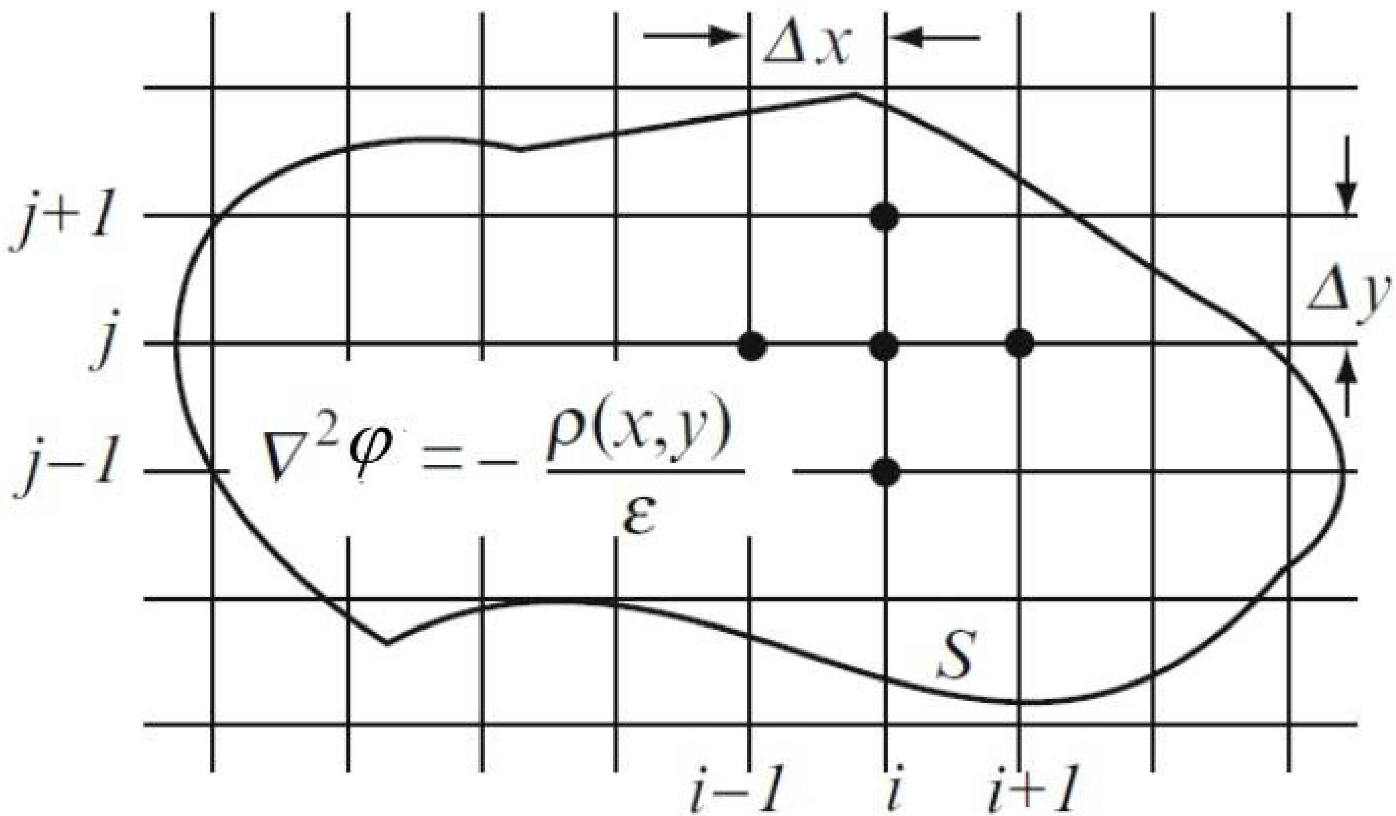
ESCOLA POLITÉCNICA DA UNIVERSIDADE DE SÃO PAULO

Solução da Equação de Laplace

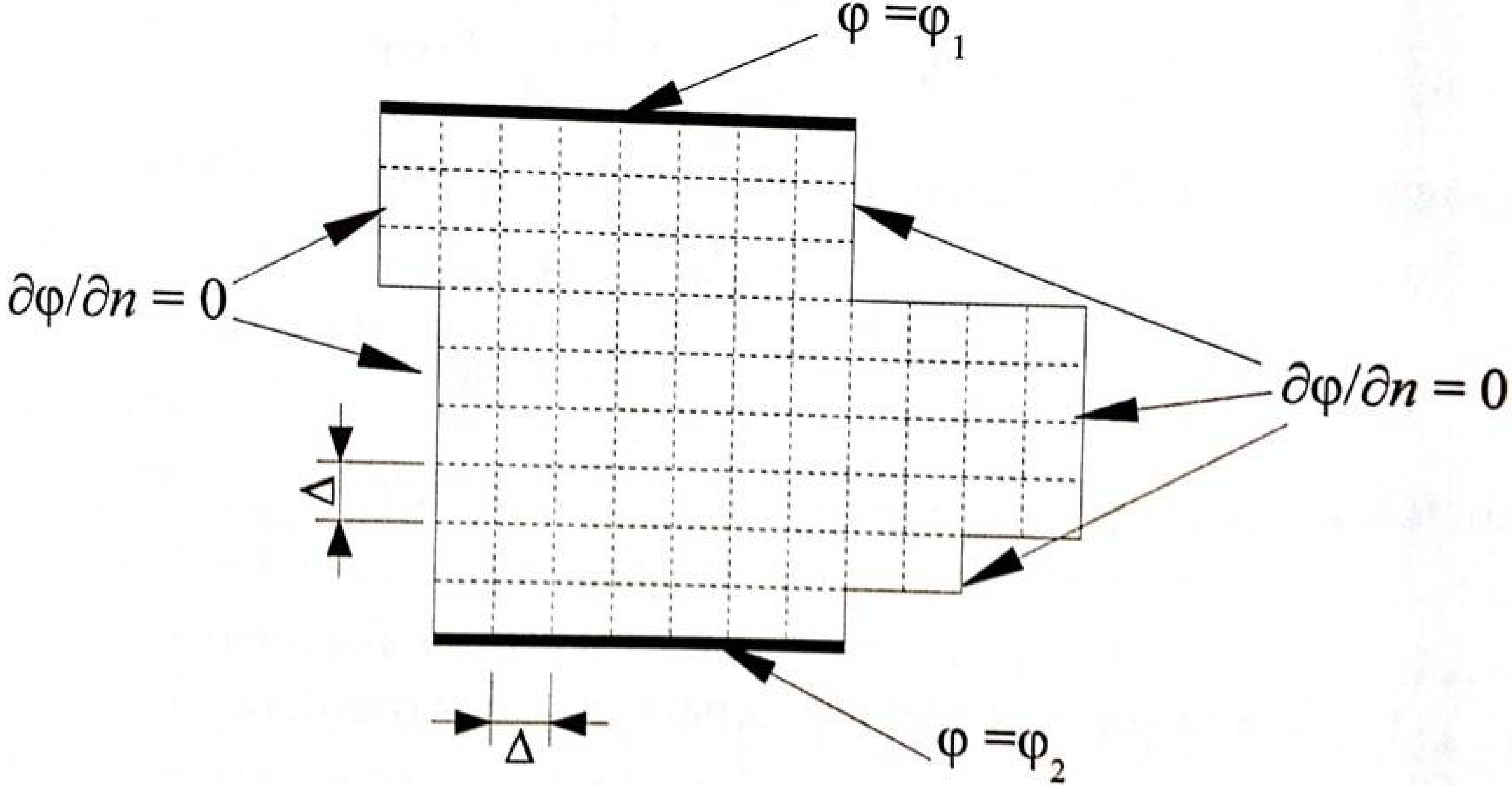
Método das Diferenças Finitas

2D

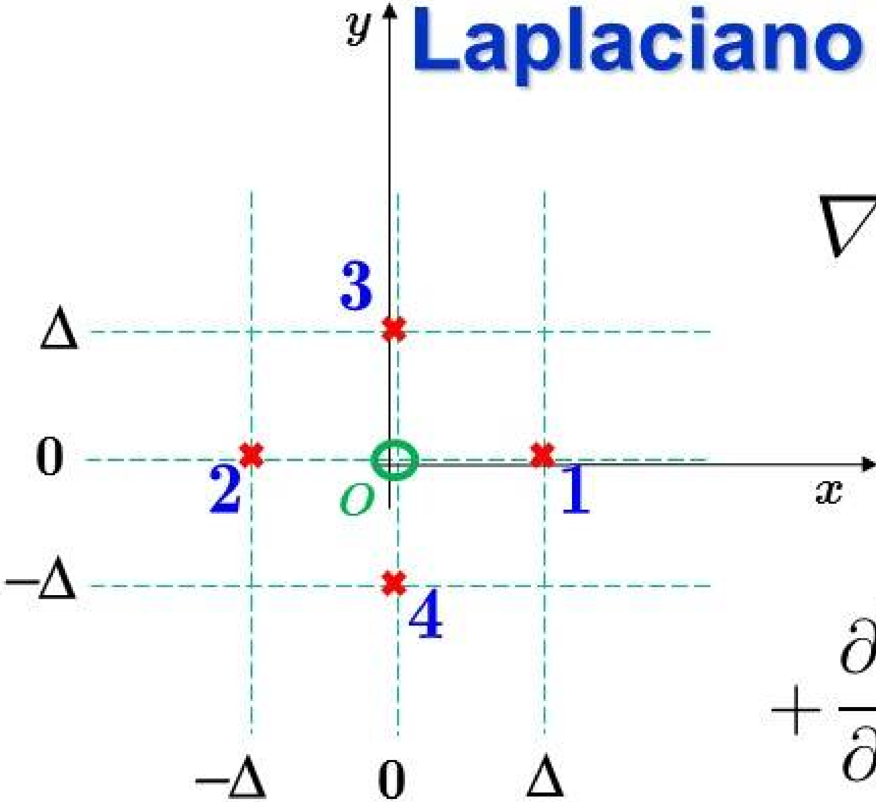
Método de Diferenças Finitas – Grade Regular



Reticulado de resolução dx=dy=Δ



Laplaciano aproximado em 2D



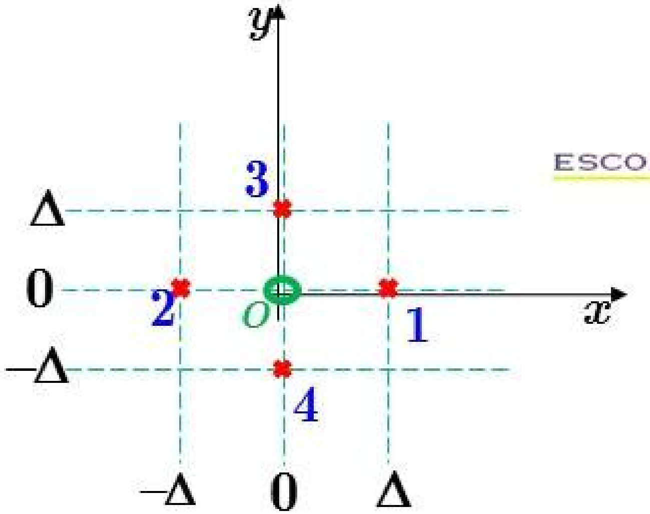
$$\nabla^2\varphi = 0 \quad \Rightarrow \quad \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0$$

$$\varphi(x,y) = \varphi(0) + \frac{\partial\varphi}{\partial x}\Big|_0 x + \frac{\partial\varphi}{\partial y}\Big|_0 y +$$

$$+ \frac{\partial^2\varphi}{\partial x^2}\Big|_0 \frac{x^2}{2} + \frac{\partial^2\varphi}{\partial y^2}\Big|_0 \frac{y^2}{2} + \frac{\partial^2\varphi}{\partial x\partial y}\Big|_0 xy +$$

$$+ \frac{\partial^3\varphi}{\partial x^3}\Big|_0 \frac{x^3}{6} + \frac{\partial^3\varphi}{\partial y^3}\Big|_0 \frac{y^3}{6} + \frac{\partial^3\varphi}{\partial x^2\partial y}\Big|_0 \frac{x^2 y}{2} + \frac{\partial^2\varphi}{\partial x\partial y^2}\Big|_0 \frac{xy^2}{2} +$$

$$+ \text{ termos de ordem } \geq 4$$



Laplaciano aproximado em *O*

$$\varphi_1 = \varphi(\Delta, 0) = \varphi(O) + \left.\frac{\partial \varphi}{\partial x}\right|_0 \Delta + \left.\frac{\partial^2 \varphi}{\partial x^2}\right|_0 \frac{\Delta^2}{2} + \left.\frac{\partial^3 \varphi}{\partial x^3}\right|_0 \frac{\Delta^3}{6} + O(\Delta^4)$$

$$\varphi_2 = \varphi(-\Delta, 0) = \varphi(O) - \left.\frac{\partial \varphi}{\partial x}\right|_0 \Delta + \left.\frac{\partial^2 \varphi}{\partial x^2}\right|_0 \frac{\Delta^2}{2} - \left.\frac{\partial^3 \varphi}{\partial x^3}\right|_0 \frac{\Delta^3}{6} + O(\Delta^4)$$

$$\varphi_3 = \varphi(0, \Delta) = \varphi(O) + \left.\frac{\partial \varphi}{\partial y}\right|_0 \Delta + \left.\frac{\partial^2 \varphi}{\partial y^2}\right|_0 \frac{\Delta^2}{2} + \left.\frac{\partial^3 \varphi}{\partial y^3}\right|_0 \frac{\Delta^3}{6} + O(\Delta^4)$$

$$\varphi_4 = \varphi(0, -\Delta) = \varphi(O) - \left.\frac{\partial \varphi}{\partial y}\right|_0 \Delta + \left.\frac{\partial^2 \varphi}{\partial y^2}\right|_0 \frac{\Delta^2}{2} - \left.\frac{\partial^3 \varphi}{\partial y^3}\right|_0 \frac{\Delta^3}{6} + O(\Delta^4)$$

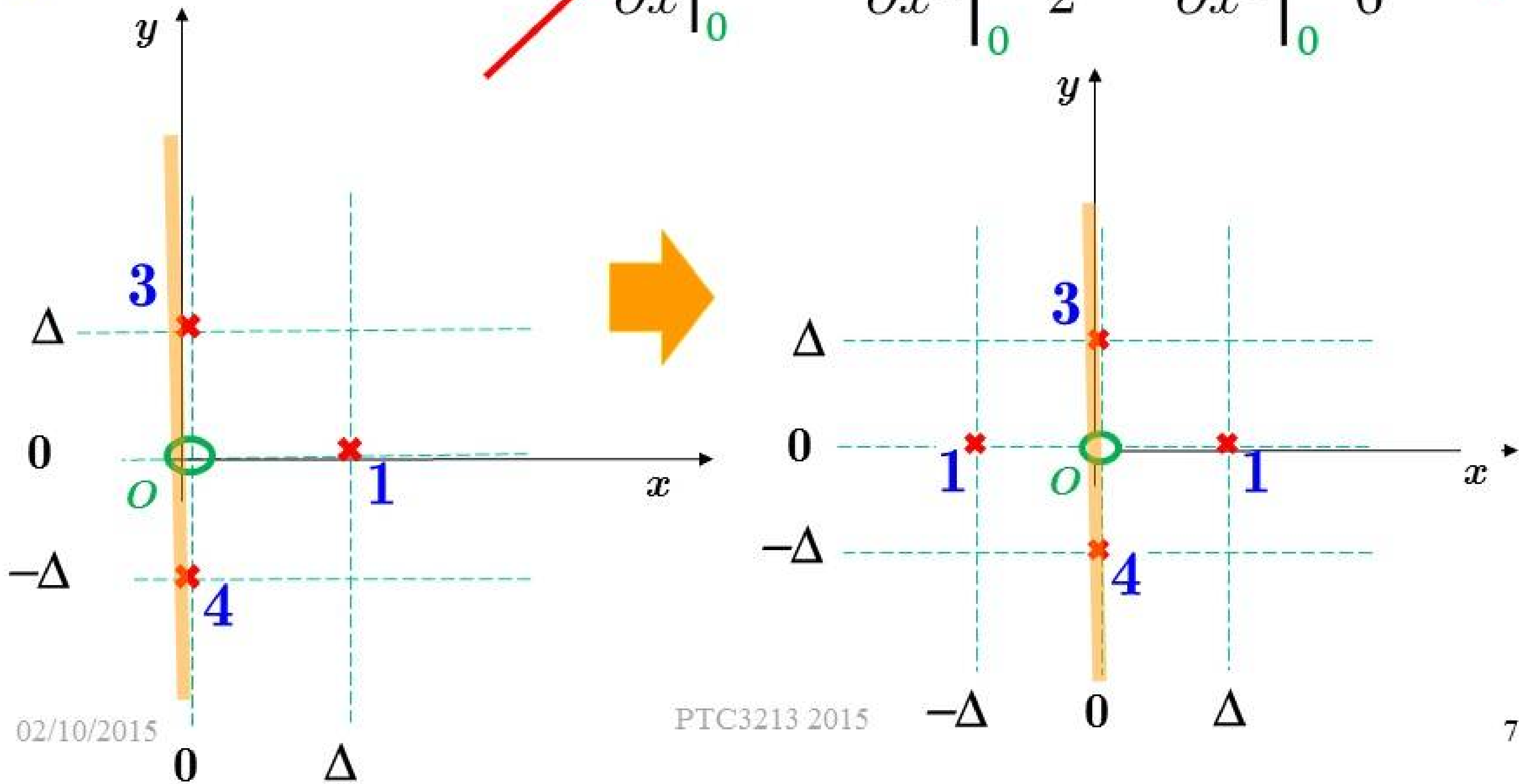
Laplaciano aproximado

$$\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 = 4\varphi(0) + \Delta^2 \cancelto{0}{\nabla^2 \varphi} + O(\xcancel{\Delta^4})$$

$$\varphi(0) = \frac{\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4}{4}$$

Pontos de Fronteira – Cond. Neumann

$$\varphi_1 = \varphi(\Delta, 0) = \varphi(O) + \cancelto{0}{\left.\frac{\partial \varphi}{\partial x}\right|_0 \Delta} + \left.\frac{\partial^2 \varphi}{\partial x^2}\right|_0 \frac{\Delta^2}{2} + \left.\frac{\partial^3 \varphi}{\partial x^3}\right|_0 \frac{\Delta^3}{6} + \cancel{O(\Delta^4)}$$



Pontos de Fronteira – Cond. Neumann

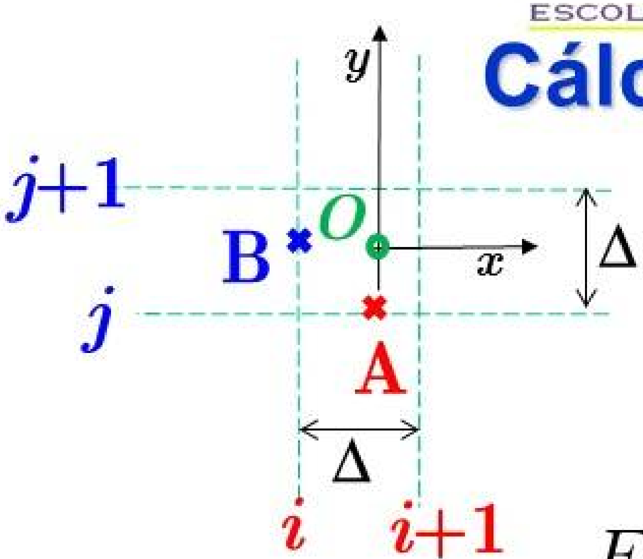
$$\varphi_1 = \varphi(\Delta, 0) = \varphi(O) + \left.\frac{\partial^2 \varphi}{\partial x^2}\right|_0 \frac{\Delta^2}{2} + \left.\frac{\partial^3 \varphi}{\partial x^3}\right|_0 \frac{\Delta^3}{6} + O(\Delta^4)$$

$$2\varphi_1 + \varphi_3 + \varphi_4$$

$$= 4\varphi(0) + \Delta^2 \cancelto{0}{\nabla^2\varphi} + O(\Delta^4)$$

$$\varphi(O) = \varphi_O = \frac{2\varphi_1 + \varphi_3 + \varphi_4}{4}$$

Cálculo do Campo Elétrico



$$\overrightarrow{E} = -\nabla\varphi = -\frac{\partial\varphi}{\partial x}\widehat{u}_x - \frac{\partial\varphi}{\partial y}\widehat{u}_y$$

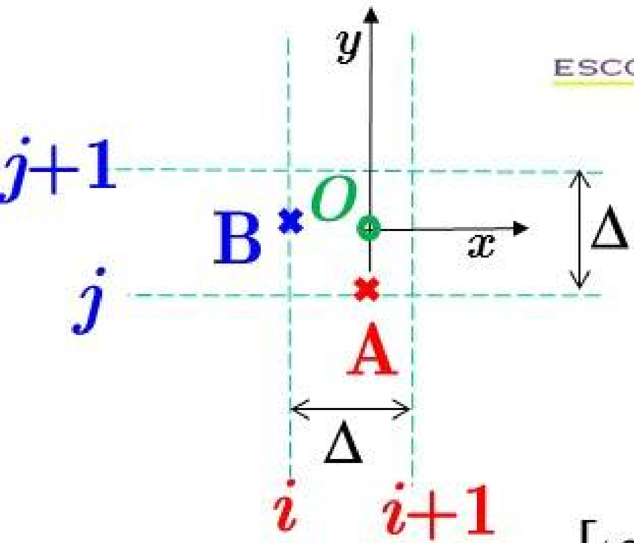
$$E_x(A) = -\left.\frac{\partial\varphi}{\partial x}\right|_A = -\frac{[\varphi(i+1,j) - \varphi(i,j)]}{\Delta} =$$

$$= \frac{\varphi(i,j) - \varphi(i+1,j)}{\Delta}$$

$$E_y(B) = -\left.\frac{\partial\varphi}{\partial y}\right|_B = -\frac{[\varphi(i,j+1) - \varphi(i,j)]}{\Delta} =$$

$$= \frac{\varphi(i,j) - \varphi(i,j+1)}{\Delta}$$

Cálculo do Campo Elétrico



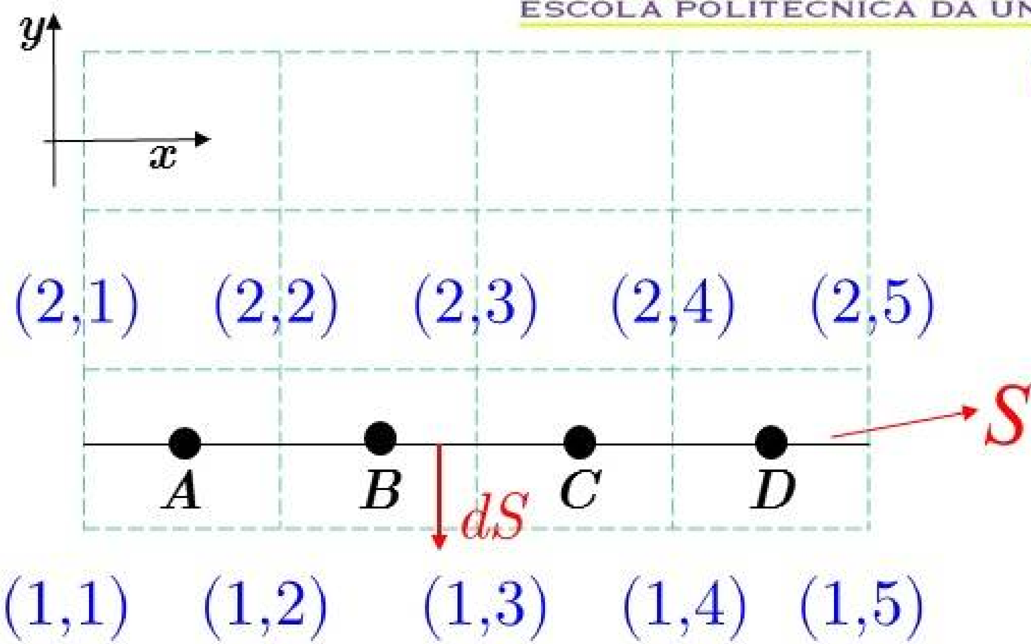
$$E_x(O) = \frac{\dfrac{[\varphi(i,j) + \varphi(i,j+1)]}{2} - \dfrac{[\varphi(i+1,j) + \varphi(i+1,j+1)]}{2}}{\Delta}$$

$$E_x(O) = \frac{\varphi(i,j) + \varphi(i,j+1) - \varphi(i+1,j) - \varphi(i+1,j+1)}{2\Delta}$$

$$E_y(O) = \frac{\dfrac{[\varphi(i,j) + \varphi(i+1,j)]}{2} - \dfrac{[\varphi(i,j+1) + \varphi(i+1,j+1)]}{2}}{\Delta}$$

$$E_y(O) = \frac{\varphi(i,j) + \varphi(i+1,j) - \varphi(i,j+1) - \varphi(i+1,j+1)}{2\Delta}$$

Cálculo da Corrente



$$\vec{J} = \sigma\,\vec{E}$$

$$I = \iint_S \vec{J}\cdot d\vec{S}$$

$$I_S = \sigma L\Big[-\Delta.E_y(A) - \Delta.E_y(B) - \Delta.E_y(C) - \Delta.E_y(D)\Big] =$$

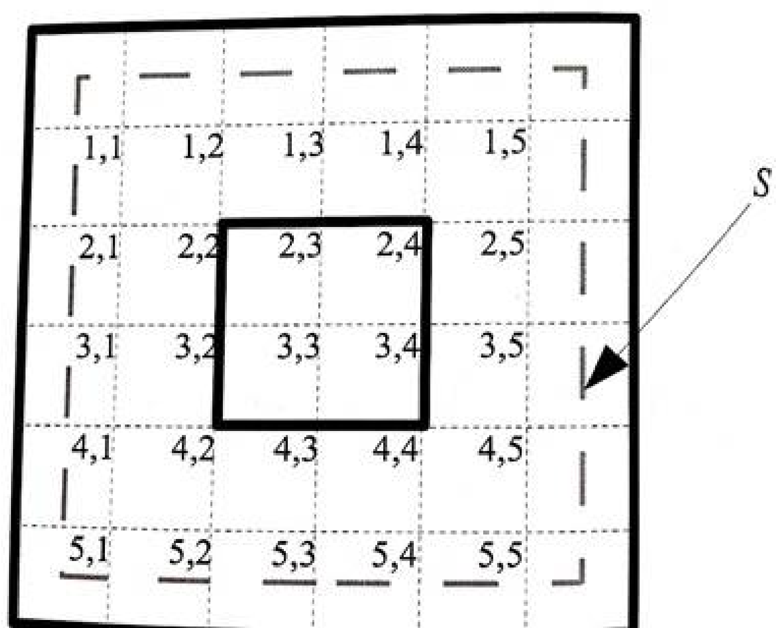
$$= \sigma L\,[\sum_{i=1}^{5}\Delta\,\frac{\varphi(\mathbf{2},i)+\varphi(\mathbf{2},i+1)-\varphi(\mathbf{1},i)-\varphi(\mathbf{1},i+1)}{2\Delta}] =$$

$$I_S = \sigma L\left[\frac{\varphi(2{,}1)}{2} + \varphi(2{,}2) + \varphi(2{,}3) + \varphi(2{,}4) + \frac{\varphi(2{,}5)}{2} - \right.$$
$$\left. - \frac{\varphi(1{,}1)}{2} - \varphi(1{,}2) - \varphi(1{,}3) - \varphi(1{,}4) - \frac{\varphi(1{,}5)}{2}\right]$$

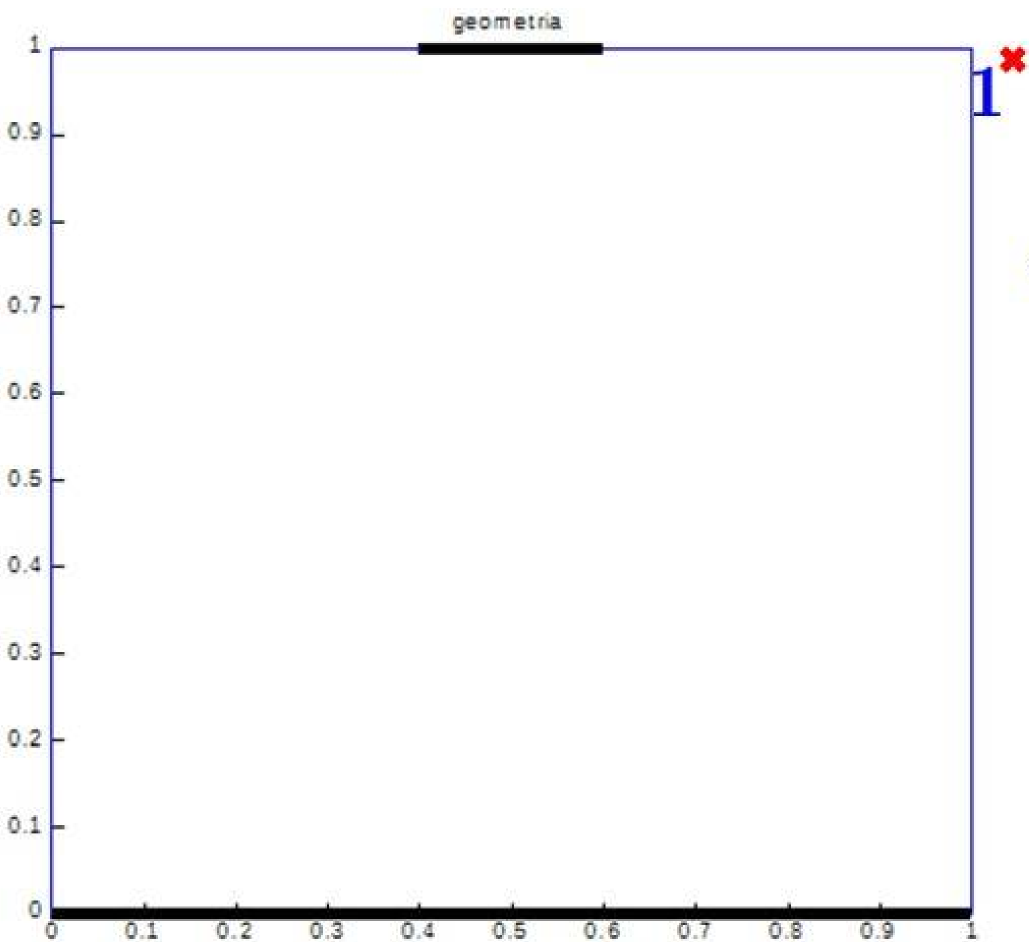
Cálculo da Corrente

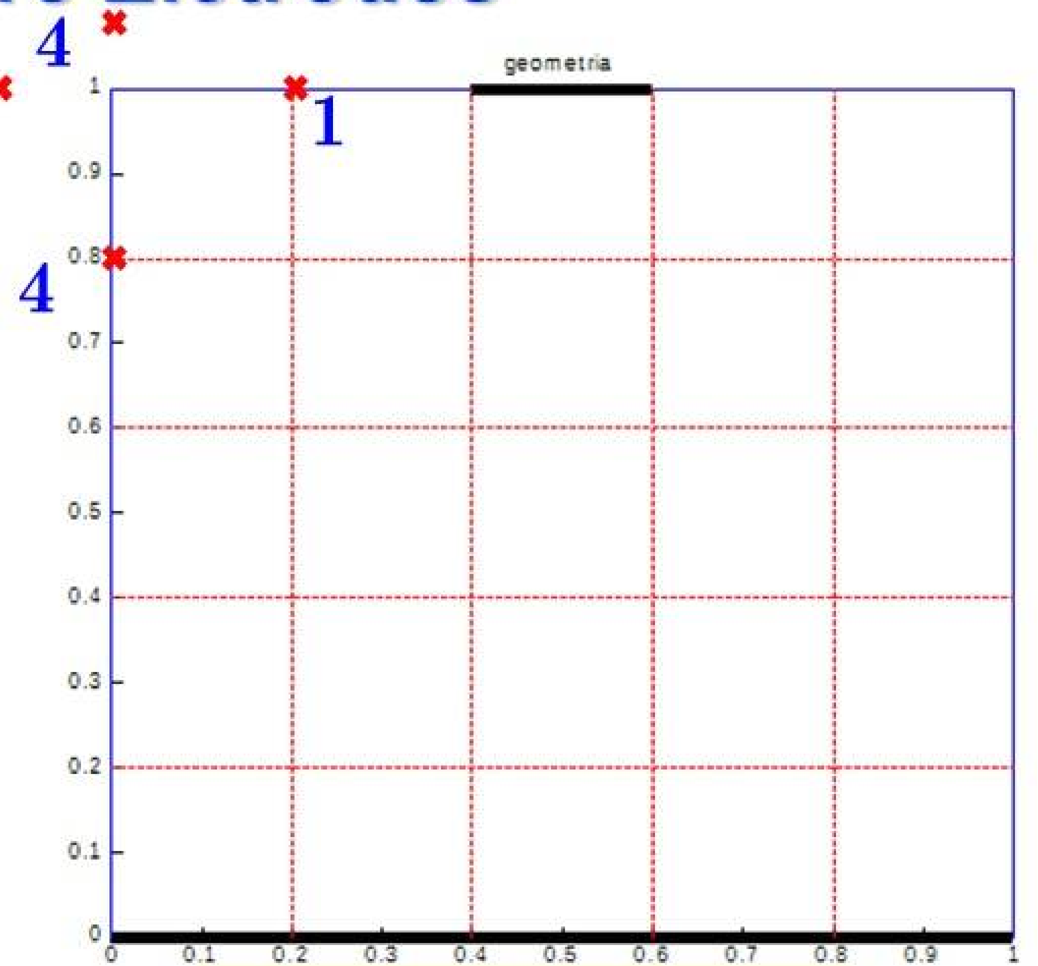
$$\vec{J} = \sigma\,\vec{E} \qquad\qquad I = \iint_S \vec{J}\cdot d\vec{S}$$

$$I_S = \sigma L\left[\sum_{i=1}^{5}\varphi(i,1) + \sum_{i=1}^{5}\varphi(i,5) + \sum_{j=1}^{5}\varphi(5,j) + \sum_{j=1}^{5}\varphi(1,j)\right] =$$



Resistência entre Eletrodos





$\sigma = 0{,}01$ S/m $\quad \Delta = 0{,}2$ m $\quad L = 1$ m

$$\varphi_O = \frac{2\varphi_1 + 2\varphi_4}{4}$$

ESCOLA POLITÉCNICA DA UNIVERSIDADE DE SÃO PAULO

Resistência entre Eletrodos

inicial:

0	0	100	100	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

1a iteração:

0	25	100	100	25	13
0	6	27	32	14	10
0	2	7	10	6	6
0	0	2	3	2	2
0	0	0	1	1	1
0	0	0	0	0	0

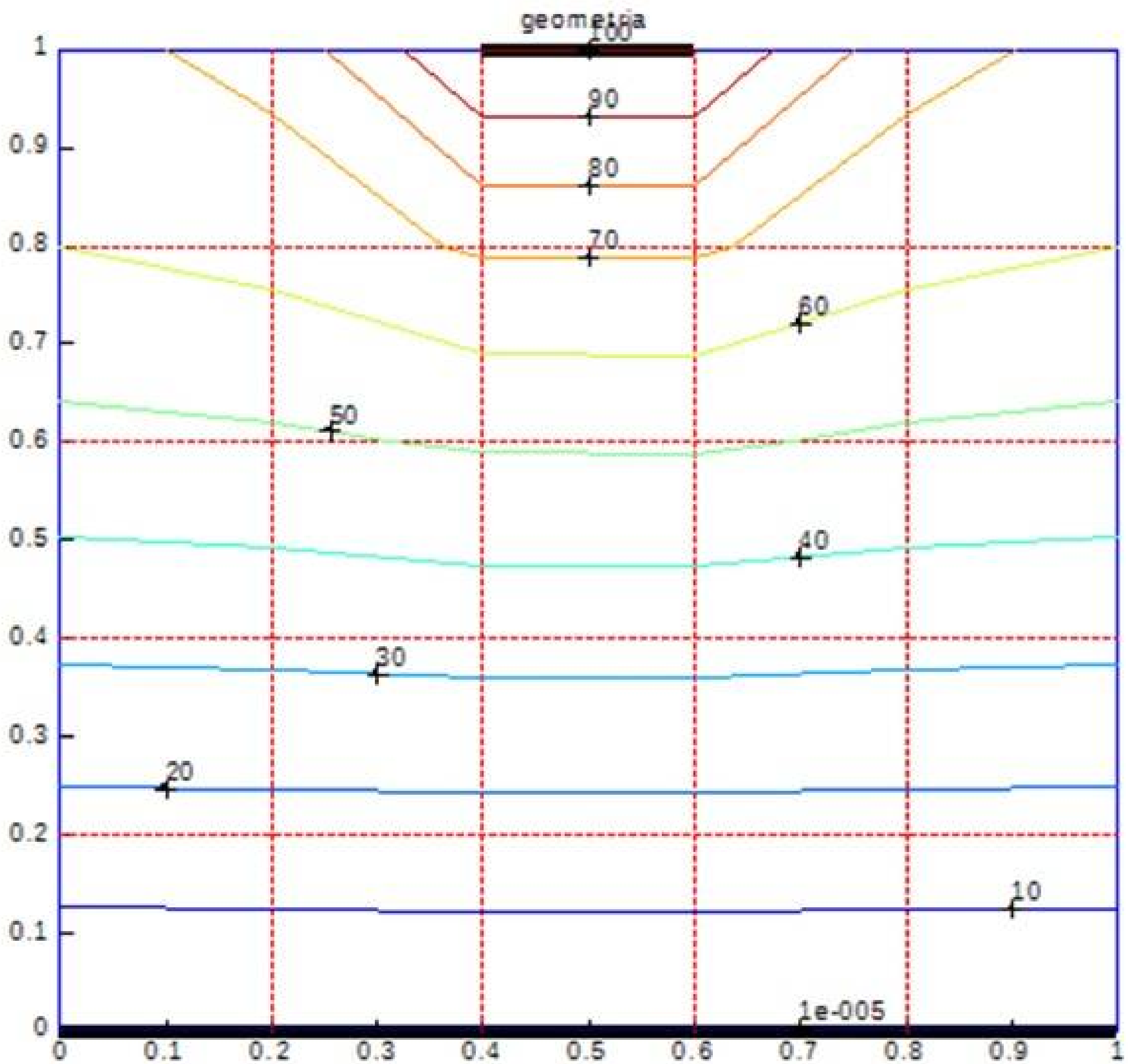
2a. Iteração:

13	31	100	100	35	23
6	16	39	41	23	19
2	7	14	16	12	11
1	2	5	6	5	6
0	1	2	2	2	2
0	0	0	0	0	0

36a. Iteração:

67	73	100	100	73	67
60	63	71	71	63	60
47	49	51	51	49	47
32	33	33	33	33	32
16	16	17	17	16	16
0	0	0	0	0	0

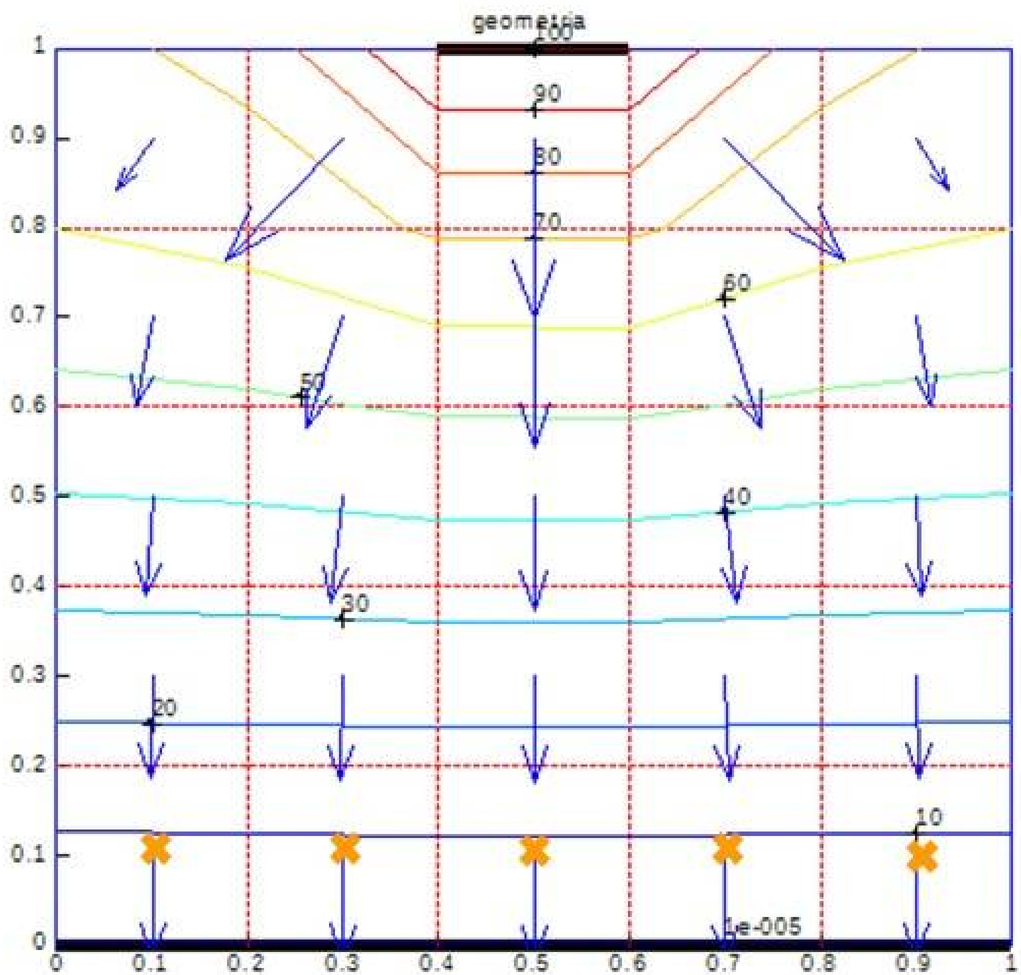
Equipotenciais



Ex =
-24.7084
-87.0787,
-0.0450,
86.7753,
24.3644
-11.2270
-26.7290,
-0.1021,
26.4566,
10.9252
,-4.3024,
-8.4591,
-0.1049,
8.2266,
4.0562
,-1.5078,
-2.6674,
-0.0740,
2.5120,
1.3468,
-0.3649,
-0.6125,
-0.0261,
0.5585,
0.3094

Ey =
-41.2442 -
96.2463 -
142.6303 -
96.1770 -
41.2171
-68.9248 -
87.7537 -
101.6624 -
87.7400 -
68.9421
-77.8153 -
84.1808 -
88.5391 -
84.2286 -
77.9276
-80.5375 -
82.7582 -
84.2225 -
82.8493 -
80.7263
-81.3298 -
82.3071 -
82.9457 -
82.4133 -
81.5454

Cálculo de I e R



Jy=σ*Ey

,-0.4124,-0.9625,-1.4263,-0.9618,-0.4122
,-0.6892,-0.8775,-1.0166,-0.8774,-0.6894
,-0.7782,-0.8418,-0.8854,-0.8423,-0.7793
,-0.8054,-0.8276,-0.8422,-0.8285,-0.8073
,-0.8133,-0.8231,-0.8295,-0.8241,-0.8155

$$I = -\sigma\times(-81{,}33 - 82{,}31 - 82{,}94 - 82{,}41 - 81{,}54)\times L\times\Delta$$

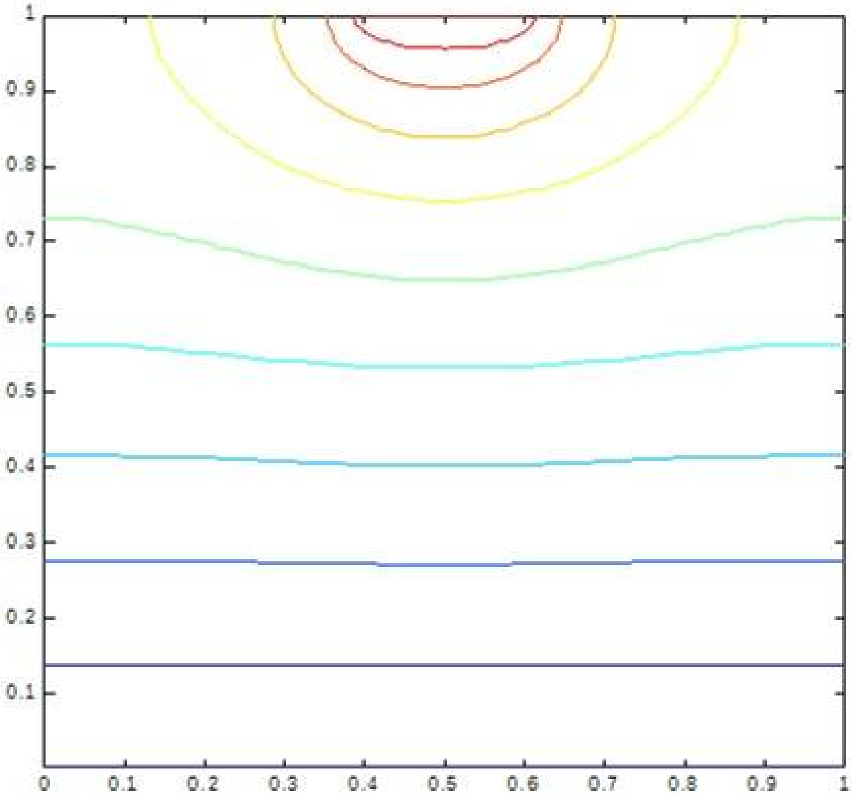
I=0,8211 A

R =100/0,8211

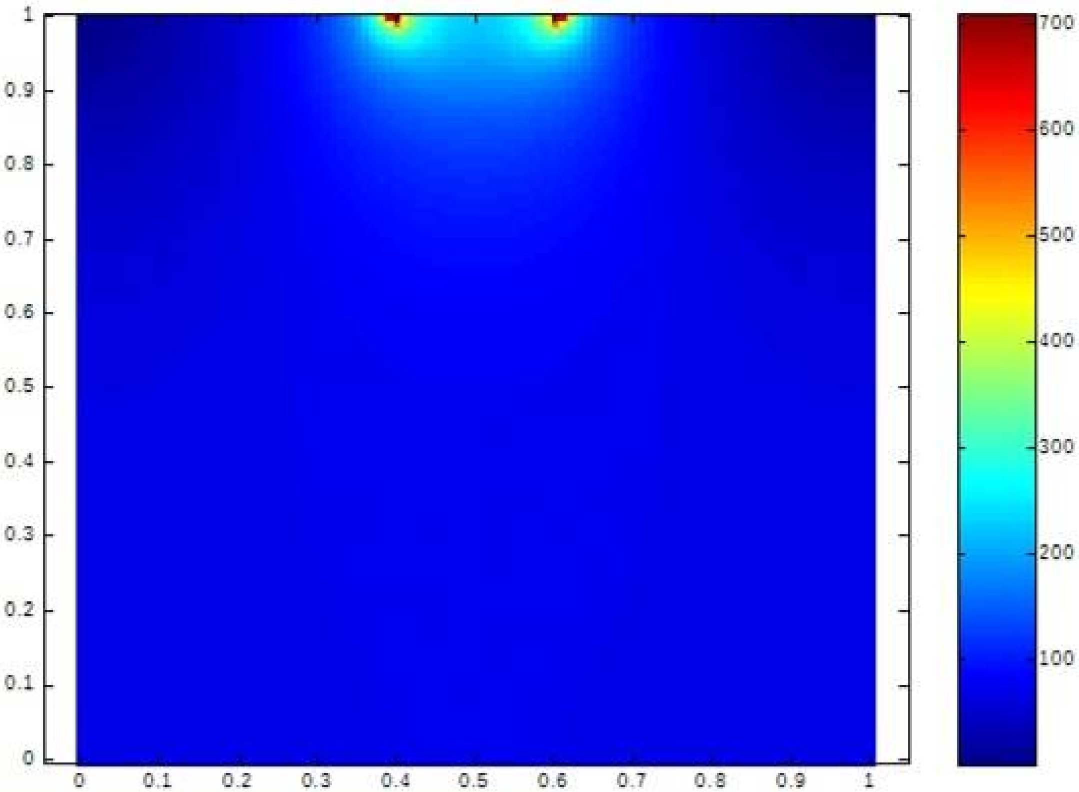
R =121.7904 Ω

Solução mais discretizada

$dx = dy = \Delta = 0.01\ \mathrm{m}$



campo elétrico (módulo):



$R = 136.3089\,\Omega$