

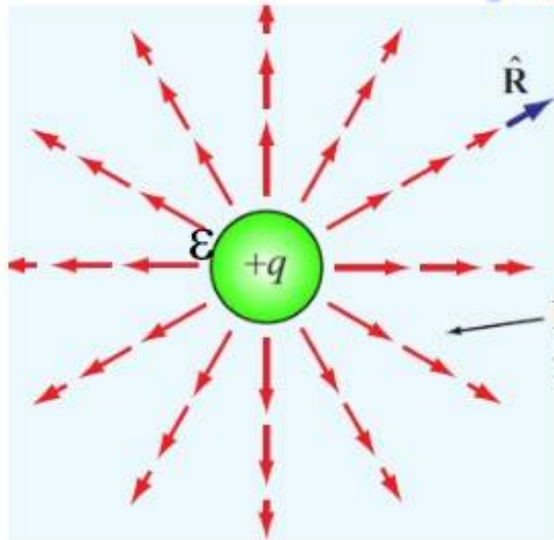


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**Equações de Laplace
e Poisson
Dipolo Elétrico
Distribuição de Cargas
Condutor Cilíndrico**



Potencial a partir de Q, ρ



$$\nabla^2 \varphi = 0 \quad \Rightarrow \quad \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \varphi}{\partial r} \right) = 0$$

Simetria Esférica \rightarrow 1D

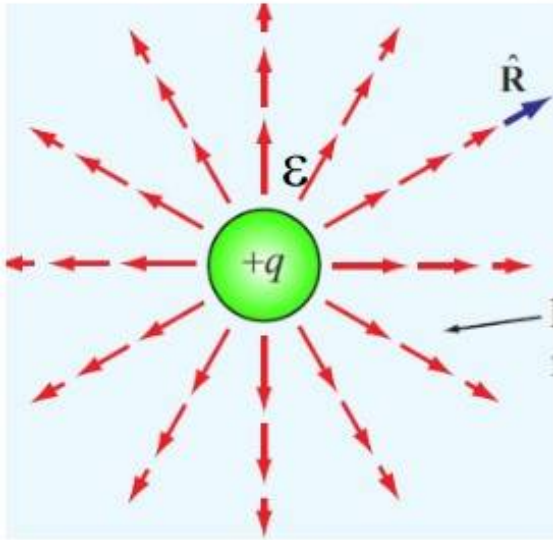
$$r^2 \frac{\partial \varphi}{\partial r} = k_1 \quad \Rightarrow \quad \varphi(r) = -\frac{k_1}{r} + k_2$$

$$\varphi(\infty) = \lim_{r \rightarrow \infty} \left(-\frac{k_1}{r} + k_2 \right) = 0 \quad \Rightarrow \quad k_2 = 0$$

$$\varphi(r) = -\frac{k_1}{r}$$



Potencial a partir de q, ρ



$$\vec{E} = -\nabla\varphi \Rightarrow \vec{E} = -\frac{\partial\varphi}{\partial r}\hat{u}_r$$

$$\vec{E} = -\frac{k_1}{r^2}\hat{u}_r \Rightarrow \vec{D} = \epsilon\vec{E} = -\epsilon\frac{k_1}{r^2}\hat{u}_r$$

$$q = \oiint_S \vec{D} \cdot d\vec{S} = -\epsilon\frac{k_1}{r^2} \oiint_S dS = -4\pi\epsilon k_1 \Rightarrow k_1 = -\frac{q}{4\pi\epsilon}$$

$$\varphi(r) = \frac{q}{4\pi\epsilon r}$$

$$\vec{E} = \frac{q}{4\pi\epsilon r^2}\hat{u}_r$$

$$\vec{D} = \frac{q}{4\pi r^2}\hat{u}_r$$



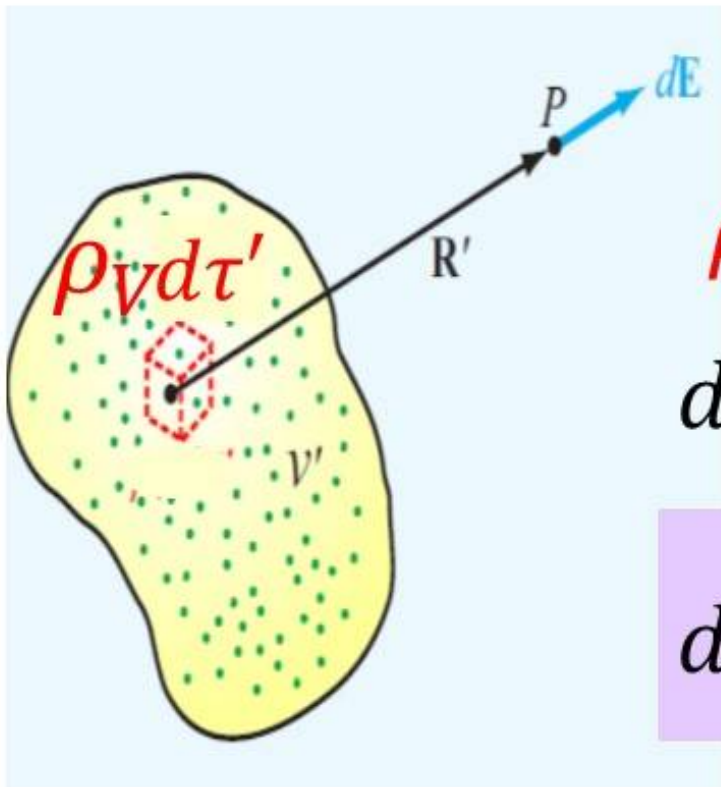
Potencial a partir de ρ

$$\varphi(P) = \sum_{i=1}^n \frac{q_i}{4\pi\epsilon r_i}$$

$$\rho_V(x', y', z') \quad d\tau' = dx' dy' dz'$$

$$dq = \rho_V(x', y', z') d\tau' \quad P(x, y, z)$$

$$d\varphi(x, y, z) = \frac{\rho_V(x', y', z') d\tau'}{4\pi\epsilon R}$$





Potencial a partir de ρ

$$d\varphi(x, y, z) = \frac{\rho_V(x', y', z') d\tau'}{4\pi\epsilon R}$$

$$R = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

$$\varphi(x, y, z) = \iiint_{\tau} \frac{\rho_V(x', y', z') d\tau'}{4\pi\epsilon R}$$

$$\frac{\rho}{\epsilon} = -\nabla^2 \varphi$$



$$\varphi(x, y, z) = - \iiint_{\tau} \frac{\nabla^2 \varphi d\tau'}{4\pi R}$$



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Potencial a partir de ρ_S e ρ_l

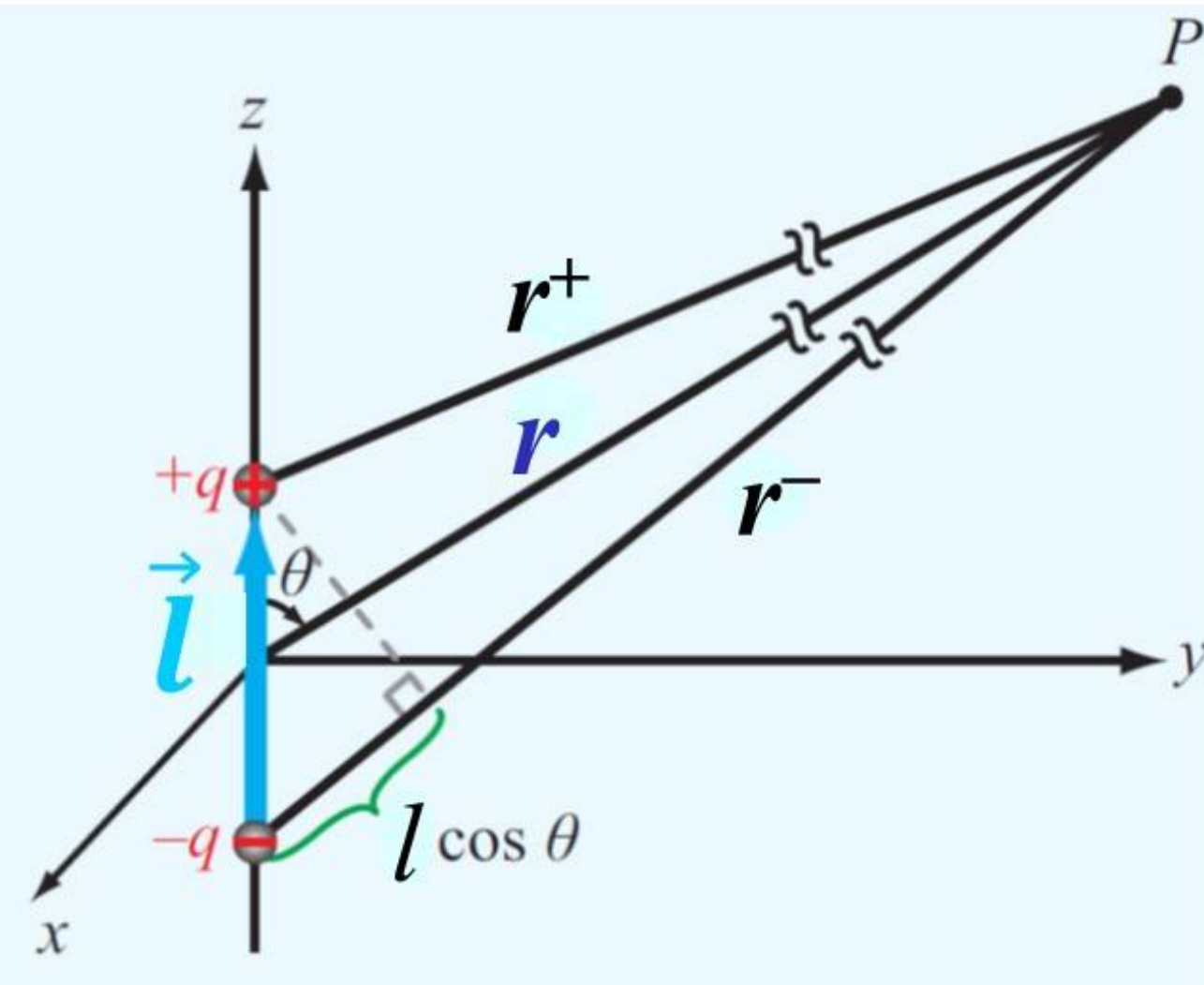
$$\varphi(x, y, z) = \iint_S \frac{\rho_S(x', y', z') dS'}{4\pi\epsilon R}$$

$$\varphi(x, y, z) = \int_l \frac{\rho_l(x', y', z') dl'}{4\pi\epsilon R}$$



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Campo do Dipolo Elétrico



$$\varphi = \frac{q}{4\pi\epsilon} \left(\frac{1}{r^+} - \frac{1}{r^-} \right)$$

Se $r \gg l$

$$\vec{r}^+ \parallel \vec{r}^- \parallel \vec{r}$$

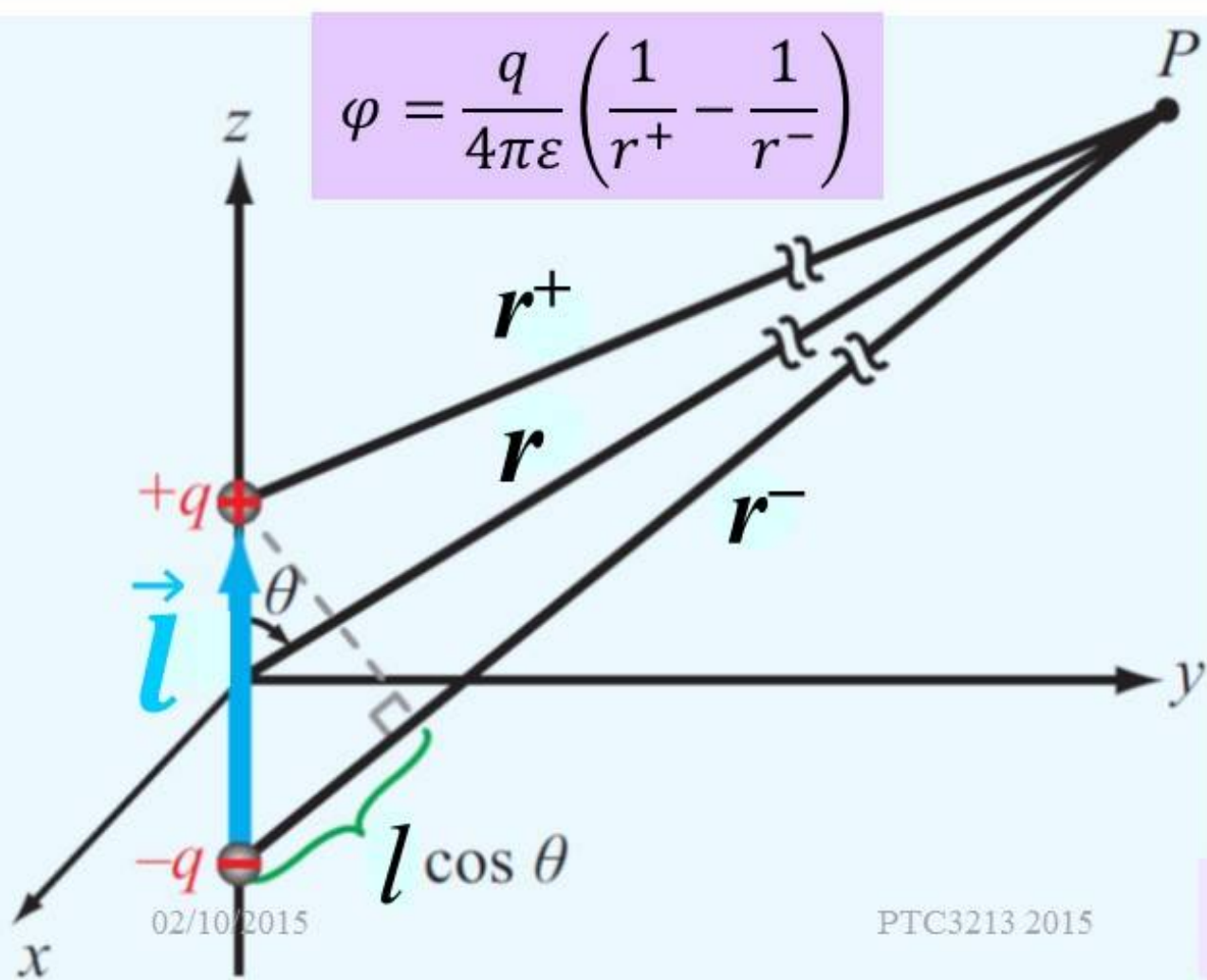
$$\vec{r}^+ \approx r - \frac{l}{2} \cos \theta$$

$$\vec{r}^- \approx r + \frac{l}{2} \cos \theta$$



Campo do Dipolo Elétrico

$$\varphi = \frac{q}{4\pi\epsilon} \left(\frac{1}{r^+} - \frac{1}{r^-} \right)$$



$$\left(\frac{1}{r^+} - \frac{1}{r^-} \right) \approx \frac{l \cos \theta}{r^2}$$

$$\vec{l} = l \hat{u}_z$$

$$\varphi = \frac{q}{4\pi\epsilon} \frac{l \cos \theta}{r^2}$$

$$\varphi = \frac{1}{4\pi\epsilon r^3} q \vec{l} \cdot \vec{r}$$

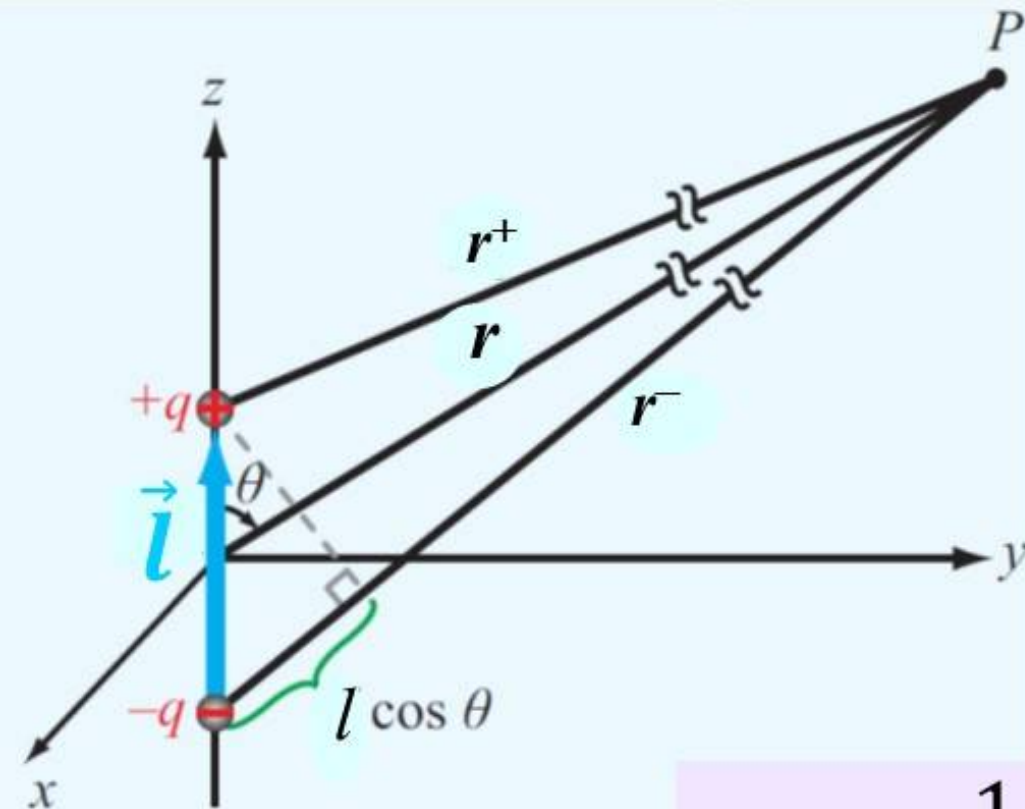
$$\vec{p} = q \vec{l}$$

Momento do Dipolo⁸



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Campo do Dipolo Elétrico



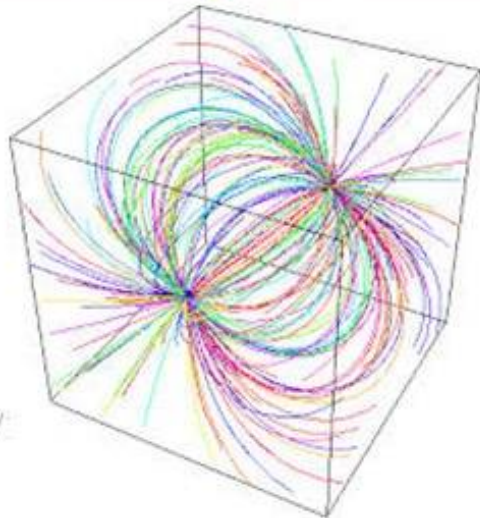
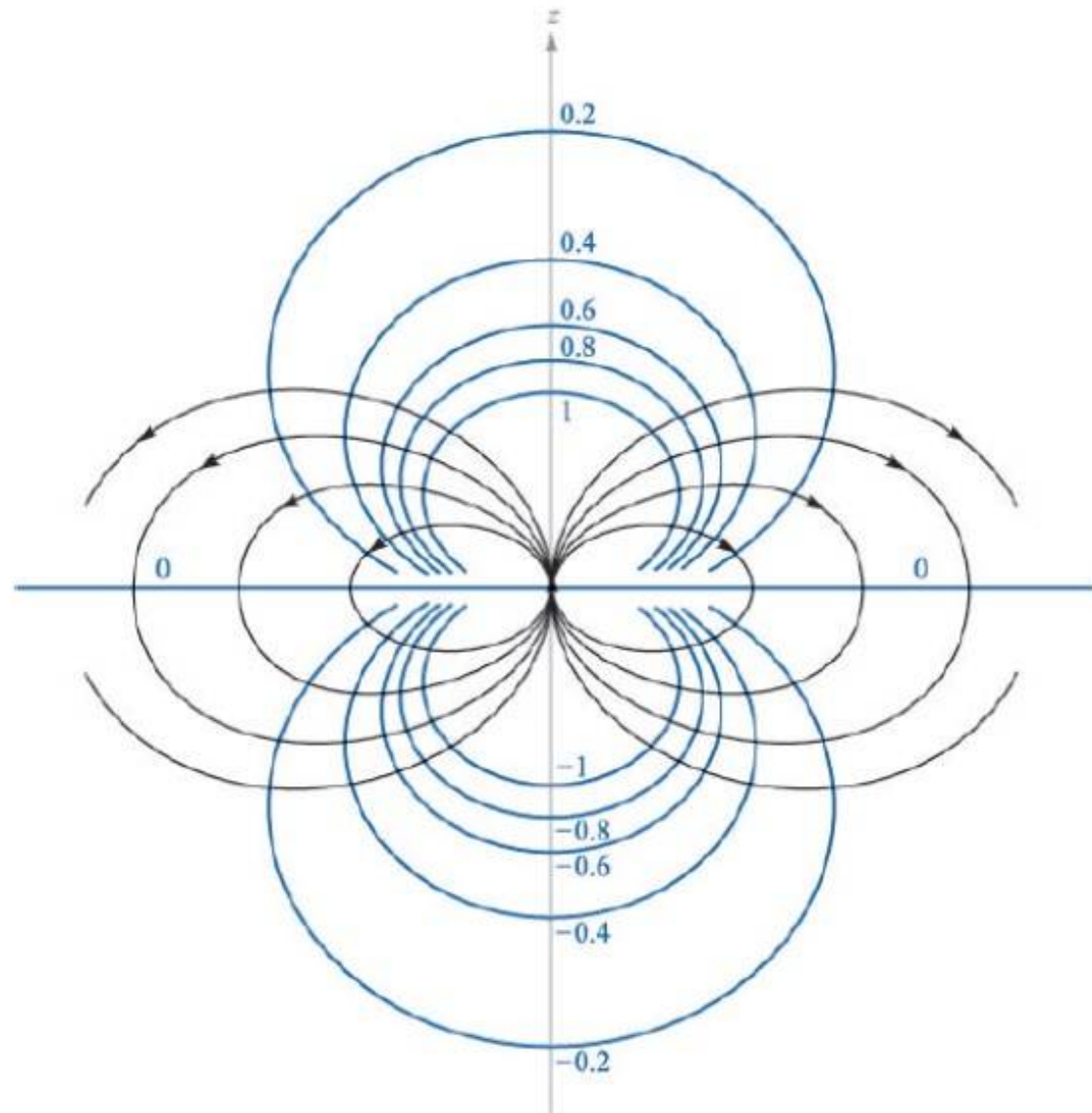
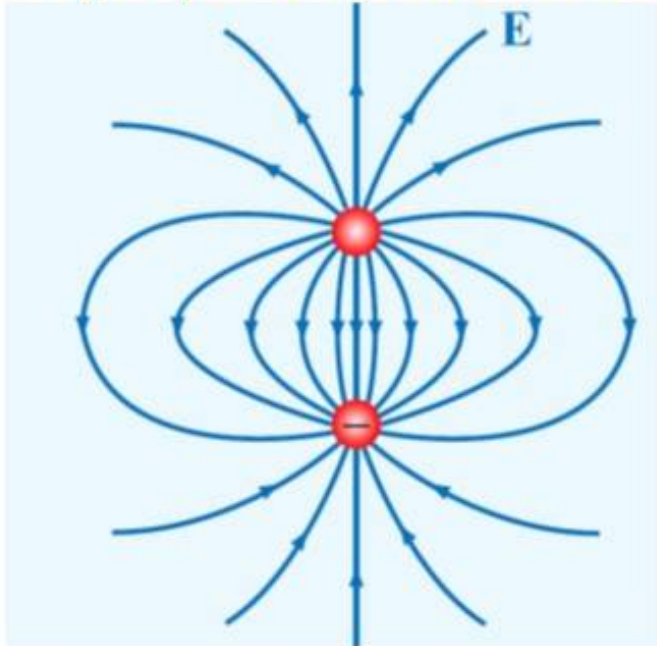
$$\varphi = \frac{1}{4\pi\epsilon r^3} \vec{p} \cdot \vec{r}$$

$$\vec{E} = \frac{1}{4\pi\epsilon r^3} (2 \cos \theta \hat{u}_r + \sin \theta \hat{u}_\theta)$$



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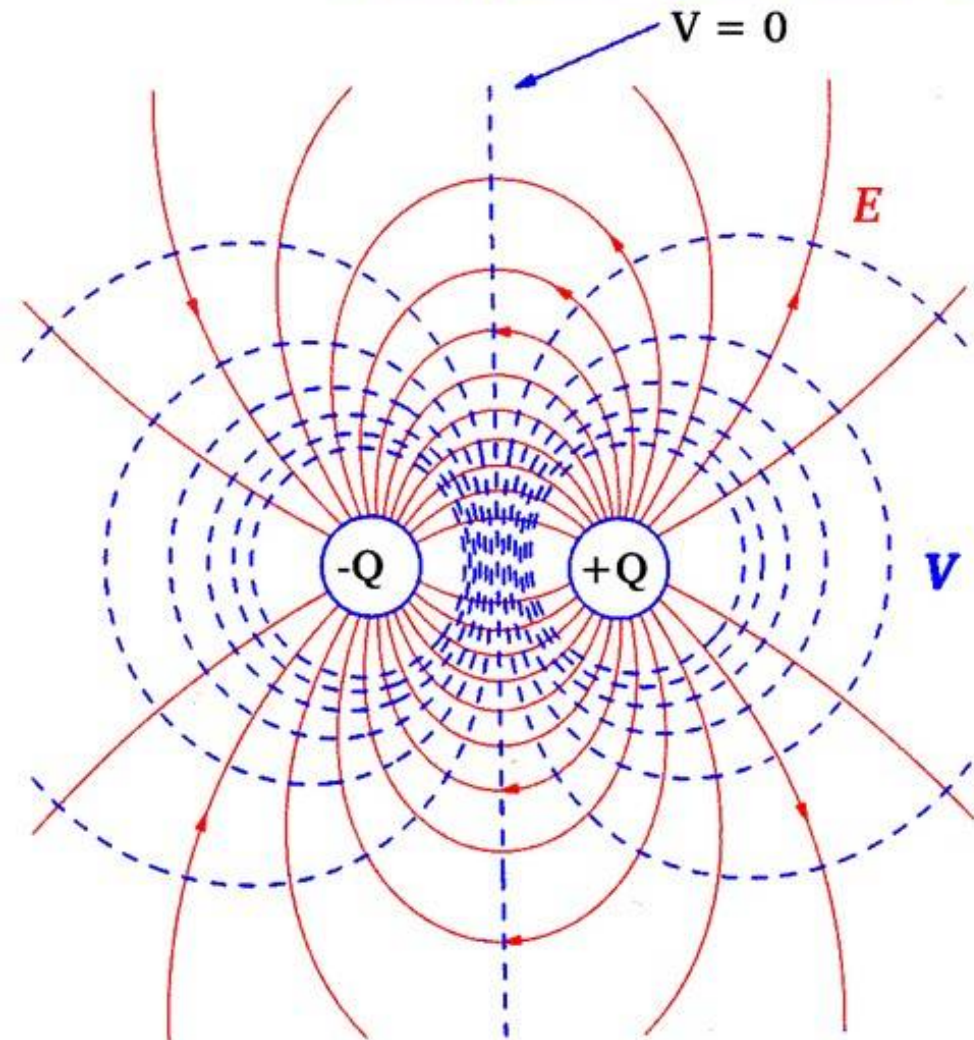
Equipotenciais e Linhas de Campo do Dipolo Elétrico



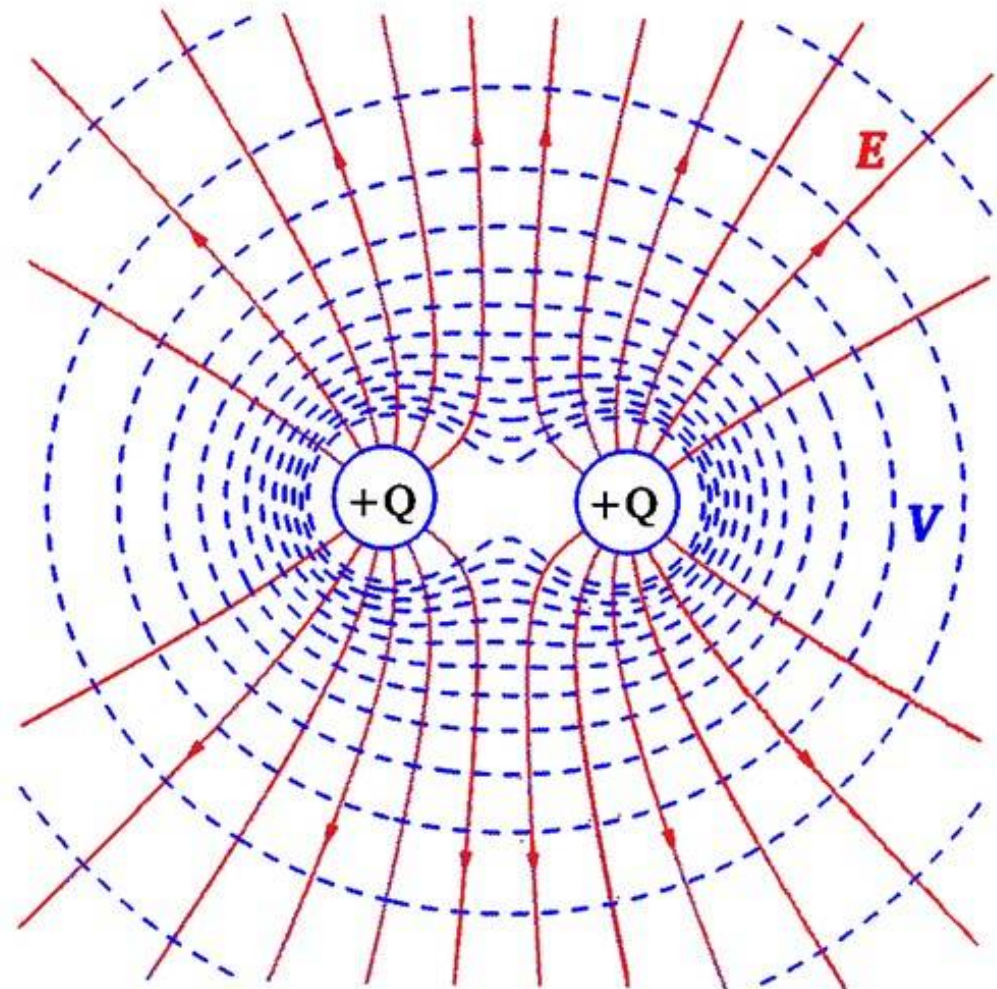


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Equipotenciais e Linhas de Campo Comparação entre Dipolo e cargas iguais



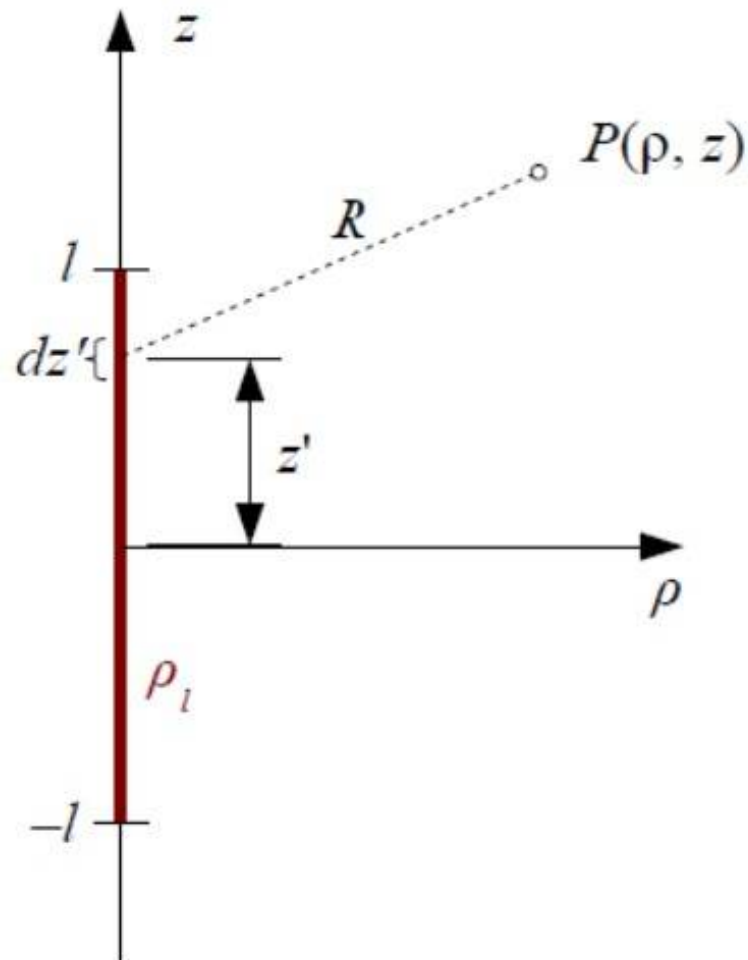
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Campo de Fonte Linear Condutor Cilíndrico - Aterramento





Campo de Fonte Linear Condutor Cilíndrico - Aterramento

$$\begin{aligned}\varphi(\rho, z) &= \int_{-l}^l \frac{\rho_l dz'}{4\pi\epsilon R} = \int_{-l}^l \frac{\rho_l dz'}{4\pi\epsilon \sqrt{(z-z')^2 + \rho^2}} = \\ &= \frac{-\rho_l}{4\pi\epsilon} \ln \left[(z-z') + \sqrt{(z-z')^2 + \rho^2} \right] \Big|_{z'=-l}^{z'=l} \\ \varphi(\rho, z) &= \frac{\rho_l}{4\pi\epsilon} \ln \left[\frac{(z+l) + \sqrt{(z+l)^2 + \rho^2}}{(z-l) + \sqrt{(z-l)^2 + \rho^2}} \right]\end{aligned}$$



Superfícies Equipotenciais

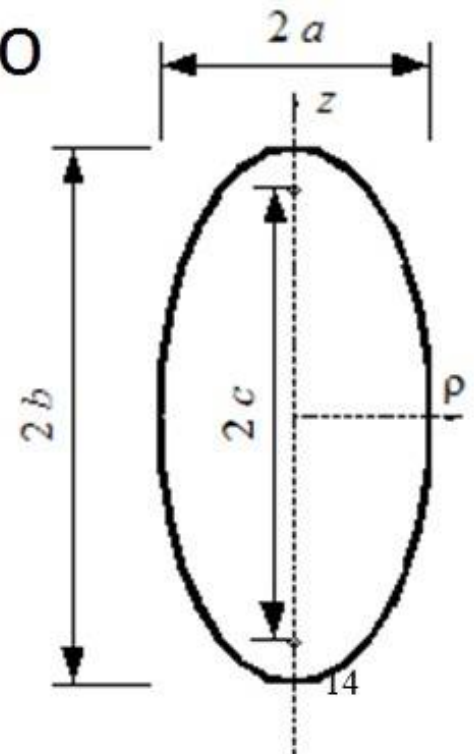
$$\varphi = V \Rightarrow \left[\frac{(z+l) + \sqrt{(z+l)^2 + \rho^2}}{(z-l) + \sqrt{(z-l)^2 + \rho^2}} \right] = e^{\frac{4\pi\epsilon V}{\rho_l}} = \alpha ; 1 < \alpha < \infty$$

Que pode ser reescrita como

$$\frac{z^2}{b^2} + \frac{\rho^2}{a^2} = 1 \quad b^2 = l^2 \frac{(\alpha+1)^2}{(\alpha-1)^2}, \quad a^2 = l^2 \frac{4\alpha}{(\alpha-1)^2}$$

Elipsoide de raios a e b e distância focal c :

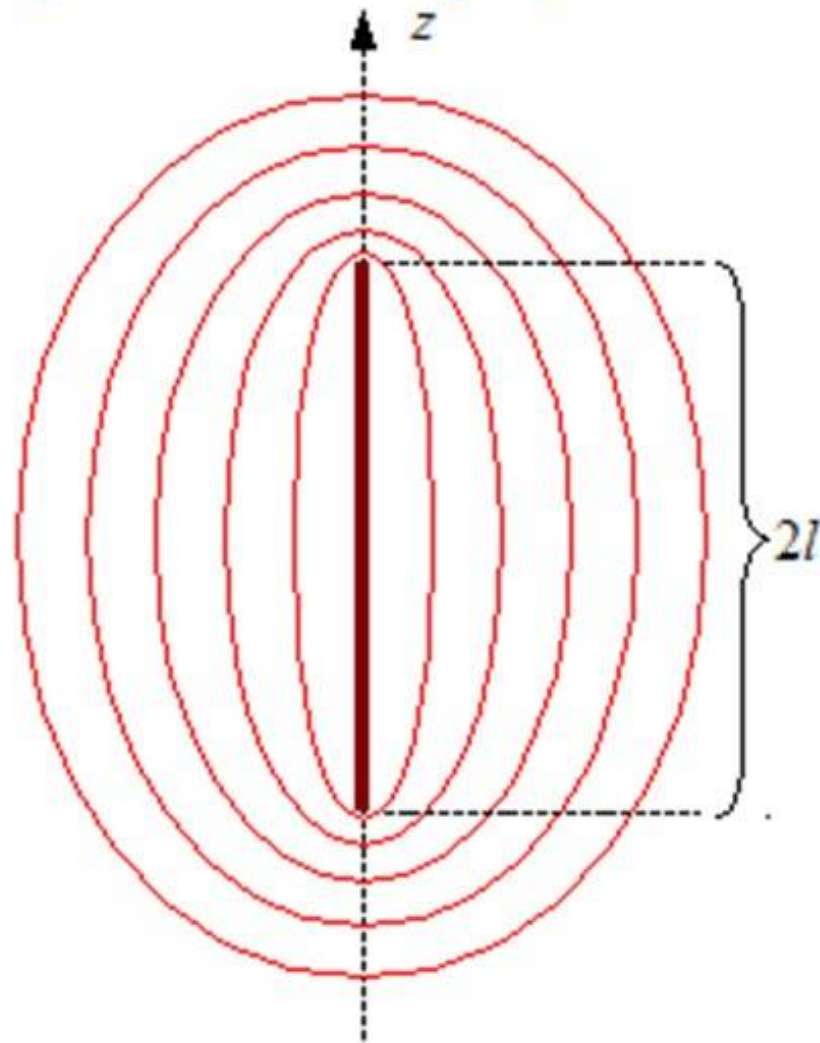
$$c^2 = b^2 - a^2 = l^2$$





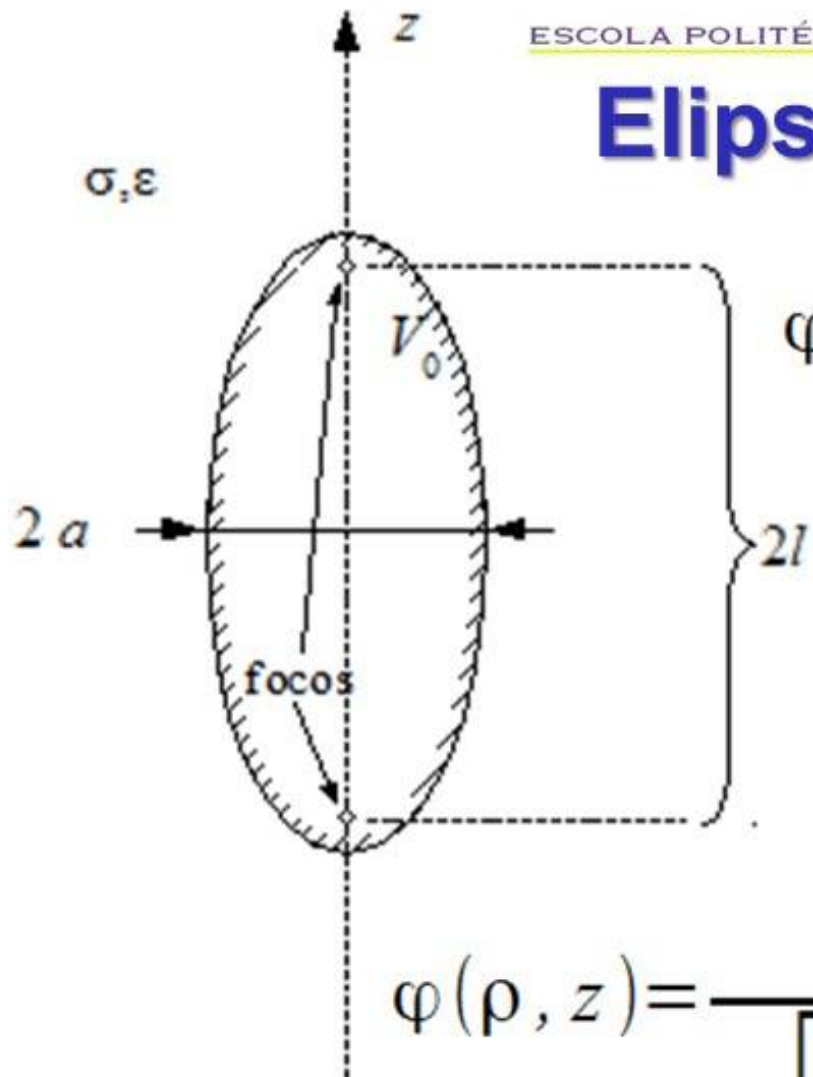
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Superfícies Equipotenciais





Elipsoide condutor



$$\varphi(a, 0) = \frac{\rho_l}{4\pi\epsilon} \ln \left[\frac{\sqrt{l^2 + a^2} + l}{\sqrt{l^2 + a^2} - l} \right] = V_0 \Rightarrow$$

$$\Rightarrow \rho_l = \frac{4\pi\epsilon V_0}{\ln \left[\frac{\sqrt{l^2 + a^2} + l}{\sqrt{l^2 + a^2} - l} \right]}$$

$$\varphi(\rho, z) = \frac{V_0}{\ln \left[\frac{\sqrt{l^2 + a^2} + l}{\sqrt{l^2 + a^2} - l} \right]} \ln \left[\frac{(z+l) + \sqrt{(z+l)^2 + \rho^2}}{(z-l) + \sqrt{(z-l)^2 + \rho^2}} \right]$$



Elipsoide condutor

Carga do elipsoide:

$$Q = 2l \frac{4\pi\epsilon V_0}{\ln \left[\frac{\sqrt{l^2 + a^2} + l}{\sqrt{l^2 + a^2} - l} \right]} = \frac{8\pi\epsilon l V_0}{\ln \left[\frac{\sqrt{l^2 + a^2} + l}{\sqrt{l^2 + a^2} - l} \right]}$$

Capacitância:

$$C = \frac{8\pi\epsilon l}{\ln \left[\frac{\sqrt{l^2 + a^2} + l}{\sqrt{l^2 + a^2} - l} \right]}$$

Resistência:

$$R = \frac{1}{8\pi\sigma l} \ln \left[\frac{\sqrt{l^2 + a^2} + l}{\sqrt{l^2 + a^2} - l} \right]$$



Elipsoide condutor semienterrado

- Condições de contorno:

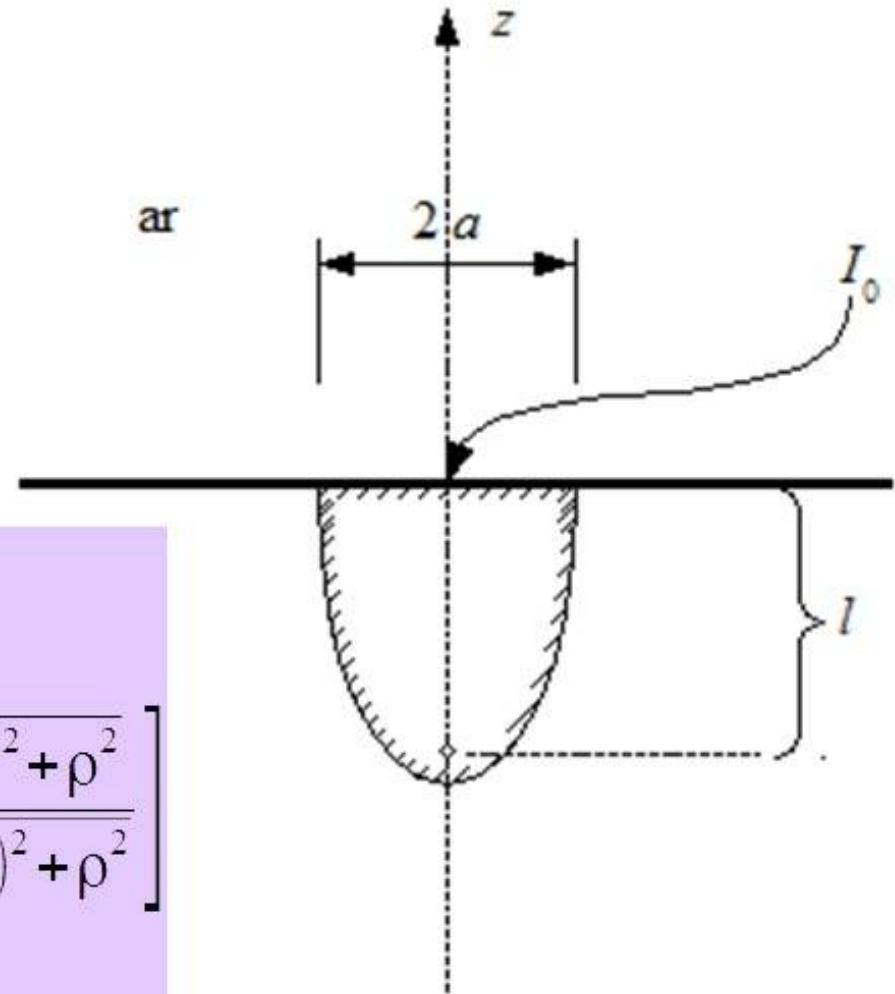
$$\varphi(S_{\text{elipsoide}}) = V_0$$

$$\varphi(z, \rho \rightarrow \infty) = 0$$

$$\left. \frac{\partial \varphi}{\partial z} \right|_{z=0} = 0$$

- Solução:

$$\varphi(\rho, z) = \frac{V_0}{\ln \left[\frac{\sqrt{l^2 + a^2} + l}{\sqrt{l^2 + a^2} - l} \right]} \ln \left[\frac{(z+l) + \sqrt{(z+l)^2 + \rho^2}}{(z-l) + \sqrt{(z-l)^2 + \rho^2}} \right]$$





Elipsoide condutor semienterrado

- Para mesmo V_0 :
 - Mesmo campo elétrico E
 - Mesma densidade de corrente J
 - Metade da corrente
 - Dobro da resistência:

$$R = \frac{1}{4\pi\sigma l} \ln \left[\frac{\sqrt{l^2 + a^2} + l}{\sqrt{l^2 + a^2} - l} \right]$$



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Elipsoide condutor semienterrado

Aproximação para $a \ll l$:

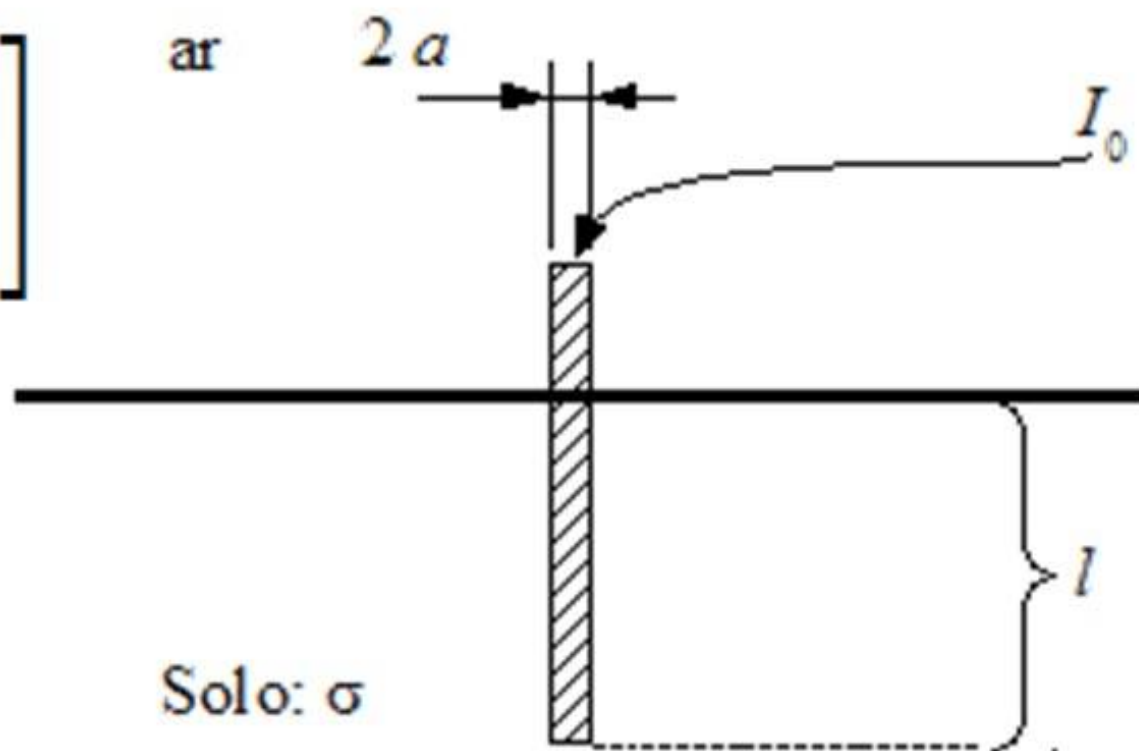
$$\begin{aligned} R &= \frac{1}{4\pi\sigma l} \ln \left[\frac{\sqrt{1+(a/l)^2} + 1}{\sqrt{1+(a/l)^2} - 1} \right] \approx \\ &\approx \frac{1}{4\pi\sigma l} \ln \left[\frac{1+(a/l)^2/2 + 1}{1+(a/l)^2/2 - 1} \right] \approx \\ &\approx \frac{1}{4\pi\sigma l} \ln \left[\frac{2}{(a/l)^2/2} \right] = \\ &= \frac{1}{4\pi\sigma l} \ln \left[\frac{4l^2}{a^2} \right] = \frac{1}{2\pi\sigma l} \ln \left[\frac{2l}{a} \right] \end{aligned}$$



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Estaca Cilíndrica Condutora

$$R \approx \frac{1}{2\pi\sigma l} \ln \left[\frac{2l}{a} \right]$$



$2a = 5 \text{ cm}; l = 1 \text{ m}; \sigma = 0,01 \text{ S/m};$
 $R \approx 70 \Omega$