The concepts of linear span, linear independence, linear dependence, and basis, are defined for $V_n(\mathbf{C})$ exactly as in the real case. Theorems 12.7 through 12.10 and their proofs are also valid without change for $V_n(\mathbf{C})$.

12.17 Exercises

1. Let $A = (1, i)$, $B = (i, -i)$, and $C = (2i, 1)$ be three vectors in $V_2(\mathbf{C})$. Compute each of the following dot products:
(a) $A \cdot B$; (b) $B \cdot A$; (c) $(iA) \cdot B$; (d) $A \cdot (iB)$; (e) $(iA) \cdot (iB)$;
(f) $B \cdot C$; (g) $A \cdot C$; (h) $(B + C) \cdot A$; (i) $(A - C) \cdot B$;
(j) $(A - iB) \cdot (A + iB)$.
2. If $A = (2, 1, -i)$ and $B = (i, -1, 2i)$, find a nonzero vector C in $V_3(\mathbf{C})$ orthogonal to both A and B.
3. Prove that for any two vectors A and B in $V_n(\mathbf{C})$, we have the identity
$$\|A + B\|^2 = \|A\|^2 + \|B\|^2 + A \cdot B + \overline{A \cdot B}.$$
4. Prove that for any two vectors A and B in $V_n(\mathbf{C})$, we have the identity
$$\|A + B\|^2 - \|A - B\|^2 = 2(A \cdot B + \overline{A \cdot B}).$$
5. Prove that for any two vectors A and B in $V_n(\mathbf{C})$, we have the identity
$$\|A + B\|^2 + \|A - B\|^2 = 2\,\|A\|^2 + 2\,\|B\|^2.$$
6. (a) Prove that for any two vectors A and B in $V_n(\mathbf{C})$, the sum $\overline{A \cdot B} + A \cdot B$ is real.
(b) If A and B are nonzero vectors in $V_n(\mathbf{C})$, prove that
$$-2 \le \frac{A \cdot B + \overline{A \cdot B}}{\|A\|\,\|B\|} \le 2.$$
7. We define the angle θ between two nonzero vectors A and B in $V_n(\mathbf{C})$ by the equation
$$\theta = \arccos \frac{\frac{1}{2}(A \cdot B + \overline{A \cdot B})}{\|A\|\,\|B\|}.$$
The inequality in Exercise 6 shows that there is always a unique angle θ in the closed interval $0 \le \theta \le \pi$ satisfying this equation. Prove that we have
$$\|A - B\|^2 = \|A\|^2 + \|B\|^2 - 2\,\|A\|\,\|B\| \cos \theta.$$
8. Use the definition in Exercise 7 to compute the angle between the following two vectors in $V_5(\mathbf{C})$: $A = (1, 0, i, i, i)$, and $B = (i, i, i, 0, i)$.
9. (a) Prove that the following three vectors form a basis for $V_3(\mathbf{C})$: $A = (1, 0, 0)$, $B = (0, i, 0)$, $C = (1, 1, i)$.
(b) Express the vector $(5, 2 - i, 2i)$ as a linear combination of A, B, C.
10. Prove that the basis of unit coordinate vectors $E_1, \ldots, E_n$ in V_n is also a basis for $V_n(\mathbf{C})$.

13

APPLICATIONS OF VECTOR ALGEBRA TO ANALYTIC GEOMETRY

13.1 Introduction

This chapter discusses applications of vector algebra to the study of lines, planes, and conic sections. In Chapter 14 vector algebra is combined with the methods of calculus, and further applications are given to the study of curves and to some problems in mechanics.

The study of geometry as a deductive system, as conceived by Euclid around 300 B.C., begins with a set of axioms or postulates which describe properties of points and lines. The concepts "point" and "line" are taken as primitive notions and remain undefined. Other concepts are defined in terms of points and lines, and theorems are systematically deduced from the axioms. Euclid listed ten axioms from which he attempted to deduce all his theorems. It has since been shown that these axioms are not adequate for the theory. For example, in the proof of his very first theorem Euclid made a tacit assumption concerning the intersection of two circles that is not covered by his axioms. Since then other lists of axioms have been formulated that do give all of Euclid's theorems. The most famous of these is a list given by the German mathematician David Hilbert (1862–1943) in his now classic *Grundlagen der Geometrie*, published in 1899. (An English translation exists: *The Foundations of Geometry*, Open Court Publishing Co., 1947.) This work, which went through seven German editions in Hilbert's lifetime, is said to have inaugurated the abstract mathematics of the twentieth century.

Hilbert starts his treatment of plane geometry with five undefined concepts: *point*, *line*, *on* (a relation holding between a point and a line), *between* (a relation between a point and a pair of points), and *congruence* (a relation between pairs of points). He then gives fifteen axioms from which he develops all of plane Euclidean geometry. His treatment of solid geometry is based on twenty-one axioms involving six undefined concepts.

The approach in analytic geometry is somewhat different. We define concepts such as point, line, on, between, etc., but we do so in terms of real numbers, which are left undefined. The resulting mathematical structure is called an *analytic model* of Euclidean geometry. In this model, properties of real numbers are used to deduce Hilbert's axioms. We shall not attempt to describe all of Hilbert's axioms. Instead, we shall merely indicate how the primitive concepts may be defined in terms of numbers and give a few proofs to illustrate the methods of analytic geometry.

13.2 Lines in n-space

In this section we use real numbers to define the concepts of *point*, *line*, and *on*. The definitions are formulated to fit our intuitive ideas about three-dimensional Euclidean geometry, but they are meaningful in n-space for any $n \geq 1$.

A point is simply a vector in V_n, that is, an ordered n-tuple of real numbers; we shall use the words "point" and "vector" interchangeably. The vector space V_n is called an analytic model of n-dimensional Euclidean space or simply *Euclidean n-space*. To define "line," we employ the algebraic operations of addition and multiplication by scalars in V_n.

DEFINITION. *Let P be a given point and A a given nonzero vector. The set of all points of the form $P + tA$, where t runs through all real numbers, is called a line through P parallel to A. We denote this line by $L(P; A)$ and write*

$$L(P; A) = \{P + tA \mid t \text{ real}\} \quad \textit{or, more briefly,} \quad L(P; A) = \{P + tA\}.$$

A point Q is said to be on the line $L(P; A)$ if $Q \in L(P; A)$.

In the symbol $L(P; A)$, the point P which is written first is on the line since it corresponds to $t = 0$. The second point, A, is called a *direction vector* for the line. The line $L(O; A)$ through the origin O is the linear span of A; it consists of all scalar multiples of A. The line through P parallel to A is obtained by adding P to each vector in the linear span of A.

Figure 13.1 shows the geometric interpretation of this definition in V_3. Each point $P + tA$ can be visualized as the tip of a geometric vector drawn from the origin. As t varies over all the real numbers, the corresponding point $P + tA$ traces out a line through P parallel to the vector A. Figure 13.1 shows points corresponding to a few values of t on both lines $L(P; A)$ and $L(O; A)$.

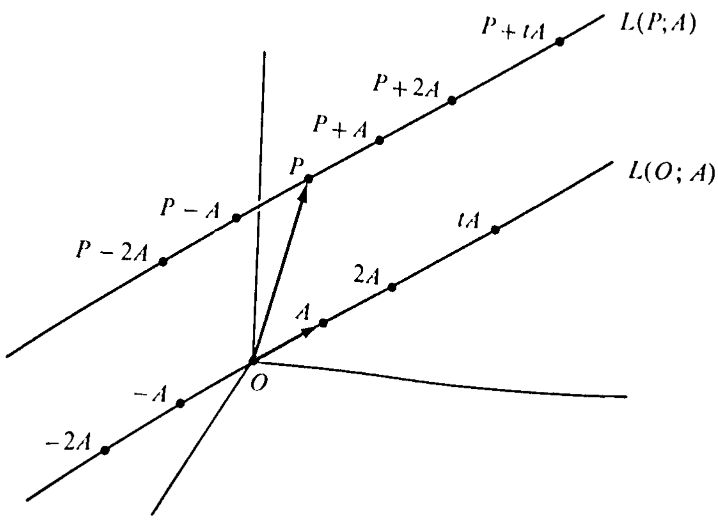


FIGURE 13.1 The line $L(P; A)$ through P parallel to A and its geometric relation to the line $L(O; A)$ through O parallel to A.

13.3 Some simple properties of straight lines

First we show that the direction vector A which occurs in the definition of $L(P; A)$ can be replaced by any vector parallel to A. (We recall that two vectors A and B are called parallel if $A = cB$ for some nonzero scalar c.)

THEOREM 13.1. *Two lines $L(P; A)$ and $L(P; B)$ through the same point P are equal if and only if the direction vectors A and B are parallel.*

Proof. Assume first that $L(P; A) = L(P; B)$. Take a point on $L(P; A)$ other than P, for example, $P + A$. This point is also on $L(P; B)$ so $P + A = P + cB$ for some scalar c. Hence, we have $A = cB$ and $c \neq 0$ since $A \neq O$. Therefore, A and B are parallel.

Now we prove the converse. Assume A and B are parallel, say $A = cB$ for some $c \neq 0$. If Q is on $L(P; A)$, then we have $Q = P + tA = P + t(cB) = P + (ct)B$, so Q is on $L(P; B)$. Therefore $L(P; A) \subseteq L(P; B)$. Similarly, $L(P; B) \subseteq L(P; A)$, so $L(P; A) = L(P; B)$.

Next we show that the point P which occurs in the definition of $L(P; A)$ can be replaced by any other point Q on the same line.

THEOREM 13.2. *Two lines $L(P; A)$ and $L(Q; A)$ with the same direction vector A are equal if and only if Q is on $L(P; A)$.*

Proof. Assume $L(P; A) = L(Q; A)$. Since Q is on $L(Q; A)$, Q is also on $L(P; A)$. To prove the converse, assume that Q is on $L(P; A)$, say $Q = P + cA$. We wish to prove that $L(P; A) = L(Q; A)$. If $X \in L(P; A)$, then $X = P + tA$ for some t. But $P = Q - cA$, so $X = Q - cA + tA = Q + (t - c)A$, and hence X is also on $L(Q; A)$. Therefore $L(P; A) \subseteq L(Q; A)$. Similarly, we find $L(Q; A) \subseteq L(P; A)$, so the two lines are equal.

One of Euclid's famous postulates is the *parallel postulate* which is logically equivalent to the statement that "through a given point there exists one and only one line parallel to a given line." We shall deduce this property as an easy consequence of Theorem 13.1. First we need to define parallelism of lines.

DEFINITION. *Two lines $L(P; A)$ and $L(Q; B)$ are called parallel if their direction vectors A and B are parallel.*

THEOREM 13.3. *Given a line L and a point Q not on L, then there is one and only one line L' containing Q and parallel to L.*

Proof. Suppose the given line has direction vector A. Consider the line $L' = L(Q; A)$. This line contains Q and is parallel to L. Theorem 13.1 tells us that this is the only line with these two properties.

Note: For a long time mathematicians suspected that the parallel postulate could be deduced from the other Euclidean postulates, but all attempts to prove this resulted in failure. Then in the early 19th century the mathematicians Karl F. Gauss (1777–1855),

J. Bolyai (1802–1860), and N. I. Lobatchevski (1793–1856) became convinced that the parallel postulate could not be derived from the others and proceeded to develop non-Euclidean geometries, that is to say, geometries in which the parallel postulate does not hold. The work of these men inspired other mathematicians and scientists to enlarge their points of view about "accepted truths" and to challenge other axioms that had been considered sacred for centuries.

It is also easy to deduce the following property of lines which Euclid stated as an axiom.

THEOREM 13.4. *Two distinct points determine a line. That is, if $P \neq Q$, there is one and only one line containing both P and Q. It can be described as the set $\{P + t(Q - P)\}$.*

Proof. Let L be the line through P parallel to $Q - P$, that is, let

$$L = L(P; Q - P) = \{P + t(Q - P)\}.$$

This line contains both P and Q (take $t = 0$ to get P and $t = 1$ to get Q). Now let L' be any line containing both P and Q. We shall prove that $L' = L$. Since L' contains P, we have $L' = L(P; A)$ for some $A \neq O$. But L' also contains Q so $P + cA = Q$ for some c. Hence we have $Q - P = cA$, where $c \neq 0$ since $Q \neq P$. Therefore $Q - P$ is parallel to A so, by Theorem 13.2, we have $L' = L(P; A) = L(P; Q - P) = L$.

EXAMPLE. Theorem 13.4 gives us an easy way to test if a point Q is on a given line $L(P; A)$. It tells us that Q is on $L(P; A)$ if and only if $Q - P$ is parallel to A. For example, consider the line $L(P; A)$, where $P = (1, 2, 3)$ and $A = (2, -1, 5)$. To test if the point $Q = (1, 1, 4)$ is on this line, we examine $Q - P = (0, -1, 1)$. Since $Q - P$ is not a scalar multiple of A, the point $(1, 1, 4)$ is not on this line. On the other hand, if $Q = (5, 0, 13)$, we find that $Q - P = (4, -2, 10) = 2A$, so this Q is on the line.

Linear dependence of two vectors in V_n can be expressed in geometric language.

THEOREM 13.5. *Two vectors A and B in V_n are linearly dependent if and only if they lie on the same line through the origin.*

Proof. If either A or B is zero, the result holds trivially. If both are nonzero, then A and B are dependent if and only if $B = tA$ for some scalar t. But $B = tA$ if and only if B lies on the line through the origin parallel to A.

13.4 Lines and vector-valued functions

The concept of a line can be related to the function concept. The correspondence which associates to each real t the vector $P + tA$ on the line $L(P; A)$ is an example of a function whose domain is the set of real numbers and whose range is the line $L(P; A)$. If we denote the function by the symbol X, then the function value $X(t)$ at t is given by the equation

$$X(t) = P + tA. \tag{13.1}$$

We call this a vector-valued function of a real variable.

The function point of view is important because, as we shall see in Chapter 14, it provides a natural method for describing more general space curves as well.

The scalar t in Equation (13.1) is often called a *parameter*, and Equation (13.1) is called a *vector parametric equation* or, simply a *vector equation* of the line. Occasionally it is convenient to think of the line as the track of a moving particle, in which case the parameter t is referred to as *time* and the vector $X(t)$ is called the *position vector.*

Note that two points $X(a)$ and $X(b)$ on a given line $L(P; A)$ are equal if and only if we have $P + aA = P + bA$, or $(a - b)A = O$. Since $A \neq O$, this last relation holds if and only if $a = b$. Thus, distinct values of the parameter t lead to distinct points on the line.

Now consider three distinct points on a given line, say $X(a)$, $X(b)$, and $X(c)$, where $a > b$. We say that $X(c)$ is *between* $X(a)$ and $X(b)$ if c is between a and b, that is, if $a < c < b$.

Congruence can be defined in terms of norms. A pair of points P, Q is called *congruent* to another pair P', Q' if $\|P - Q\| = \|P' - Q'\|$. The norm $\|P - Q\|$ is also called the distance between P and Q.

This completes the definitions of the concepts of *point, line, on, between,* and *congruence* in our analytic model of Euclidean n-space. We conclude this section with some further remarks concerning parametric equations for lines in 3-space.

If a line passes through two distinct points P and Q, we can use $Q - P$ for the direction vector A in Equation (13.1); the vector equation of the line then becomes

$$X(t) = P + t(Q - P) \qquad \text{or} \qquad X(t) = tQ + (1 - t)P.$$

Vector equations can also be expressed in terms of components. For example, if we write $P = (p, q, r)$, $A = (a, b, c)$, and $X(t) = (x, y, z)$, Equation (13.1) is equivalent to the three scalar equations

$$x = p + ta, \qquad y = q + tb, \qquad z = r + tc. \tag{13.2}$$

These are called *scalar parametric equations* or simply *parametric equations* for the line; they are useful in computations involving components. The vector equation is simpler and more natural for studying general properties of lines.

If all the vectors are in 2-space, only the first two parametric equations in (13.2) are needed. In this case, we can eliminate t from the two parametric equations to obtain the relation

$$b(x - p) - a(y - q) = 0, \tag{13.3}$$

which is called a *Cartesian equation* for the line. If $a \neq 0$, this can be written in the *point-slope form*

$$y - q = \frac{b}{a}(x - p).$$

The point (p, q) is on the line; the number b/a is the slope of the line.

The Cartesian equation (13.3) can also be written in terms of dot products. If we let $N = (b, -a)$, $X = (x, y)$, and $P = (p, q)$, Equation (13.3) becomes

$$(X - P) \cdot N = 0 \qquad \text{or} \qquad X \cdot N = P \cdot N.$$

The vector N is perpendicular to the direction vector A since $N \cdot A = ba - ab = 0$; the vector N is called a *normal vector* to the line. The line consists of all points X satisfying the relation $(X - P) \cdot N = 0$.

The geometric meaning of this relation is shown in Figure 13.2. The points P and X are on the line and the normal vector N is orthogonal to $X - P$. The figure suggests that among all points X on the line, the smallest length $\|X\|$ occurs when X is the projection of P along N. We now give an algebraic proof of this fact.

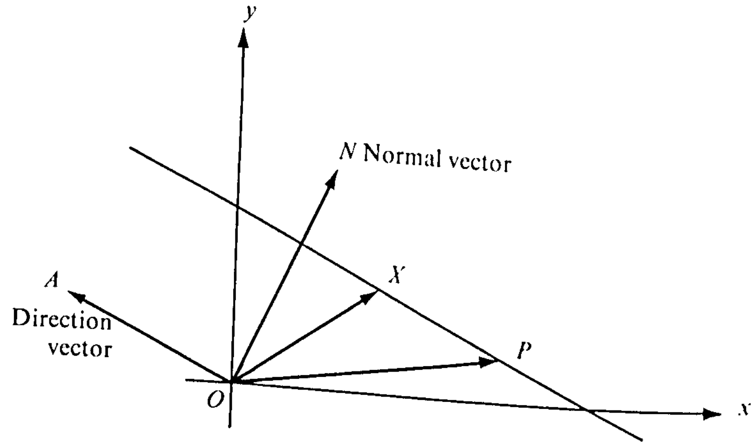


FIGURE 13.2 A line in the xy-plane through P with normal vector N. Each point X on the line satisfies $(X - P) \cdot N = 0$.

THEOREM 13.6. *Let L be the line in V_2 consisting of all points X satisfying*

$$X \cdot N = P \cdot N,$$

where P is on the line and N is a nonzero vector normal to the line. Let

$$d = \frac{|P \cdot N|}{\|N\|}.$$

Then every X on L has length $\|X\| \geq d$. Moreover, $\|X\| = d$ if and only if X is the projection of P along N:

$$X = tN, \quad \text{where} \quad t = \frac{P \cdot N}{N \cdot N}.$$

Proof. If $X \in L$, we have $X \cdot N = P \cdot N$. By the Cauchy-Schwarz inequality, we have

$$|P \cdot N| = |X \cdot N| \leq \|X\| \, \|N\|,$$

which implies $\|X\| \geq |P \cdot N|/\|N\| = d$. The equality sign holds if and only if $X = tN$ for some scalar t, in which case $P \cdot N = X \cdot N = tN \cdot N$, so $t = P \cdot N/N \cdot N$. This completes the proof.

In the same way we can prove that if Q is a given point in V_2 not on the line L, then for all X on L the smallest value of $\|X - Q\|$ is $|(P - Q) \cdot N| / \|N\|$, and this occurs when $X - Q$ is the projection of $P - Q$ along the normal vector N. The number

$$\frac{|(P - Q) \cdot N|}{\|N\|}$$

is called the *distance from the point Q to the line L.* The reader should illustrate these concepts on a figure similar to that in Figure 13.2.

13.5 Exercises

1. A line L in V_2 contains the two points $P = (-3, 1)$ and $Q = (1, 1)$. Determine which of the following points are on L. (a) $(0, 0)$; (b) $(0, 1)$; (c) $(1, 2)$; (d) $(2, 1)$; (e) $(-2, 1)$.
2. Solve Exercise 1 if $P = (2, -1)$ and $Q = (-4, 2)$.
3. A line L in V_3 contains the point $P = (-3, 1, 1)$ and is parallel to the vector $(1, -2, 3)$. Determine which of the following points are on L. (a) $(0, 0, 0)$; (b) $(2, -1, 4)$; (c) $(-2, -1, 4)$; (d) $(-4, 3, -2)$; (e) $(2, -9, 16)$.
4. A line L contains the two points $P = (-3, 1, 1)$ and $Q = (1, 2, 7)$. Determine which of the following points are on L. (a) $(-7, 0, 5)$; (b) $(-7, 0, -5)$; (c) $(-11, 1, 11)$; (d) $(-11, -1, 11)$; (e) $(-1, \frac{3}{2}, 4)$; (f) $(-\frac{5}{3}, \frac{4}{3}, 3)$; (g) $(-1, \frac{3}{2}, -4)$.
5. In each case, determine if all three points P, Q, R lie on a line.
   (a) $P = (2, 1, 1)$, $Q = (4, 1, -1)$, $R = (3, -1, 1)$.
   (b) $P = (2, 2, 3)$, $Q = (-2, 3, 1)$, $R = (-6, 4, 1)$.
   (c) $P = (2, 1, 1)$, $Q = (-2, 3, 1)$, $R = (5, -1, 1)$.
6. Among the following eight points, the three points A, B, and C lie on a line. Determine all subsets of three or more points which lie on a line: $A = (2, 1, 1)$, $B = (6, -1, 1)$, $C = (-6, 5, 1)$, $D = (-2, 3, 1)$, $E = (1, 1, 1)$, $F = (-4, 4, 1)$, $G = (-13, 9, 1)$, $H = (14, -6, 1)$.
7. A line through the point $P = (1, 1, 1)$ is parallel to the vector $A = (1, 2, 3)$. Another line through $Q = (2, 1, 0)$ is parallel to the vector $B = (3, 8, 13)$. Prove that the two lines intersect and determine the point of intersection.
8. (a) Prove that two lines $L(P; A)$ and $L(Q; B)$ in V_n intersect if and only if $P - Q$ is in the linear span of A and B.
   (b) Determine whether or not the following two lines in V_3 intersect:
$$L = \{(1, 1, -1) + t(-2, 1, 3)\}, \qquad L' = \{(3, -4, 1) + t(-1, 5, 2)\}.$$
9. Let $X(t) = P + tA$ be an arbitrary point on the line $L(P; A)$, where $P = (1, 2, 3)$ and $A = (1, -2, 2)$, and let $Q = (3, 3, 1)$.
   (a) Compute $\|Q - X(t)\|^2$, the square of the distance between Q and $X(t)$.
   (b) Prove that there is exactly one point $X(t_0)$ for which the distance $\|Q - X(t)\|$ is a minimum, and compute this minimum distance.
   (c) Prove that $Q - X(t_0)$ is orthogonal to A.
10. Let Q be a point not on the line $L(P; A)$ in V_n.
   (a) Let $f(t) = \|Q - X(t)\|^2$, where $X(t) = P + tA$. Prove that $f(t)$ is a quadratic polynomial in t and that this polynomial takes on its minimum value at exactly one t, say at $t = t_0$.
   (b) Prove that $Q - X(t_0)$ is orthogonal to A.
11. Given two parallel lines $L(P; A)$ and $L(Q; A)$ in V_n. Prove that either $L(P; A) = L(Q; A)$ or the intersection $L(P; A) \cap L(Q; A)$ is empty.
12. Given two lines $L(P; A)$ and $L(Q; B)$ in V_n which are not parallel. Prove that the intersection is either empty or consists of exactly one point.

13.6 Planes in Euclidean n-space

A line in n-space was defined to be a set of the form $\{P + tA\}$ obtained by adding to a given point P all vectors in the linear span of a nonzero vector A. A plane is defined in a similar fashion except that we add to P all vectors in the linear span of two linearly independent vectors A and B. To make certain that V_n contains two linearly independent vectors, we assume at the outset that $n \geq 2$. Most of our applications will be concerned with the case $n = 3$.

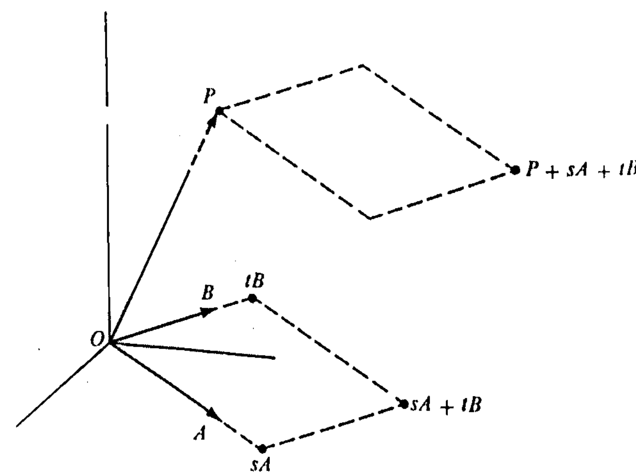


FIGURE 13.3 The plane through P spanned by A and B, and its geometric relation to the plane through O spanned by A and B.

DEFINITION. *A set M of points in V_n is called a plane if there is a point P and two linearly independent vectors A and B such that*

$$M = \{P + sA + tB \mid s, t \text{ real}\}.$$

We shall denote the set more briefly by writing $M = \{P + sA + tB\}$. Each point of M is said to be *on* the plane. In particular, taking $s = t = 0$, we see that P is on the plane. The set $\{P + sA + tB\}$ is also called the plane through P spanned by A and B. When P is the origin, the plane is simply the linear span of A and B. Figure 13.3 shows a plane in V_3 through the origin spanned by A and B and also a plane through a nonzero point P spanned by the same two vectors.

Now we shall deduce some properties of planes analogous to the properties of lines given in Theorems 13.1 through 13.4. The first of these shows that the vectors A and B in the definition of the plane $\{P + sA + tB\}$ can be replaced by any other pair which has the same linear span.

THEOREM 13.7. *Two planes $M = \{P + sA + tB\}$ and $M' = \{P + sC + tD\}$ through the same point P are equal if and only if the linear span of A and B is equal to the linear span of C and D.*

Proof. If the linear span of A and B is equal to that of C and D, then it is clear that $M = M'$. Conversely, assume that $M = M'$. Plane M contains both $P + A$ and $P + B$. Since both these points are also on M', each of A and B must be in the linear span of C and D. Similarly, each of C and D is in the linear span of A and B. Therefore the linear span of A and B is equal to that of C and D.

The next theorem shows that the point P which occurs in the definition of the plane $\{P + sA + tB\}$ can be replaced by any other point Q on the same plane.

THEOREM 13.8. *Two planes $M = \{P + sA + tB\}$ and $M' = \{Q + sA + tB\}$ spanned by the same vectors A and B are equal if and only if Q is on M.*

Proof. If $M = M'$, then Q is certainly on M. To prove the converse, assume Q is on M, say $Q = P + aA + bB$. Take any point X in M. Then $X = P + sA + tB$ for some scalars s and t. But $P = Q - aA - bB$, so $X = Q + (s - a)A + (t - b)B$. Therefore X is in M', so $M \subseteq M'$. Similarly, we find that $M' \subseteq M$, so the two planes are equal.

Euclid's parallel postulate (Theorem 13.3) has an analog for planes. Before we state this theorem we need to define parallelism of two planes. The definition is suggested by the geometric representation in Figure 13.3.

DEFINITION. *Two planes $M = \{P + sA + tB\}$ and $M' = \{Q + sC + tD\}$ are said to be parallel if the linear span of A and B is equal to the linear span of C and D. We also say that a vector X is parallel to the plane M if X is in the linear span of A and B.*

THEOREM 13.9. *Given a plane M and a point Q not on M, there is one and only one plane M' which contains Q and is parallel to M.*

Proof. Let $M = \{P + sA + tB\}$ and consider the plane $M' = \{Q + sA + tB\}$. This plane contains Q and is spanned by the same vectors A and B which span M. Therefore M' is parallel to M. If M'' is another plane through Q parallel to M, then

$$M'' = \{Q + sC + tD\}$$

where the linear span of C and D is equal to that of A and B. By Theorem 13.7, we must have $M'' = M'$. Therefore M' is the only plane through Q which is parallel to M.

Theorem 13.4 tells us that two distinct points determine a line. The next theorem shows that three distinct points determine a plane, provided that the three points are not collinear.

THEOREM 13.10. *If P, Q, and R are three points not on the same line, then there is one and only one plane M containing these three points. It can be described as the set*

$$M = \{P + s(Q - P) + t(R - P)\}. \tag{13.4}$$

Proof. We assume first that one of the points, say P, is the origin. Then Q and R are not on the same line through the origin so they are linearly independent. Therefore, they span a plane through the origin, say the plane

$$M' = \{sQ + tR\}.$$

This plane contains all three points O, Q, and R.

Now we prove that M' is the only plane which contains all three points O, Q, and R. Any other plane through the origin has the form

$$M'' = \{sA + tB\},$$

where A and B are linearly independent. If M'' contains Q and R, we have

$$Q = aA + bB, \qquad R = cA + dB, \tag{13.5}$$

for some scalars a, b, c, d. Hence, every linear combination of Q and R is also a linear combination of A and B, so $M' \subseteq M''$.

To prove that $M'' \subseteq M'$, it suffices to prove that each of A and B is a linear combination of Q and R. Multiplying the first equation in (13.5) by d and the second by b and subtracting, we eliminate B and get

$$(ad - bc)A = dQ - bR.$$

Now $ad - bc$ cannot be zero, otherwise Q and R would be dependent. Therefore we can divide by $ad - bc$ and express A as a linear combination of Q and R. Similarly, we can express B as a linear combination of Q and R, so we have $M'' \subseteq M'$. This proves the theorem when one of the three points P, Q, R is the origin.

To prove the theorem in the general case, let M be the set in (13.4), and let $C = Q - P$, $D = R - P$. First we show that C and D are linearly independent. If not we would have $D = tC$ for some scalar t, giving us $R - P = t(Q - P)$, or $R = P + t(Q - P)$, contradicting the fact that P, Q, R are not on the same line. Therefore the set M is a plane through P spanned by the linearly independent pair C and D. This plane contains all three points P, Q, and R (take $s = 1$, $t = 0$ to get Q, and $s = 0$, $t = 1$ to get R). Now we must prove that this is the only plane containing P, Q, and R.

Let M' be any plane containing P, Q, and R. Since M' is a plane containing P, we have

$$M' = \{P + sA + tB\}$$

for some linearly independent pair A and B. Let $M_0' = \{sA + tB\}$ be the plane through the origin spanned by the same pair A and B. Clearly, M' contains a vector X if and only if M_0' contains $X - P$. Since M' contains Q and R, the plane M_0' contains $C = Q - P$ and $D = R - P$. But we have just shown that there is one and only one plane containing O, C, and D since C and D are linearly independent. Therefore $M_0' = \{sC + tD\}$, so $M' = \{P + sC + tD\} = M$. This completes the proof.

In Theorem 13.5 we proved that two vectors in V_n are linearly dependent if and only if they lie on a line through the origin. The next theorem is the corresponding result for three vectors.

THEOREM 13.11. *Three vectors A, B, C in V_n are linearly dependent if and only if they lie on the same plane through the origin.*

Proof. Assume A, B, C are dependent. Then we can express one of the vectors as a linear combination of the other two, say $C = sA + tB$. If A and B are independent, they span a plane through the origin and C is on this plane. If A and B are dependent, then A, B, and C lie on a line through the origin, and hence they lie on any plane through the origin which contains all three points A, B, and C.

To prove the converse, assume that A, B, C lie on the same plane through the origin, say the plane M. If A and B are dependent, then A, B, and C are dependent, and there is nothing more to prove. If A and B are independent, they span a plane M' through the origin. By Theorem 13.10, there is one and only one plane through O containing A and B. Therefore $M' = M$. Since C is on this plane, we must have $C = sA + tB$, so A, B, and C are dependent.

13.7 Planes and vector-valued functions

The correspondence which associates to each pair of real numbers s and t the vector $P + sA + tB$ on the plane $M = \{P + sA + tB\}$ is another example of a vector-valued function. In this case, the domain of the function is the set of all pairs of real numbers (s, t) and its range is the plane M. If we denote the function by X and the function values by $X(s, t)$, then for each pair (s, t) we have

$$X(s, t) = P + sA + tB. \tag{13.6}$$

We call X a vector-valued function of two real variables. The scalars s and t are called parameters, and the equation (13.6) is called a parametric or vector equation of the plane. This is analogous to the representation of a line by a vector-valued function of one real variable. The presence of two parameters in Equation (13.6) gives the plane a two-dimensional quality. When each vector is in V_3 and is expressed in terms of its components, say

$$P = (p_1, p_2, p_3), \qquad A = (a_1, a_2, a_3), \qquad B = (b_1, b_2, b_3), \qquad \text{and} \qquad X(s, t) = (x, y, z),$$

the vector equation (13.6) can be replaced by three scalar equations,

$$x = p_1 + sa_1 + tb_1, \qquad y = p_2 + sa_2 + tb_2, \qquad z = p_3 + sa_3 + tb_3.$$

The parameters s and t can always be eliminated from these three equations to give one linear equation of the form $ax + by + cz = d$, called a Cartesian equation of the plane. We illustrate with an example.

EXAMPLE. Let $M = \{P + sA + tB\}$, where $P = (1, 2, 3)$, $A = (1, 2, 1)$, and $B = (1, -4, -1)$. The corresponding vector equation is

$$X(s, t) = (1, 2, 3) + s(1, 2, 1) + t(1, -4, -1).$$

From this we obtain the three scalar parametric equations

$$x = 1 + s + t, \qquad y = 2 + 2s - 4t, \qquad z = 3 + s - t.$$

To obtain a Cartesian equation, we rewrite the first and third equations in the form $x - 1 = s + t$, $z - 3 = s - t$. Adding and then subtracting these equations, we find that $2s = x + z - 4$, $2t = x - z + 2$. Substituting in the equation for y, we are led to the Cartesian equation $x + y - 3z = -6$. We shall return to a further study of linear Cartesian equations in Section 13.16.

13.8 Exercises

1. Let $M = \{P + sA + tB\}$, where $P = (1, 2, -3)$, $A = (3, 2, 1)$, and $B = (1, 0, 4)$. Determine which of the following points are on M.
(a) $(1, 2, 0)$; (b) $(1, 2, 1)$; (c) $(6, 4, 6)$; (d) $(6, 6, 6)$; (e) $(6, 6, -5)$.
2. The three points $P = (1, 1, -1)$, $Q = (3, 3, 2)$, and $R = (3, -1, -2)$ determine a plane M. Determine which of the following points are on M.
(a) $(2, 2, \frac{1}{2})$; (b) $(4, 0, -\frac{1}{2})$; (c) $(-3, 1, -3)$; (d) $(3, 1, 3)$; (e) $(0, 0, 0)$.
3. Determine scalar parametric equations for each of the following planes.
(a) The plane through $(1, 2, 1)$ spanned by the vectors $(0, 1, 0)$ and $(1, 1, 4)$.
(b) The plane through $(1, 2, 1)$, $(0, 1, 0)$, and $(1, 1, 4)$.
4. A plane M has scalar parametric equations

$$x = 1 + s - 2t, \qquad y = 2 + s + 4t, \qquad z = 2s + t.$$

(a) Determine which of the following points are on M: $(0, 0, 0)$, $(1, 2, 0)$, $(2, -3, -3)$.
(b) Find vectors P, A, and B such that $M = \{P + sA + tB\}$.
5. Let M be the plane determined by three points P, Q, R not on the same line.
(a) If p, q, r are three scalars such that $p + q + r = 1$, prove that $pP + qQ + rR$ is on M.
(b) Prove that every point on M has the form $pP + qQ + rR$, where $p + q + r = 1$.
6. Determine a linear Cartesian equation of the form $ax + by + cz = d$ for each of the following planes.
(a) The plane through $(2, 3, 1)$ spanned by $(3, 2, 1)$ and $(-1, -2, -3)$.
(b) The plane through $(2, 3, 1)$, $(-2, -1, -3)$, and $(4, 3, -1)$.
(c) The plane through $(2, 3, 1)$ parallel to the plane through the origin spanned by $(2, 0, -2)$ and $(1, 1, 1)$.
7. A plane M has the Cartesian equation $3x - 5y + z = 9$.
(a) Determine which of the following points are on M: $(0, -2, -1)$, $(-1, -2, 2)$, $(3, 1, -5)$.
(b) Find vectors P, A, and B such that $M = \{P + sA + tB\}$.
8. Consider the two planes $M = \{P + sA + tB\}$ and $M' = \{Q + sC + tD\}$, where $P = (1, 1, 1)$, $A = (2, -1, 3)$, $B = (-1, 0, 2)$, $Q = (2, 3, 1)$, $C = (1, 2, 3)$, and $D = (3, 2, 1)$. Find two distinct points on the intersection $M \cap M'$.
9. Given a plane $M = \{P + sA + tB\}$, where $P = (2, 3, 1)$, $A = (1, 2, 3)$, and $B = (3, 2, 1)$, and another plane M' with Cartesian equation $x - 2y + z = 0$.
(a) Determine whether M and M' are parallel.

(b) Find two points on the intersection $M' \cap M''$ if M'' has the Cartesian equation

$$x + 2y + z = 0.$$

10. Let L be the line through $(1, 1, 1)$ parallel to the vector $(2, -1, 3)$, and let M be the plane through $(1, 1, -2)$ spanned by the vectors $(2, 1, 3)$ and $(0, 1, 1)$. Prove that there is one and only one point on the intersection $L \cap M$ and determine this point.
11. A line with direction vector X is said to be *parallel* to a plane M if X is parallel to M. Let L be the line through $(1, 1, 1)$ parallel to the vector $(2, -1, 3)$. Determine whether L is parallel to each of the following planes.
    (a) The plane through $(1, 1, -2)$ spanned by $(2, 1, 3)$ and $(\frac{3}{4}, 1, 1)$.
    (b) The plane through $(1, 1, -2)$, $(3, 5, 2)$, and $(2, 4, -1)$.
    (c) The plane with Cartesian equation $x + 2y + 3z = -3$.
12. Two distinct points P and Q lie on a plane M. Prove that every point on the line through P and Q also lies on M.
13. Given the line L through $(1, 2, 3)$ parallel to the vector $(1, 1, 1)$, and given a point $(2, 3, 5)$ which is not on L. Find a Cartesian equation for the plane M through $(2, 3, 5)$ which contains every point on L.
14. Given a line L and a point P not on L. Prove that there is one and only one plane through P which contains every point on L.

13.9 The cross product

In many applications of vector algebra to problems in geometry and mechanics it is helpful to have an easy method for constructing a vector perpendicular to each of two given vectors A and B. This is accomplished by means of the cross product $A \times B$ (read "A cross B") which is defined as follows:

DEFINITION. *Let $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ be two vectors in V_3. Their cross product $A \times B$ (in that order) is defined to be the vector*

$$A \times B = (a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1).$$

The following properties are easily deduced from this definition.

THEOREM 13.12. *For all vectors A, B, C in V_3 and for all real c we have:*

(a) $A \times B = -(B \times A)$ *(skew symmetry)*,
(b) $A \times (B + C) = (A \times B) + (A \times C)$ *(distributive law)*,
(c) $c(A \times B) = (cA) \times B$,
(d) $A \cdot (A \times B) = 0$ *(orthogonality to A)*,
(e) $B \cdot (A \times B) = 0$ *(orthogonality to B)*,
(f) $\|A \times B\|^2 = \|A\|^2\|B\|^2 - (A \cdot B)^2$ *(Lagrange's identity)*,
(g) $A \times B = O$ *if and only if A and B are linearly dependent.*

Proof. Parts (a), (b), and (c) follow quickly from the definition and are left as exercises for the reader. To prove (d), we note that

$$A \cdot (A \times B) = a_1(a_2b_3 - a_3b_2) + a_2(a_3b_1 - a_1b_3) + a_3(a_1b_2 - a_2b_1) = 0.$$

Part (e) follows in the same way, or it can be deduced from (a) and (d). To prove (f), we write

$$\|A \times B\|^2 = (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2$$

and

$$\|A\|^2\|B\|^2 - (A \cdot B)^2 = (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$$

and then verify by brute force that the two right-hand members are identical.

Property (f) shows that $A \times B = O$ if and only if $(A \cdot B)^2 = \|A\|^2\|B\|^2$. By the Cauchy-Schwarz inequality (Theorem 12.3), this happens if and only if one of the vectors is a scalar multiple of the other. In other words, $A \times B = O$ if and only if A and B are linearly dependent, which proves (g).

EXAMPLES. Both (a) and (g) show that $A \times A = O$. From the definition of cross product we find that

$$i \times j = k, \qquad j \times k = i, \qquad k \times i = j.$$

The cross product is *not* associative. For example, we have

$$i \times (i \times j) = i \times k = -j \qquad \text{but} \qquad (i \times i) \times j = O \times j = O.$$

The next theorem describes two more fundamental properties of the cross product.

THEOREM 13.13. *Let A and B be linearly independent vectors in V_3. Then we have the following:*

(a) *The vectors A, B, $A \times B$ are linearly independent.*

(b) *Every vector N in V_3 orthogonal to both A and B is a scalar multiple of $A \times B$.*

Proof. Let $C = A \times B$. Then $C \neq O$ since A and B are linearly independent. Given scalars a, b, c such that $aA + bB + cC = O$, we take the dot product of each member with C and use the relations $A \cdot C = B \cdot C = 0$ to find $c = 0$. This gives $aA + bB = O$, so $a = b = 0$ since A and B are independent. This proves (a).

Let N be any vector orthogonal to both A and B, and let $C = A \times B$. We shall prove that

$$(N \cdot C)^2 = (N \cdot N)(C \cdot C).$$

Then from the Cauchy-Schwarz inequality (Theorem 12.3) it follows that N is a scalar multiple of C.

Since A, B, and C are linearly independent, we know, by Theorem 12.10(c), that they span V_3. In particular, they span N, so we can write

$$N = aA + bB + cC$$

for some scalars a, b, c. This gives us

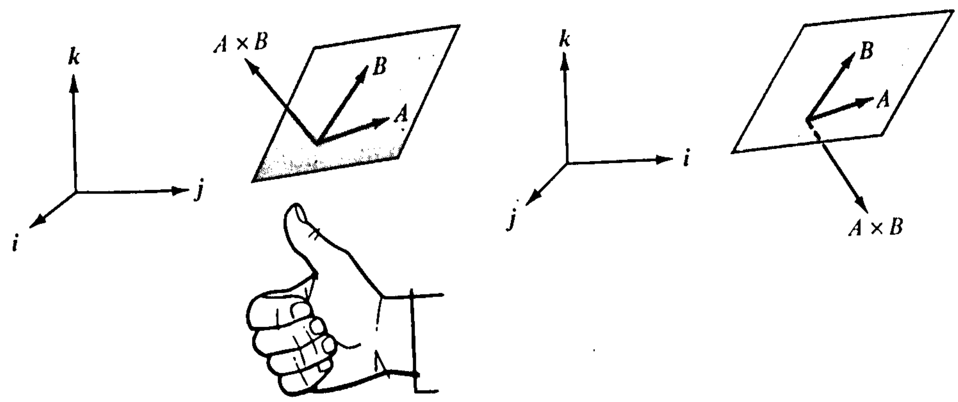
$$N \cdot N = N \cdot (aA + bB + cC) = cN \cdot C$$

since $N \cdot A = N \cdot B = 0$. Also, since $C \cdot A = C \cdot B = 0$, we have

$$C \cdot N = C \cdot (aA + bB + cC) = cC \cdot C.$$

Therefore, $(N \cdot N)(C \cdot C) = (cN \cdot C)(C \cdot C) = (N \cdot C)(cC \cdot C) = (N \cdot C)^2$, which completes the proof.

Theorem 13.12 helps us visualize the cross product geometrically. From properties (d) and (e), we know that $A \times B$ is perpendicular to both A and B. When the vector $A \times B$ is represented geometrically by an arrow, the direction of the arrow depends on the relative



(a) A right-handed coordinate system

(b) A left-handed coordinate system

FIGURE 13.4 Illustrating the relative positions of A, B, and $A \times B$.

positions of the three unit coordinate vectors. If $i, j,$ and k are arranged as shown in Figure 13.4(a), they are said to form a *right-handed coordinate system*. In this case, the direction of $A \times B$ is determined by the "right-hand rule." That is to say, when A is rotated into B in such a way that the fingers of the right hand point in the direction of rotation, then the thumb indicates the direction of $A \times B$ (assuming, for the sake of the discussion, that the thumb is perpendicular to the other fingers). In a left-handed coordinate system, as shown in Figure 13.4(b), the direction of $A \times B$ is reversed and may be determined by a corresponding left-hand rule.

The length of $A \times B$ has an interesting geometric interpretation. If A and B are nonzero vectors making an angle θ with each other, where $0 \le \theta \le \pi$, we may write $A \cdot B = \|A\| \, \|B\| \cos \theta$ in property (f) of Theorem 13.12 to obtain

$$\|A \times B\|^2 = \|A\|^2\|B\|^2(1 - \cos^2 \theta) = \|A\|^2\|B\|^2 \sin^2 \theta ,$$

from which we find

$$\|A \times B\| = \|A\| \, \|B\| \sin \theta .$$

Since $\|B\| \sin \theta$ is the altitude of the parallelogram determined by A and B (see Figure 13.5), we see that *the length of $A \times B$ is equal to the area of this parallelogram.*

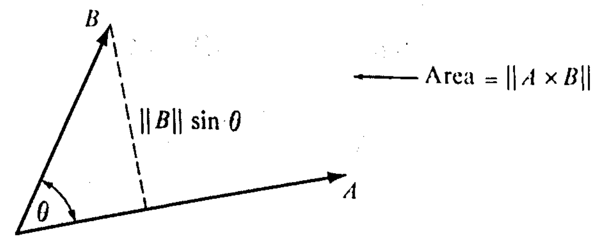


FIGURE 13.5 The length of $A \times B$ is the area of the parallelogram determined by A and B.

13.10 The cross product expressed as a determinant

The formula which defines the cross product can be put in a more compact form with the aid of determinants. If a, b, c, d are four numbers, the difference $ad - bc$ is often denoted by the symbol

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

and is called a *determinant* (of order two). The numbers a, b, c, d are called its *elements*, and they are said to be arranged in two horizontal *rows*, a, b and c, d, and in two vertical *columns*, a, c and b, d. Note that an interchange of two rows or of two columns only changes the sign of the determinant. For example, since $ad - bc = -(bc - ad)$, we have

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = -\begin{vmatrix} b & a \\ d & c \end{vmatrix}.$$

If we express each of the components of the cross product as a determinant of order two, the formula defining $A \times B$ becomes

$$A \times B = \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\right).$$

This can also be expressed in terms of the unit coordinate vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ as follows:

$$(13.7) \qquad A \times B = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \boldsymbol{i} + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \boldsymbol{k}.$$

Determinants of order three are written with three rows and three columns and they may be defined in terms of second-order determinants by the formula

$$(13.8) \qquad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

This is said to be an "expansion" of the determinant along its first row. Note that the determinant on the right that multiplies a_1 may be obtained from that on the left by deleting the row and column in which a_1 appears. The other two determinants on the right are obtained similarly.

Determinants of order greater than three are discussed in Volume II. Our only purpose in introducing determinants of order two and three at this stage is to have a useful device for writing certain formulas in a compact form that makes them easier to remember.

Determinants are meaningful if the elements in the first row are vectors. For example, if we write the determinant

$$\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

and "expand" this according to the rule prescribed in (13.8), we find that the result is equal to the right member of (13.7). In other words, we may write the definition of the cross product $A \times B$ in the following compact form:

$$A \times B = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

For example, to compute the cross product of $A = 2i - 8j + 3k$ and $B = 4j + 3k$, we write

$$A \times B = \begin{vmatrix} i & j & k \\ 2 & -8 & 3 \\ 0 & 4 & 3 \end{vmatrix} = \begin{vmatrix} -8 & 3 \\ 4 & 3 \end{vmatrix} i - \begin{vmatrix} 2 & 3 \\ 0 & 3 \end{vmatrix} j + \begin{vmatrix} 2 & -8 \\ 0 & 4 \end{vmatrix} k = -36i - 6j + 8k$$

13.11 Exercises

1. Let $A = -i + 2k$, $B = 2i + j - k$, $C = i + 2j + 2k$. Compute each of the following vectors in terms of i, j, k:
   (a) $A \times B$; (d) $A \times (C \times A)$; (g) $(A \times C) \times B$;
   (b) $B \times C$; (e) $(A \times B) \times C$; (h) $(A + B) \times (A - C)$;
   (c) $C \times A$; (f) $A \times (B \times C)$; (i) $(A \times B) \times (A \times C)$.
2. In each case find a vector of length 1 in V_3 orthogonal to both A and B:
   (a) $A = i + j + k$, $B = 2i + 3j - k$;
   (b) $A = 2i - 3j + 4k$, $B = -i + 5j + 7k$;
   (c) $A = i - 2j + 3k$, $B = -3i + 2j - k$.
3. In each case use the cross product to compute the area of the triangle with vertices A, B, C:
   (a) $A = (0, 2, 2)$, $B = (2, 0, -1)$, $C = (3, 4, 0)$;
   (b) $A = (-2, 3, 1)$, $B = (1, -3, 4)$, $C = (1, 2, 1)$;
   (c) $A = (0, 0, 0)$, $B = (0, 1, 1)$, $C = (1, 0, 1)$.
4. If $A = 2i + 5j + 3k$, $B = 2i + 7j + 4k$, and $C = 3i + 3j + 6k$, express the cross product $(A - C) \times (B - A)$ in terms of i, j, k.
5. Prove that $\|A \times B\| = \|A\| \, \|B\|$ if and only if A and B are orthogonal.
6. Given two linearly independent vectors A and B in V_3. Let $C = (B \times A) - B$.
   (a) Prove that A is orthogonal to $B + C$.

(b) Prove that the angle θ between B and C satisfies $\frac{1}{2}\pi < \theta < \pi$.
(c) If $\|B\| = 1$ and $\|B \times A\| = 2$, compute the length of C.

7. Let A and B be two orthogonal vectors in V_3, each having length 1.
(a) Prove that $A, B, A \times B$ is an orthonormal basis for V_3.
(b) Let $C = (A \times B) \times A$. Prove that $\|C\| = 1$.
(c) Draw a figure showing the geometric relation between A, B, and $A \times B$, and use this figure to obtain the relations

$$(A \times B) \times A = B, \qquad (A \times B) \times B = -A.$$

(d) Prove the relations in part (c) algebraically.

8. (a) If $A \times B = O$ and $A \cdot B = 0$, then at least one of A or B is zero. Prove this statement and give its geometric interpretation.
(b) Given $A \neq O$. If $A \times B = A \times C$ and $A \cdot B = A \cdot C$, prove that $B = C$.

9. Let $A = 2i - j + 2k$ and $C = 3i + 4j - k$.
(a) Find a vector B such that $A \times B = C$. Is there more than one solution?
(b) Find a vector B such that $A \times B = C$ and $A \cdot B = 1$. Is there more than one solution?

10. Given a nonzero vector A and a vector C orthogonal to A, both vectors in V_3. Prove that there is exactly one vector B such that $A \times B = C$ and $A \cdot B = 1$.

11. Three vertices of a parallelogram are at the points $A = (1, 0, 1)$, $B = (-1, 1, 1)$, $C = (2, -1, 2)$.
(a) Find all possible points D which can be the fourth vertex of the parallelogram.
(b) Compute the area of triangle ABC.

12. Given two nonparallel vectors A and B in V_3 with $A \cdot B = 2$, $\|A\| = 1$, $\|B\| = 4$. Let $C = 2(A \times B) - 3B$. Compute $A \cdot (B + C)$, $\|C\|$, and the cosine of the angle θ between B and C.

13. Given two linearly independent vectors A and B in V_3. Determine whether each of the following statements is true or false.
(a) $A + B, A - B, A \times B$ are linearly independent.
(b) $A + B, A + (A \times B), B + (A \times B)$ are linearly independent.
(c) $A, B, (A + B) \times (A - B)$ are linearly independent.

14. (a) Prove that three vectors A, B, C in V_3 lie on a line if and only if $(B - A) \times (C - A) = O$.
(b) If $A \neq B$, prove that the line through A and B consists of the set of all vectors P such that $(P - A) \times (P - B) = O$.

15. Given two orthogonal vectors A, B in V_3, each of length 1. Let P be a vector satisfying the equation $P \times B = A - P$. Prove each of the following statements.
(a) P is orthogonal to B and has length $\frac{1}{2}\sqrt{2}$.
(b) $P, B, P \times B$ form a basis for V_3.
(c) $(P \times B) \times B = -P$.
(d) $P = \frac{1}{2}A - \frac{1}{2}(A \times B)$.

13.12 The scalar triple product

The dot and cross products can be combined to form the *scalar triple product* $A \cdot B \times C$, which can only mean $A \cdot (B \times C)$. Since this is a dot product of two vectors, its value is a scalar. We can compute this scalar by means of determinants. Write $A = (a_1, a_2, a_3)$, $B = (b_1, b_2, b_3)$, $C = (c_1, c_2, c_3)$ and express $B \times C$ according to Equation (13.7). Forming the dot product with A, we obtain

$$A \cdot B \times C = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Thus, $A \cdot B \times C$ is equal to the determinant whose rows are the components of the factors A, B, and C.

In Theorem 13.12 we found that two vectors A and B are linearly dependent if and only if their cross product $A \times B$ is the zero vector. The next theorem gives a corresponding criterion for linear dependence of three vectors.

THEOREM 13.14. *Three vectors A, B, C in V_3 are linearly dependent if and only if*

$$A \cdot B \times C = 0.$$

Proof. Assume first that A, B, and C are dependent. If B and C are dependent, then $B \times C = O$, and hence $A \cdot B \times C = 0$. Suppose, then, that B and C are independent. Since all three are dependent, there exist scalars a, b, c, not all zero, such that $aA + bB + cC = O$. We must have $a \neq 0$ in this relation, otherwise B and C would be dependent. Therefore, we can divide by a and express A as a linear combination of B and C, say $A = tB + sC$. Taking the dot product of each member with $B \times C$, we find

$$A \cdot (B \times C) = tB \cdot B \times C + sC \cdot B \times C = 0,$$

since each of B and C is orthogonal to $B \times C$. Therefore dependence of A, B, and C implies $A \cdot B \times C = 0$.

To prove the converse, assume that $A \cdot B \times C = 0$. If B and C are dependent, then so are A, B, and C, and there is nothing more to prove. Assume then, that B and C are linearly independent. Then, by Theorem 13.13, the three vectors B, C, and $B \times C$ are linearly independent. Hence, they span A so we can write

$$A = aB + bC + c(B \times C)$$

for some scalars a, b, c. Taking the dot product of each member with $B \times C$ and using the fact that $A \cdot (B \times C) = 0$, we find $c = 0$, so $A = aB + bC$. This proves that A, B, and C are linearly dependent.

EXAMPLE. To determine whether the three vectors $(2, 3, -1)$, $(3, -7, 5)$, and $(1, -5, 2)$ are dependent, we form their scalar triple product, expressing it as the determinant

$$\begin{vmatrix} 2 & 3 & -1 \\ 3 & -7 & 5 \\ 1 & -5 & 2 \end{vmatrix} = 2(-14 + 25) - 3(6 - 5) - 1(-15 + 7) = 27.$$

Since the scalar triple product is nonzero, the vectors are linearly independent.

The scalar triple product has an interesting geometric interpretation. Figure 13.6 shows a parallelepiped determined by three geometric vectors A, B, C not in the same plane. Its altitude is $\|C\| \cos \phi$, where ϕ is the angle between $A \times B$ and C. In this figure, $\cos \phi$ is positive because $0 \leq \phi < \frac{1}{2}\pi$. The area of the parallelogram which forms the base is $\|A \times B\|$, and this is also the area of each cross section parallel to the base. Integrating the cross-sectional area from 0 to $\|C\| \cos \phi$, we find that the volume of the parallelepiped is $\|A \times B\| (\|C\| \cos \phi)$, the area of the base times the altitude. But we have

$$\|A \times B\| (\|C\| \cos \phi) = (A \times B) \cdot C .$$

In other words, the scalar triple product $A \times B \cdot C$ is equal to the volume of the parallelepiped determined by A, B, C. When $\frac{1}{2}\pi < \phi \leq \pi$, $\cos \phi$ is negative and the product $A \times B \cdot C$ is the negative of the volume. If A, B, C are on a plane through the origin, they are linearly dependent and their scalar triple product is zero. In this case, the parallelepiped degenerates and has zero volume.

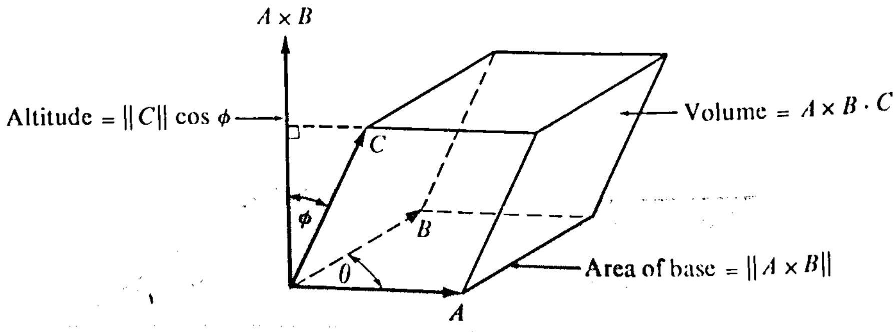


FIGURE 13.6 Geometric interpretation of the scalar triple product as the volume of a parallelepiped.

This geometric interpretation of the scalar triple product suggests certain algebraic properties of this product. For example, a cyclic permutation of the vectors A, B, C leaves the scalar triple product unchanged. By this we mean that

$$A \times B \cdot C = B \times C \cdot A = C \times A \cdot B . \tag{13.9}$$

An algebraic proof of this property is outlined in Exercise 7 of Section 13.14. This property implies that the dot and cross are interchangeable in a scalar triple product. In fact, the commutativity of the dot product implies $(B \times C) \cdot A = A \cdot (B \times C)$ and when this is combined with the first equation in (13.9), we find that

$$A \times B \cdot C = A \cdot B \times C . \tag{13.10}$$

The scalar triple product $A \cdot B \times C$ is often denoted by the symbol $[ABC]$ without indicating the dot or cross. Because of Equation (13.10), there is no ambiguity in this notation—the product depends only on the order of the factors A, B, C and not on the positions of the dot and cross.

13.13 Cramer's rule for solving a system of three linear equations

The scalar triple product may be used to solve a system of three simultaneous linear equations in three unknowns x, y, z. Suppose the system is written in the form

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 , \\ a_2x + b_2y + c_2z &= d_2 , \\ a_3x + b_3y + c_3z &= d_3 . \end{aligned} \tag{13.11}$$

Let A be the vector with components a_1, a_2, a_3 and define B, C, and D similarly. Then the three equations in (13.11) are equivalent to the single vector equation

(13.12) $$xA + yB + zC = D\,.$$

If we dot multiply both sides of this equation with $B \times C$, writing $[ABC]$ for $A \cdot B \times C$, we find that

$$x[ABC] + y[BBC] + z[CBC] = [DBC]\,.$$

Since $[BBC] = [CBC] = 0$, the coefficients of y and z drop out and we obtain

(13.13) $$x = \frac{[DBC]}{[ABC]} \quad \text{if} \quad [ABC] \neq 0\,.$$

A similar argument yields analogous formulas for y and z. Thus we have

(13.14) $$y = \frac{[ADC]}{[ABC]} \quad \text{and} \quad z = \frac{[ABD]}{[ABC]} \quad \text{if} \quad [ABC] \neq 0\,.$$

The condition $[ABC] \neq 0$ means that the three vectors A, B, C are linearly independent. In this case, (13.12) shows that every vector D in 3-space is spanned by A, B, C and the multipliers x, y, z are uniquely determined by the formulas in (13.13) and (13.14). When the scalar triple products that occur in these formulas are written as determinants, the result is known as *Cramer's rule* for solving the system (13.11):

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

If $[ABC] = 0$, then A, B, C lie on a plane through the origin and the system has no solution unless D lies in the same plane. In this latter case, it is easy to show that there are infinitely many solutions of the system. In fact, the vectors A, B, C are linearly dependent so there exist scalars u, v, w not all zero such that $uA + vB + wC = O$. If the triple (x, y, z) satisfies (13.12), then so does the triple $(x + tu, y + tv, z + tw)$ for all real t, since we have

$$(x + tu)A + (y + tv)B + (z + tw)C$$
$$= xA + yB + zC + t(uA + vB + wC) = xA + yB + zC\,.$$

13.14 Exercises

1. Compute the scalar triple product $A \cdot B \times C$ in each case.
   (a) $A = (3, 0, 0)$, $\quad B = (0, 4, 0)$, $\quad C = (0, 0, 8)$.
   (b) $A = (2, 3, -1)$, $\quad B = (3, -7, 5)$, $\quad C = (1, -5, 2)$.
   (c) $A = (2, 1, 3)$, $\quad B = (-3, 0, 6)$, $\quad C = (4, 5, -1)$.

2. Find all real t for which the three vectors $(1, t, 1)$, $(t, 1, 0)$, $(0, 1, t)$ are linearly dependent.
3. Compute the volume of the parallelepiped determined by the vectors $i + j$, $j + k$, $k + i$.
4. Prove that $A \times B = A \cdot (B \times i)i + A \cdot (B \times j)j + A \cdot (B \times k)k$.
5. Prove that $i \times (A \times i) + j \times (A \times j) + k \times (A \times k) = 2A$.
6. (a) Find all vectors $ai + bj + ck$ which satisfy the relation

$$(ai + bj + ck) \cdot k \times (6i + 3j + 4k) = 3 .$$

(b) Find that vector $ai + bj + ck$ of shortest length which satisfies the relation in (a).
7. Use algebraic properties of the dot and cross products to derive the following properties of the scalar triple product.
(a) $(A + B) \cdot (A + B) \times C = 0$.
(b) $A \cdot B \times C = -B \cdot A \times C$. This shows that switching the first two vectors reverses the sign. [*Hint:* Use part (a) and distributive laws.]
(c) $A \cdot B \times C = -A \cdot C \times B$. This shows that switching the second and third vectors reverses the sign. [*Hint:* Use skew-symmetry.]
(d) $A \cdot B \times C = -C \cdot B \times A$. This shows that switching the first and third vectors reverses the sign. [*Hint:* Use (b) and (c).]
Equating the right members of (b), (c), and (d), we find that

$$A \cdot B \times C = B \cdot C \times A = C \cdot A \times B ,$$

which shows that a cyclic permutation of A, B, C leaves their scalar triple product unchanged.
9. This exercise outlines a proof of the vector identity

$$A \times (B \times C) = (C \cdot A)B - (B \cdot A)C , \tag{13.15}$$

sometimes referred to as the "cab minus bac" formula. Let $B = (b_1, b_2, b_3)$, $C = (c_1, c_2, c_3)$ and prove that

$$i \times (B \times C) = c_1 B - b_1 C .$$

This proves (13.15) in the special case $A = i$. Prove corresponding formulas for $A = j$ and $A = k$, and then combine them to obtain (13.15).
10. Use the "cab minus bac" formula of Exercise 9 to derive the following vector identities.
(a) $(A \times B) \times (C \times D) = (A \times B \cdot D)C - (A \times B \cdot C)D$.
(b) $A \times (B \times C) + B \times (C \times A) + C \times (A \times B) = O$.
(c) $A \times (B \times C) = (A \times B) \times C$ if and only if $B \times (C \times A) = O$.
(d) $(A \times B) \cdot (C \times D) = (B \cdot D)(A \cdot C) - (B \cdot C)(A \cdot D)$.
11. Four vectors A, B, C, D in V_3 satisfy the relations $A \times C \cdot B = 5$, $A \times D \cdot B = 3$, $C + D = i + 2j + k$, $C - D = i - k$. Compute $(A \times B) \times (C \times D)$ in terms of i, j, k.
12. Prove that $(A \times B) \cdot (B \times C) \times (C \times A) = (A \cdot B \times C)^2$.
13. Prove or disprove the formula $A \times [A \times (A \times B)] \cdot C = -\|A\|^2 A \cdot B \times C$.
14. (a) Prove that the volume of the tetrahedron whose vertices are A, B, C, D is

$$\tfrac{1}{6} |(B - A) \cdot (C - A) \times (D - A)| .$$

(b) Compute this volume when $A = (1, 1, 1)$, $B = (0, 0, 2)$, $C = (0, 3, 0)$, and $D = (4, 0, 0)$.
15. (a) If $B \neq C$, prove that the perpendicular distance from A to the line through B and C is

$$\|(A - B) \times (C - B)\| / \|B - C\| .$$

(b) Compute this distance when $A = (1, -2, -5)$, $B = (-1, 1, 1)$, and $C = (4, 5, 1)$.

16. Heron's formula for computing the area S of a triangle whose sides have lengths a, b, c states that $S = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = (a+b+c)/2$. This exercise outlines a vectorial proof of this formula.

Assume the triangle has vertices at O, A, and B, with $\|A\| = a$, $\|B\| = b$, $\|B - A\| = c$.

(a) Combine the two identities

$$\|A \times B\|^2 = \|A\|^2\|B\|^2 - (A \cdot B)^2, \qquad -2A \cdot B = \|A - B\|^2 - \|A\|^2 - \|B\|^2$$

to obtain the formula

$$4S^2 = a^2b^2 - \tfrac{1}{4}(c^2 - a^2 - b^2)^2 = \tfrac{1}{4}(2ab - c^2 + a^2 + b^2)(2ab + c^2 - a^2 - b^2).$$

(b) Rewrite the formula in part (a) to obtain

$$S^2 = \tfrac{1}{16}(a + b + c)(a + b - c)(c - a + b)(c + a - b),$$

and thereby deduce Heron's formula.

Use Cramer's rule to solve the system of equations in each of Exercises 17, 18, and 19.

17. $x + 2y + 3z = 5, \quad 2x - y + 4z = 11, \quad -y + z = 3.$
18. $x + y + 2z = 4, \quad 3x - y - z = 2, \quad 2x + 5y + 3z = 3.$
19. $x + y = 5, \quad x + z = 2, \quad y + z = 5.$
20. If $P = (1, 1, 1)$ and $A = (2, 1, -1)$, prove that each point (x, y, z) on the line $\{P + tA\}$ satisfies the system of linear equations $x - y + z = 1$, $x + y + 3z = 5$, $3x + y + 7z = 11$.

13.15 Normal vectors to planes

A plane was defined in Section 13.6 as a set of the form $\{P + sA + tB\}$, where A and B are linearly independent vectors. Now we show that planes in V_3 can be described in an entirely different way, using the concept of a normal vector.

DEFINITION. *Let $M = \{P + sA + tB\}$ be the plane through P spanned by A and B. A vector N in V_3 is said to be perpendicular to M if N is perpendicular to both A and B. If, in addition, N is nonzero, then N is called a normal vector to the plane.*

Note: If $N \cdot A = N \cdot B = 0$, then $N \cdot (sA + tB) = 0$, so a vector perpendicular to both A and B is perpendicular to every vector in the linear span of A and B. Also, if N is normal to a plane, so is tN for every real $t \neq 0$.

THEOREM 13.15. *Given a plane $M = \{P + sA + tB\}$ through P spanned by A and B. Let $N = A \times B$. Then we have the following:*

(a) *N is a normal vector to M.*

(b) *M is the set of all X in V_3 satisfying the equation*

$$(X - P) \cdot N = 0. \tag{13.16}$$

Proof. Since M is a plane, A and B are linearly independent, so $A \times B \neq O$. This proves (a) since $A \times B$ is orthogonal to both A and B.

To prove (b), let M' be the set of all X in V_3 satisfying Equation (13.16). If $X \in M$, then $X - P$ is in the linear span of A and B, so $X - P$ is orthogonal to N. Therefore $X \in M'$ which proves that $M \subseteq M'$. Conversely, suppose $X \in M'$. Then X satisfies (13.16). Since A, B, N are linearly independent (Theorem 13.13), they span every vector in V_3 so, in particular, we have

$$X - P = sA + tB + uN$$

for some scalars s, t, u. Taking the dot product of each member with N, we find $u = 0$, so $X - P = sA + tB$. This shows that $X \in M$. Hence, $M' \subseteq M$, which completes the proof of (b).

The geometric meaning of Theorem 13.15 is shown in Figure 13.7. The points P and X are on the plane and the normal vector N is orthogonal to $X - P$. This figure suggests the following theorem.

THEOREM 13.16. *Given a plane M through a point P, and given a nonzero vector N normal to M, let*

$$d = \frac{|P \cdot N|}{\|N\|}. \tag{13.17}$$

Then every X on M has length $\|X\| \geq d$. Moreover, we have $\|X\| = d$ if and only if X is the projection of P along N:

$$X = tN, \quad \text{where} \quad t = \frac{P \cdot N}{N \cdot N}.$$

Proof. The proof follows from the Cauchy-Schwarz inequality in exactly the same way as we proved Theorem 13.6, the corresponding result for lines in V_2.

By the same argument we find that if Q is a point not on M, then among all points X on M the smallest length $\|X - Q\|$ occurs when $X - Q$ is the projection of $P - Q$ along N. This minimum length is $|(P - Q) \cdot N|/\|N\|$ and is called the *distance from Q to the plane*. The number d in (13.17) is the distance from the origin to the plane.

13.16 Linear Cartesian equations for planes

The results of Theorems 13.15 and 13.16 can also be expressed in terms of components. If we write $N = (a, b, c)$, $P = (x_1, y_1, z_1)$, and $X = (x, y, z)$, Equation (13.16) becomes

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0. \tag{13.18}$$

This is called a Cartesian equation for the plane, and it is satisfied by those and only those points (x, y, z) which lie on the plane. The set of points satisfying (13.18) is not altered if we multiply each of a, b, c by a nonzero scalar t. This simply amounts to a different choice of normal vector in (13.16).

We may transpose the terms not involving x, y, and z, and write (13.18) in the form

$$ax + by + cz = d_1, \tag{13.19}$$

where $d_1 = ax_1 + by_1 + cz_1$. An equation of this type is said to be *linear* in x, y, and z. We have just shown that every point (x, y, z) on a plane satisfies a linear Cartesian equation (13.19) in which not all three of a, b, c are zero. Conversely, every linear equation with this property represents a plane. (The reader may verify this as an exercise.)

The number d_1 in Equation (13.19) bears a simple relation to the distance d of the plane from the origin. Since $d_1 = P \cdot N$, we have $|d_1| = |P \cdot N| = d\|N\|$. In particular $|d_1| = d$ if the normal N has length 1. The plane passes through the origin if and only if $d_1 = 0$.

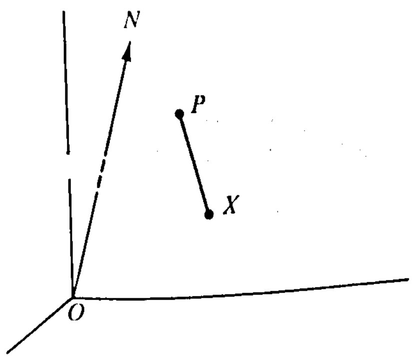


FIGURE 13.7 A plane through P and X with normal vector N.

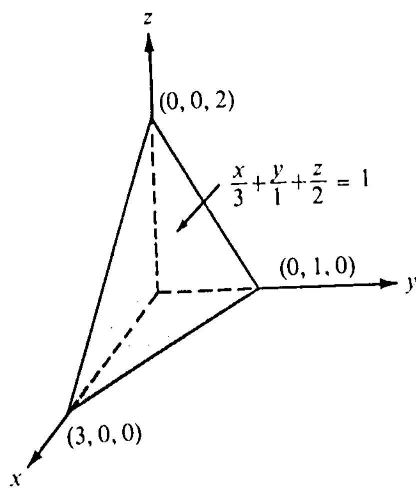


FIGURE 13.8 A plane with intercepts 3, 1, 2.

EXAMPLE. The Cartesian equation $2x + 6y + 3z = 6$ represents a plane with normal vector $N = 2i + 6j + 3k$. We rewrite the Cartesian equation in the form

$$\frac{x}{3} + \frac{y}{1} + \frac{z}{2} = 1$$

from which it is apparent that the plane intersects the coordinate axes at the points $(3, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 2)$. The numbers 3, 1, 2 are called, respectively, the x-, y-, and z-*intercepts* of the plane. A knowledge of the intercepts makes it possible to sketch the plane quickly. A portion of the plane is shown in Figure 13.8. Its distance d from the origin is $d = 6/\|N\| = 6/7$.

Two parallel planes will have a common normal N. If $N = (a, b, c)$, the Cartesian equations of two parallel planes can be written as follows:

$$ax + by + cz = d_1, \qquad ax + by + cz = d_2,$$

the only difference being in the right-hand members. The number $|d_1 - d_2|/\|N\|$ is called the perpendicular distance between the two planes, a definition suggested by Theorem 13.16.

Two planes are called perpendicular if a normal of one is perpendicular to a normal of the other. More generally, if the normals of two planes make an angle θ with each other, then we say that θ is an angle between the two planes.

13.17 Exercises

1. Given vectors $A = 2i + 3j - 4k$ and $B = j + k$.
   (a) Find a nonzero vector N perpendicular to both A and B.
   (b) Give a Cartesian equation for the plane through the origin spanned by A and B.
   (c) Give a Cartesian equation for the plane through $(1, 2, 3)$ spanned by A and B.
2. A plane has Cartesian equation $x + 2y - 2z + 7 = 0$. Find the following:
   (a) a normal vector of unit length;
   (b) the intercepts of the plane;
   (c) the distance of the plane from the origin;
   (d) the point Q on the plane nearest the origin.
3. Find a Cartesian equation of the plane which passes through $(1, 2, -3)$ and is parallel to the plane given by $3x - y + 2z = 4$. What is the distance between the two planes?
4. Four planes have Cartesian equations $x + 2y - 2z = 5$, $3x - 6y + 3z = 2$, $2x + y + 2z = -1$, and $x - 2y + z = 7$.
   (a) Show that two of them are parallel and the other two are perpendicular.
   (b) Find the distance between the two parallel planes.
5. The three points $(1, 1, -1)$, $(3, 3, 2)$, and $(3, -1, -2)$ determine a plane. Find (a) a vector normal to the plane; (b) a Cartesian equation for the plane; (c) the distance of the plane from the origin.
6. Find a Cartesian equation for the plane determined by $(1, 2, 3)$, $(2, 3, 4)$, and $(-1, 7, -2)$.
7. Determine an angle between the planes with Cartesian equations $x + y = 1$ and $y + z = 2$.
8. A line parallel to a nonzero vector N is said to be perpendicular to a plane M if N is normal to M. Find a Cartesian equation for the plane through $(2, 3, -7)$, given that the line through $(1, 2, 3)$ and $(2, 4, 12)$ is perpendicular to this plane.
9. Find a vector parametric equation for the line which contains the point $(2, 1, -3)$ and is perpendicular to the plane given by $4x - 3y + z = 5$.
10. A point moves in space in such a way that at time t its position is given by the vector $X(t) = (1 - t)i + (2 - 3t)j + (2t - 1)k$.
    (a) Prove that the point moves along a line. (Call it L.)
    (b) Find a vector N parallel to L.
    (c) At what time does the point strike the plane given by $2x + 3y + 2z + 1 = 0$?
    (d) Find a Cartesian equation for that plane parallel to the one in part (c) which contains the point $X(3)$.
    (e) Find a Cartesian equation for that plane perpendicular to L which contains the point $X(2)$.
11. Find a Cartesian equation for the plane through $(1, 1, 1)$ if a normal vector N makes angles $\frac{1}{3}\pi$, $\frac{1}{4}\pi$, $\frac{1}{3}\pi$, with i, j, k, respectively.
12. Compute the volume of the tetrahedron whose vertices are at the origin and at the points where the coordinate axes intersect the plane given by $x + 2y + 3z = 6$.
13. Find a vector A of length 1 perpendicular to $i + 2j - 3k$ and parallel to the plane with Cartesian equation $x - y + 5z = 1$.
14. Find a Cartesian equation of the plane which is parallel to both vectors $i + j$ and $j + k$ and intersects the x-axis at $(2, 0, 0)$.
15. Find all points which lie on the intersection of the three planes given by $3x + y + z = 5$, $3x + y + 5z = 7$, $x - y + 3z = 3$.
16. Prove that three planes whose normals are linearly independent intersect in one and only one point.

17. A line with direction vector A is said to be parallel to a plane M if A is parallel to M. A line containing $(1, 2, 3)$ is parallel to each of the planes given by $x + 2y + 3z = 4$, $2x + 3y + 4z = 5$. Find a vector parametric equation for this line.
18. Given a line L not parallel to a plane M, prove that the intersection $L \cap M$ contains exactly one point.
19. (a) Prove that the distance from the point (x_0, y_0, z_0) to the plane with Cartesian equation $ax + by + cz + d = 0$ is

$$\frac{|ax_0 + by_0 + cz_0 + d|}{(a^2 + b^2 + c^2)^{1/2}}.$$

(b) Find the point P on the plane given by $5x - 14y + 2z + 9 = 0$ which is nearest to the point $Q = (-2, 15, -7)$.
20. Find a Cartesian equation for the plane parallel to the plane given by $2x - y + 2z + 4 = 0$ if the point $(3, 2, -1)$ is equidistant from both planes.
21. (a) If three points A, B, C determine a plane, prove that the distance from a point Q to this plane is $|(Q - A) \cdot (B - A) \times (C - A)| / \|(B - A) \times (C - A)\|$.
(b) Compute this distance if $Q = (1, 0, 0)$, $A = (0, 1, 1)$, $B = (1, -1, 1)$, and $C = (2, 3, 4)$.
22. Prove that if two planes M and M' are not parallel, their intersection $M \cap M'$ is a line.
23. Find a Cartesian equation for the plane which is parallel to j and which passes through the intersection of the planes described by the equations $x + 2y + 3z = 4$, and $2x + y + z = 2$.
24. Find a Cartesian equation for the plane parallel to the vector $3i - j + 2k$ if it contains every point on the line of intersection of the planes with equations $x + y = 3$ and $2y + 3z = 4$.

13.18 The conic sections

A moving line G which intersects a fixed line A at a given point P, making a constant angle θ with A, where $0 < \theta < \frac{1}{2}\pi$, generates a surface in 3-space called a right circular cone. The line G is called a *generator* of the cone, A is its *axis*, and P its *vertex*. Each of the cones shown in Figure 13.9 has a vertical axis. The upper and lower portions of the cone meeting at the vertex are called *nappes* of the cone. The curves obtained by slicing the cone with a plane not passing through the vertex are called *conic sections*, or simply *conics*. If the cutting plane is parallel to a line of the cone through the vertex, the conic is called a

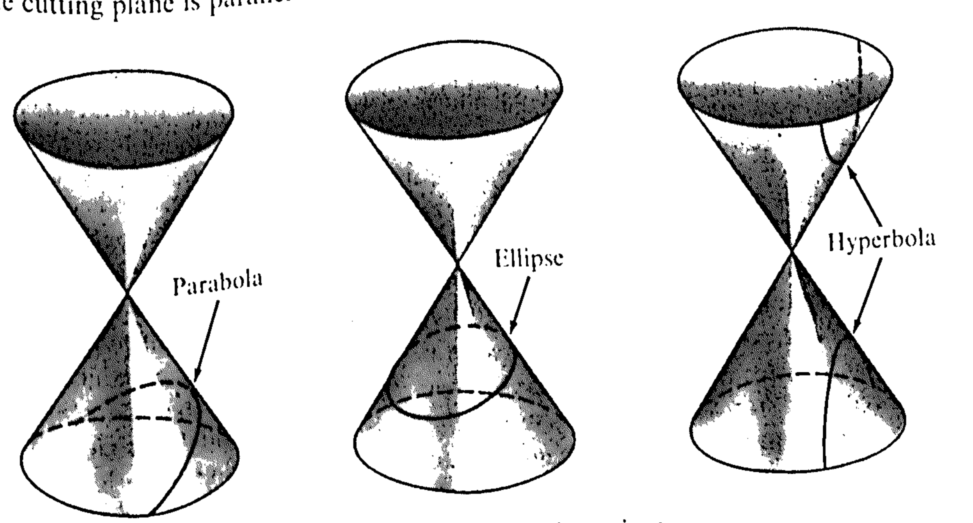


FIGURE 13.9 The conic sections.

parabola. Otherwise the intersection is called an *ellipse* or a *hyperbola*, according as the plane cuts just one or both nappes. (See Figure 13.9.) The hyperbola consists of two "branches," one on each nappe.

Many important discoveries in both pure and applied mathematics have been related to the conic sections. Appolonius' treatment of conics as early as the 3rd century B.C. was one of the most profound achievements of classical Greek geometry. Nearly 2000 years later, Galileo discovered that a projectile fired horizontally from the top of a tower falls to earth along a parabolic path (if air resistance is neglected and if the motion takes place above a part of the earth that can be regarded as a flat plane). One of the turning points in the history of astronomy occurred around 1600 when Kepler suggested that all planets move in elliptical orbits. Some 80 years later, Newton was able to demonstrate that an elliptical planetary path implies an inverse-square law of gravitational attraction. This led Newton to formulate his famous theory of universal gravitation which has often been referred to as the greatest scientific discovery ever made. Conic sections appear not only as orbits of planets and satellites but also as trajectories of elementary atomic particles. They are used in the design of lenses and mirrors, and in architecture. These examples and many others show that the importance of the conic sections can hardly be overestimated.

There are other equivalent definitions of the conic sections. One of these refers to special points known as *foci* (singular: *focus*). An ellipse may be defined as the set of all points in a plane the sum of whose distances d_1 and d_2 from two fixed points F_1 and F_2 (the foci) is

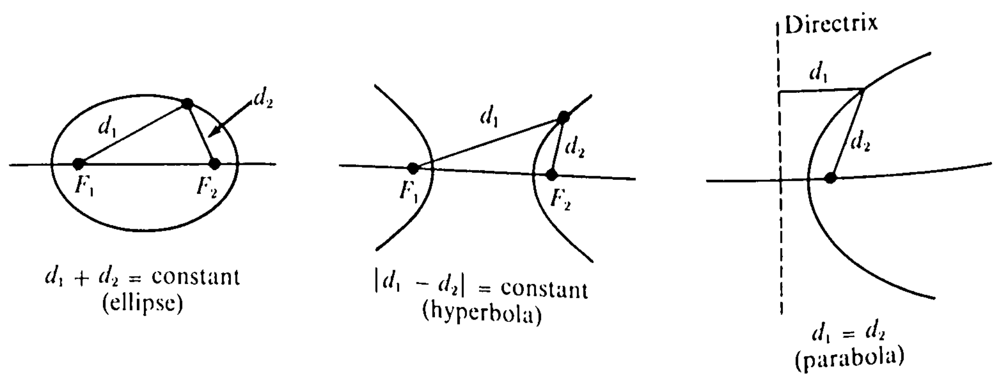


FIGURE 13.10 Focal definitions of the conic sections.

constant. (See Figure 13.10.) If the foci coincide, the ellipse reduces to a circle. A hyperbola is the set of all points for which the difference $|d_1 - d_2|$ is constant. A parabola is the set of all points in a plane for which the distance to a fixed point F (called the focus) is equal to the distance to a given line (called the directrix).

There is a very simple and elegant argument which shows that the focal property of an ellipse is a consequence of its definition as a section of a cone. This proof, which we may refer to as the "ice-cream-cone proof," was discovered in 1822 by a Belgian mathematician, G. P. Dandelin (1794–1847), and makes use of the two spheres S_1 and S_2 which are drawn so as to be tangent to the cutting plane and the cone, as illustrated in Figure 13.11. These spheres touch the cone along two parallel circles C_1 and C_2. We shall prove that the points F_1 and F_2, where the spheres contact the plane, can serve as foci of the ellipse.

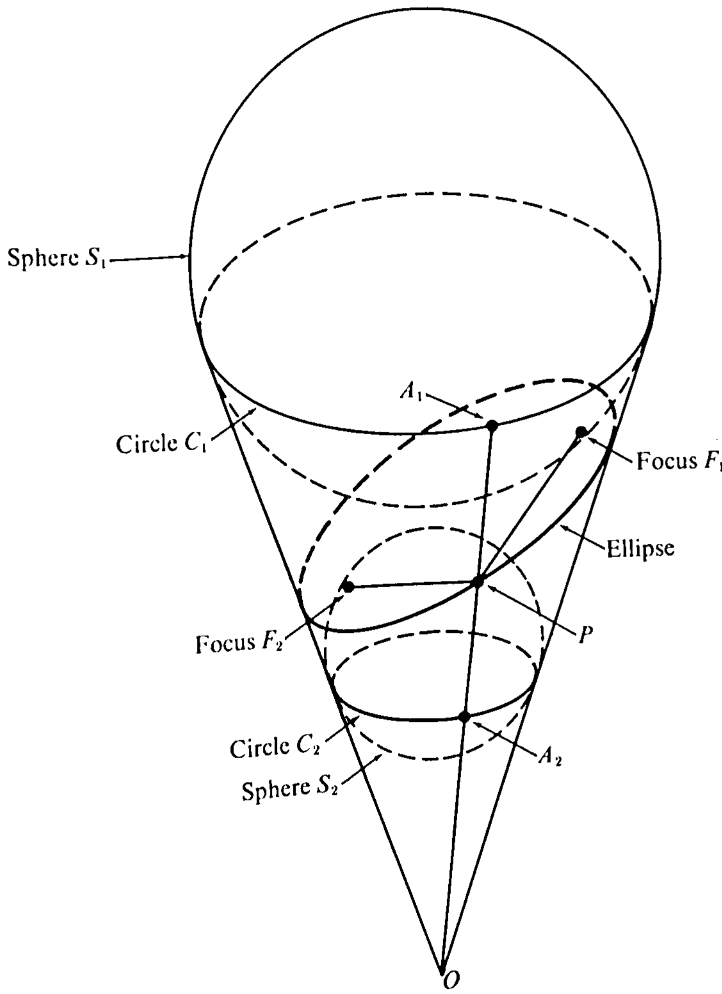


FIGURE 13.11 The ice-cream-cone proof.

Let P be an arbitrary point of the ellipse. The problem is to prove that $\|\overrightarrow{PF_1}\| + \|\overrightarrow{PF_2}\|$ is constant, that is, independent of the choice of P. For this purpose, draw that line on the cone from the vertex O to P and let A_1 and A_2 be its intersections with the circles C_1 and C_2, respectively. Then $\overrightarrow{PF_1}$ and $\overrightarrow{PA_1}$ are two tangents to S_1 from P, and hence $\|\overrightarrow{PF_1}\| = \|\overrightarrow{PA_1}\|$. Similarly $\|\overrightarrow{PF_2}\| = \|\overrightarrow{PA_2}\|$, and therefore we have

$$\|\overrightarrow{PF_1}\| + \|\overrightarrow{PF_2}\| = \|\overrightarrow{PA_1}\| + \|\overrightarrow{PA_2}\| .$$

But $\|\overrightarrow{PA_1}\| + \|\overrightarrow{PA_2}\| = \|\overrightarrow{A_1A_2}\|$, which is the distance between the parallel circles C_1 and C_2 measured along the surface of the cone. This proves that F_1 and F_2 can serve as foci of the ellipse, as asserted.

Modifications of this proof work also for the hyperbola and the parabola. In the case of the hyperbola, the proof employs one sphere in each portion of the cone. For the parabola one sphere tangent to the cutting plane at the focus F is used. This sphere touches the cone along a circle which lies in a plane whose intersection with the cutting plane is the directrix of the parabola. With these hints the reader should be able to show that the focal properties of the hyperbola and parabola may be deduced from their definitions as sections of a cone.

13.19 Eccentricity of conic sections

Another characteristic property of conic sections involves a concept called eccentricity. A conic section can be defined as a curve traced out by a point moving in a plane in such a way that the ratio of its distances from a fixed point and a fixed line is constant. This constant ratio is called the *eccentricity* of the curve and is denoted by e. (This should not be confused with the Euler number e.) The curve is an *ellipse* if $0 < e < 1$, a *parabola* if $e = 1$, and a *hyperbola* if $e > 1$. The fixed point is called a *focus* and the fixed line a *directrix*.

We shall adopt this definition as the basis for our study of the conic sections since it permits a simultaneous treatment of all three types of conics and lends itself to the use of vector methods. In this discussion it is understood that all points and lines are in the same plane.

DEFINITION. *Given a line L, a point F not on L, and a positive number e. Let $d(X, L)$ denote the distance from a point X to L. The set of all X satisfying the relation*

$$\|X - F\| = e\, d(X, L) \tag{13.20}$$

is called a conic section with eccentricity e. The conic is called an ellipse if $e < 1$, a parabola if $e = 1$, and a hyperbola if $e > 1$.

If N is a vector normal to L and if P is any point on L the distance $d(X, L)$ from any point X to L is given by the formula

$$d(X, L) = \frac{|(X - P) \cdot N|}{\|N\|}.$$

When N has length 1, this simplifies to $d(X, L) = |(X - P) \cdot N|$, and the basic equation (13.20) for the conic sections becomes

$$\|X - F\| = e\,|(X - P) \cdot N|. \tag{13.21}$$

The line L separates the plane into two parts which we shall arbitrarily label as "positive" and "negative" according to the choice of N. If $(X - P) \cdot N > 0$, we say that X is in the positive half-plane, and if $(X - P) \cdot N < 0$, we say that X is in the negative half-plane. On the line L itself we have $(X - P) \cdot N = 0$. In Figure 13.12 the choice of the normal vector N dictates that points to the right of L are in the positive half-plane and those to the left are in the negative half-plane.

Now we place the focus F in the negative half-plane, as indicated in Figure 13.12, and choose P to be that point on L nearest to F. Then $P - F = dN$, where $|d| = \|P - F\|$ is

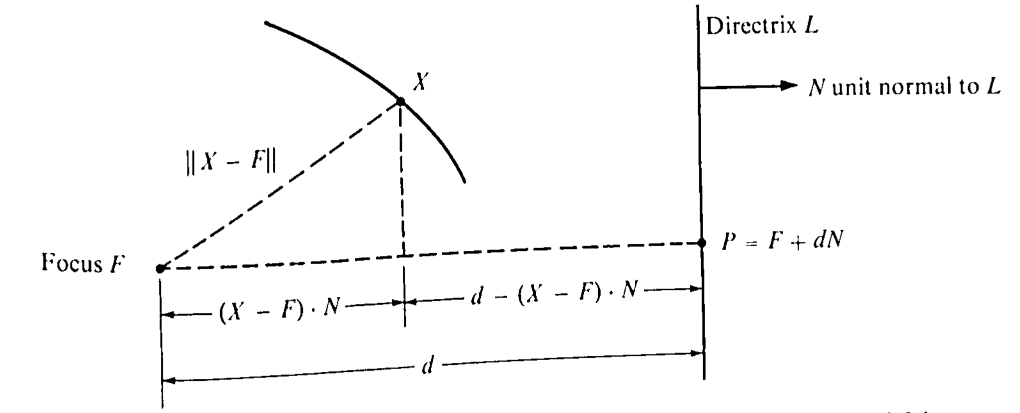


FIGURE 13.12 A conic section with eccentricity e is the set of all X satisfying $\|X - F\| = e\,|(X - F) \cdot N - d|$.

the distance from the focus to the directrix. Since F is in the negative half-plane, we have $(F - P) \cdot N = -d < 0$, so d is positive. Replacing P by $F + dN$ in (13.21), we obtain the following theorem, which is illustrated in Figure 13.12.

THEOREM 13.17. *Let C be a conic section with eccentricity e, focus F, and directrix L at a distance d from F. If N is a unit normal to L and if F is in the negative half-plane determined by N, then C consists of all points X satisfying the equation*

$$\|X - F\| = e\,|(X - F) \cdot N - d|\,. \tag{13.22}$$

13.20 Polar equations for conic sections

The equation in Theorem 13.17 can be simplified if we place the focus in a special position. For example, if the focus is at the origin the equation becomes

$$\|X\| = e\,|X \cdot N - d|\,. \tag{13.23}$$

This form is especially useful if we wish to express X in terms of polar coordinates. Take the directrix L to be vertical, as shown in Figure 13.13, and let $N = i$. If X has polar coordinates r and θ, we have $\|X\| = r$, $X \cdot N = r\cos\theta$, and Equation (13.23) becomes

$$r = e\,|r\cos\theta - d|\,. \tag{13.24}$$

If X lies to the left of the directrix, we have $r\cos\theta < d$, so $|r\cos\theta - d| = d - r\cos\theta$ and (13.24) becomes $r = e(d - r\cos\theta)$, or, solving for r, we obtain

$$r = \frac{ed}{e\cos\theta + 1}\,. \tag{13.25}$$

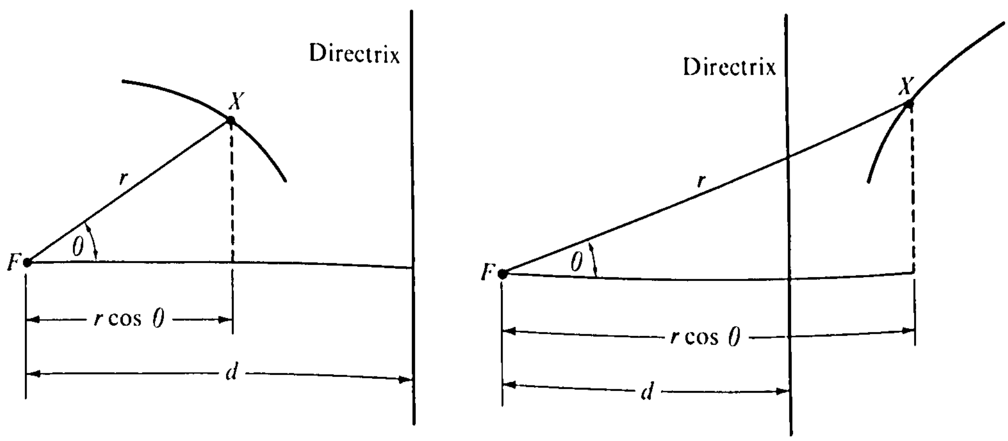
If X lies to the right of the directrix, we have $r \cos \theta > d$, so (13.24) becomes

$$r = e(r \cos \theta - d),$$

giving us

(13.26)
$$r = \frac{ed}{e \cos \theta - 1}.$$

Since $r > 0$, this last equation implies $e > 1$. In other words, there are points to the right of the directrix only for the hyperbola. Thus, we have proved the following theorem which is illustrated in Figure 13.13.



(a) $r \cos \theta < d$ on the ellipse, parabola, and left branch of the hyperbola

(b) $r \cos \theta > d$ on the right branch of the hyperbola

FIGURE 13.13 Conic sections with polar equation $r = e\,|r \cos \theta - d|$. The focus F is at the origin and lies to the left of the directrix.

THEOREM 13.18. *Let C be a conic section with eccentricity e, with a focus F at the origin, and with a vertical directrix L at a distance d to the right of F. If* $0 < e \leq 1$, *the conic C is an ellipse or a parabola; every point on C lies to the left of L and satisfies the polar equation*

(13.27)
$$r = \frac{ed}{e \cos \theta + 1}.$$

If $e > 1$, *the curve is a hyperbola with a branch on each side of L. Points on the left branch satisfy* (13.27) *and points on the right branch satisfy*

(13.28)
$$r = \frac{ed}{e \cos \theta - 1}.$$

Polar equations corresponding to other positions of the directrix are discussed in the next set of exercises.

13.21 Exercises

1. Prove that Equation (13.22) in Theorem 13.17 must be replaced by

$$\|X - F\| = e\,|(X - F)\cdot N + d|$$

if F is in the positive half-plane determined by N.

2. Let C be a conic section with eccentricity e, with a focus at the origin, and with a vertical directrix L at a distance d to the left of F.
(a) Prove that if C is an ellipse or parabola, every point of C lies to the right of L and satisfies the polar equation

$$r = \frac{ed}{1 - e\cos\theta}.$$

(b) Prove that if C is a hyperbola, points on the right branch satisfy the equation in part (a) and points on the left branch satisfy $r = -ed/(1 + e\cos\theta)$. Note that $1 + e\cos\theta$ is always negative in this case.

3. If a conic section has a horizontal directrix at a distance d above a focus at the origin, prove that its points satisfy the polar equations obtained from those in Theorem 13.18 by replacing $\cos\theta$ by $\sin\theta$. What are the corresponding polar equations if the directrix is horizontal and lies below the focus?

Each of Exercises 4 through 9 gives a polar equation for a conic section with a focus F at the origin and a vertical directrix lying to the right of F. In each case, determine the eccentricity e and the distance d from the focus to the directrix. Make a sketch showing the relation of the curve to its focus and directrix.

4. $r = \dfrac{2}{1 + \cos\theta}.$

5. $r = \dfrac{3}{1 + \frac{1}{2}\cos\theta}.$

6. $r = \dfrac{6}{3 + \cos\theta}.$

7. $r = \dfrac{1}{-\frac{1}{2} + \cos\theta}.$

8. $r = \dfrac{4}{1 + 2\cos\theta}.$

9. $r = \dfrac{4}{1 + \cos\theta}.$

In each of Exercises 10 through 12, a conic section of eccentricity e has a focus at the origin and a directrix with the given Cartesian equation. In each case, compute the distance d from the focus to the directrix and determine a polar equation for the conic section. For a hyperbola, give a polar equation for each branch. Make a sketch showing the relation of the curve to its focus and directrix.

10. $e = \frac{1}{2}$; directrix: $3x + 4y = 25$.

11. $e = 1$; directrix: $4x + 3y = 25$.

12. $e = 2$; directrix: $x + y = 1$.

13. A comet moves in a parabolic orbit with the sun at the focus. When the comet is 10^8 miles from the sun, a vector from the focus to the comet makes an angle of $\pi/3$ with a unit vector N from the focus perpendicular to the directrix, the focus being in the negative half-plane determined by N.
(a) Find a polar equation for the orbit, taking the origin at the focus, and compute the smallest distance from the comet to the sun.
(b) Solve part (a) if the focus is in the positive half-plane determined by N.

13.22 Conic sections symmetric about the origin

A set of points is said to be *symmetric about the origin* if $-X$ is in the set whenever X is in the set. We show next that the focus of an ellipse or hyperbola can always be placed so the conic section will be symmetric about the origin. To do this we rewrite the basic equation (13.22) as follows:

$$(13.29)\qquad \|X - F\| = e\,|(X - F)\cdot N - d| = e\,|X\cdot N - F\cdot N - d| = |eX\cdot N - a|,$$

where $a = ed + eF\cdot N$. Squaring both members, we obtain

$$(13.30)\qquad \|X\|^2 - 2F\cdot X + \|F\|^2 = e^2(X\cdot N)^2 - 2eaX\cdot N + a^2.$$

If we are to have symmetry about the origin, this equation must also be satisfied when X is replaced by $-X$, giving us

$$(13.31)\qquad \|X\|^2 + 2F\cdot X + \|F\|^2 = e^2(X\cdot N)^2 + 2eaX\cdot N + a^2.$$

Subtracting (13.31) from (13.30), we have symmetry if and only if

$$F\cdot X = eaX\cdot N \qquad \text{or} \qquad (F - eaN)\cdot X = 0.$$

This equation can be satisfied for all X on the curve if and only if F and N are related by the equation

$$(13.32)\qquad F = eaN, \qquad \text{where} \quad a = ed + eF\cdot N.$$

The relation $F = eaN$ implies $F\cdot N = ea$, giving us $a = ed + e^2a$. If $e = 1$, this last equation cannot be satisfied since d, the distance from the focus to the directrix, is nonzero. This means there is no symmetry about the origin for a parabola. If $e \neq 1$, we can always satisfy the relations in (13.32) by taking

$$(13.33)\qquad a = \frac{ed}{1 - e^2}, \qquad F = \frac{e^2d}{1 - e^2}N.$$

Note that $a > 0$ if $e < 1$ and $a < 0$ if $e > 1$. Putting $F = eaN$ in (13.30) we obtain the following.

THEOREM 13.19. *Let C be a conic section with eccentricity $e \neq 1$ and with a focus F at a distance d from a directrix L. If N is a unit normal to L and if $F = eaN$, where $a = ed/(1 - e^2)$, then C is the set of all points X satisfying the equation*

$$(13.34)\qquad \|X\|^2 + e^2a^2 = e^2(X\cdot N)^2 + a^2.$$

This equation displays the symmetry about the origin since it is unchanged when X is replaced by $-X$. Because of this symmetry, the ellipse and the hyperbola each have two foci, symmetrically located about the center, and two directrices, also symmetrically located about the center.

Equation (13.34) is satisfied when $X = \pm aN$. These two points are called *vertices* of the conic. The segment joining them is called the *major axis* if the conic is an ellipse, the *transverse axis* if the conic is a hyperbola.

Let N' be a unit vector orthogonal to N. If $X = bN'$, then $X \cdot N = 0$, so Equation (13.34) is satisfied by $X = bN'$ if and only if $b^2 + e^2a^2 = a^2$. This requires $e < 1$, $b^2 = a^2(1 - e^2)$. The segment joining the points $X = \pm bN'$, where $b = a\sqrt{1 - e^2}$ is called the *minor axis* of the ellipse.

Note: If we put $e = 0$ in (13.34), it becomes $\|X\| = a$, the equation of a circle of radius a and center at the origin. In view of (13.33), we can consider such a circle as a limiting case of an ellipse in which $e \to 0$ and $d \to \infty$ in such a way that $ed \to a$.

13.23 Cartesian equations for the conic sections

To obtain Cartesian equations for the ellipse and hyperbola, we simply write (13.34) in terms of the rectangular coordinates of X. Choose $N = \boldsymbol{i}$ (which means the directrices are vertical) and let $X = (x, y)$. Then $\|X\|^2 = x^2 + y^2$, $X \cdot N = x$, and (13.34) becomes $x^2 + y^2 + e^2a^2 = e^2x^2 + a^2$, or $x^2(1 - e^2) + y^2 = a^2(1 - e^2)$, which gives us

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1 . \tag{13.35}$$

This Cartesian equation represents both the ellipse ($e < 1$) and the hyperbola ($e > 1$) and is said to be in *standard form*. The foci are at the points $(ae, 0)$ and $(-ae, 0)$; the directrices are the vertical lines $x = a/e$ and $x = -a/e$.

If $e < 1$, we let $b = a\sqrt{1 - e^2}$ and write the equation of the ellipse in the standard form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 . \tag{13.36}$$

Its foci are located at $(c, 0)$ and $(-c, 0)$, where $c = ae = \sqrt{a^2 - b^2}$. An example is shown in Figure 13.14(a).

If $e > 1$, we let $b = |a|\sqrt{e^2 - 1}$ and write the equation of the hyperbola in the standard form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 . \tag{13.37}$$

Its foci are at the points $(c, 0)$ and $(-c, 0)$, where $c = |a|\, e = \sqrt{a^2 + b^2}$. An example is shown in Figure 13.14(b).

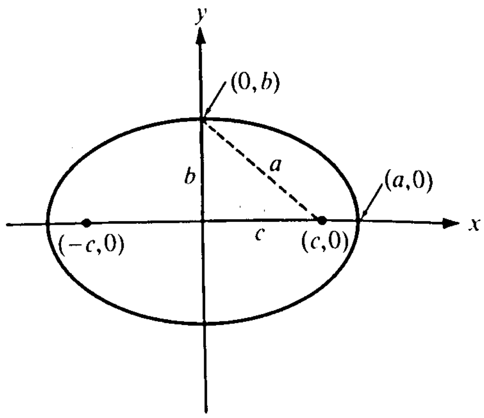
Note: Solving for y in terms of x in (13.37), we obtain two solutions

$$y = \pm \frac{b}{|a|}\sqrt{x^2 - a^2}. \tag{13.38}$$

For large positive x, the number $\sqrt{x^2 - a^2}$ is nearly equal to x, so the right member of (13.38) is nearly $\pm bx/|a|$. It is easy to prove that the difference between $y_1 = bx/|a|$ and $y_2 = b\sqrt{x^2 - a^2}/|a|$ approaches 0 as $x \to +\infty$. This difference is

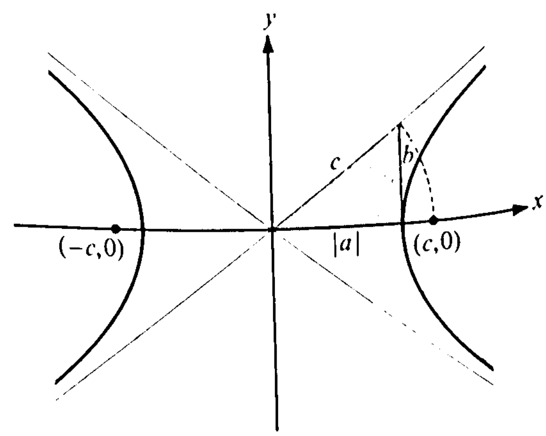
$$y_1 - y_2 = \frac{b}{|a|}(x - \sqrt{x^2 - a^2}) = \frac{b}{|a|}\frac{x^2 - (x^2 - a^2)}{x + \sqrt{x^2 - a^2}} = \frac{|a|\, b}{x + \sqrt{x^2 - a^2}} < \frac{|a|\, b}{x},$$

so $y_1 - y_2 \to 0$ as $x \to +\infty$. Therefore, the line $y = bx/|a|$ is an asymptote of the hyperbola. The line $y = -bx/|a|$ is another asymptote. The hyperbola is said to approach these lines asymptotically. The asymptotes are shown in Figure 13.14(b).



(a) Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1; \quad b^2 = a^2 - c^2$$



(b) Hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1; \quad b^2 = c^2 - a^2$$

FIGURE 13.14 Conic sections of eccentricity $e \neq 1$, symmetric about the origin. The foci are at $(\pm c, 0)$, where $c = |a|\, e$. The triangles relate a, b, c geometrically.

The Cartesian equation for the ellipse and hyperbola will take a different form if the directrices are not vertical. For example, if the directrices are taken to be horizontal, we may take $N = j$ in Equation (13.34). Since $X \cdot N = X \cdot j = y$, we obtain a Cartesian equation like (13.35), except that x and y are interchanged. The standard form in this case is

$$\frac{y^2}{a^2} + \frac{x^2}{a^2(1 - e^2)} = 1. \tag{13.39}$$

If the conic is translated by adding a vector $X_0 = (x_0, y_0)$ to each of its points, the center will be at (x_0, y_0) instead of at the origin. The corresponding Cartesian equations may be obtained from (13.35) or (13.39) by replacing x by $x - x_0$ and y by $y - y_0$.

To obtain a Cartesian equation for the parabola, we return to the basic equation (13.20) with $e = 1$. Take the directrix to be the vertical line $x = -c$ and place the focus at $(c, 0)$. If $X = (x, y)$, we have $X - F = (x - c, y)$, and Equation (13.20) gives us $(x - c)^2 + y^2 = |x + c|^2$. This simplifies to the standard form

$$y^2 = 4cx. \tag{13.40}$$

The point midway between the focus and directrix (the origin in Figure 13.15) is called the *vertex* of the parabola, and the line passing through the vertex and focus is the *axis* of the parabola. The parabola is symmetric about its axis. If $c > 0$, the parabola lies to the right of the y-axis, as in Figure 13.15. When $c < 0$, the curve lies to the left of the y-axis.

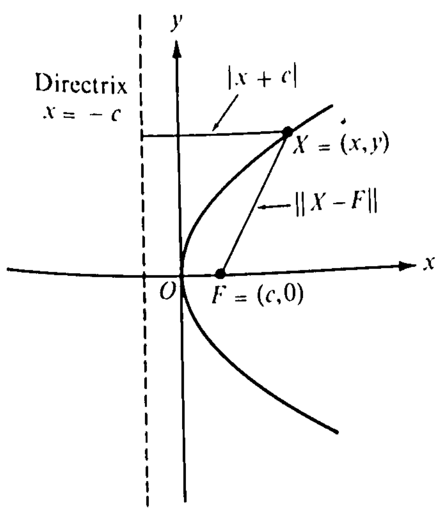


FIGURE 13.15 The parabola $y^2 = 4cx$.

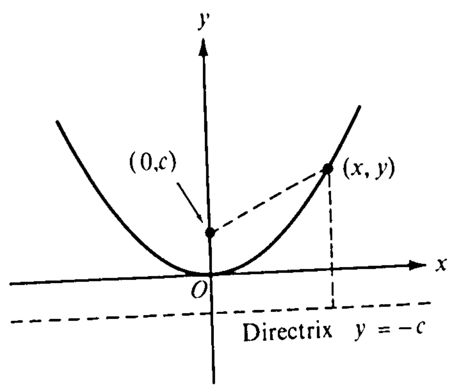


FIGURE 13.16 The parabola $x^2 = 4cy$.

If the axes are chosen so the focus is on the y-axis at the point $(0, c)$ and if the horizontal line $y = -c$ is taken as directrix, the standard form of the Cartesian equation becomes

$$x^2 = 4cy.$$

When $c > 0$ the parabola opens upward as shown in Figure 13.16. When $c < 0$, it opens downward.

If the parabola in Figure 13.15 is translated so that its vertex is at the point (x_0, y_0), the corresponding equation becomes

$$(y - y_0)^2 = 4c(x - x_0).$$

The focus is now at the point $(x_0 + c, y_0)$ and the directrix is the line $x = x_0 - c$. The axis of the parabola is the line $y = y_0$.

Similarly, a translation of the parabola in Figure 13.16 leads to the equation

$$(x - x_0)^2 = 4c(y - y_0),$$

with focus at $(x_0, y_0 + c)$. The line $y = y_0 - c$ is its directrix, the line $x = x_0$ its axis.

The reader may find it amusing to prove that a parabola does not have any asymptotes.

13.24 Exercises

Each of the equations in Exercises 1 through 6 represents an ellipse. Find the coordinates of the center, the foci, and the vertices, and sketch each curve. Also determine the eccentricity.

1. $\dfrac{x^2}{100} + \dfrac{y^2}{36} = 1.$
2. $\dfrac{y^2}{100} + \dfrac{x^2}{36} = 1.$
3. $\dfrac{(x-2)^2}{16} + \dfrac{(y+3)^2}{9} = 1.$
4. $9x^2 + 25y^2 = 25.$
5. $4y^2 + 3x^2 = 1.$
6. $\dfrac{(x+1)^2}{16} + \dfrac{(y+2)^2}{25} = 1.$

In each of Exercises 7 through 12, find a Cartesian equation (in the appropriate standard form) for the ellipse that satisfies the conditions given. Sketch each curve.

7. Center at (0, 0), one focus at $(\frac{3}{4}, 0)$, one vertex at (1, 0).
8. Center at (−3, 4), semiaxes of lengths 4 and 3, major axis parallel to the x-axis.
9. Same as Exercise 8, except with major axis parallel to the y-axis.
10. Vertices at (−1, 2), (−7, 2), minor axis of length 2.
11. Vertices at (3, −2), (13, −2), foci at (4, −2), (12, −2).
12. Center at (2, 1), major axis parallel to the x-axis, the curve passing through the points (6, 1) and (2, 3).

Each of the equations in Exercises 13 through 18 represents a hyperbola. Find the coordinates of the center, the foci, and the vertices. Sketch each curve and show the positions of the asymptotes. Also, compute the eccentricity.

13. $\dfrac{x^2}{100} - \dfrac{y^2}{64} = 1.$
14. $\dfrac{y^2}{100} - \dfrac{x^2}{64} = 1.$
15. $\dfrac{(x+3)^2}{4} - (y-3)^2 = 1.$
16. $9x^2 - 16y^2 = 144.$
17. $4x^2 - 5y^2 + 20 = 0.$
18. $\dfrac{(x-1)^2}{4} - \dfrac{(y+2)^2}{9} = 1.$

In each of Exercises 19 through 23, find a Cartesian equation (in the appropriate standard form) for the hyperbola which satisfies the conditions given. Sketch each curve and show the positions of the asymptotes.

19. Center at (0, 0), one focus at (4, 0), one vertex at (2, 0).
20. Foci at $(0, \pm\sqrt{2})$, vertices at $(0, \pm 1)$.
21. Vertices at $(\pm 2, 0)$, asymptotes $y = \pm 2x$.
22. Center at (−1, 4), one focus at (−1, 2), one vertex at (−1, 3).
23. Center at (2, −3), transverse axis parallel to one of the coordinate axes, the curve passing through (3, −1) and (−1, 0).
24. For what value (or values) of C will the line $3x - 2y = C$ be tangent to the hyperbola $x^2 - 3y^2 = 1$?
25. The asymptotes of a hyperbola are the lines $2x - y = 0$ and $2x + y = 0$. Find a Cartesian equation for the curve if it passes through the point (3, −5).

Each of the equations in Exercises 26 through 31 represents a parabola. Find the coordinates of the vertex, an equation for the directrix, and an equation for the axis. Sketch each of the curves.

26. $y^2 = -8x.$
27. $y^2 = 3x.$
28. $(y-1)^2 = 12x - 6.$
29. $x^2 = 6y.$
30. $x^2 + 8y = 0.$
31. $(x+2)^2 = 4y + 9.$

In each of Exercises 32 through 37, find a Cartesian equation (in appropriate standard form) for the parabola that satisfies the conditions given and sketch the curve.

32. Focus at $(0, -\frac{1}{4})$; equation of directrix, $y = \frac{1}{4}$.
33. Vertex at $(0, 0)$; equation of directrix, $x = -2$.
34. Vertex at $(-4, 3)$; focus at $(-4, 1)$.
35. Focus at $(3, -1)$; equation of directrix, $x = \frac{1}{2}$.
36. Axis is parallel to the y-axis; passes through $(0, 1)$, $(1, 0)$, and $(2, 0)$.
37. Axis is parallel to the x-axis; vertex at $(1, 3)$; passes through $(-1, -1)$.
38. Proceeding directly from the focal definition, find a Cartesian equation for the parabola whose focus is the origin and whose directrix is the line $2x + y = 10$.

13.25 Miscellaneous exercises on conic sections

1. Show that the area of the region bounded by the ellipse $x^2/a^2 + y^2/b^2 = 1$ is ab times the area of a circle of radius 1.

   Note: This statement can be proved from general properties of the integral, without performing any integrations.

2. (a) Show that the volume of the solid of revolution generated by rotating the ellipse $x^2/a^2 + y^2/b^2 = 1$ about its major axis is ab^2 times the volume of a unit sphere.

   Note: This statement can be proved from general properties of the integral, without performing any integrations.

   (b) What is the result if the ellipse is rotated about its minor axis?
3. Find all positive numbers A and B, $A > B$, such that the area of the region enclosed by the ellipse $Ax^2 + By^2 = 3$ is equal to the area of the region enclosed by the ellipse
$$(A + B)x^2 + (A - B)y^2 = 3 .$$
4. A parabolic arch has a base of length b and altitude h. Determine the area of the region bounded by the arch and the base.
5. The region bounded by the parabola $y^2 = 8x$ and the line $x = 2$ is rotated about the x-axis. Find the volume of the solid of revolution so generated.
6. Two parabolas having the equations $y^2 = 2(x - 1)$ and $y^2 = 4(x - 2)$ enclose a plane region R.
   (a) Compute the area of R by integration.
   (b) Find the volume of the solid of revolution generated by revolving R about the x-axis.
   (c) Same as (b), but revolve R about the y-axis.
7. Find a Cartesian equation for the conic section consisting of all points (x, y) whose distance from the point $(0, 2)$ is half the distance from the line $y = 8$.
8. Find a Cartesian equation for the parabola whose focus is at the origin and whose directrix is the line $x + y + 1 = 0$.
9. Find a Cartesian equation for a hyperbola passing through the origin, given that its asymptotes are the lines $y = 2x + 1$ and $y = -2x + 3$.
10. (a) For each $p > 0$, the equation $px^2 + (p + 2)y^2 = p^2 + 2p$ represents an ellipse. Find (in terms of p) the eccentricity and the coordinates of the foci.
    (b) Find a Cartesian equation for the hyperbola which has the same foci as the ellipse of part (a) and which has eccentricity $\sqrt{3}$.
11. In Section 13.22 we proved that a conic symmetric about the origin satisfies the equation $\|X - F\| = |eX \cdot N - a|$, where $a = ed + eF \cdot N$. Use this relation to prove that $\|X - F\| + \|X + F\| = 2a$ if the conic is an ellipse. In other words, the sum of the distances from any point on an ellipse to its foci is constant.

12. Refer to Exercise 11. Prove that on each branch of a hyperbola the difference $\|X - F\| - \|X + F\|$ is constant.
13. (a) Prove that a similarity transformation (replacing x by tx and y by ty) carries an ellipse with center at the origin into another ellipse with the same eccentricity. In other words, similar ellipses have the same eccentricity.
(b) Prove also the converse. That is, if two concentric ellipses have the same eccentricity and major axes on the same line, then they are related by a similarity transformation.
(c) Prove results corresponding to (a) and (b) for hyperbolas.
14. Use the Cartesian equation which represents all conics of eccentricity e and center at the origin to prove that these conics are integral curves of the differential equation $y' = (e^2 - 1)x/y$.

Note: Since this is a homogeneous differential equation (Section 8.25), the set of all such conics of eccentricity e is invariant under a similarity transformation. (Compare with Exercise 13.)

15. (a) Prove that the collection of all parabolas is invariant under a similarity transformation. That is, a similarity transformation carries a parabola into a parabola.
(b) Find all the parabolas similar to $y = x^2$.
16. The line $x - y + 4 = 0$ is tangent to the parabola $y^2 = 16x$. Find the point of contact.
17. (a) Given $a \neq 0$. If the two parabolas $y^2 = 4p(x - a)$ and $x^2 = 4qy$ are tangent to each other, show that the x-coordinate of the point of contact is determined by a alone.
(b) Find a condition on a, p, and q which expresses the fact that the two parabolas are tangent to each other.
18. Consider the locus of the points P in the plane for which the distance of P from the point $(2, 3)$ is equal to the sum of the distances of P from the two coordinate axes.
(a) Show that the part of this locus which lies in the first quadrant is part of a hyperbola. Locate the asymptotes and make a sketch.
(b) Sketch the graph of the locus in the other quadrants.
19. Two parabolas have the same point as focus and the same line as axis, but their vertices lie on opposite sides of the focus. Prove that the parabolas intersect orthogonally (i.e., their tangent lines are perpendicular at the points of intersection).
20. (a) Prove that the Cartesian equation

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$$

represents all conics symmetric about the origin with foci at $(c, 0)$ and $(-c, 0)$.
(b) Keep c fixed and let S denote the set of all such conics obtained as a^2 varies over all positive numbers $\neq c^2$. Prove that every curve in S satisfies the differential equation

$$xy\left(\frac{dy}{dx}\right)^2 + (x^2 - y^2 - c^2)\frac{dy}{dx} - xy = 0 .$$

(c) Prove that S is self-orthogonal; that is, the set of all orthogonal trajectories of curves in S is S itself. [*Hint:* Replace y' by $-1/y'$ in the differential equation in (b).]
21. Show that the locus of the centers of a family of circles, all of which pass through a given point and are tangent to a given line, is a parabola.
22. Show that the locus of the centers of a family of circles, all of which are tangent (externally) to a given circle and also to a given straight line, is a parabola. (Exercise 21 can be considered to be a special case.)

23. (a) A chord of length $8|c|$ is drawn perpendicular to the axis of the parabola $y^2 = 4cx$. Let P and Q be the points where the chord meets the parabola. Show that the vector from O to P is perpendicular to that from O to Q.
(b) The chord of a parabola drawn through the focus and parallel to the directrix is called the *latus rectum*. Show first that the length of the latus rectum is twice the distance from the focus to the directrix, and then show that the tangents to the parabola at both ends of the latus rectum intersect the axis of the parabola on the directrix.
24. Two points P and Q are said to be symmetric with respect to a circle if P and Q are collinear with the center, if the center is not between them, and if the product of their distances from the center is equal to the square of the radius. Given that Q describes the straight line $x + 2y - 5 = 0$, find the locus of the point P symmetric to Q with respect to the circle $x^2 + y^2 = 4$.