2. Note that

$$1 = 1\,,$$
$$1 - 4 = -(1 + 2)\,,$$
$$1 - 4 + 9 = 1 + 2 + 3\,,$$
$$1 - 4 + 9 - 16 = -(1 + 2 + 3 + 4)\,.$$

Guess the general law suggested and prove it by induction.

3. Note that

$$1 + \tfrac{1}{2} = 2 - \tfrac{1}{2}\,,$$
$$1 + \tfrac{1}{2} + \tfrac{1}{4} = 2 - \tfrac{1}{4}\,,$$
$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} = 2 - \tfrac{1}{8}\,.$$

Guess the general law suggested and prove it by induction.

4. Note that

$$1 - \tfrac{1}{2} = \tfrac{1}{2}\,,$$
$$(1 - \tfrac{1}{2})(1 - \tfrac{1}{3}) = \tfrac{1}{3}\,,$$
$$(1 - \tfrac{1}{2})(1 - \tfrac{1}{3})(1 - \tfrac{1}{4}) = \tfrac{1}{4}\,.$$

Guess the general law suggested and prove it by induction.

5. Guess a general law which simplifies the product

$$\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right)\left(1 - \frac{1}{16}\right) \cdots \left(1 - \frac{1}{n^2}\right)$$

and prove it by induction.

6. Let $A(n)$ denote the statement: $1 + 2 + \cdots + n = \frac{1}{8}(2n + 1)^2$.
   (a) Prove that if $A(k)$ is true for an integer k, then $A(k + 1)$ is also true.
   (b) Criticize the statement: "By induction it follows that $A(n)$ is true for all n."
   (c) Amend $A(n)$ by changing the equality to an inequality that is true for all positive integers n.
7. Let n_1 be the smallest positive integer n for which the inequality $(1 + x)^n > 1 + nx + nx^2$ is true for all $x > 0$. Compute n_1, and prove that the inequality is true for all integers $n \geq n_1$.
8. Given positive real numbers $a_1, a_2, a_3, \ldots$, such that $a_n \leq ca_{n-1}$ for all $n \geq 2$, where c is a fixed positive number, use induction to prove that $a_n \leq a_1 c^{n-1}$ for all $n \geq 1$.
9. Prove the following statement by induction: If a line of unit length is given, then a line of length $\sqrt{n}$ can be constructed with straightedge and compass for each positive integer n.
10. Let b denote a fixed positive integer. Prove the following statement by induction: For every integer $n \geq 0$, there exist nonnegative integers q and r such that

$$n = qb + r\,, \qquad 0 \leq r < b\,.$$

11. Let n and d denote integers. We say that d is a *divisor* of n if $n = cd$ for some integer c. An integer n is called a *prime* if $n > 1$ and if the only positive divisors of n are 1 and n. Prove, by induction, that every integer $n > 1$ is either a prime or a product of primes.
12. Describe the fallacy in the following "proof" by induction:

Statement. Given any collection of n blonde girls. If at least one of the girls has blue eyes, then all n of them have blue eyes.

"*Proof*." The statement is obviously true when $n = 1$. The step from k to $k + 1$ can be illustrated by going from $n = 3$ to $n = 4$. Assume, therefore, that the statement is true when $n = 3$ and let G_1, G_2, G_3, G_4 be four blonde girls, at least one of which, say G_1, has blue eyes. Taking G_1, G_2, and G_3 together and using the fact that the statement is true when $n = 3$, we find that G_2 and G_3 also have blue eyes. Repeating the process with G_1, G_2, and G_4, we find that G_4 has blue eyes. Thus all four have blue eyes. A similar argument allows us to make the step from k to $k + 1$ in general.

Corollary. All blonde girls have blue eyes.

Proof. Since there exists at least one blonde girl with blue eyes, we can apply the foregoing result to the collection consisting of all blonde girls.

Note: This example is from G. Pólya, who suggests that the reader may want to test the validity of the statement by experiment.

⋆I 4.5 Proof of the well-ordering principle

In this section we deduce the well-ordering principle from the principle of induction.

Let T be a nonempty collection of positive integers. We want to prove that T has a smallest member, that is, that there is a positive integer t_0 in T such that $t_0 \le t$ for all t in T.

Suppose T has no smallest member. We shall show that this leads to a contradiction. The integer 1 cannot be in T (otherwise it would be the smallest member of T). Let S denote the collection of all positive integers n such that $n < t$ for all t in T. Now 1 is in S because $1 < t$ for all t in T. Next, let k be a positive integer in S. Then $k < t$ for all t in T. We shall prove that $k + 1$ is also in S. If this were not so, then for some t_1 in T we would have $t_1 \le k + 1$. Since T has no smallest member, there is an integer t_2 in T such that $t_2 < t_1$, and hence $t_2 < k + 1$. But this means that $t_2 \le k$, contradicting the fact that $k < t$ for all t in T. Therefore $k + 1$ is in S. By the induction principle, S contains all positive integers. Since T is nonempty, there is a positive integer t in T. But this t must also be in S (since S contains all positive integers). It follows from the definition of S that $t < t$, which is a contradiction. Therefore, the assumption that T has no smallest member leads to a contradiction. It follows that T must have a smallest member, and in turn this proves that the well-ordering principle is a consequence of the principle of induction.

I 4.6 The summation notation

In the calculations for the area of the parabolic segment, we encountered the sum

$$1^2 + 2^2 + 3^2 + \cdots + n^2 . \tag{I.20}$$

Note that a typical term in this sum is of the form k^2, and we get all the terms by letting k run through the values $1, 2, 3, \ldots, n$. There is a very useful and convenient notation which enables us to write sums like this in a more compact form. This is called the *summation notation* and it makes use of the Greek letter sigma, $\sum$. Using summation notation, we can write the sum in (I.20) as follows:

$$\sum_{k=1}^{n} k^2.$$

This symbol is read: "The sum of k^2 for k running from 1 to n." The numbers appearing under and above the sigma tell us the range of values taken by k. The letter k itself is referred to as the *index of summation*. Of course, it is not important that we use the letter k; any other convenient letter may take its place. For example, instead of $\sum_{k=1}^{n} k^2$ we could write $\sum_{i=1}^{n} i^2$, $\sum_{j=1}^{n} j^2$, $\sum_{m=1}^{n} m^2$, etc., all of which are considered as alternative notations for the same thing. The letters i, j, k, m, etc. that are used in this way are called *dummy indices*. It would not be a good idea to use the letter n for the dummy index in this particular example because n is already being used for the number of terms.

More generally, when we want to form the sum of several real numbers, say $a_1, a_2, \ldots, a_n$, we denote such a sum by the symbol

(I.21) $$a_1 + a_2 + \cdots + a_n$$

which, using summation notation, can be written as follows:

(I.22) $$\sum_{k=1}^{n} a_k .$$

For example, we have

$$\sum_{k=1}^{4} a_k = a_1 + a_2 + a_3 + a_4 ,$$

$$\sum_{i=1}^{5} x_i = x_1 + x_2 + x_3 + x_4 + x_5 .$$

Sometimes it is convenient to begin summations from 0 or from some value of the index beyond 1. For example, we have

$$\sum_{i=0}^{4} x_i = x_0 + x_1 + x_2 + x_3 + x_4 ,$$

$$\sum_{n=2}^{5} n^3 = 2^3 + 3^3 + 4^3 + 5^3.$$

Other uses of the summation notation are illustrated below:

$$\sum_{m=0}^{4} x^{m+1} = x + x^2 + x^3 + x^4 + x^5,$$

$$\sum_{j=1}^{6} 2^{j-1} = 1 + 2 + 2^2 + 2^3 + 2^4 + 2^5.$$

To emphasize once more that the choice of dummy index is unimportant, we note that the last sum may also be written in each of the following forms:

$$\sum_{q=1}^{6} 2^{q-1} = \sum_{r=0}^{5} 2^r = \sum_{n=0}^{5} 2^{5-n} = \sum_{k=1}^{6} 2^{6-k}.$$

Note: From a strictly logical standpoint, the symbols in (I.21) and (I.22) do not appear among the primitive symbols for the real-number system. In a more careful treatment, we could define these new symbols in terms of the primitive undefined symbols of our system.

This may be done by a process known as *definition by induction* which, like proof by induction, consists of two parts:

(a) We define

$$\sum_{k=1}^{1} a_k = a_1 .$$

(b) Assuming that we have defined $\sum_{k=1}^{n} a_k$ for a fixed $n \geq 1$, we further define

$$\sum_{k=1}^{n+1} a_k = \left(\sum_{k=1}^{n} a_k \right) + a_{n+1} .$$

To illustrate, we may take $n = 1$ in (b) and use (a) to obtain

$$\sum_{k=1}^{2} a_k = \sum_{k=1}^{1} a_k + a_2 = a_1 + a_2 .$$

Now, having defined $\sum_{k=1}^{2} a_k$, we can use (b) again with $n = 2$ to obtain

$$\sum_{k=1}^{3} a_k = \sum_{k=1}^{2} a_k + a_3 = (a_1 + a_2) + a_3 .$$

By the associative law for addition (Axiom 2), the sum $(a_1 + a_2) + a_3$ is the same as $a_1 + (a_2 + a_3)$, and therefore there is no danger of confusion if we drop the parentheses and simply write $a_1 + a_2 + a_3$ for $\sum_{k=1}^{3} a_k$. Similarly, we have

$$\sum_{k=1}^{4} a_k = \sum_{k=1}^{3} a_k + a_4 = (a_1 + a_2 + a_3) + a_4 .$$

In this case we can *prove* that the sum $(a_1 + a_2 + a_3) + a_4$ is the same as $(a_1 + a_2) + (a_3 + a_4)$ or $a_1 + (a_2 + a_3 + a_4)$, and therefore the parentheses can be dropped again without danger of ambiguity, and we agree to write

$$\sum_{k=1}^{4} a_k = a_1 + a_2 + a_3 + a_4 .$$

Continuing in this way, we find that (a) and (b) together give us a complete definition of the symbol in (I.22). The notation in (I.21) is considered to be merely an alternative way of writing (I.22). It is justified by a general associative law for addition which we shall not attempt to state or to prove here.

The reader should notice that *definition by induction* and *proof by induction* involve the same underlying idea. A definition by induction is also called a *recursive definition*.

I 4.7 Exercises

1. Find the numerical values of the following sums:

(a) $\sum_{k=1}^{4} k$, (c) $\sum_{r=0}^{3} 2^{2r+1}$, (e) $\sum_{i=0}^{5} (2i + 1)$,

(b) $\sum_{n=2}^{5} 2^{n-2}$, (d) $\sum_{n=1}^{4} n^n$, (f) $\sum_{k=1}^{5} \frac{1}{k(k+1)}$.

2. Establish the following properties of the summation notation:

(a) $\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$ (additive property).

(b) $\sum_{k=1}^{n}(ca_k) = c\sum_{k=1}^{n} a_k$ (homogeneous property).

(c) $\sum_{k=1}^{n}(a_k - a_{k-1}) = a_n - a_0$ (telescoping property).

Use the properties in Exercise 2 whenever possible to derive the formulas in Exercises 3 through 8.

3. $\sum_{k=1}^{n} 1 = n$. (This means $\sum_{k=1}^{n} a_k$, where each $a_k = 1$.)

4. $\sum_{k=1}^{n}(2k - 1) = n^2$. [*Hint:* $2k - 1 = k^2 - (k-1)^2$.]

5. $\sum_{k=1}^{n} k = \frac{n^2}{2} + \frac{n}{2}$. [*Hint:* Use Exercises 3 and 4.]

6. $\sum_{k=1}^{n} k^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$. [*Hint:* $k^3 - (k-1)^3 = 3k^2 - 3k + 1$.]

7. $\sum_{k=1}^{n} k^3 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}$.

8. (a) $\sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}$ if $x \neq 1$. *Note:* x^0 is defined to be 1.
[*Hint:* Apply Exercise 2 to $(1 - x)\sum_{k=0}^{n} x^k$.]
(b) What is the sum equal to when $x = 1$?

9. Prove, by induction, that the sum $\sum_{k=1}^{2n}(-1)^k(2k+1)$ is proportional to n, and find the constant of proportionality.

10. (a) Give a reasonable definition of the symbol $\sum_{k=m}^{m+n} a_k$.
(b) Prove, by induction, that for $n \geq 1$ we have

$$\sum_{k=n+1}^{2n} \frac{1}{k} = \sum_{m=1}^{2n} \frac{(-1)^{m+1}}{m}.$$

11. Determine whether each of the following statements is true or false. In each case give a reason for your decision.

(a) $\sum_{n=0}^{100} n^4 = \sum_{n=1}^{100} n^4$.

(b) $\sum_{j=0}^{100} 2 = 200$.

(c) $\sum_{k=0}^{100}(2 + k) = 2 + \sum_{k=0}^{100} k$.

(d) $\sum_{i=1}^{100}(i+1)^2 = \sum_{i=0}^{99} i^2$.

(e) $\sum_{k=1}^{100} k^3 = \left(\sum_{k=1}^{100} k\right) \cdot \left(\sum_{k=1}^{100} k^2\right)$.

(f) $\sum_{k=0}^{100} k^3 = \left(\sum_{k=0}^{100} k\right)^3$.

12. Guess and prove a general rule which simplifies the sum

$$\sum_{k=1}^{n} \frac{1}{k(k+1)}.$$

13. Prove that $2(\sqrt{n+1} - \sqrt{n}) < \frac{1}{\sqrt{n}} < 2(\sqrt{n} - \sqrt{n-1})$ if $n \geq 1$. Then use this to prove that

$$2\sqrt{m} - 2 < \sum_{n=1}^{m} \frac{1}{\sqrt{n}} < 2\sqrt{m} - 1$$

if $m \geq 2$. In particular, when $m = 10^6$, the sum lies between 1998 and 1999.

I 4.8 Absolute values and the triangle inequality

Calculations with inequalities arise quite frequently in calculus. They are of particular importance in dealing with the notion of *absolute value*. If x is a real number, the absolute value of x is a nonnegative real number denoted by $|x|$ and defined as follows:

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x \leq 0. \end{cases}$$

Note that $-|x| \leq x \leq |x|$. When real numbers are represented geometrically on a real axis, the number $|x|$ is called the *distance* of x from 0. If $a > 0$ and if a point x lies between $-a$ and a, then $|x|$ is nearer to 0 than a is. The analytic statement of this fact is given by the following theorem.

THEOREM I.38. *If $a \geq 0$, then $|x| \leq a$ if and only if $-a \leq x \leq a$.*

Proof. There are two statements to prove: first, that the inequality $|x| \leq a$ implies the two inequalities $-a \leq x \leq a$ and, conversely, that $-a \leq x \leq a$ implies $|x| \leq a$.

Suppose $|x| \leq a$. Then we also have $-a \leq -|x|$. But either $x = |x|$ or $x = -|x|$ and hence $-a \leq -|x| \leq x \leq |x| \leq a$. This proves the first statement.

To prove the converse, assume $-a \leq x \leq a$. Then if $x \geq 0$, we have $|x| = x \leq a$, whereas if $x \leq 0$, we have $|x| = -x \leq a$. In either case we have $|x| \leq a$, and this completes the proof.

Figure I.9 illustrates the geometrical significance of this theorem.

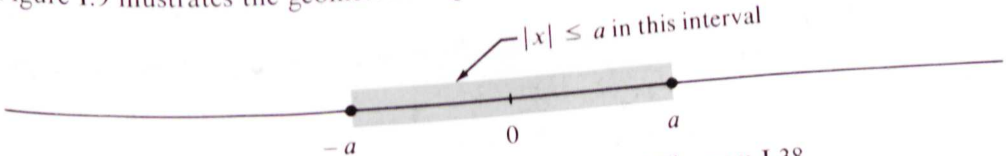

FIGURE I.9 Geometrical significance of Theorem I.38.

As a consequence of Theorem I.38, it is easy to derive an important inequality which states that the absolute value of a sum of two real numbers cannot exceed the sum of their absolute values.

THEOREM I.39. *For arbitrary real numbers x and y, we have*

$$|x + y| \le |x| + |y| .$$

Note: This property is called the *triangle inequality*, because when it is generalized to vectors it states that the length of any side of a triangle is less than or equal to the sum of the lengths of the other two sides.

Proof. Adding the inequalities $-|x| \le x \le |x|$ and $-|y| \le y \le |y|$, we obtain

$$-(|x| + |y|) \le x + y \le |x| + |y| ,$$

and hence, by Theorem I.38, we conclude that $|x + y| \le |x| + |y|$.

If we take $x = a - c$ and $y = c - b$, then $x + y = a - b$ and the triangle inequality becomes

$$|a - b| \le |a - c| + |b - c| .$$

This form of the triangle inequality is often used in practice.

Using mathematical induction, we may extend the triangle inequality as follows:

THEOREM I.40. *For arbitrary real numbers $a_1, a_2, \ldots, a_n$, we have*

$$\left|\sum_{k=1}^{n} a_k\right| \le \sum_{k=1}^{n} |a_k| .$$

Proof. When $n = 1$ the inequality is trivial, and when $n = 2$ it is the triangle inequality. Assume, then, that it is true for n real numbers. Then for $n + 1$ real numbers $a_1, a_2, \ldots, a_{n+1}$, we have

$$\left|\sum_{k=1}^{n+1} a_k\right| = \left|\sum_{k=1}^{n} a_k + a_{n+1}\right| \le \left|\sum_{k=1}^{n} a_k\right| + |a_{n+1}| \le \sum_{k=1}^{n} |a_k| + |a_{n+1}| = \sum_{k=1}^{n+1} |a_k| .$$

Hence the theorem is true for $n + 1$ numbers if it is true for n. By induction, it is true for every positive integer n.

The next theorem describes an important inequality that we shall use later in connection with our study of vector algebra.

THEOREM I.41. THE CAUCHY–SCHWARZ INEQUALITY. *If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are arbitrary real numbers, we have*

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \le \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) . \tag{I.23}$$

The equality sign holds if and only if there is a real number x such that $a_k x + b_k = 0$ for each $k = 1, 2, \ldots, n$.

Proof. We have $\sum_{k=1}^{n} (a_k x + b_k)^2 \geq 0$ for every real x because a sum of squares can never be negative. This may be written in the form

$$Ax^2 + 2Bx + C \geq 0\,, \tag{I.24}$$

where

$$A = \sum_{k=1}^{n} a_k^2\,, \qquad B = \sum_{k=1}^{n} a_k b_k\,, \qquad C = \sum_{k=1}^{n} b_k^2\,.$$

We wish to prove that $B^2 \leq AC$. If $A = 0$, then each $a_k = 0$, so $B = 0$ and the result is trivial. If $A \neq 0$, we may complete the square and write

$$Ax^2 + 2Bx + C = A\left(x + \frac{B}{A}\right)^2 + \frac{AC - B^2}{A}\,.$$

The right side has its smallest value when $x = -B/A$. Putting $x = -B/A$ in (I.24), we obtain $B^2 \leq AC$. This proves (I.23). The reader should verify that the equality sign holds if and only if there is an x such that $a_k x + b_k = 0$ for each k.

I 4.9 Exercises

1. Prove each of the following properties of absolute values.
   (a) $|x| = 0$ if and only if $x = 0$.
   (b) $|-x| = |x|$.
   (c) $|x - y| = |y - x|$.
   (d) $|x|^2 = x^2$.
   (e) $|x| = \sqrt{x^2}$.
   (f) $|xy| = |x|\,|y|$.
   (g) $|x/y| = |x|/|y|$ if $y \neq 0$.
   (h) $|x - y| \leq |x| + |y|$.
   (i) $|x| - |y| \leq |x - y|$.
   (j) $\big|\,|x| - |y|\,\big| \leq |x - y|$.
2. Each inequality (a_i), listed below, is equivalent to exactly one inequality (b_j). For example, $|x| < 3$ if and only if $-3 < x < 3$, and hence (a_1) is equivalent to (b_2). Determine all equivalent pairs.
   (a_1) $|x| < 3$.
   (a_2) $|x - 1| < 3$.
   (a_3) $|3 - 2x| < 1$.
   (a_4) $|1 + 2x| \leq 1$.
   (a_5) $|x - 1| > 2$.
   (a_6) $|x + 2| \geq 5$.
   (a_7) $|5 - x^{-1}| < 1$.
   (a_8) $|x - 5| < |x + 1|$.
   (a_9) $|x^2 - 2| \leq 1$.
   (a_{10}) $x < x^2 - 12 < 4x$.
   (b_1) $4 < x < 6$.
   (b_2) $-3 < x < 3$.
   (b_3) $x > 3$ or $x < -1$.
   (b_4) $x > 2$.
   (b_5) $-2 < x < 4$.
   (b_6) $-\sqrt{3} \leq x \leq -1$ or $1 \leq x \leq \sqrt{3}$.
   (b_7) $1 < x < 2$.
   (b_8) $x \leq -7$ or $x \geq 3$.
   (b_9) $\frac{1}{6} < x < \frac{1}{4}$.
   (b_{10}) $-1 \leq x \leq 0$.
3. Determine whether each of the following is true or false. In each case give a reason for your decision.
   (a) $x < 5$ implies $|x| < 5$.
   (b) $|x - 5| < 2$ implies $3 < x < 7$.
   (c) $|1 + 3x| \leq 1$ implies $x \geq -\frac{2}{3}$.
   (d) There is no real x for which $|x - 1| = |x - 2|$.
   (e) For every $x > 0$ there is a $y > 0$ such that $|2x + y| = 5$.
4. Show that the equality sign holds in the Cauchy–Schwarz inequality if and only if there is a real number x such that $a_k x + b_k = 0$ for every $k = 1, 2, \ldots, n$.

⋆I 4.10 Miscellaneous exercises involving induction

In this section we assemble a number of miscellaneous facts whose proofs are good exercises in the use of mathematical induction. Some of these exercises may serve as a basis for supplementary classroom discussion.

Factorials and binomial coefficients. The symbol $n!$ (read "n *factorial*") may be defined by induction as follows: $0! = 1$, $n! = (n-1)!\,n$ if $n \ge 1$. Note that $n! = 1 \cdot 2 \cdot 3 \cdots n$.

If $0 \le k \le n$, the *binomial coefficient* $\binom{n}{k}$ is defined as follows:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

Note: Sometimes ${}_nC_k$ is written for $\binom{n}{k}$. These numbers appear as coefficients in the binomial theorem. (See Exercise 4 below.)

1. Compute the values of the following binomial coefficients:
   (a) $\binom{5}{3}$, (b) $\binom{7}{0}$, (c) $\binom{7}{1}$, (d) $\binom{7}{2}$, (e) $\binom{17}{14}$, (f) $\binom{0}{0}$.
2. (a) Show that $\binom{n}{k} = \binom{n}{n-k}$.
   (b) Find n, given that $\binom{n}{10} = \binom{n}{7}$.
   (c) Find k, given that $\binom{14}{k} = \binom{14}{k-4}$.
   (d) Is there a k such that $\binom{12}{k} = \binom{12}{k-3}$?
3. Prove that $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$. This is called the *law of Pascal's triangle* and it provides a rapid way of computing binomial coefficients successively. Pascal's triangle is illustrated here for $n \le 6$.

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

4. Use induction to prove the binomial theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

Then use the theorem to derive the formulas

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \quad \text{and} \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0, \quad \text{if} \quad n > 0.$$

The product notation. The product of n real numbers $a_1, a_2, \ldots, a_n$ is denoted by the symbol $\prod_{k=1}^{n} a_k$, which may be defined by induction. The symbol $a_1 a_2 \cdots a_n$ is an alternative notation for this product. Note that

$$n! = \prod_{k=1}^{n} k.$$

5. Give a definition by induction for the product $\prod_{k=1}^{n} a_k$.

Prove the following properties of products by induction:

6. $\prod_{k=1}^{n} (a_k b_k) = \left(\prod_{k=1}^{n} a_k\right)\left(\prod_{k=1}^{n} b_k\right)$ (multiplicative property).

An important special case is the relation $\prod_{k=1}^{n} (ca_k) = c^n \prod_{k=1}^{n} a_k$.

7. $\prod_{k=1}^{n} \frac{a_k}{a_{k-1}} = \frac{a_n}{a_0}$ if each $a_k \neq 0$ (telescoping property).

8. If $x \neq 1$, show that

$$\prod_{k=1}^{n} (1 + x^{2^{k-1}}) = \frac{1 - x^{2^n}}{1 - x}.$$

What is the value of the product when $x = 1$?

9. If $a_k < b_k$ for each $k = 1, 2, \ldots, n$, it is easy to prove by induction that $\sum_{k=1}^{n} a_k < \sum_{k=1}^{n} b_k$. Discuss the corresponding inequality for products:

$$\prod_{k=1}^{n} a_k < \prod_{k=1}^{n} b_k.$$

Some special inequalities

10. If $x > 1$, prove by induction that $x^n > x$ for every integer $n \geq 2$. If $0 < x < 1$, prove that $x^n < x$ for every integer $n \geq 2$.
11. Determine all positive integers n for which $2^n < n!$.
12. (a) Use the binomial theorem to prove that for n a positive integer we have

$$\left(1 + \frac{1}{n}\right)^n = 1 + \sum_{k=1}^{n} \left\{\frac{1}{k!} \prod_{r=0}^{k-1} \left(1 - \frac{r}{n}\right)\right\}.$$

(b) If $n > 1$, use part (a) and Exercise 11 to deduce the inequalities

$$2 < \left(1 + \frac{1}{n}\right)^n < 1 + \sum_{k=1}^{n} \frac{1}{k!} < 3.$$

13. (a) Let p be a positive integer. Prove that

$$b^p - a^p = (b - a)(b^{p-1} + b^{p-2}a + b^{p-3}a^2 + \cdots + ba^{p-2} + a^{p-1}).$$

[*Hint:* Use the telescoping property for sums.]

(b) Let p and n denote positive integers. Use part (a) to show that

$$n^p < \frac{(n + 1)^{p+1} - n^{p+1}}{p + 1} < (n + 1)^p.$$

*I 4.10 Miscellaneous exercises involving induction

In this section we assemble a number of miscellaneous facts whose proofs are good exercises in the use of mathematical induction. Some of these exercises may serve as a basis for supplementary classroom discussion.

Factorials and binomial coefficients. The symbol $n!$ (read "n *factorial*") may be defined by induction as follows: $0! = 1$, $n! = (n-1)!\,n$ if $n \geq 1$. Note that $n! = 1 \cdot 2 \cdot 3 \cdots n$.

If $0 \leq k \leq n$, the *binomial coefficient* $\binom{n}{k}$ is defined as follows:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

Note: Sometimes ${}_nC_k$ is written for $\binom{n}{k}$. These numbers appear as coefficients in the binomial theorem. (See Exercise 4 below.)

1. Compute the values of the following binomial coefficients:
   (a) $\binom{5}{3}$, (b) $\binom{7}{0}$, (c) $\binom{7}{1}$, (d) $\binom{7}{2}$, (e) $\binom{17}{14}$, (f) $\binom{0}{0}$.
2. (a) Show that $\binom{n}{k} = \binom{n}{n-k}$. (c) Find k, given that $\binom{14}{k} = \binom{14}{k-4}$.
   (b) Find n, given that $\binom{n}{10} = \binom{n}{7}$. (d) Is there a k such that $\binom{12}{k} = \binom{12}{k-3}$?
3. Prove that $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$. This is called the *law of Pascal's triangle* and it provides a rapid way of computing binomial coefficients successively. Pascal's triangle is illustrated here for $n \leq 6$.

$$\begin{array}{ccccccccccccc} &&&&&&1&&&&&& \\ &&&&&1&&1&&&&& \\ &&&&1&&2&&1&&&& \\ &&&1&&3&&3&&1&&& \\ &&1&&4&&6&&4&&1&& \\ &1&&5&&10&&10&&5&&1& \\ 1&&6&&15&&20&&15&&6&&1 \end{array}$$

4. Use induction to prove the binomial theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

Then use the theorem to derive the formulas

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \quad \text{and} \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0, \quad \text{if } n > 0.$$

The product notation. The product of n real numbers $a_1, a_2, \ldots, a_n$ is denoted by the symbol $\prod_{k=1}^{n} a_k$, which may be defined by induction. The symbol $a_1 a_2 \cdots a_n$ is an alternative notation for this product. Note that

$$n! = \prod_{k=1}^{n} k.$$

5. Give a definition by induction for the product $\prod_{k=1}^{n} a_k$.

Prove the following properties of products by induction:

6. $\prod_{k=1}^{n} (a_k b_k) = \left(\prod_{k=1}^{n} a_k\right)\left(\prod_{k=1}^{n} b_k\right)$ (multiplicative property).

An important special case is the relation $\prod_{k=1}^{n} (ca_k) = c^n \prod_{k=1}^{n} a_k$.

7. $\prod_{k=1}^{n} \frac{a_k}{a_{k-1}} = \frac{a_n}{a_0}$ if each $a_k \neq 0$ (telescoping property).

8. If $x \neq 1$, show that

$$\prod_{k=1}^{n} (1 + x^{2^{k-1}}) = \frac{1 - x^{2^n}}{1 - x}.$$

What is the value of the product when $x = 1$?

9. If $a_k < b_k$ for each $k = 1, 2, \ldots, n$, it is easy to prove by induction that $\sum_{k=1}^{n} a_k < \sum_{k=1}^{n} b_k$. Discuss the corresponding inequality for products:

$$\prod_{k=1}^{n} a_k < \prod_{k=1}^{n} b_k.$$

Some special inequalities

10. If $x > 1$, prove by induction that $x^n > x$ for every integer $n \geq 2$. If $0 < x < 1$, prove that $x^n < x$ for every integer $n \geq 2$.
11. Determine all positive integers n for which $2^n < n!$.
12. (a) Use the binomial theorem to prove that for n a positive integer we have

$$\left(1 + \frac{1}{n}\right)^n = 1 + \sum_{k=1}^{n} \left\{\frac{1}{k!} \prod_{r=0}^{k-1} \left(1 - \frac{r}{n}\right)\right\}.$$

(b) If $n > 1$, use part (a) and Exercise 11 to deduce the inequalities

$$2 < \left(1 + \frac{1}{n}\right)^n < 1 + \sum_{k=1}^{n} \frac{1}{k!} < 3.$$

13. (a) Let p be a positive integer. Prove that

$$b^p - a^p = (b - a)(b^{p-1} + b^{p-2}a + b^{p-3}a^2 + \cdots + ba^{p-2} + a^{p-1}).$$

[*Hint:* Use the telescoping property for sums.]

(b) Let p and n denote positive integers. Use part (a) to show that

$$n^p < \frac{(n + 1)^{p+1} - n^{p+1}}{p + 1} < (n + 1)^p.$$

(c) Use induction to prove that

$$\sum_{k=1}^{n-1} k^p < \frac{n^{p+1}}{p+1} < \sum_{k=1}^{n} k^p .$$

Part (b) will assist in making the inductive step from n to $n + 1$.

14. Let $a_1, \ldots, a_n$ be n real numbers, all having the same sign and all greater than -1. Use induction to prove that

$$(1 + a_1)(1 + a_2) \cdots (1 + a_n) \geq 1 + a_1 + a_2 + \cdots + a_n .$$

In particular, when $a_1 = a_2 = \cdots = a_n = x$, where $x > -1$, this yields

(I.25) $$(1 + x)^n \geq 1 + nx \qquad \textit{(Bernoulli's inequality)}.$$

Show that when $n > 1$ the equality sign holds in (I.25) only for $x = 0$.

15. If $n \geq 2$, prove that $n!/n^n \leq (\frac{1}{2})^k$, where k is the greatest integer $\leq n/2$.

16. The numbers 1, 2, 3, 5, 8, 13, 21, . . . , in which each term after the second is the sum of its two predecessors, are called *Fibonacci numbers*. They may be defined by induction as follows:

$$a_1 = 1, \qquad a_2 = 2, \qquad a_{n+1} = a_n + a_{n-1} \quad \text{if} \quad n \geq 2 .$$

Prove that

$$a_n < \left(\frac{1 + \sqrt{5}}{2}\right)^n$$

for every $n \geq 1$.

Inequalities relating different types of averages. Let $x_1, x_2, \ldots, x_n$ be n positive real numbers. If p is a nonzero integer, the *pth-power mean* M_p of the n numbers is defined as follows:

$$M_p = \left(\frac{x_1^p + \cdots + x_n^p}{n}\right)^{1/p} .$$

The number M_1 is also called the *arithmetic mean*, M_2 the *root mean square*, and M_{-1} the *harmonic mean*.

17. If $p > 0$, prove that $M_p < M_{2p}$ when $x_1, x_2, \ldots, x_n$ are not all equal.

[*Hint:* Apply the Cauchy–Schwarz inequality with $a_k = x_k^p$ and $b_k = 1$.]

18. Use the result of Exercise 17 to prove that

$$a^4 + b^4 + c^4 \geq \tfrac{64}{3}$$

if $a^2 + b^2 + c^2 = 8$ and $a > 0$, $b > 0$, $c > 0$.

19. Let $a_1, \ldots, a_n$ be n positive real numbers whose product is equal to 1. Prove that $a_1 + \cdots + a_n \geq n$ and that the equality sign holds only if every $a_k = 1$.

[*Hint:* Consider two cases: (a) All $a_k = 1$; (b) not all $a_k = 1$. Use induction. In case (b) notice that if $a_1 a_2 \cdots a_{n+1} = 1$, then at least one factor, say a_1, exceeds 1 and at least one factor, say a_{n+1}, is less than 1. Let $b_1 = a_1 a_{n+1}$ and apply the induction hypothesis to the product $b_1 a_2 \cdots a_n$, using the fact that $(a_1 - 1)(a_{n+1} - 1) < 0$.]

20. The *geometric mean* G of n positive real numbers $x_1, \ldots, x_n$ is defined by the formula $G = (x_1 x_2 \cdots x_n)^{1/n}$.
(a) Let M_p denote the pth power mean. Prove that $G \leq M_1$ and that $G = M_1$ only when $x_1 = x_2 = \cdots = x_n$.
(b) Let p and q be integers, $q < 0 < p$. From part (a) deduce that $M_q < G < M_p$ when $x_1, x_2, \ldots, x_n$ are not all equal.

21. Use the result of Exercise 20 to prove the following statement: If a, b, and c are positive real numbers such that $abc = 8$, then $a + b + c \geq 6$ and $ab + ac + bc \geq 12$.

22. If $x_1, \ldots, x_n$ are positive numbers and if $y_k = 1/x_k$, prove that

$$\left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} y_k\right) \geq n^2 .$$

23. If a, b, and c are positive and if $a + b + c = 1$, prove that $(1 - a)(1 - b)(1 - c) \geq 8abc$.