute rank A and verify (1) of Theorem 2.

(a) $A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 0 & 1 \\ 4 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

◆(b) $A = \begin{bmatrix} 3 & 5 & 5 & 2 & 0 \\ 1 & 0 & 2 & 2 & 1 \\ 1 & 1 & 1 & -2 & -2 \\ -2 & 0 & -4 & -4 & -2 \end{bmatrix}$

8. Let $A = \mathbf{c}R$ where $\mathbf{c} \ne \mathbf{0}$ is a column in $\mathbb{R}^m$ and $\mathbf{r} \ne \mathbf{0}$ is a row in $\mathbb{R}^n$.

(a) Show that col $A = \text{span}\{\mathbf{c}\}$ and row $A = \text{span}\{\mathbf{r}\}$.

◆(b) Find $\dim(\text{null } A)$.

(c) Show that null A = null $\mathbf{r}$.

9. Let A be $m \times n$ with columns $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n$.

(a) If $\{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ is independent, show null $A = \{\mathbf{0}\}$.

◆(b) If null $A = \{\mathbf{0}\}$, show that $\{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ is independent.

10. Let A be an $n \times n$ matrix.

(a) Show that $A^2 = 0$ if and only if col $A \subseteq$ null A.

◆(b) Conclude that if $A^2 = 0$, then rank $A \le \frac{n}{2}$.

(c) Find a matrix A for which col A = null A.

11. Let B be $m \times n$ and let AB be $k \times n$. If rank $B = \text{rank}(AB)$, show that null $B = \text{null}(AB)$. [*Hint*: Theorem 1.]

◆12. Give a careful argument why $\text{rank}(A^T) = \text{rank } A$.

13. Let A be an $m \times n$ matrix with columns $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n$. If rank $A = n$, show that $\{A^T\mathbf{c}_1, A^T\mathbf{c}_2, \dots, A^T\mathbf{c}_n\}$ is a basis of $\mathbb{R}^n$.

14. If A is $m \times n$ and $\mathbf{b}$ is $m \times 1$, show that $\mathbf{b}$ lies in the column space of A if and only if $\text{rank}[A\ \mathbf{b}] = \text{rank } A$.

15. (a) Show that $A\mathbf{x} = \mathbf{b}$ has a solution if and only if rank $A = \text{rank}[A\ \mathbf{b}]$. [*Hint*: Exercises 12 and 14.]

◆(b) If $A\mathbf{x} = \mathbf{b}$ has no solution, show that $\text{rank}[A\ \mathbf{b}] = 1 + \text{rank } A$.

16. Let X be a $k \times m$ matrix. If I is the $m \times m$ identity matrix, show that $I + X^TX$ is invertible. [*Hint*: $I + X^TX = A^TA$ where $A = \begin{bmatrix} I \\ X \end{bmatrix}$ in block form.]

17. If A is $m \times n$ of rank r, show that A can be factored as $A = PQ$ where P is $m \times r$ with r independent columns, and Q is $r \times n$ with r independent rows. [*Hint*: Let $UAV = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ by Theorem 3, Section 2.5, and write $U^{-1} = \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix}$ and $V^{-1} = \begin{bmatrix} V_1 & V_2 \\ V_3 & V_4 \end{bmatrix}$ in block form, where U_1 and V_1 are $r \times r$.]

18. (a) Show that if A and B have independent columns, so does AB.

(b) Show that if A and B have independent rows, so does AB.

19. A matrix obtained from A by deleting rows and columns is called a **submatrix** of A. If A has an invertible $k \times k$ submatrix, show that rank $A \ge k$. [*Hint*: Show that row and column operations carry $A \to \begin{bmatrix} I_k & P \\ 0 & Q \end{bmatrix}$ in block form.] *Remark*: It can be shown that rank A is the largest integer r such that A has an invertible $r \times r$ submatrix.

SECTION 5.5

Similarity and Diagonalization

In Section 3.3 we studied diagonalization of a square matrix A, and found important applications (for example to linear dynamical systems). We can now utilize the concepts of subspace, basis, and dimension to clarify the diagonalization process, reveal some new results, and prove some theorems which could not be demonstrated in Section 3.3.

Before proceeding, we introduce a notion that simplifies the discussion of diagonalization, and is used throughout the book.

Similar Matrices

Definition 5.11 *If A and B are $n \times n$ matrices, we say that A and B are* **similar**, *and write $A \sim B$, if $B = P^{-1}AP$ for some invertible matrix P.*

Note that $A \sim B$ if and only if $B = QAQ^{-1}$ where Q is invertible (write $P^{-1} = Q$). The language of similarity is used throughout linear algebra. For example, a matrix A is diagonalizable if and only if it is similar to a diagonal matrix.

If $A \sim B$, then necessarily $B \sim A$. To see why, suppose that $B = P^{-1}AP$. Then $A = PBP^{-1} = Q^{-1}BQ$ where $Q = P^{-1}$ is invertible. This proves the second of the following properties of similarity (the others are left as an exercise):

1. *$A \sim A$ for all square matrices A.*
2. *If $A \sim B$, then $B \sim A$.* $(*)$
3. *If $A \sim B$ and $B \sim C$, then $A \sim C$.*

These properties are often expressed by saying that the similarity relation $\sim$ is an **equivalence relation** on the set of $n \times n$ matrices. Here is an example showing how these properties are used.

EXAMPLE 1

If A is similar to B and either A or B is diagonalizable, show that the other is also diagonalizable.

Solution ▶ We have $A \sim B$. Suppose that A is diagonalizable, say $A \sim D$ where D is diagonal. Since $B \sim A$ by (2) of $(*)$, we have $B \sim A$ and $A \sim D$. Hence $B \sim D$ by (3) of $(*)$, so B is diagonalizable too. An analogous argument works if we assume instead that B is diagonalizable.

Similarity is compatible with inverses, transposes, and powers:

$$\text{If } A \sim B \quad \text{then} \quad A^{-1} \sim B^{-1},\ A^T \sim B^T, \text{ and } A^k \sim B^k \text{ for all integers } k \geq 1.$$

The proofs are routine matrix computations using Theorem 1 Section 3.3. Thus, for example, if A is diagonalizable, so also are A^T, A^{-1} (if it exists), and A^k (for each $k \geq 1$). Indeed, if $A \sim D$ where D is a diagonal matrix, we obtain $A^T \sim D^T$, $A^{-1} \sim D^{-1}$, and $A^k \sim D^k$, and each of the matrices D^T, D^{-1}, and D^k is diagonal.

We pause to introduce a simple matrix function that will be referred to later.

Definition 5.12 *The* **trace** $\operatorname{tr} A$ *of an* $n \times n$ *matrix* A *is defined to be the sum of the main diagonal elements of* A.

In other words:

$$\text{If } A = [a_{ij}], \text{ then } \operatorname{tr} A = a_{11} + a_{22} + \cdots + a_{nn}.$$

It is evident that $\operatorname{tr}(A + B) = \operatorname{tr} A + \operatorname{tr} B$ and that $\operatorname{tr}(cA) = c \operatorname{tr} A$ holds for all $n \times n$ matrices A and B and all scalars c. The following fact is more surprising.

Lemma 1

Let A *and* B *be* $n \times n$ *matrices. Then* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

PROOF

Write $A = [a_{ij}]$ and $B = [b_{ij}]$. For each i, the (i, i)-entry d_i of the matrix AB is $d_i = a_{i1}b_{1i} + a_{i2}b_{2i} + \cdots + a_{in}b_{ni} = \sum_j a_{ij}b_{ji}$. Hence

$$\operatorname{tr}(AB) = d_1 + d_2 + \cdots + d_n = \sum_i d_i = \sum_i \left(\sum_j a_{ij}b_{ji}\right).$$

Similarly we have $\operatorname{tr}(BA) = \sum_i \left(\sum_j b_{ij}a_{ji}\right)$. Since these two double sums are the same, Lemma 1 is proved.

As the name indicates, similar matrices share many properties, some of which are collected in the next theorem for reference.

Theorem 1

If A *and* B *are similar* $n \times n$ *matrices, then* A *and* B *have the same determinant, rank, trace, characteristic polynomial, and eigenvalues.*

PROOF

Let $B = P^{-1}AP$ for some invertible matrix P. Then we have

$$\det B = \det(P^{-1}) \det A \det P = \det A \quad \text{because} \quad \det(P^{-1}) = 1/\det P.$$

Similarly, $\operatorname{rank} B = \operatorname{rank}(P^{-1}AP) = \operatorname{rank} A$ by Corollary 3 of Theorem 1 Section 5.4. Next Lemma 1 gives

$$\operatorname{tr}(P^{-1}AP) = \operatorname{tr}[P^{-1}(AP)] = \operatorname{tr}[(AP)P^{-1}] = \operatorname{tr} A.$$

As to the characteristic polynomial,

$$\begin{aligned} c_B(x) = \det(xI - B) &= \det\{x(P^{-1}IP) - P^{-1}AP\} \\ &= \det\{P^{-1}(xI - A)P\} \\ &= \det(xI - A) \\ &= c_A(x). \end{aligned}$$

Finally, this shows that A and B have the same eigenvalues because the eigenvalues of a matrix are the roots of its characteristic polynomial.

EXAMPLE 2

Sharing the five properties in Theorem 1 does not guarantee that two matrices are similar. The matrices $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ have the same determinant, rank, trace, characteristic polynomial, and eigenvalues, but they are not similar because $P^{-1}IP = I$ for any invertible matrix P.

Diagonalization Revisited

Recall that a square matrix A is **diagonalizable** if there exists an invertible matrix P such that $P^{-1}AP = D$ is a diagonal matrix, that is if A is similar to a diagonal matrix D. Unfortunately, not all matrices are diagonalizable, for example $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ (see Example 10 Section 3.3). Determining whether A is diagonalizable is closely related to the eigenvalues and eigenvectors of A. Recall that a number λ is called an **eigenvalue** of A if $A\mathbf{x} = \lambda\mathbf{x}$ for some nonzero column $\mathbf{x}$ in $\mathbb{R}^n$, and any such nonzero vector $\mathbf{x}$ is called an **eigenvector** of A corresponding to λ (or simply a λ-eigenvector of A). The eigenvalues and eigenvectors of A are closely related to the **characteristic polynomial** $c_A(x)$ of A, defined by

$$c_A(x) = \det(xI - A).$$

If A is $n \times n$ this is a polynomial of degree n, and its relationship to the eigenvalues is given in the following theorem (a repeat of Theorem 2 Section 3.3).

Theorem 2

Let A be an $n \times n$ matrix.

1. *The eigenvalues λ of A are the roots of the characteristic polynomial $c_A(x)$ of A.*
2. *The λ-eigenvectors $\mathbf{x}$ are the nonzero solutions to the homogeneous system*

$$(\lambda I - A)\mathbf{x} = \mathbf{0}$$

of linear equations with $\lambda I - A$ as coefficient matrix.

EXAMPLE 3

Show that the eigenvalues of a triangular matrix are the main diagonal entries.

Solution ▶ Assume that A is triangular. Then the matrix $xI - A$ is also triangular and has diagonal entries $(x - a_{11}), (x - a_{22}), \dots, (x - a_{nn})$ where $A = [a_{ij}]$. Hence Theorem 4 Section 3.1 gives

$$c_A(x) = (x - a_{11})(x - a_{22})\cdots(x - a_{nn})$$

and the result follows because the eigenvalues are the roots of $c_A(x)$.

Theorem 4 Section 3.3 asserts (in part) that an $n \times n$ matrix A is diagonalizable if and only if it has n eigenvectors $\mathbf{x}_1, \dots, \mathbf{x}_n$ such that the matrix $P = [\mathbf{x}_1 \ \cdots \ \mathbf{x}_n]$ with the $\mathbf{x}_i$ as columns is invertible. This is equivalent to requiring that $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ is a basis of $\mathbb{R}^n$ consisting of eigenvectors of A. Hence we can restate Theorem 4 Section 3.3 as follows:

Theorem 3

Let A be an $n \times n$ matrix.

1. *A is diagonalizable if and only if $\mathbb{R}^n$ has a basis $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ consisting of eigenvectors of A.*
2. *When this is the case, the matrix $P = [\mathbf{x}_1 \ \mathbf{x}_2 \ \cdots \ \mathbf{x}_n]$ is invertible and $P^{-1}AP = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ where, for each i, λ_i is the eigenvalue of A corresponding to $\mathbf{x}_i$.*

The next result is a basic tool for determining when a matrix is diagonalizable. It reveals an important connection between eigenvalues and linear independence: Eigenvectors corresponding to distinct eigenvalues are necessarily linearly independent.

Theorem 4

Let $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ be eigenvectors corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$ of an $n \times n$ matrix A. Then $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ is a linearly independent set.

PROOF

We use induction on k. If $k = 1$, then $\{\mathbf{x}_1\}$ is independent because $\mathbf{x}_1 \neq \mathbf{0}$. In general, suppose the theorem is true for some $k \geq 1$. Given eigenvectors $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{k+1}\}$, suppose a linear combination vanishes:

$$t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_{k+1}\mathbf{x}_{k+1} = \mathbf{0}. \qquad (*)$$

We must show that each $t_i = 0$. Left multiply $(*)$ by A and use the fact that $A\mathbf{x}_i = \lambda_i\mathbf{x}_i$ to get

$$t_1\lambda_1\mathbf{x}_1 + t_2\lambda_2\mathbf{x}_2 + \cdots + t_{k+1}\lambda_{k+1}\mathbf{x}_{k+1} = \mathbf{0}. \qquad (**)$$

If we multiply $(*)$ by λ_1 and subtract the result from $(**)$, the first terms cancel and we obtain

$$t_2(\lambda_2 - \lambda_1)\mathbf{x}_2 + t_3(\lambda_3 - \lambda_1)\mathbf{x}_3 + \cdots + t_{k+1}(\lambda_{k+1} - \lambda_1)\mathbf{x}_{k+1} = \mathbf{0}.$$

Since $\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_{k+1}$ correspond to distinct eigenvalues $\lambda_2, \lambda_3, \dots, \lambda_{k+1}$, the set $\{\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_{k+1}\}$ is independent by the induction hypothesis. Hence,

$$t_2(\lambda_2 - \lambda_1) = 0, \quad t_3(\lambda_3 - \lambda_1) = 0, \quad \dots, \quad t_{k+1}(\lambda_{k+1} - \lambda_1) = 0,$$

and so $t_2 = t_3 = \cdots = t_{k+1} = 0$ because the λ_i are distinct. Hence $(*)$ becomes $t_1\mathbf{x}_1 = \mathbf{0}$, which implies that $t_1 = 0$ because $\mathbf{x}_1 \neq \mathbf{0}$. This is what we wanted.

Theorem 4 will be applied several times; we begin by using it to give a useful condition for when a matrix is diagonalizable.

Theorem 5

If A is an $n \times n$ matrix with n distinct eigenvalues, then A is diagonalizable.

PROOF

Choose one eigenvector for each of the n distinct eigenvalues. Then these eigenvectors are independent by Theorem 4, and so are a basis of $\mathbb{R}^n$ by Theorem 7 Section 5.2. Now use Theorem 3.

EXAMPLE 4

Show that $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 3 \\ -1 & 1 & 0 \end{bmatrix}$ is diagonalizable.

Solution ▶ A routine computation shows that $c_A(x) = (x-1)(x-3)(x+1)$ and so has distinct eigenvalues 1, 3, and -1. Hence Theorem 5 applies.

However, a matrix can have multiple eigenvalues as we saw in Section 3.3. To deal with this situation, we prove an important lemma which formalizes a technique that is basic to diagonalization, and which will be used three times below.

Lemma 2

Let $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ be a linearly independent set of eigenvectors of an $n \times n$ matrix A, extend it to a basis $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k, \dots, \mathbf{x}_n\}$ of $\mathbb{R}^n$, and let

$$P = [\mathbf{x}_1 \ \mathbf{x}_2 \ \cdots \ \mathbf{x}_n]$$

be the (invertible) $n \times n$ matrix with the $\mathbf{x}_i$ as its columns. If $\lambda_1, \lambda_2, \dots, \lambda_k$ are the (not necessarily distinct) eigenvalues of A corresponding to $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ respectively, then $P^{-1}AP$ has block form

$$P^{-1}AP = \begin{bmatrix} \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_k) & B \\ 0 & A_1 \end{bmatrix}$$

where B has size $k \times (n-k)$ and A_1 has size $(n-k) \times (n-k)$.

PROOF

If $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is the standard basis of $\mathbb{R}^n$, then

$$\begin{aligned}[\mathbf{e}_1 \ \mathbf{e}_2 \ \dots \ \mathbf{e}_n] = I_n = P^{-1}P &= P^{-1}[\mathbf{x}_1 \ \mathbf{x}_2 \ \cdots \ \mathbf{x}_n] \\ &= [P^{-1}\mathbf{x}_1 \ P^{-1}\mathbf{x}_2 \ \cdots \ P^{-1}\mathbf{x}_n]\end{aligned}$$

Comparing columns, we have $P^{-1}\mathbf{x}_i = \mathbf{e}_i$ for each $1 \le i \le n$. On the other hand, observe that

$$P^{-1}AP = P^{-1}A[\mathbf{x}_1 \ \mathbf{x}_2 \ \cdots \ \mathbf{x}_n] = [(P^{-1}A)\mathbf{x}_1 \ (P^{-1}A)\mathbf{x}_2 \ \cdots \ (P^{-1}A)\mathbf{x}_n].$$

Hence, if $1 \le i \le k$, column i of $P^{-1}AP$ is

$$(P^{-1}A)\mathbf{x}_i = P^{-1}(\lambda_i\mathbf{x}_i) = \lambda_i(P^{-1}\mathbf{x}_1) = \lambda_i\mathbf{e}_i.$$

This describes the first k columns of $P^{-1}AP$, and Lemma 2 follows.

Note that Lemma 2 (with $k = n$) shows that an $n \times n$ matrix A is diagonalizable if $\mathbb{R}^n$ has a basis of eigenvectors of A, as in (1) of Theorem 3.

Definition 5.13

If λ is an eigenvalue of an $n \times n$ matrix A, define the **eigenspace** *of A corresponding to λ by*

$$E_\lambda(A) = \{\mathbf{x} \text{ in } \mathbb{R}^n \mid A\mathbf{x} = \lambda\mathbf{x}\}.$$

This is a subspace of $\mathbb{R}^n$ and the eigenvectors corresponding to λ are just the nonzero vectors in $E_\lambda(A)$. In fact $E_\lambda(A)$ is the null space of the matrix $(\lambda I - A)$:

$$E_\lambda(A) = \{\mathbf{x} \mid (\lambda I - A)\mathbf{x} = \mathbf{0}\} = \text{null}(\lambda I - A).$$

Hence, by Theorem 2 Section 5.4, the basic solutions of the homogeneous system $(\lambda I - A)\mathbf{x} = \mathbf{0}$ given by the gaussian algorithm form a basis for $E_\lambda(A)$. In particular

$$\dim E_\lambda(A) \text{ is the number of basic solutions } \mathbf{x} \text{ of } (\lambda I - A)\mathbf{x} = \mathbf{0}. \qquad (***)$$

Now recall (Definition 3.7) that the **multiplicity**[11] of an eigenvalue λ of A is the number of times λ occurs as a root of the characteristic polynomial $c_A(x)$ of A. In other words, the multiplicity of λ is the largest integer $m \geq 1$ such that

$$c_A(x) = (x - \lambda)^m g(x)$$

for some polynomial $g(x)$. Because of $(***)$, the assertion (without proof) in Theorem 5 Section 3.3 can be stated as follows: A square matrix is diagonalizable if and only if the multiplicity of each eigenvalue λ equals $\dim[E_\lambda(A)]$. We are going to prove this, and the proof requires the following result which is valid for *any* square matrix, diagonalizable or not.

Lemma 3

Let λ be an eigenvalue of multiplicity m of a square matrix A. Then $\dim[E_\lambda(A)] \leq m$.

PROOF

Write $\dim[E_\lambda(A)] = d$. It suffices to show that $c_A(x) = (x - \lambda)^d g(x)$ for some polynomial $g(x)$, because m is the highest power of $(x - \lambda)$ that divides $c_A(x)$. To this end, let $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_d\}$ be a basis of $E_\lambda(A)$. Then Lemma 2 shows that an invertible $n \times n$ matrix P exists such that

$$P^{-1}AP = \begin{bmatrix} \lambda I_d & B \\ 0 & A_1 \end{bmatrix}$$

in block form, where I_d denotes the $d \times d$ identity matrix. Now write $A' = P^{-1}AP$ and observe that $c_{A'}(x) = c_A(x)$ by Theorem 1. But Theorem 5 Section 3.1 gives

$$\begin{aligned} c_A(x) = c_{A'}(x) = \det(xI_n - A') &= \det\begin{bmatrix} (x - \lambda)I_d & -B \\ 0 & xI_{n-d} - A_1 \end{bmatrix} \\ &= \det[(x - \lambda)I_d] \det[(xI_{n-d} - A_1)] \\ &= (x - \lambda)^d g(x). \end{aligned}$$

where $g(x) = c_{A_1}(x)$. This is what we wanted.

It is impossible to ignore the question when equality holds in Lemma 3 for each eigenvalue λ. It turns out that this characterizes the diagonalizable $n \times n$ matrices A for which $c_A(x)$ **factors completely** over $\mathbb{R}$. By this we mean that $c_A(x) = (x - \lambda_1)(x - \lambda_2)\cdots(x - \lambda_n)$, where the λ_i are *real* numbers (not necessarily

11 This is often called the *algebraic* multiplicity of λ.

distinct); in other words, every eigenvalue of A is real. This need not happen (consider $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$), and we investigate the general case below.

Theorem 6

The following are equivalent for a square matrix A for which $c_A(x)$ factors completely.

1. *A is diagonalizable.*
2. *$\dim[E_\lambda(A)]$ equals the multiplicity of λ for every eigenvalue λ of the matrix A.*

PROOF

Let A be $n \times n$ and let $\lambda_1, \lambda_2, \dots, \lambda_k$ be the distinct eigenvalues of A. For each i, let m_i denote the multiplicity of λ_i and write $d_i = \dim[E_{\lambda_i}(A)]$. Then

$$c_A(x) = (x - \lambda_1)^{m_1}(x - \lambda_2)^{m_2}\cdots(x - \lambda_n)^{m_k}$$

so $m_1 + \cdots + m_k = n$ because $c_A(x)$ has degree n. Moreover, $d_i \le m_i$ for each i by Lemma 3.

(1) $\Rightarrow$ (2). By (1), $\mathbb{R}^n$ has a basis of n eigenvectors of A, so let t_i of them lie in $E_{\lambda_i}(A)$ for each i. Since the subspace spanned by these t_i eigenvectors has dimension t_i, we have $t_i \le d_i$ for each i by Theorem 4 Section 5.2. Hence

$$n = t_1 + \cdots + t_k \le d_1 + \cdots + d_k \le m_1 + \cdots + m_k = n.$$

It follows that $d_1 + \cdots + d_k = m_1 + \cdots + m_k$ so, since $d_i \le m_i$ for each i, we must have $d_i = m_i$. This is (2).

(2) $\Rightarrow$ (1). Let B_i denote a basis of $E_{\lambda_i}(A)$ for each i, and let $B = B_1 \cup \cdots \cup B_k$. Since each B_i contains m_i vectors by (2), and since the B_i are pairwise disjoint (the λ_i are distinct), it follows that B contains n vectors. So it suffices to show that B is linearly independent (then B is a basis of $\mathbb{R}^n$). Suppose a linear combination of the vectors in B vanishes, and let $\mathbf{y}_i$ denote the sum of all terms that come from B_i. Then $\mathbf{y}_i$ lies in $E_{\lambda_i}(A)$ for each i, so the nonzero $\mathbf{y}_i$ are independent by Theorem 4 (as the λ_i are distinct). Since the sum of the $\mathbf{y}_i$ is zero, it follows that $\mathbf{y}_i = \mathbf{0}$ for each i. Hence all coefficients of terms in $\mathbf{y}_i$ are zero (because B_i is independent). Since this holds for each i, it shows that B is independent.

EXAMPLE 5

If $A = \begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & -2 \end{bmatrix}$, show that A is diagonalizable but B is not.

Solution ▸ We have $c_A(x) = (x + 3)^2(x - 1)$ so the eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = 1$. The corresponding eigenspaces are $E_{\lambda_1}(A) = \operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2\}$ and $E_{\lambda_2}(A) = \operatorname{span}\{\mathbf{x}_3\}$ where

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

as the reader can verify. Since $\{\mathbf{x}_1, \mathbf{x}_2\}$ is independent, we have $\dim(E_{\lambda_1}(A)) = 2$ which is the multiplicity of λ_1. Similarly, $\dim(E_{\lambda_2}(A)) = 1$ equals the multiplicity of λ_2. Hence A is diagonalizable by Theorem 6, and a diagonalizing matrix is $P = [\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3]$.

Turning to B, $c_B(x) = (x + 1)^2(x - 3)$ so the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 3$. The corresponding eigenspaces are $E_\lambda 1(B) = \text{span}\{\mathbf{y}_1\}$ and $E_\lambda 2(B) = \text{span}\{\mathbf{y}_2\}$ where

$$\mathbf{y}_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} 5 \\ 6 \\ -1 \end{bmatrix}.$$

Here $\dim(E_{\lambda_1}(B)) = 1$ is *smaller* than the multiplicity of λ_1, so the matrix B is *not* diagonalizable, again by Theorem 6. The fact that $\dim(E_{\lambda_1}(B)) = 1$ means that there is no possibility of finding *three* linearly independent eigenvectors.

Complex Eigenvalues

All the matrices we have considered have had real eigenvalues. But this need not be the case: The matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ has characteristic polynomial $c_A(x) = x^2 + 1$ which has no real roots. Nonetheless, this matrix is diagonalizable; the only difference is that we must use a larger set of scalars, the complex numbers. The basic properties of these numbers are outlined in Appendix A.

Indeed, nearly everything we have done for real matrices can be done for complex matrices. The methods are the same; the only difference is that the arithmetic is carried out with complex numbers rather than real ones. For example, the gaussian algorithm works in exactly the same way to solve systems of linear equations with complex coefficients, matrix multiplication is defined the same way, and the matrix inversion algorithm works in the same way.

But the complex numbers are better than the real numbers in one respect: While there are polynomials like $x^2 + 1$ with real coefficients that have no real root, this problem does not arise with the complex numbers: *Every* nonconstant polynomial with complex coefficients has a complex root, and hence factors completely as a product of linear factors. This fact is known as the fundamental theorem of algebra.[12]

EXAMPLE 6

Diagonalize the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Solution ▶ The characteristic polynomial of A is

$$c_A(x) = \det(xI - A) = x^2 + 1 = (x - i)(x + i)$$

where $i^2 = -1$. Hence the eigenvalues are $\lambda_1 = i$ and $\lambda_2 = -i$, with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$. Hence A is diagonalizable by the complex version of Theorem 5, and the complex version of Theorem 3 shows that $P = [\mathbf{x}_1\ \ \mathbf{x}_2] = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$ is invertible and $P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$. Of course, this can be checked directly.

We shall return to complex linear algebra in Section 8.6.

12 This was a famous open problem in 1799 when Gauss solved it at the age of 22 in his Ph.D. dissertation.

Symmetric Matrices[13]

On the other hand, many of the applications of linear algebra involve a real matrix A and, while A will have complex eigenvalues by the fundamental theorem of algebra, it is always of interest to know when the eigenvalues are, in fact, real. While this can happen in a variety of ways, it turns out to hold whenever A is symmetric. This important theorem will be used extensively later. Surprisingly, the theory of *complex* eigenvalues can be used to prove this useful result about *real* eigenvalues.

Let $\overline{z}$ denote the conjugate of a complex number z. If A is a complex matrix, the **conjugate matrix** $\overline{A}$ is defined to be the matrix obtained from A by conjugating every entry. Thus, if $A = [z_{ij}]$, then $\overline{A} = [\overline{z}_{ij}]$. For example,

$$\text{If } A = \begin{bmatrix} -i+2 & 5 \\ i & 3+4i \end{bmatrix} \quad \text{then} \quad \overline{A} = \begin{bmatrix} i+2 & 5 \\ -i & 3-4i \end{bmatrix}$$

Recall that $\overline{z+w} = \overline{z} + \overline{w}$ and $\overline{zw} = \overline{z}\,\overline{w}$ hold for all complex numbers z and w. It follows that if A and B are two complex matrices, then

$$\overline{A+B} = \overline{A} + \overline{B}, \quad \overline{AB} = \overline{A}\,\overline{B} \quad \text{and } \overline{\lambda A} = \overline{\lambda}\,\overline{B}$$

hold for all complex scalars λ. These facts are used in the proof of the following theorem.

Theorem 7

Let A be a symmetric real matrix. If λ is any complex eigenvalue of A, then λ is real.[14]

PROOF

Observe that $\overline{A} = A$ because A is real. If λ is an eigenvalue of A, we show that λ is real by showing that $\overline{\lambda} = \lambda$. Let $\mathbf{x}$ be a (possibly complex) eigenvector corresponding to λ, so that $\mathbf{x} \neq \mathbf{0}$ and $A\mathbf{x} = \lambda\mathbf{x}$. Define $c = \mathbf{x}^T\overline{\mathbf{x}}$.

If we write $\mathbf{x} = (z_1, z_2, \ldots, z_n)$ where the z_i are complex numbers, we have

$$c = \mathbf{x}^T\overline{\mathbf{x}} = z_1\overline{z_1} + z_2\overline{z_2} + \cdots + z_n\overline{z_n} = |\overline{z_1}|^2 + |\overline{z_2}|^2 + \cdots + |\overline{z_n}|^2.$$

Thus c is a real number, and $c > 0$ because at least one of the $z_i \neq 0$ (as $\mathbf{x} \neq \mathbf{0}$). We show that $\overline{\lambda} = \lambda$ by verifying that $\lambda c = \overline{\lambda} c$. We have

$$\lambda c = \lambda(\mathbf{x}^T\overline{\mathbf{x}}) = (\lambda\mathbf{x})^T\overline{\mathbf{x}} = (A\mathbf{x})^T\overline{\mathbf{x}} = \mathbf{x}^T A^T \overline{\mathbf{x}}.$$

At this point we use the hypothesis that A is symmetric and real. This means $A^T = A = \overline{A}$, so we continue the calculation:

$$\begin{aligned} \lambda c = \mathbf{x}^T A^T \overline{\mathbf{x}} = \mathbf{x}^T(\overline{A}\,\overline{\mathbf{x}}) = \mathbf{x}^T(\overline{A\mathbf{x}}) &= \mathbf{x}^T(\overline{\lambda\mathbf{x}}) \\ &= \mathbf{x}^T(\overline{\lambda}\,\overline{\mathbf{x}}) \\ &= \overline{\lambda}\mathbf{x}^T\overline{\mathbf{x}} \\ &= \overline{\lambda} c \end{aligned}$$

as required.

The technique in the proof of Theorem 7 will be used again when we return to complex linear algebra in Section 8.6.

13 This discussion uses complex conjugation and absolute value. These topics are discussed in Appendix A.

14 This theorem was first proved in 1829 by the great French mathematician Augustin Louis Cauchy (1789–1857).

EXAMPLE 7

Verify Theorem 7 for every real, symmetric 2×2 matrix A.

Solution ▶ If $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ we have $c_A(x) = x^2 - (a + c)x + (ac - b^2)$, so the eigenvalues are given by $\lambda = \frac{1}{2}\left[(a + c) \pm \sqrt{(a + c)^2 - 4(ac - b^2)}\right]$. But here

$$(a + c)^2 - 4(ac - b^2) = (a - c)^2 + 4b^2 \geq 0$$

for any choice of a, b, and c. Hence, the eigenvalues are real numbers.

EXERCISES 5.5

1. By computing the trace, determinant, and rank, show that A and B are *not* similar in each case.

   (a) $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

   ◆(b) $A = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$

   (c) $A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 0 \\ 1 & -1 \end{bmatrix}$

   ◆(d) $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$

   (e) $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ -3 & 6 & -3 \end{bmatrix}$

   ◆(f) $A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & -1 & 2 \\ 0 & 3 & -5 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 1 & 3 \\ 6 & -3 & -9 \\ 0 & 0 & 0 \end{bmatrix}$

2. Show that $\begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & -1 \\ 4 & 3 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & -1 & 3 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & -1 & 4 & 1 \\ 5 & -1 & -1 & -4 \end{bmatrix}$ are *not* similar.

3. If $A \sim B$, show that:

   (a) $A^T \sim B^T$ ◆(b) $A^{-1} \sim B^{-1}$

   (c) $rA \sim rB$ for r in $\mathbb{R}$ (d) $A^n \sim B^n$ for $n \geq 1$

4. In each case, decide whether the matrix A is diagonalizable. If so, find P such that $P^{-1}AP$ is diagonal.

   (a) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ ◆(b) $\begin{bmatrix} 3 & 0 & 6 \\ 0 & -3 & 0 \\ 5 & 0 & 2 \end{bmatrix}$

   (c) $\begin{bmatrix} 3 & 1 & 6 \\ 2 & 1 & 0 \\ -1 & 0 & -3 \end{bmatrix}$ ◆(d) $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 2 \\ 2 & 3 & 1 \end{bmatrix}$

5. If A is invertible, show that AB is similar to BA for all B.

6. Show that the only matrix similar to a scalar matrix $A = rI$, r in $\mathbb{R}$, is A itself.

7. Let λ be an eigenvalue of A with corresponding eigenvector $\mathbf{x}$. If $B = P^{-1}AP$ is similar to A, show that $P^{-1}\mathbf{x}$ is an eigenvector of B corresponding to λ.

8. If $A \sim B$ and A has any of the following properties, show that B has the same property.

   (a) Idempotent, that is $A^2 = A$.

   ◆(b) Nilpotent, that is $A^k = 0$ for some $k \geq 1$.

   (c) Invertible.

9. Let A denote an $n \times n$ upper triangular matrix.

   (a) If all the main diagonal entries of A are distinct, show that A is diagonalizable.

   ◆(b) If all the main diagonal entries of A are equal, show that A is diagonalizable only if it is *already* diagonal.

   (c) Show that $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is diagonalizable but that $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is not diagonalizable.

10. Let A be a diagonalizable $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ (including multiplicities). Show that:

 (a) $\det A = \lambda_1\lambda_2\cdots\lambda_n$

 ◆(b) $\operatorname{tr} A = \lambda_1 + \lambda_2 + \cdots + \lambda_n$

11. Given a polynomial $p(x) = r_0 + r_1x + \cdots + r_nx^n$ and a square matrix A, the matrix $p(A) = r_0I + r_1A + \cdots + r_nA^n$ is called the **evaluation** of $p(x)$ at A. Let $B = P^{-1}AP$. Show that $p(B) = P^{-1}p(A)P$ for all polynomials $p(x)$.

12. Let P be an invertible $n \times n$ matrix. If A is any $n \times n$ matrix, write $T_P(A) = P^{-1}AP$. Verify that:

 (a) $T_P(I) = I$

 ◆(b) $T_P(AB) = T_P(A)T_P(B)$

 (c) $T_P(A + B) = T_P(A) + T_P(B)$

 (d) $T_P(rA) = rT_P(A)$

 (e) $T_P(A^k) = [T_P(A)]^k$ for $k \geq 1$

 (f) If A is invertible, $T_P(A^{-1}) = [T_P(A)]^{-1}$.

 (g) If Q is invertible, $T_Q[T_P(A)] = T_{PQ}(A)$.

13. (a) Show that two diagonalizable matrices are similar if and only if they have the same eigenvalues with the same multiplicities.

 ◆(b) If A is diagonalizable, show that $A \sim A^T$.

 (c) Show that $A \sim A^T$ if $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

14. If A is 2×2 and diagonalizable, show that $C(A) = \{X \mid XA = AX\}$ has dimension 2 or 4. [*Hint*: If $P^{-1}AP = D$, show that X is in $C(A)$ if and only if $P^{-1}XP$ is in $C(D)$.]

15. If A is diagonalizable and $p(x)$ is a polynomial such that $p(\lambda) = 0$ for all eigenvalues λ of A, show that $p(A) = 0$ (see Example 9 Section 3.3). In particular, show $c_A(A) = 0$. [*Remark*: $c_A(A) = 0$ for *all* square matrices A—this is the Cayley-Hamilton theorem (see Theorem 2 Section 9.4).]

16. Let A be $n \times n$ with n distinct real eigenvalues. If $AC = CA$, show that C is diagonalizable.

17. Let $A = \begin{bmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{bmatrix}$ and $B = \begin{bmatrix} c & a & b \\ a & b & c \\ b & c & a \end{bmatrix}$.

 (a) Show that $x^3 - (a^2 + b^2 + c^2)x - 2abc$ has real roots by considering A.

 ◆(b) Show that $a^2 + b^2 + c^2 \geq ab + ac + bc$ by considering B.

18. Assume the 2×2 matrix A is similar to an upper triangular matrix. If $\operatorname{tr} A = 0 = \operatorname{tr} A^2$, show that $A^2 = 0$.

19. Show that A is similar to A^T for all 2×2 matrices A. [*Hint*: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $c = 0$, treat the cases $b = 0$ and $b \neq 0$ separately. If $c \neq 0$, reduce to the case $c = 1$ using Exercise 12(d).]

20. Refer to Section 3.4 on linear recurrences. Assume that the sequence $x_0, x_1, x_2, \dots$ satisfies

$$x_{n+k} = r_0x_n + r_1x_{n+1} + \cdots + r_{k-1}x_{n+k-1}$$

for all $n \geq 0$. Define

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ r_0 & r_1 & r_2 & \cdots & r_{k-1} \end{bmatrix}, \quad V_n = \begin{bmatrix} x_n \\ x_{n+1} \\ \vdots \\ x_{n+k-1} \end{bmatrix}.$$

Then show that:

 (a) $V_n = A^nV_0$ for all n.

 (b) $c_A(x) = x^k - r_{k-1}x^{k-1} - \cdots - r_1x - r_0$.

 (c) If λ is an eigenvalue of A, the eigenspace E_λ has dimension 1, and $\mathbf{x} = (1, \lambda, \lambda^2, \dots, \lambda^{k-1})^T$ is an eigenvector. [*Hint*: Use $c_A(\lambda) = 0$ to show that $E_\lambda = \mathbb{R}\mathbf{x}$.]

 (d) A is diagonalizable if and only if the eigenvalues of A are distinct. [*Hint*: See part (c) and Theorem 4.]

 (e) If $\lambda_1, \lambda_2, \dots, \lambda_k$ are distinct real eigenvalues, there exist constants $t_1, t_2, \dots, t_k$ such that $x_n = t_1\lambda_1^n + \cdots + t_k\lambda_k^n$ holds for all n. [*Hint*: If D is diagonal with $\lambda_1, \lambda_2, \dots, \lambda_k$ as the main diagonal entries, show that $A^n = PD^nP^{-1}$ has entries that are linear combinations of $\lambda_1^n, \lambda_2^n, \dots, \lambda_k^n$.]

SECTION 5.6 Best Approximation and Least Squares

Often an exact solution to a problem in applied mathematics is difficult to obtain. However, it is usually just as useful to find arbitrarily close approximations to a solution. In particular, finding “linear approximations” is a potent technique in applied mathematics. One basic case is the situation where a system of linear equations has no solution, and it is desirable to find a “best approximation” to a solution to the system. In this section best approximations are defined and a method for finding them is described. The result is then applied to “least squares” approximation of data.

Suppose A is an $m \times n$ matrix and $\mathbf{b}$ is a column in $\mathbb{R}^m$, and consider the system

$$A\mathbf{x} = \mathbf{b}$$

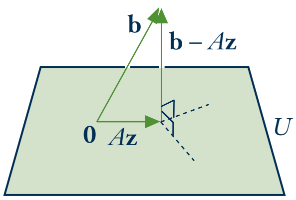
of m linear equations in n variables. This need not have a solution. However, given any column $\mathbf{z}$ in $\mathbb{R}^n$, the distance $\|\mathbf{b} - A\mathbf{z}\|$ is a measure of how far $A\mathbf{z}$ is from $\mathbf{b}$. Hence it is natural to ask whether there is a column $\mathbf{z}$ in $\mathbb{R}^n$ that is as close as possible to a solution in the sense that

$$\|\mathbf{b} - A\mathbf{z}\|$$

is the minimum value of $\|\mathbf{b} - A\mathbf{x}\|$ as $\mathbf{x}$ ranges over all columns in $\mathbb{R}^n$.

The answer is “yes”, and to describe it define

$$U = \{A\mathbf{x} \mid \mathbf{x} \text{ lies in } \mathbb{R}^n\}.$$



This is a subspace of $\mathbb{R}^n$ (verify) and we want a vector $A\mathbf{z}$ in U as close as possible to $\mathbf{b}$. That there is such a vector is clear geometrically if $n = 3$ by the diagram. In general such a vector $A\mathbf{z}$ exists by a general result called the *projection theorem* that will be proved in Chapter 8 (Theorem 3 Section 8.1). Moreover, the projection theorem gives a simple way to compute $\mathbf{z}$ because it also shows that the vector $\mathbf{b} - A\mathbf{z}$ is *orthogonal* to every vector $A\mathbf{x}$ in U. Thus, for all $\mathbf{x}$ in $\mathbb{R}^n$,

$$\begin{aligned} 0 = (A\mathbf{x}) \cdot (\mathbf{b} - A\mathbf{z}) = (A\mathbf{x})^T(\mathbf{b} - A\mathbf{z}) &= \mathbf{x}^T A^T(\mathbf{b} - A\mathbf{z}) \\ &= \mathbf{x} \cdot [A^T(\mathbf{b} - A\mathbf{z})] \end{aligned}$$

In other words, the vector $A^T(\mathbf{b} - A\mathbf{z})$ in $\mathbb{R}^n$ is orthogonal to *every* vector in $\mathbb{R}^n$ and so must be zero (being orthogonal to itself). Hence $\mathbf{z}$ satisfies

$$(A^T A)\mathbf{z} = A^T\mathbf{b}.$$

Definition 5.14 *This is a system of linear equations called the* **normal equations** *for* $\mathbf{z}$.

Note that this system can have more than one solution (see Exercise 5). However, the $n \times n$ matrix $A^T A$ is invertible if (and only if) the columns of A are linearly independent (Theorem 3 Section 5.4); so, in this case, $\mathbf{z}$ is uniquely determined and is given explicitly by $\mathbf{z} = (A^T A)^{-1} A^T \mathbf{b}$. However, the most efficient way to find $\mathbf{z}$ is to apply gaussian elimination to the normal equations.

This discussion is summarized in the following theorem.

Theorem 1

Best Approximation Theorem
Let A be an $m \times n$ matrix, let $\mathbf{b}$ be any column in $\mathbb{R}^m$, and consider the system

$$A\mathbf{x} = \mathbf{b}$$

of m equations in n variables.

(1) *Any solution* $\mathbf{z}$ *to the normal equations*

$$(A^T A)\mathbf{z} = A^T\mathbf{b}$$

is a best approximation to a solution to $A\mathbf{x} = \mathbf{b}$ *in the sense that* $\|\mathbf{b} - A\mathbf{z}\|$ *is the minimum value of* $\|\mathbf{b} - A\mathbf{x}\|$ *as* $\mathbf{x}$ *ranges over all columns in* $\mathbb{R}^n$.

(2) *If the columns of* A *are linearly independent, then* $A^T A$ *is invertible and* $\mathbf{z}$ *is given uniquely by* $\mathbf{z} = (A^T A)^{-1} A^T \mathbf{b}$.

We note in passing that if A is $n \times n$ and invertible, then

$$\mathbf{z} = (A^T A)^{-1} A^T \mathbf{b} = A^{-1}\mathbf{b}$$

is the solution to the system of equations, and $\|\mathbf{b} - A\mathbf{z}\| = 0$. Hence if A has independent columns, then $(A^T A)^{-1} A^T$ is playing the role of the inverse of the nonsquare matrix A. The matrix $A^T(AA^T)^{-1}$ plays a similar role when the rows of A are linearly independent. These are both special cases of the **generalized inverse** of a matrix A (see Exercise 14). However, we shall not pursue this topic here.

EXAMPLE 1

The system of linear equations

$$\begin{aligned} 3x - y &= 4 \\ x + 2y &= 0 \\ 2x + y &= 1 \end{aligned}$$

has no solution. Find the vector $\mathbf{z} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ that best approximates a solution.

Solution ▶ In this case,

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad \text{so } A^T A = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 1 \\ 1 & 6 \end{bmatrix}$$

is invertible. The normal equations $(A^T A)\mathbf{z} = A^T\mathbf{b}$ are

$$\begin{bmatrix} 14 & 1 \\ 1 & 6 \end{bmatrix}\mathbf{z} = \begin{bmatrix} 14 \\ -3 \end{bmatrix}, \quad \text{so } \mathbf{z} = \tfrac{1}{83}\begin{bmatrix} 87 \\ -56 \end{bmatrix}.$$

Thus $x_0 = \frac{87}{83}$ and $y_0 = \frac{-56}{83}$. With these values of x and y, the left sides of the equations are, approximately,

$$\begin{aligned} 3x_0 - y_0 &= \tfrac{317}{83} = 3.82 \\ x_0 + 2y_0 &= \tfrac{-25}{83} = -0.30 \\ 2x_0 + y_0 &= \tfrac{118}{83} = 1.42 \end{aligned}$$

This is as close as possible to a solution.

EXAMPLE 2

The average number g of goals per game scored by a hockey player seems to be related linearly to two factors: the number x_1 of years of experience and the number x_2 of goals in the preceding 10 games. The data on the following page were collected on four players. Find the linear function $g = a_0 + a_1x_1 + a_2x_2$ that best fits these data.

g	x_1	x_2
0.8	5	3
0.8	3	4
0.6	1	5
0.4	2	1

Solution ▶ If the relationship is given by $g = r_0 + r_1x_1 + r_2x_2$, then the data can be described as follows:

$$\begin{bmatrix} 1 & 5 & 3 \\ 1 & 3 & 4 \\ 1 & 1 & 5 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.8 \\ 0.6 \\ 0.4 \end{bmatrix}$$

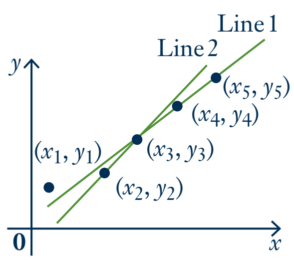
Using the notation in Theorem 1, we get

$$\begin{aligned} \mathbf{z} &= (A^TA)^{-1}A^T\mathbf{b} \\ &= \tfrac{1}{42}\begin{bmatrix} 119 & -17 & -19 \\ -17 & 5 & 1 \\ -19 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 5 & 3 & 1 & 2 \\ 3 & 4 & 5 & 1 \end{bmatrix} \begin{bmatrix} 0.8 \\ 0.8 \\ 0.6 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.14 \\ 0.09 \\ 0.08 \end{bmatrix} \end{aligned}$$

Hence the best-fitting function is $g = 0.14 + 0.09x_1 + 0.08x_2$. The amount of computation would have been reduced if the normal equations had been constructed and then solved by gaussian elimination.

Least Squares Approximation

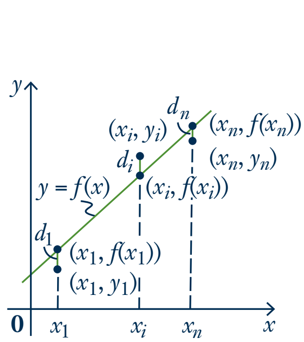
In many scientific investigations, data are collected that relate two variables. For example, if x is the number of dollars spent on advertising by a manufacturer and y is the value of sales in the region in question, the manufacturer could generate data by spending $x_1, x_2, \dots, x_n$ dollars at different times and measuring the corresponding sales values $y_1, y_2, \dots, y_n$.



Suppose it is known that a linear relationship exists between the variables x and y—in other words, that $y = a + bx$ for some constants a and b. If the data are plotted, the points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ may appear to lie on a straight line and estimating a and b requires finding the "best-fitting" line through these data points. For example, if five data points occur as shown in the diagram, line 1 is clearly a better fit than line 2. In general, the problem is to find the values of the constants a and b such that the line $y = a + bx$ best approximates the data in question. Note that an exact fit would be obtained if a and b were such that $y_i = a + bx_i$ were true for each data point (x_i, y_i). But this is too much to expect. Experimental errors in measurement are bound to occur, so the choice of a and b should be made in such a way that the errors between the observed values y_i and the corresponding fitted values $a + bx_i$ are in some sense minimized. Least squares approximation is a way to do this.

The first thing we must do is explain exactly what we mean by the *best fit* of a line $y = a + bx$ to an observed set of data points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$. For convenience, write the linear function $r_0 + r_1x$ as

$$f(x) = r_0 + r_1x$$



so that the fitted points (on the line) have coordinates $(x_1, f(x_1)), \dots, (x_n, f(x_n))$. The second diagram is a sketch of what the line $y = f(x)$ might look like. For each i the observed data point (x_i, y_i) and the fitted point $(x_i, f(x_i))$ need not be the same, and the distance d_i between them measures how far the line misses the observed point. For this reason d_i is often called the **error** at x_i, and a natural measure of how close the line $y = f(x)$ is to the observed data points is the sum $d_1 + d_2 + \cdots + d_n$ of all these errors. However, it turns out to be better to use the sum of squares

$$S = d_1^2 + d_2^2 + \cdots + d_n^2$$

as the measure of error, and the line $y = f(x)$ is to be chosen so as to make this sum as small as possible. This line is said to be the **least squares approximating line** for the data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.

The square of the error d_i is given by $d_i^2 = [y_i - f(x_i)]^2$ for each i, so the quantity S to be minimized is the sum:

$$S = [y_1 - f(x_1)]^2 + [y_2 - f(x_2)]^2 + \cdots + [y_n - f(x_n)]^2.$$

Note that all the numbers x_i and y_i are *given* here; what is required is that the *function* f be chosen in such a way as to minimize S. Because $f(x) = r_0 + r_1x$, this amounts to choosing r_0 and r_1 to minimize S. This problem can be solved using Theorem 1. The following notation is convenient.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \quad \text{and} \quad f(\mathbf{x}) = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{bmatrix} = \begin{bmatrix} r_0 + r_1x_1 \\ r_0 + r_1x_2 \\ \vdots \\ r_0 + r_1x_n \end{bmatrix}$$

Then the problem takes the following form: Choose r_0 and r_1 such that

$$S = [y_1 - f(x_1)]^2 + [y_2 - f(x_2)]^2 + \cdots + [y_n - f(x_n)]^2 = \|\mathbf{y} - f(\mathbf{x})\|^2$$

is as small as possible. Now write

$$M = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \quad \text{and} \quad \mathbf{r} = \begin{bmatrix} r_0 \\ r_1 \end{bmatrix}.$$

Then $M\mathbf{r} = f(\mathbf{x})$, so we are looking for a column $\mathbf{r} = \begin{bmatrix} r_0 \\ r_1 \end{bmatrix}$ such that $\|\mathbf{y} - M\mathbf{r}\|^2$ is as small as possible. In other words, we are looking for a best approximation $\mathbf{z}$ to the system $M\mathbf{r} = \mathbf{y}$. Hence Theorem 1 applies directly, and we have

Theorem 2

Suppose that n data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ are given, where at least two of $x_1, x_2, \ldots, x_n$ are distinct. Put

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \quad M = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}$$

Then the least squares approximating line for these data points has equation

$$y = z_0 + z_1x$$

where $\mathbf{z} = \begin{bmatrix} z_0 \\ z_1 \end{bmatrix}$ is found by gaussian elimination from the normal equations

$$(M^TM)\mathbf{z} = M^T\mathbf{y}.$$

The condition that at least two of $x_1, x_2, \ldots, x_n$ are distinct ensures that M^TM is an invertible matrix, so $\mathbf{z}$ is unique:

$$\mathbf{z} = (M^TM)^{-1}M^T\mathbf{y}.$$

EXAMPLE 3

Let data points $(x_1, y_1), (x_2, y_2), \ldots, (x_5, y_5)$ be given as in the accompanying table. Find the least squares approximating line for these data.

x	y
1	1
3	2
4	3
6	4
7	5

Solution ▶ In this case we have

$$M^TM = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_5 \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_5 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & x_1 + \cdots + x_5 \\ x_1 + \cdots + x_5 & x_1^2 + \cdots + x_5^2 \end{bmatrix} = \begin{bmatrix} 5 & 21 \\ 21 & 111 \end{bmatrix},$$

$$\text{and } M^T\mathbf{y} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_5 \end{bmatrix}$$

$$= \begin{bmatrix} y_1 + y_2 + \cdots + y_5 \\ x_1y_1 + x_2y_2 + \cdots + x_5y_5 \end{bmatrix} = \begin{bmatrix} 15 \\ 78 \end{bmatrix},$$

so the normal equations $(M^TM)\mathbf{z} = M^T\mathbf{y}$ for $\mathbf{z} = \begin{bmatrix} z_0 \\ z_1 \end{bmatrix}$ become

$$\begin{bmatrix} 5 & 21 \\ 21 & 111 \end{bmatrix} = \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = \begin{bmatrix} 15 \\ 78 \end{bmatrix}$$

The solution (using gaussian elimination) is $\mathbf{z} = \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0.24 \\ 0.66 \end{bmatrix}$ to two decimal places, so the least squares approximating line for these data is $y = 0.24 + 0.66x$. Note that M^TM is indeed invertible here (the determinant is 114), and the exact solution is

$$\mathbf{z} = (M^TM)^{-1}M^T\mathbf{y} = \tfrac{1}{114}\begin{bmatrix} 111 & -21 \\ -21 & 5 \end{bmatrix}\begin{bmatrix} 15 \\ 78 \end{bmatrix} = \tfrac{1}{114}\begin{bmatrix} 27 \\ 75 \end{bmatrix} = \tfrac{1}{38}\begin{bmatrix} 9 \\ 25 \end{bmatrix}.$$

Least Squares Approximating Polynomials

Suppose now that, rather than a straight line, we want to find a polynomial

$$y = f(x) = r_0 + r_1x + r_2x^2 + \cdots + r_mx^m$$

of degree m that best approximates the data pairs $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$. As before, write

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \quad \text{and} \quad f(\mathbf{x}) = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{bmatrix}$$

For each x_i we have two values of the variable y, the observed value y_i, and the computed value $f(x_i)$. The problem is to choose $f(x)$—that is, choose $r_0, r_1, \ldots, r_m$—such that the $f(x_i)$ are as close as possible to the y_i. Again we define "as close as possible" by the least squares condition: We choose the r_i such that

$$\|\mathbf{y} - f(\mathbf{x})\|^2 = [y_1 - f(x_1)]^2 + [y_2 - f(x_2)]^2 + \cdots + [y_n - f(x_n)]^2$$

is as small as possible.

Definition 5.15 *A polynomial $f(x)$ satisfying this condition is called a* **least squares approximating polynomial** *of degree m for the given data pairs.*

If we write

$$M = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^m \\ 1 & x_2 & x_2^2 & \cdots & x_2^m \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^m \end{bmatrix} \quad \text{and} \quad \mathbf{r} = \begin{bmatrix} r_0 \\ r_1 \\ \vdots \\ r_m \end{bmatrix}$$

we see that $f(\mathbf{x}) = M\mathbf{r}$. Hence we want to find R such that $\|\mathbf{y} - M\mathbf{r}\|^2$ is as small as possible; that is, we want a best approximation $\mathbf{z}$ to the system $M\mathbf{r} = \mathbf{y}$. Theorem 1 gives the first part of Theorem 3.

Theorem 3

Let n data pairs $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ be given, and write

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \quad M = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^m \\ 1 & x_2 & x_2^2 & \cdots & x_2^m \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^m \end{bmatrix} \quad \mathbf{z} = \begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_m \end{bmatrix}$$

1. *If $\mathbf{z}$ is any solution to the normal equations*

$$(M^TM)\mathbf{z} = M^T\mathbf{y}$$

then the polynomial

$$z_0 + z_1x + z_2x^2 + \cdots + z_mx^m$$

is a least squares approximating polynomial of degree m for the given data pairs.

2. *If at least $m + 1$ of the numbers $x_1, x_2, \ldots, x_n$ are distinct (so $n \geq m + 1$), the matrix M^TM is invertible and $\mathbf{z}$ is uniquely determined by*

$$\mathbf{z} = (M^TM)^{-1}M^T\mathbf{y}$$

PROOF

It remains to prove (2), and for that we show that the columns of M are linearly independent (Theorem 3 Section 5.4). Suppose a linear combination of the columns vanishes:

$$r_0 \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} + r_1 \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \cdots + r_m \begin{bmatrix} x_1^m \\ x_2^m \\ \vdots \\ x_n^m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

If we write $q(x) = r_0 + r_1x + \cdots + r_mx^m$, equating coefficients shows that $q(x_1) = q(x_2) = \cdots = q(x_n) = 0$. Hence $q(x)$ is a polynomial of degree m with at least $m + 1$ distinct roots, so $q(x)$ must be the zero polynomial (see Appendix D or Theorem 4 Section 6.5). Thus $r_0 = r_1 = \cdots = r_m = 0$ as required.

EXAMPLE 4

Find the least squares approximating quadratic $y = z_0 + z_1x + z_2x^2$ for the following data points.

$$(-3, 3), (-1, 1), (0, 1), (1, 2), (3, 4)$$

Solution ▶ This is an instance of Theorem 3 with $m = 2$. Here

$$\mathbf{y} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 2 \\ 4 \end{bmatrix} \quad M = \begin{bmatrix} 1 & -3 & 9 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \end{bmatrix}$$

Hence,

$$M^TM = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -3 & -1 & 0 & 1 & 3 \\ 9 & 1 & 0 & 1 & 9 \end{bmatrix} \begin{bmatrix} 1 & -3 & 9 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 20 \\ 0 & 20 & 0 \\ 20 & 0 & 164 \end{bmatrix}$$

$$M^T\mathbf{y} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -3 & -1 & 0 & 1 & 3 \\ 9 & 1 & 0 & 1 & 9 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 11 \\ 4 \\ 66 \end{bmatrix}$$

The normal equations for $\mathbf{z}$ are

$$\begin{bmatrix} 5 & 0 & 20 \\ 0 & 20 & 0 \\ 20 & 0 & 164 \end{bmatrix} \mathbf{z} = \begin{bmatrix} 11 \\ 4 \\ 66 \end{bmatrix} \quad \text{whence } \mathbf{z} = \begin{bmatrix} 1.15 \\ 0.20 \\ 0.26 \end{bmatrix}$$

This means that the least squares approximating quadratic for these data is $y = 1.15 + 0.20x + 0.26x^2$.

Other Functions

There is an extension of Theorem 3 that should be mentioned. Given data pairs $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$, that theorem shows how to find a polynomial

$$f(x) = r_0 + r_1x + \cdots + r_mx^m$$

such that $\|\mathbf{y} - f(\mathbf{x})\|^2$ is as small as possible, where $\mathbf{x}$ and $f(\mathbf{x})$ are as before. Choosing the appropriate polynomial $f(x)$ amounts to choosing the coefficients $r_0, r_1, \dots, r_m$, and Theorem 3 gives a formula for the optimal choices. Here $f(x)$ is a linear combination of the functions $1, x, x^2, \dots, x^m$ where the r_i are the coefficients, and this suggests applying the method to other functions. If $f_0(x), f_1(x), \dots, f_m(x)$ are given functions, write

$$f(x) = r_0f_0(x) + r_1f_1(x) + \cdots + r_mf_m(x)$$

where the r_i are real numbers. Then the more general question is whether $r_0, r_1, \dots, r_m$ can be found such that $\|\mathbf{y} - f(\mathbf{x})\|^2$ is as small as possible where

$$f(\mathbf{x}) = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_m) \end{bmatrix}$$

Such a function $f(\mathbf{x})$ is called a **least squares best approximation** for these data pairs of the form $r_0 f_0(x) + r_1 f_1(x) + \cdots + r_m f_m(x)$, r_i in $\mathbb{R}$. The proof of Theorem 3 goes through to prove

Theorem 4

Let n data pairs $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ be given, and suppose that $m + 1$ functions $f_0(x), f_1(x), \dots, f_m(x)$ are specified. Write

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \quad M = \begin{bmatrix} f_0(x_1) & f_1(x_1) & \cdots & f_m(x_1) \\ f_0(x_2) & f_1(x_2) & \cdots & f_m(x_2) \\ \vdots & \vdots & & \vdots \\ f_0(x_n) & f_1(x_n) & \cdots & f_m(x_n) \end{bmatrix} \quad \mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{bmatrix}$$

(1) *If $\mathbf{z}$ is any solution to the normal equations*

$$(M^T M)\mathbf{z} = M^T \mathbf{y},$$

then the function

$$z_0 f_0(x) + z_1 f_1(x) + \cdots + z_m f_m(x)$$

is the best approximation for these data among all functions of the form $r_0 f_0(x) + r_1 f_1(x) + \cdots + r_m f_m(x)$ where the r_i are in $\mathbb{R}$.

(2) *If $M^T M$ is invertible (that is, if $\operatorname{rank}(M) = m + 1$), then $\mathbf{z}$ is uniquely determined; in fact, $\mathbf{z} = (M^T M)^{-1}(M^T \mathbf{y})$.*

Clearly Theorem 4 contains Theorem 3 as a special case, but there is no simple test in general for whether $M^T M$ is invertible. Conditions for this to hold depend on the choice of the functions $f_0(x), f_1(x), \dots, f_m(x)$.

EXAMPLE 5

Given the data pairs $(-1, 0)$, $(0, 1)$, and $(1, 4)$, find the least squares approximating function of the form $r_0 x + r_1 2^x$.

Solution ▶ The functions are $f_0(x) = x$ and $f_1(x) = 2^x$, so the matrix M is

$$M = \begin{bmatrix} f_0(x_1) & f_1(x_1) \\ f_0(x_2) & f_1(x_2) \\ f_0(x_3) & f_1(x_3) \end{bmatrix} = \begin{bmatrix} -1 & 2^{-1} \\ 0 & 2^0 \\ 1 & 2^1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -2 & 1 \\ 0 & 2 \\ 2 & 4 \end{bmatrix}$$

In this case $M^T M = \frac{1}{4}\begin{bmatrix} 8 & 6 \\ 6 & 21 \end{bmatrix}$ is invertible, so the normal equations

$$\frac{1}{4}\begin{bmatrix} 8 & 6 \\ 6 & 21 \end{bmatrix}\mathbf{z} = \begin{bmatrix} 4 \\ 9 \end{bmatrix} \quad \text{have a unique solution } \mathbf{z} = \frac{1}{11}\begin{bmatrix} 10 \\ 16 \end{bmatrix}$$

Hence the best-fitting function of the form $r_0x + r_12^x$ is $\overline{f}(x) = \frac{10}{11}x + \frac{16}{11}2^x$.

Note that $\overline{f}(\mathbf{x}) = \begin{bmatrix} \overline{f}(-1) \\ \overline{f}(0) \\ \overline{f}(1) \end{bmatrix} = \begin{bmatrix} \frac{-2}{11} \\ \frac{16}{11} \\ \frac{42}{11} \end{bmatrix}$, compared with $\mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$.

EXERCISES 5.6

1. Find the best approximation to a solution of each of the following systems of equations.

   (a) $$\begin{aligned} x + y - z &= 5 \\ 2x - y + 6z &= 1 \\ 3x + 2y - z &= 6 \\ -x + 4y + z &= 0 \end{aligned}$$

   ◆(b) $$\begin{aligned} 3x + y + z &= 6 \\ 2x + 3y - z &= 1 \\ 2x - y + z &= 0 \\ 3x - 3y + 3z &= 8 \end{aligned}$$

2. Find the least squares approximating line $y = z_0 + z_1x$ for each of the following sets of data points.

   (a) (1, 1), (3, 2), (4, 3), (6, 4)

   ◆(b) (2, 4), (4, 3), (7, 2), (8, 1)

   (c) $(-1, -1)$, (0, 1), (1, 2), (2, 4), (3, 6)

   ◆(d) $(-2, 3)$, $(-1, 1)$, (0, 0), $(1, -2)$, $(2, -4)$

3. Find the least squares approximating quadratic $y = z_0 + z_1x + z_2x^2$ for each of the following sets of data points.

   (a) (0, 1), (2, 2), (3, 3), (4, 5)

   ◆(b) $(-2, 1)$, (0, 0), (3, 2), (4, 3)

4. Find a least squares approximating function of the form $r_0x + r_1x^2 + r_22^x$ for each of the following sets of data pairs.

   (a) $(-1, 1)$, (0, 3), (1, 1), (2, 0)

   ◆(b) (0, 1), (1, 1), (2, 5), (3, 10)

5. Find the least squares approximating function of the form $r_0 + r_1x^2 + r_2\sin\frac{\pi x}{2}$ for each of the following sets of data pairs.

   (a) (0, 3), (1, 0), $(1, -1)$, $(-1, 2)$

   ◆(b) $(-1, \frac{1}{2})$, (0, 1), (2, 5), (3, 9)

6. If M is a square invertible matrix, show that $\mathbf{z} = M^{-1}\mathbf{y}$ (in the notation of Theorem 3).

◆7. Newton's laws of motion imply that an object dropped from rest at a height of 100 metres will be at a height $s = 100 - \frac{1}{2}gt^2$ metres t seconds later, where g is a constant called the acceleration due to gravity. The values of s and t given in the table are observed. Write $x = t^2$, find the least squares approximating line $s = a + bx$ for these data, and use b to estimate g.

   Then find the least squares approximating quadratic $s = a_0 + a_1t + a_2t^2$ and use the value of a_2 to estimate g.

t	1	2	3
s	95	80	56

8. A naturalist measured the heights y_i (in metres) of several spruce trees with trunk diameters x_i (in centimetres). The data are as given in the table. Find the least squares approximating line for these data and use it to estimate the height of a spruce tree with a trunk of diameter 10 cm.

x_i	5	7	8	12	13	16
y_i	2	3.3	4	7.3	7.9	10.1

◆9. The yield y of wheat in bushels per acre appears to be a linear function of the number of days x_1 of sunshine, the number of inches x_2 of rain, and the number of pounds x_3 of fertilizer applied per acre. Find the best fit to the data in the table by an equation of the form $y = r_0 + r_1x_1 + r_2x_2 + r_3x_3$. [*Hint*: If a calculator for inverting A^TA is not available, the inverse is given in the answer.]

y	x_1	x_2	x_3
28	50	18	10
30	40	20	16
21	35	14	10
23	40	12	12
23	30	16	14

10. (a) Use $m = 0$ in Theorem 3 to show that the best-fitting horizontal line $y = a_0$ through the data points $(x_1, y_1), \dots, (x_n, y_n)$ is $y = \frac{1}{n}(y_1 + y_2 + \cdots + y_n)$, the average of the y coordinates.

◆(b) Deduce the conclusion in (a) without using Theorem 3.

11. Assume $n = m + 1$ in Theorem 3 (so M is square). If the x_i are distinct, use Theorem 6 Section 3.2 to show that M is invertible. Deduce that $\mathbf{z} = M^{-1}\mathbf{y}$ and that the least squares polynomial is the interpolating polynomial (Theorem 6 Section 3.2) and actually passes through all the data points.

12. Let A be any $m \times n$ matrix and write $K = \{\mathbf{x} \mid A^TA\mathbf{x} = \mathbf{0}\}$. Let B be an m-column. Show that, if $\mathbf{z}$ is an n-column such that $\|\mathbf{b} - A\mathbf{z}\|$ is minimal, then *all* such vectors have the form $\mathbf{z} + \mathbf{x}$ for some $\mathbf{x}$ in K. [*Hint*: $\|\mathbf{b} - A\mathbf{y}\|$ is minimal if and only if $A^TA\mathbf{y} = A^T\mathbf{b}$.]

13. Given the situation in Theorem 4, write

$$f(x) = r_0p_0(x) + r_1p_1(x) + \cdots + r_mp_m(x)$$

Suppose that $f(x)$ has at most k roots for any choice of the coefficients $r_0, r_1, \dots, r_m$, not all zero.

(a) Show that M^TM is invertible if at least $k + 1$ of the x_i are distinct.

◆(b) If at least two of the x_i are distinct, show that there is always a best approximation of the form $r_0 + r_1e^x$.

(c) If at least three of the x_i are distinct, show that there is always a best approximation of the form $r_0 + r_1x + r_2e^x$. [Calculus is needed.]

14. If A is an $m \times n$ matrix, it can be proved that there exists a unique $n \times m$ matrix $A^\#$ satisfying the following four conditions: $AA^\#A = A$; $A^\#AA^\# = A^\#$; $AA^\#$ and $A^\#A$ are symmetric. The matrix $A^\#$ is called the **generalized inverse** of A, or the **Moore-Penrose** inverse.

(a) If A is square and invertible, show that $A^\# = A^{-1}$.

(b) If rank $A = m$, show that $A^\# = A^T(AA^T)^{-1}$.

(c) If rank $A = n$, show that $A^\# = (A^TA)^{-1}A^T$.

SECTION 5.7 An Application to Correlation and Variance

Suppose the heights $h_1, h_2, \dots, h_n$ of n men are measured. Such a data set is called a **sample** of the heights of all the men in the population under study, and various questions are often asked about such a sample: What is the average height in the sample? How much variation is there in the sample heights, and how can it be measured? What can be inferred from the sample about the heights of all men in the population? How do these heights compare to heights of men in neighbouring countries? Does the prevalence of smoking affect the height of a man?

The analysis of samples, and of inferences that can be drawn from them, is a subject called *mathematical statistics*, and an extensive body of information has been developed to answer many such questions. In this section we will describe a few ways that linear algebra can be used.

It is convenient to represent a sample $\{x_1, x_2, \dots, x_n\}$ as a **sample vector**[15] $\mathbf{x} = [x_1\ x_2\ \cdots\ x_n]$ in $\mathbb{R}^n$. This being done, the dot product in $\mathbb{R}^n$ provides a convenient tool to study the sample and describe some of the statistical concepts

15 We write vectors in $\mathbb{R}^n$ as row matrices, for convenience.

related to it. The most widely known statistic for describing a data set is the **sample mean** $\overline{x}$ defined by[16]

$$\overline{x} = \tfrac{1}{n}(x_1 + x_2 + \cdots + x_n) = \tfrac{1}{n}\sum_{i=1}^{n} x_i.$$

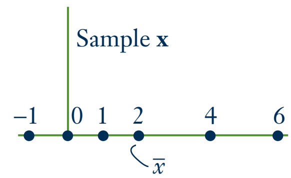
The mean $\overline{x}$ is "typical" of the sample values x_i, but may not itself be one of them. The number $x_i - \overline{x}$ is called the **deviation** of x_i from the mean $\overline{x}$. The deviation is positive if $x_i > \overline{x}$ and it is negative if $x_i < \overline{x}$. Moreover, the sum of these deviations is zero:

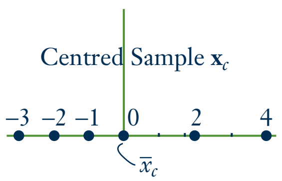
$$\sum_{i=1}^{n}(x_i - \overline{x}) = \left(\sum_{i=1}^{n} x_i\right) - n\overline{x} = n\overline{x} - n\overline{x} = 0. \qquad (*)$$

This is described by saying that the sample mean $\overline{x}$ is *central* to the sample values x_i.

If the mean $\overline{x}$ is subtracted from each data value x_i, the resulting data $x_i - \overline{x}$ are said to be **centred**. The corresponding data vector is

$$\mathbf{x}_c = [x_1 - \overline{x} \quad x_2 - \overline{x} \quad \cdots \quad x_n - \overline{x}]$$





and $(*)$ shows that the mean $\overline{x}_c = 0$. For example, the sample $\mathbf{x} = [-1 \;\; 0 \;\; 1 \;\; 4 \;\; 6]$ is plotted in the first diagram. The mean is $\overline{x} = 2$, and the centred sample $\mathbf{x}_c = [-3 \;\; -2 \;\; -1 \;\; 2 \;\; 4]$ is also plotted. Thus, the effect of centring is to shift the data by an amount $\overline{x}$ (to the left if $\overline{x}$ is positive) so that the mean moves to 0.

Another question that arises about samples is how much variability there is in the sample $\mathbf{x} = [x_1 \;\; x_2 \;\; \cdots \;\; x_n]$; that is, how widely are the data "spread out" around the sample mean $\overline{x}$. A natural measure of variability would be the sum of the deviations of the x_i about the mean, but this sum is zero by $(*)$; these deviations cancel out. To avoid this cancellation, statisticians use the *squares* $(x_i - \overline{x})^2$ of the deviations as a measure of variability. More precisely, they compute a statistic called the **sample variance** s_x^2, defined[17] as follows:

$$s_x^2 = \tfrac{1}{n-1}[(x_1 - \overline{x})^2 + (x_2 - \overline{x})^2 + \cdots + (x_n - \overline{x})^2] = \tfrac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2.$$

The sample variance will be large if there are many x_i at a large distance from the mean $\overline{x}$, and it will be small if all the x_i are tightly clustered about the mean. The variance is clearly nonnegative (hence the notation s_x^2), and the square root s_x of the variance is called the **sample standard deviation**.

The sample mean and variance can be conveniently described using the dot product. Let

$$\mathbf{1} = [1 \;\; 1 \;\; \cdots \;\; 1]$$

denote the row with every entry equal to 1. If $\mathbf{x} = [x_1 \;\; x_2 \;\; \cdots \;\; x_n]$, then $\mathbf{x} \cdot \mathbf{1} = x_1 + x_2 + \cdots + x_n$, so the sample mean is given by the formula

$$\overline{x} = \tfrac{1}{n}(\mathbf{x} \cdot \mathbf{1}).$$

Moreover, remembering that $\overline{x}$ is a scalar, we have $\overline{x}\mathbf{1} = [\overline{x} \;\; \overline{x} \;\; \cdots \;\; \overline{x}]$, so the centred sample vector $\mathbf{x}_c$ is given by

$$\mathbf{x}_c = \mathbf{x} - \overline{x}\mathbf{1} = [x_1 - \overline{x} \quad x_2 - \overline{x} \quad \cdots \quad x_n - \overline{x}].$$

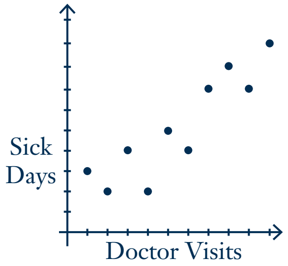
Thus we obtain a formula for the sample variance:

$$s_x^2 = \tfrac{1}{n-1}\|\mathbf{x}_c\|^2 = \tfrac{1}{n-1}\|\mathbf{x} - \overline{x}\mathbf{1}\|^2.$$

Linear algebra is also useful for comparing two different samples. To illustrate how, consider two examples.

16 The mean is often called the "average" of the sample values x_i, but statisticians use the term "mean".

17 Since there are n sample values, it seems more natural to divide by n here, rather than by $n - 1$. The reason for using $n - 1$ is that then the sample variance s_x^2 provides a better estimate of the variance of the entire population from which the sample was drawn.



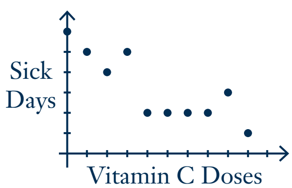
The following table represents the number of sick days at work per year and the yearly number of visits to a physician for 10 individuals.

Individual	1	2	3	4	5	6	7	8	9	10
Doctor visits	2	6	8	1	5	10	3	9	7	4
Sick days	2	4	8	3	5	9	4	7	7	2

The data are plotted in the **scatter diagram** where it is evident that, roughly speaking, the more visits to the doctor the more sick days. This is an example of a *positive correlation* between sick days and doctor visits.

Now consider the following table representing the daily doses of vitamin C and the number of sick days.

Individual	1	2	3	4	5	6	7	8	9	10
Vitamin C	1	5	7	0	4	9	2	8	6	3
Sick days	5	2	2	6	2	1	4	3	2	5



The scatter diagram is plotted as shown and it appears that the more vitamin C taken, the fewer sick days. In this case there is a *negative correlation* between daily vitamin C and sick days.

In both these situations, we have **paired samples**, that is observations of two variables are made for ten individuals: doctor visits and sick days in the first case; daily vitamin C and sick days in the second case. The scatter diagrams point to a relationship between these variables, and there is a way to use the sample to compute a number, called the correlation coefficient, that measures the degree to which the variables are associated.

To motivate the definition of the correlation coefficient, suppose two paired samples $\mathbf{x} = [x_1 \ x_2 \ \cdots \ x_n]$, and $\mathbf{y} = [y_1 \ y_2 \ \cdots \ y_n]$ are given and consider the centred samples

$$\mathbf{x}_c = [x_1 - \overline{x} \quad x_2 - \overline{x} \quad \cdots \quad x_n - \overline{x}] \quad \text{and} \quad \mathbf{y}_c = [y_1 - \overline{y} \quad y_2 - \overline{y} \quad \cdots \quad y_n - \overline{y}]$$

If x_k is large among the x_i's, then the deviation $x_k - \overline{x}$ will be positive; and $x_k - \overline{x}$ will be negative if x_k is small among the x_i's. The situation is similar for $\mathbf{y}$, and the following table displays the sign of the quantity $(x_i - \overline{x})(y_k - \overline{y})$ in all four cases:

Sign of $(x_i - \overline{x})(y_k - \overline{y})$:

	x_i large	x_i small
y_i large	positive	negative
y_i small	negative	positive

Intuitively, if $\mathbf{x}$ and $\mathbf{y}$ are positively correlated, then two things happen:

1. *Large values of the x_i tend to be associated with large values of the y_i, and*
2. *Small values of the x_i tend to be associated with small values of the y_i.*

It follows from the table that, if $\mathbf{x}$ and $\mathbf{y}$ are positively correlated, then the dot product

$$\mathbf{x}_c \cdot \mathbf{y}_c = \sum_{i=1}^{n} (x_i - \overline{x})(y_i - \overline{y})$$

is positive. Similarly $\mathbf{x}_c \cdot \mathbf{y}_c$ is negative if $\mathbf{x}$ and $\mathbf{y}$ are negatively correlated. With this in mind, the **sample correlation coefficient**[18] r is defined by

18 The idea of using a single number to measure the degree of relationship between different variables was pioneered by Francis Galton (1822–1911). He was studying the degree to which characteristics of an offspring relate to those of its parents. The idea was refined by Karl Pearson (1857–1936) and r is often referred to as the Pearson correlation coefficient.

$$r = r(\mathbf{x}, \mathbf{y}) = \frac{\mathbf{x}_c \cdot \mathbf{y}_c}{\|\mathbf{x}_c\| \, \|\mathbf{y}_c\|}.$$

Bearing the situation in $\mathbb{R}^3$ in mind, r is the cosine of the "angle" between the vectors $\mathbf{x}_c$ and $\mathbf{y}_c$, and so we would expect it to lie between -1 and 1. Moreover, we would expect r to be near 1 (or -1) if these vectors were pointing in the same (opposite) direction, that is the "angle" is near zero (or π).

This is confirmed by Theorem 1 below, and it is also borne out in the examples above. If we compute the correlation between sick days and visits to the physician (in the first scatter diagram above) the result is $r = 0.90$ as expected. On the other hand, the correlation between daily vitamin C doses and sick days (second scatter diagram) is $r = -0.84$.

However, a word of caution is in order here. We *cannot* conclude from the second example that taking more vitamin C will reduce the number of sick days at work. The (negative) correlation may arise because of some third factor that is related to both variables. For example, case it may be that less healthy people are inclined to take more vitamin C. Correlation does *not* imply causation. Similarly, the correlation between sick days and visits to the doctor does not mean that having many sick days *causes* more visits to the doctor. A correlation between two variables may point to the existence of other underlying factors, but it does not necessarily mean that there is a causality relationship between the variables.

Our discussion of the dot product in $\mathbb{R}^n$ provides the basic properties of the correlation coefficient:

Theorem 1

Let $\mathbf{x} = [x_1 \ x_2 \ \cdots \ x_n]$ and $\mathbf{y} = [y_1 \ y_2 \ \cdots \ y_n]$ be (nonzero) paired samples, and let $r = r(\mathbf{x}, \mathbf{y})$ denote the correlation coefficient. Then:

1. $-1 \leq r \leq 1$.
2. $r = 1$ *if and only if there exist a and $b > 0$ such that $y_i = a + bx_i$ for each i.*
3. $r = -1$ *if and only if there exist a and $b < 0$ such that $y_i = a + bx_i$ for each i.*

PROOF

The Cauchy inequality (Theorem 2 Section 5.3) proves (1), and also shows that $r = \pm 1$ if and only if one of $\mathbf{x}_c$ and $\mathbf{y}_c$ is a scalar multiple of the other. This in turn holds if and only if $\mathbf{y}_c = b\mathbf{x}_c$ for some $b \neq 0$, and it is easy to verify that $r = 1$ when $b > 0$ and $r = -1$ when $b < 0$.

Finally, $\mathbf{y}_c = b\mathbf{x}_c$ means $y_1 - \overline{y} = b(x_1 - \overline{x})$ for each i; that is, $y_i = a + bx_i$ where $a = \overline{y} - b\overline{x}$. Conversely, if $y_i = a + bx_i$, then $\overline{y} = a + b\overline{x}$ (verify), so $y_1 - \overline{y} = (a + bx_i) - (a + b\overline{x}) = b(x_1 - \overline{x})$ for each i. In other words, $\mathbf{y}_c = b\mathbf{x}_c$. This completes the proof.

Properties (2) and (3) in Theorem 1 show that $r(\mathbf{x}, \mathbf{y}) = 1$ means that there is a linear relation with *positive* slope between the paired data (so large x values are paired with large y values). Similarly, $r(\mathbf{x}, \mathbf{y}) = -1$ means that there is a linear relation with *negative* slope between the paired data (so small x values are paired with small y values). This is borne out in the two scatter diagrams above.

We conclude by using the dot product to derive some useful formulas for computing variances and correlation coefficients. Given samples $\mathbf{x} = [x_1\ x_2\ \cdots\ x_n]$, and $\mathbf{y} = [y_1\ y_2\ \cdots\ y_n]$, the key observation is the following formula:

$$\mathbf{x}_c \cdot \mathbf{y}_c = \mathbf{x} \cdot \mathbf{y} - n\,\overline{x}\,\overline{y}. \qquad (**)$$

Indeed, remembering that $\overline{x}$ and $\overline{y}$ are scalars:

$$\begin{aligned}\mathbf{x}_c \cdot \mathbf{y}_c &= (\mathbf{x} - \overline{x}\mathbf{1}) \cdot (\mathbf{y} - \overline{y}\mathbf{1}) \\ &= \mathbf{x} \cdot \mathbf{y} - \mathbf{x} \cdot (\overline{y}\mathbf{1}) - (\overline{x}\mathbf{1}) \cdot \mathbf{y} + (\overline{x}\mathbf{1}) \cdot (\overline{y}\mathbf{1}) \\ &= \mathbf{x} \cdot \mathbf{y} - \overline{y}(\mathbf{x} \cdot \mathbf{1}) - \overline{x}(\mathbf{1} \cdot \mathbf{y}) + \overline{x}\overline{y}(\mathbf{1} \cdot \mathbf{1}) \\ &= \mathbf{x} \cdot \mathbf{y} - \overline{y}(n\overline{x}) - \overline{x}(n\overline{y}) + \overline{x}\overline{y}(n) \\ &= \mathbf{x} \cdot \mathbf{y} - n\overline{x}\overline{y}.\end{aligned}$$

Taking $\mathbf{y} = \mathbf{x}$ in $(**)$ gives a formula for the variance $s_x^2 = \frac{1}{n-1}\|\mathbf{x}_c\|^2$ of $\mathbf{x}$.

Variance Formula

If x is a sample vector, then $s_x^2 = \frac{1}{n-1}(\|\mathbf{x}_c\|^2 - n\overline{x}^2)$.

We also get a convenient formula for the correlation coefficient, $r = r(\mathbf{x}, \mathbf{y}) = \dfrac{\mathbf{x}_c \cdot \mathbf{y}_c}{\|\mathbf{x}_c\|\ \|\mathbf{y}_c\|}$. Moreover, $(**)$ and the fact that $s_x^2 = \frac{1}{n-1}\|\mathbf{x}_c\|^2$ give:

Correlation Formula

If $\mathbf{x}$ and $\mathbf{y}$ are sample vectors, then

$$r = r(\mathbf{x}, \mathbf{y}) = \frac{\mathbf{x} \cdot \mathbf{y} - n\,\overline{x}\,\overline{y}}{(n-1)s_x s_y}.$$

Finally, we give a method that simplifies the computations of variances and correlations.

Data Scaling

Let $\mathbf{x} = [x_1\ x_2\ \cdots\ x_n]$ and $\mathbf{y} = [y_1\ y_2\ \cdots\ y_n]$ be sample vectors. Given constants a, b, c, and d, consider new samples $\mathbf{z} = [z_1\ z_2\ \cdots\ z_n]$ and $\mathbf{w} = [w_1\ w_2\ \cdots\ w_n]$ where $z_i = a + bx_i$, for each i and $w_i = c + dy_i$ for each i. Then:

(a) $\overline{z} = a + b\overline{x}$.

(b) $s_z^2 = b^2 s_x^2$, *so* $s_z = |b|s_x$.

(c) If b and d have the same sign, then $r(\mathbf{x}, \mathbf{y}) = r(\mathbf{z}, \mathbf{w})$.

The verification is left as an exercise.

For example, if $\mathbf{x} = [101\ \ 98\ \ 103\ \ 99\ \ 100\ \ 97]$, subtracting 100 yields $\mathbf{z} = [1\ \ -2\ \ 3\ \ -1\ \ 0\ \ -3]$. A routine calculation shows that $\overline{z} = -\frac{1}{3}$ and $s_z^2 = \frac{14}{3}$, so $\overline{x} = 100 - \frac{1}{3} = 99.67$, and $s_z^2 = \frac{14}{3} = 4.67$.

EXERCISES 5.7

1. The following table gives IQ scores for 10 fathers and their eldest sons. Calculate the means, the variances, and the correlation coefficient r. (The data scaling formula is useful.)

	1	2	3	4	5	6	7	8	9	10
Father's IQ	140	131	120	115	110	106	100	95	91	86
Son's IQ	130	138	110	99	109	120	105	99	100	94

◆2. The following table gives the number of years of education and the annual income (in thousands) of 10 individuals. Find the means, the variances, and the correlation coefficient. (Again the data scaling formula is useful.)

Individual	1	2	3	4	5	6	7	8	9	10
Years of education	12	16	13	18	19	12	18	19	12	14
Yearly income (1000's)	31	48	35	28	55	40	39	60	32	35

3. If $\mathbf{x}$ is a sample vector, and $\mathbf{x}_c$ is the centred sample, show that $\bar{x}_c = 0$ and the standard deviation of $\mathbf{x}_c$ is s_x.

4. Prove the data scaling formulas found on page 292: (a), ◆(b), and (c).

SUPPLEMENTARY EXERCISES FOR CHAPTER 5

1. In each case either show that the statement is true or give an example showing that it is false. Throughout, $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ denote vectors in $\mathbb{R}^n$.

   (a) If U is a subspace of $\mathbb{R}^n$ and $\mathbf{x} + \mathbf{y}$ is in U, then $\mathbf{x}$ and $\mathbf{y}$ are both in U.

   ◆(b) If U is a subspace of $\mathbb{R}^n$ and $r\mathbf{x}$ is in U, then $\mathbf{x}$ is in U.

   (c) If U is a nonempty set and $s\mathbf{x} + t\mathbf{y}$ is in U for any s and t whenever $\mathbf{x}$ and $\mathbf{y}$ are in U, then U is a subspace.

   ◆(d) If U is a subspace of $\mathbb{R}^n$ and $\mathbf{x}$ is in U, then $-\mathbf{x}$ is in U.

   (e) If $\{\mathbf{x}, \mathbf{y}\}$ is independent, then $\{\mathbf{x}, \mathbf{y}, \mathbf{x} + \mathbf{y}\}$ is independent.

   ◆(f) If $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is independent, then $\{\mathbf{x}, \mathbf{y}\}$ is independent.

   (g) If $\{\mathbf{x}, \mathbf{y}\}$ is not independent, then $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is not independent.

   ◆(h) If all of $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ are nonzero, then $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ is independent.

   (i) If one of $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ is zero, then $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ is not independent.

   ◆(j) If $a\mathbf{x} + b\mathbf{y} + c\mathbf{z} = \mathbf{0}$ where a, b, and c are in $\mathbb{R}$, then $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is independent.

   (k) If $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is independent, then $a\mathbf{x} + b\mathbf{y} + c\mathbf{z} = \mathbf{0}$ for some a, b, and c in $\mathbb{R}$.

   ◆(l) If $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ is not independent, then $t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_n\mathbf{x}_n = \mathbf{0}$ for t_i in $\mathbb{R}$ not all zero.

   (m) If $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ is independent, then $t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_n\mathbf{x}_n = \mathbf{0}$ for some t_i in $\mathbb{R}$.

   ◆(n) Every set of four non-zero vectors in $\mathbb{R}^4$ is a basis.

   (o) No basis of $\mathbb{R}^3$ can contain a vector with a component $\mathbf{0}$.

   ◆(p) $\mathbb{R}^3$ has a basis of the form $\{\mathbf{x}, \mathbf{x} + \mathbf{y}, \mathbf{y}\}$ where $\mathbf{x}$ and $\mathbf{y}$ are vectors.

   (q) Every basis of $\mathbb{R}^5$ contains one column of I_5.

   ◆(r) Every nonempty subset of a basis of $\mathbb{R}^3$ is again a basis of $\mathbb{R}^3$.

   (s) If $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ and $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4\}$ are bases of $\mathbb{R}^4$, then $\{\mathbf{x}_1 + \mathbf{y}_1, \mathbf{x}_2 + \mathbf{y}_2, \mathbf{x}_3 + \mathbf{y}_3, \mathbf{x}_4 + \mathbf{y}_4\}$ is also a basis of $\mathbb{R}^4$.