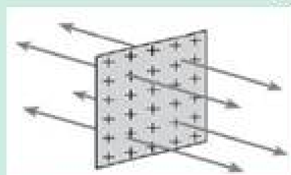
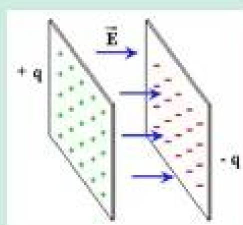
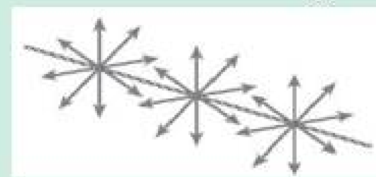
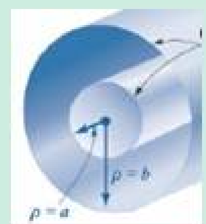
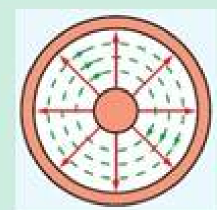
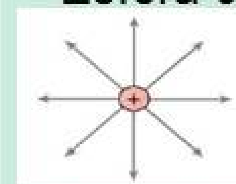
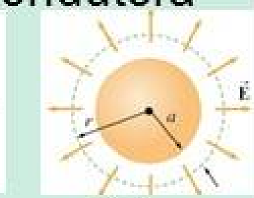
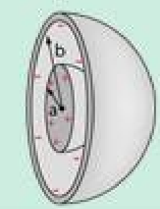
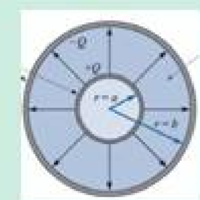
ESCOLA POLITÉCNICA DA UNIVERSIDADE DE SÃO PAULO

# Equações de Laplace e Poisson
# Solução por Integração
# Problemas com Simetria

# Eq. Laplace - Simetras com Variação 1D (distância *r*)

| Simetria | Plana | Cilíndrica | Esférica |
|---|---|---|---|
| **Configuração** | Plano carregado  Placas paralelas  | Linha carregada  Cabo coaxial   | Carga Puntiforme<br>Esfera condutora   Capacitor esférico   |
| **Variação do Potencial** | $r$ | $\ln\ r$ | $1/r$ |
| **Variação do Campo** | **constante** | $1/r$ | $1/r^2$ |

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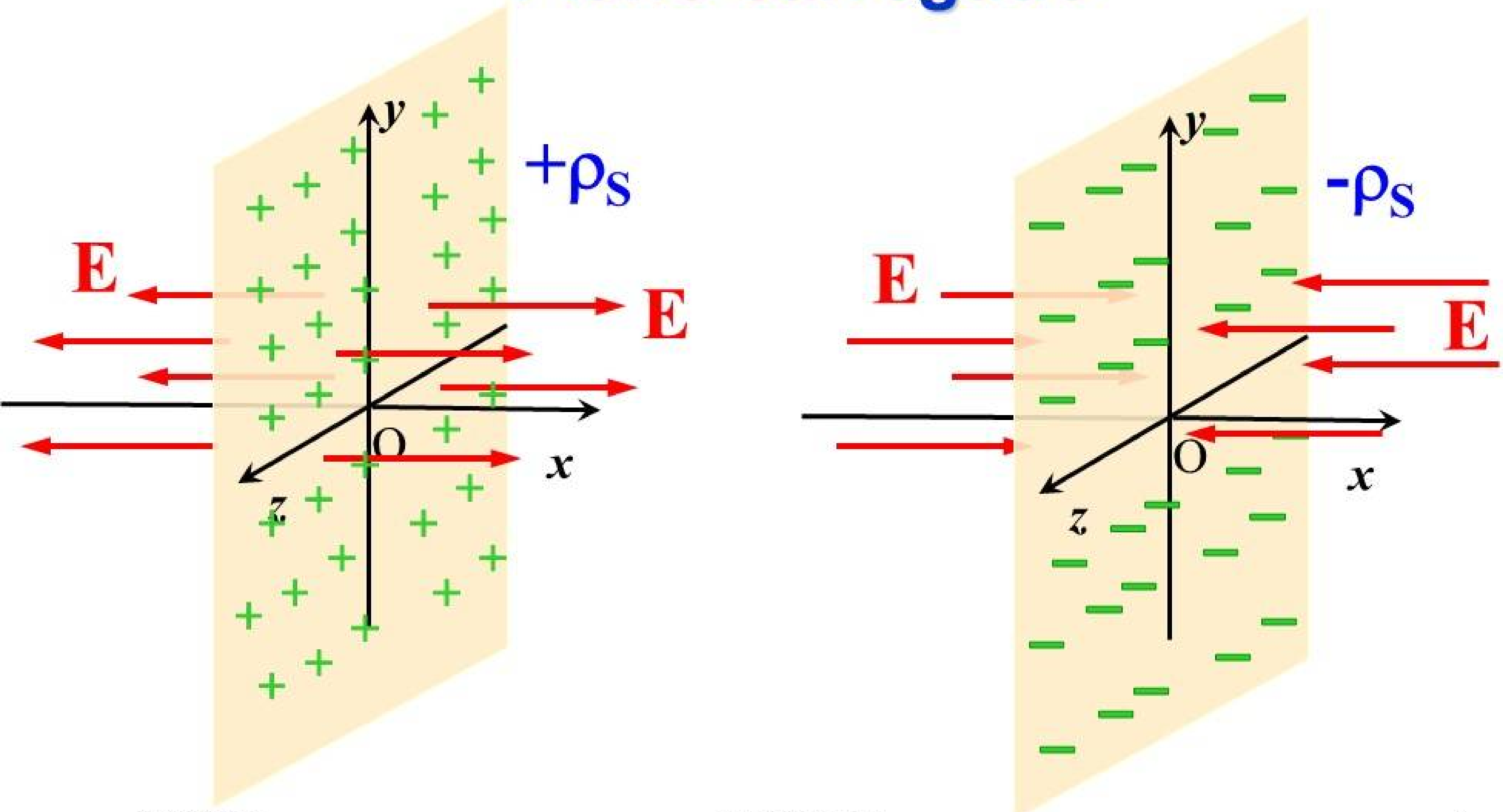
# Plano carregado



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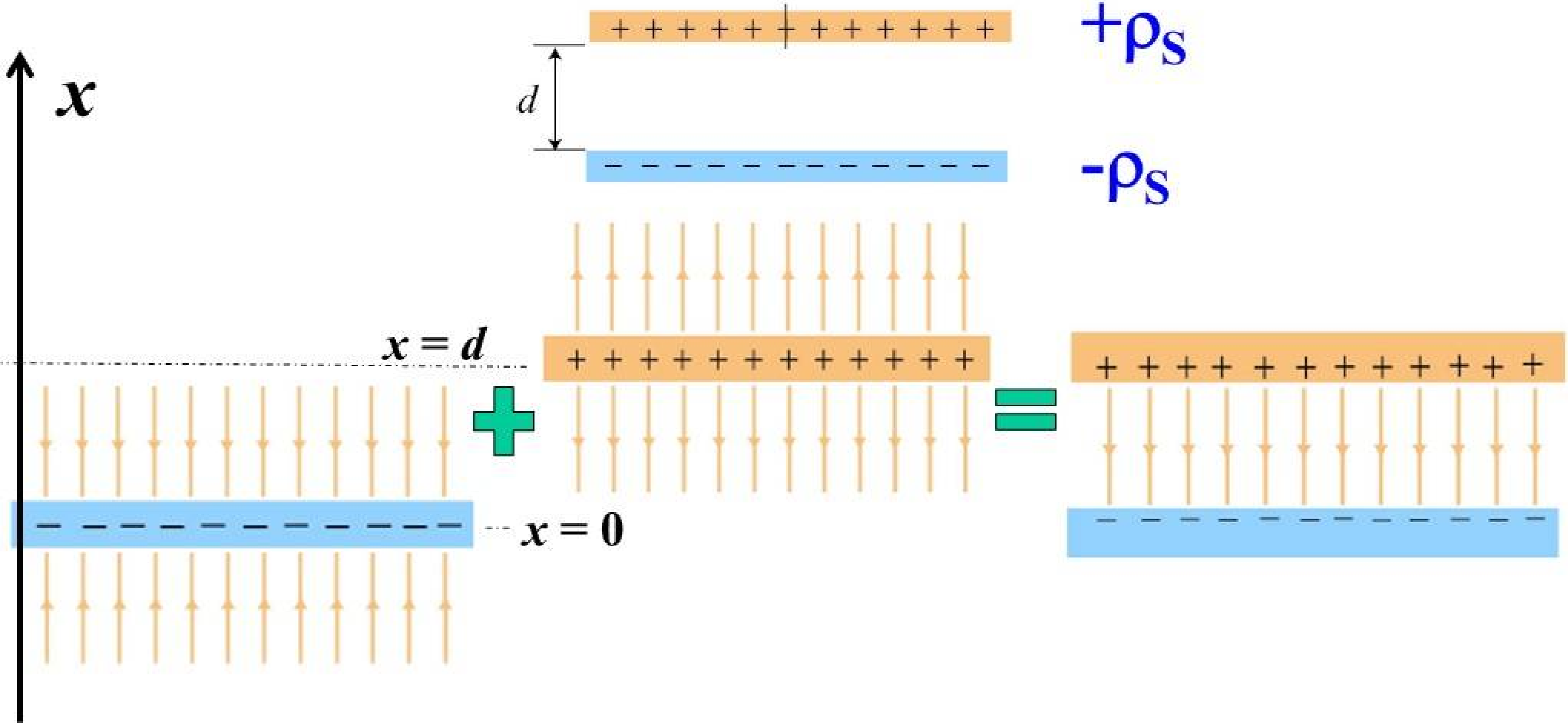
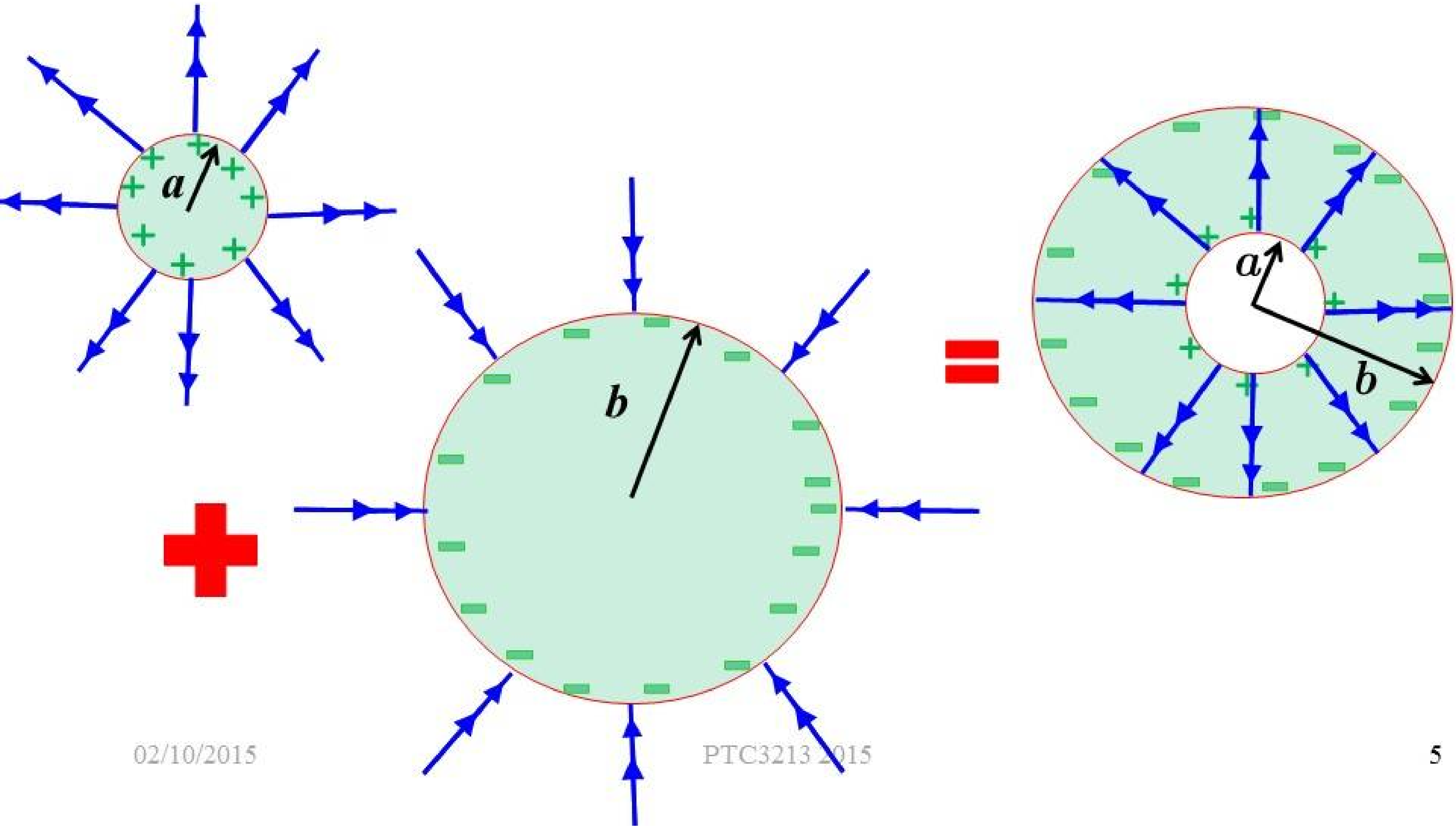
# Planos Paralelos



# Cabo Coaxial e Esferas Concêntricas



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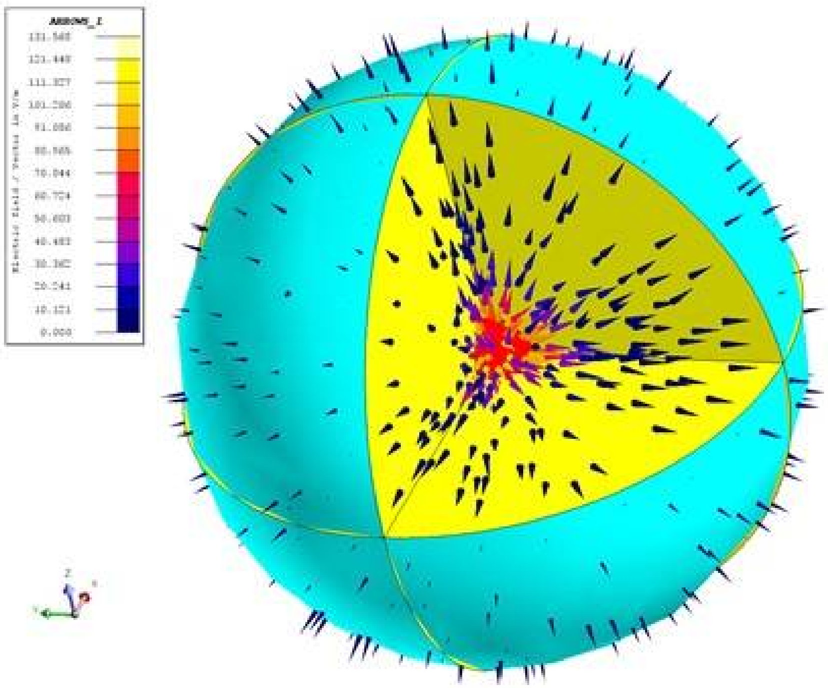
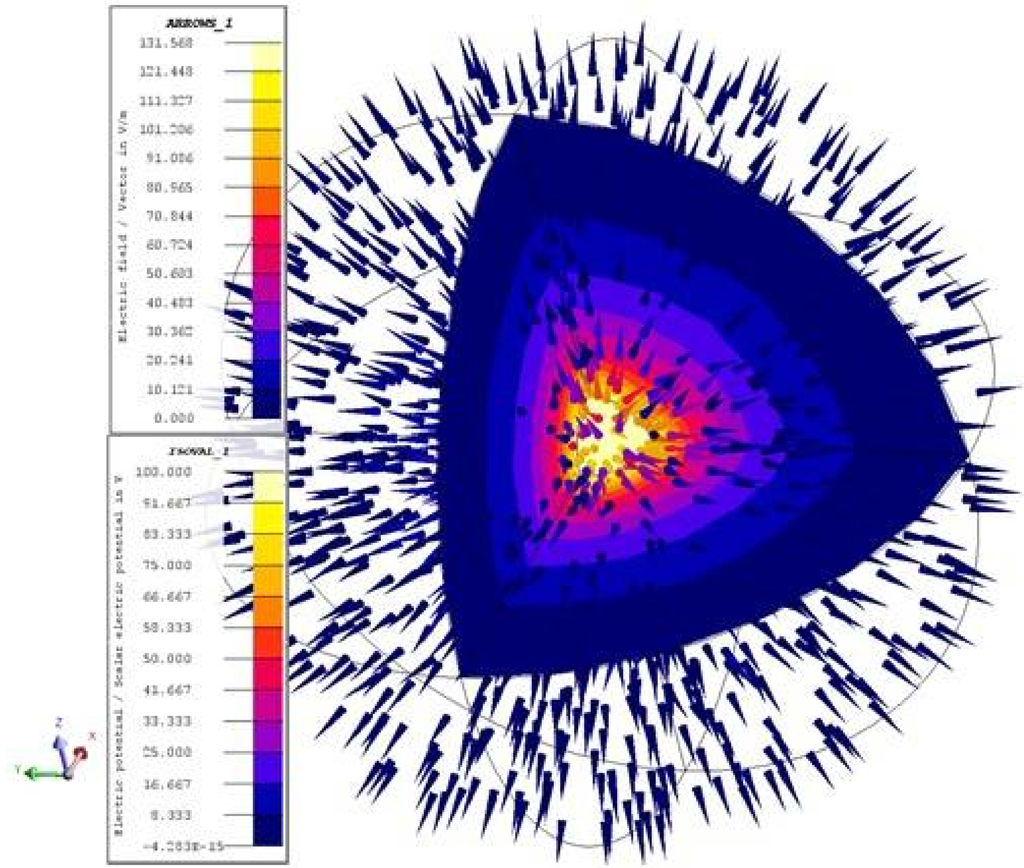
# Capacitor Esférico





# Teorema da Unicidade

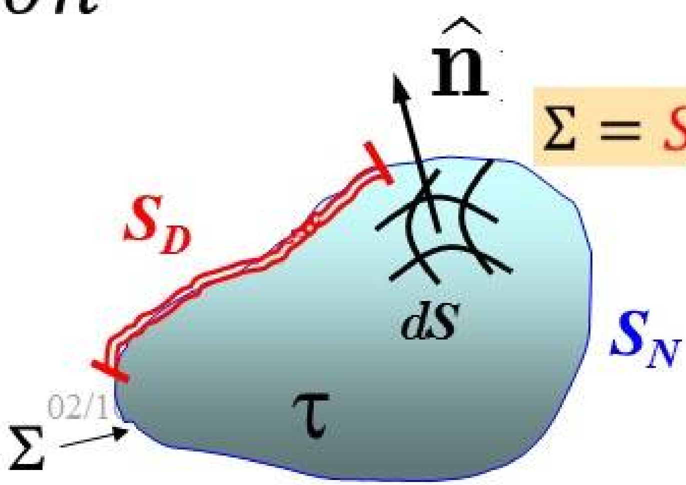
**Equação de Laplace**

$$\nabla^2 \varphi = 0$$

**Equação de Poisson**

$$\nabla^2 \varphi = -\frac{\rho_v}{\varepsilon}$$

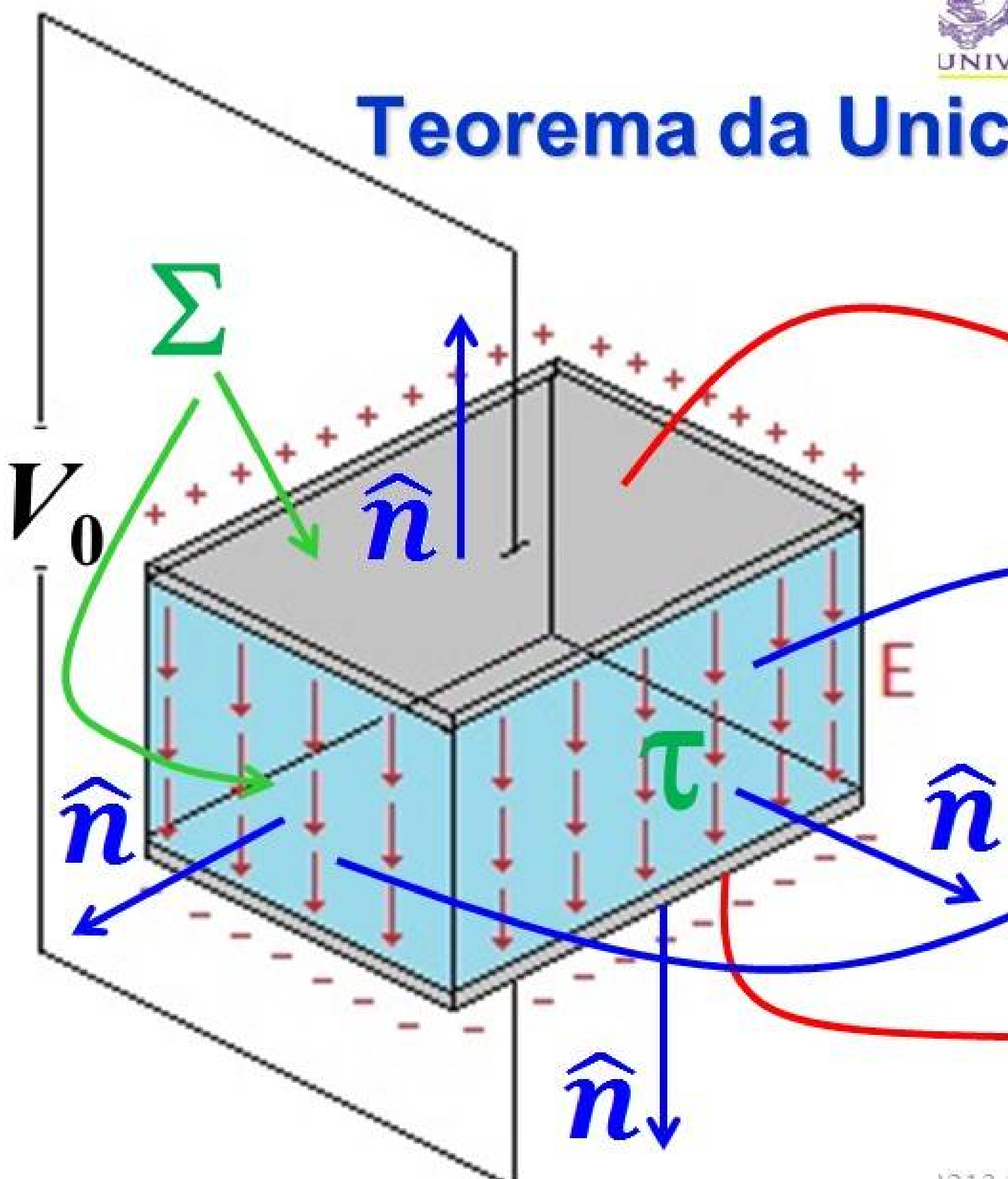
$$\begin{cases} \varphi = \varphi_0 \ \text{ em } S_D & \Rightarrow \text{Condição de Contorno de Dirichlet} \\ \dfrac{\partial \varphi}{\partial n} = \nabla\varphi \cdot \hat{n}, \text{em } S_N & \Rightarrow \text{Condição de Contorno de Neumann} \end{cases}$$




$$\Sigma = S_D \cup S_N$$

Apenas $\boldsymbol{\partial\varphi/\partial n}$ → solução única a menos de uma constante

# Teorema da Unicidade



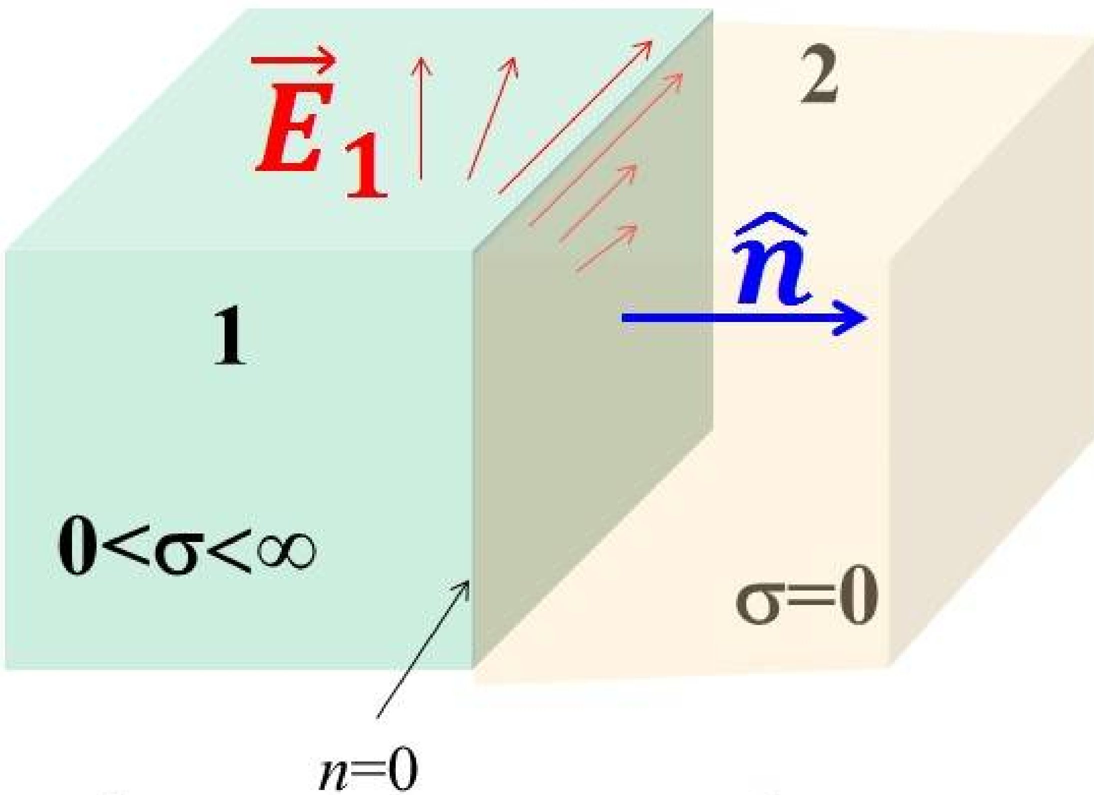

$$\Sigma(\tau) = S_D \cup S_N$$

$\varphi = V_0$ ➡ $S_D$

$E_n = \partial\varphi/\partial n = 0$ ➡ $S_N$

$\varphi = 0$ ➡ $S_D$

# Condição de Contorno de Neumann
# Interface Condutor(1)-Isolante(2)




$\partial\varphi/\partial n$

$$\vec{J}_2 = 0$$

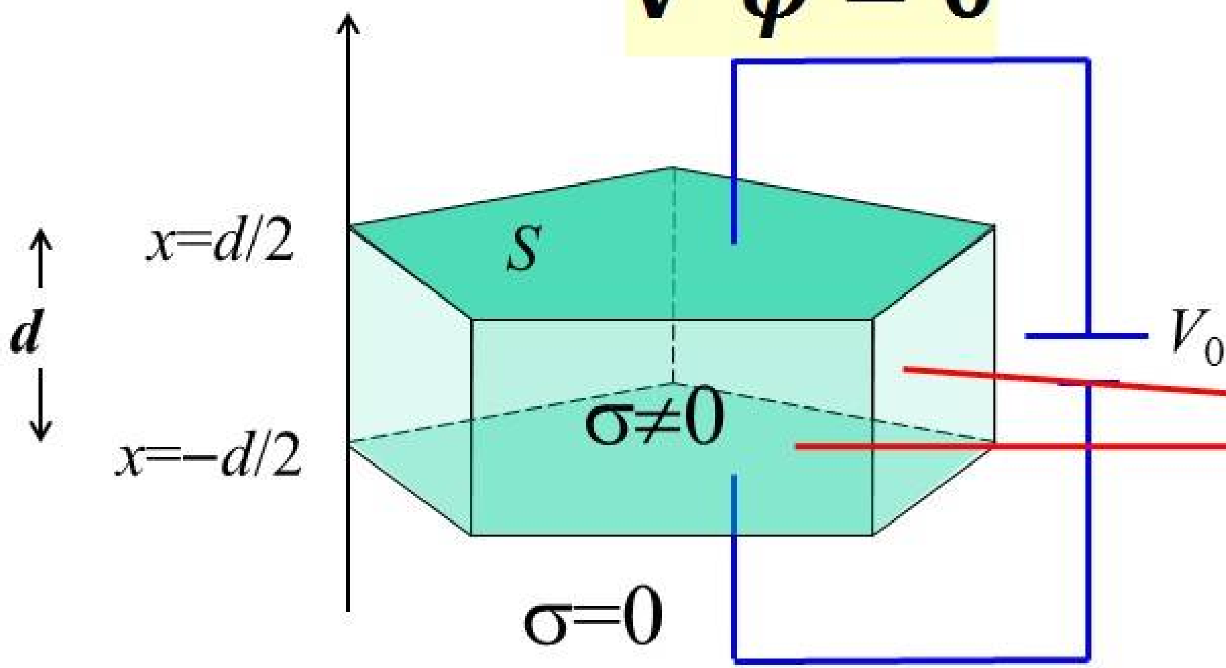
$$J_{n_1} = J_{n_2}$$

$$J_{n_1}\big|_{n=0} = 0 \Rightarrow \sigma E_{n_1}\big|_{n=0} = 0$$

$$E_{n_1}\big|_{n=0} = 0$$

# Ex. : Teorema da Unicidade - Capacitor

$$\nabla^2 \varphi = 0$$




$$\varphi(x = -d/2) = -V_0/2$$

$$\varphi(x = d/2) = V_0/2$$

$$\left.\frac{\partial \varphi}{\partial n}\right|_{\substack{\text{sup}\\ \text{lat}}} = 0$$

$$\varphi(x) = k_1 x + k_2$$

$$-k_1 d/2 + k_2 = -V_0/2 \quad \Rightarrow \quad k_1 d/2 + k_2 = V_0/2$$

$$k_1 = \frac{V_0}{d} \qquad k_2 = 0 \quad \Rightarrow \quad \varphi(x) = \frac{V_0}{d} x \quad \Rightarrow \quad R = \frac{d}{\sigma S}$$

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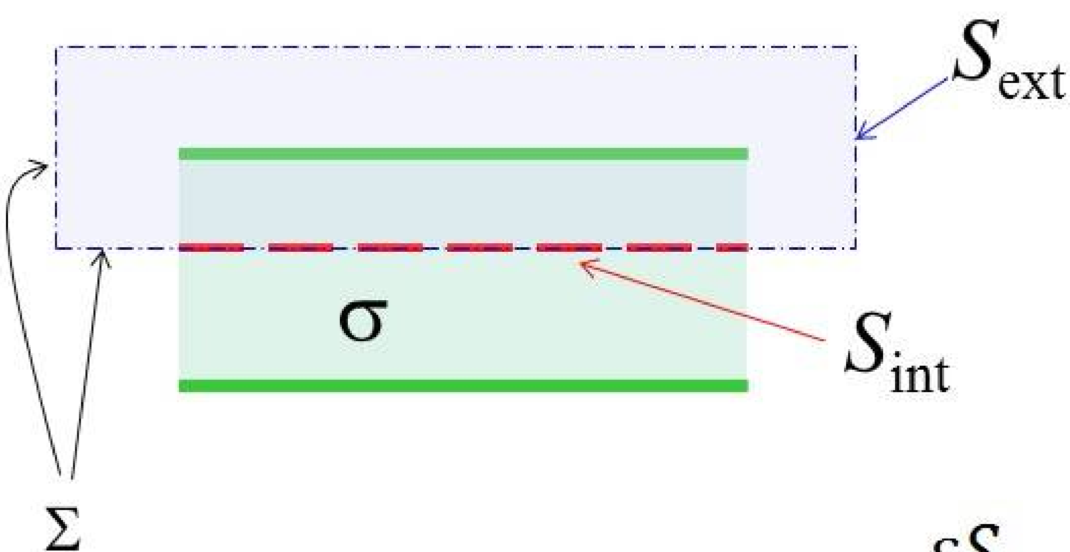
# Teorema da Unicidade - Capacitor

$$C = \frac{\oiint_{\Sigma} \vec{D} \cdot d\vec{S}}{V_0}$$

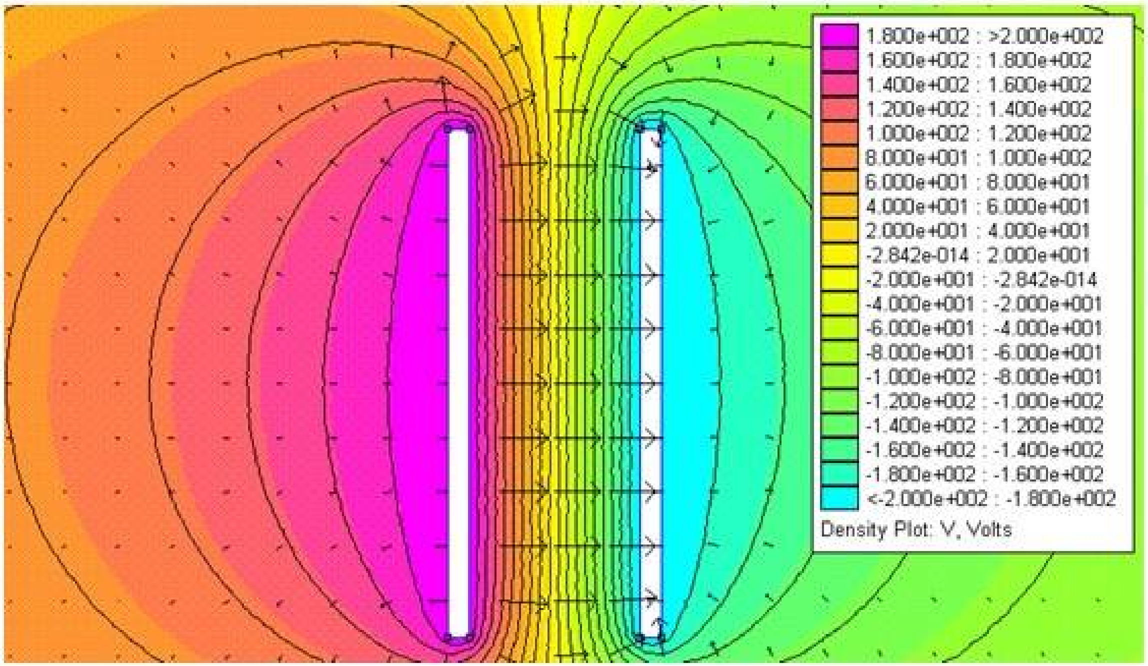
$$\oiint_{\Sigma} \vec{D} \cdot d\vec{S} = \iint_{S_{int}} \vec{D} \cdot d\vec{S} + \iint_{S_{ext}} \vec{D} \cdot d\vec{S}$$

$$= \frac{\varepsilon V_0 S}{d} + \iint_{S_{ext}} \vec{D} \cdot d\vec{S}$$



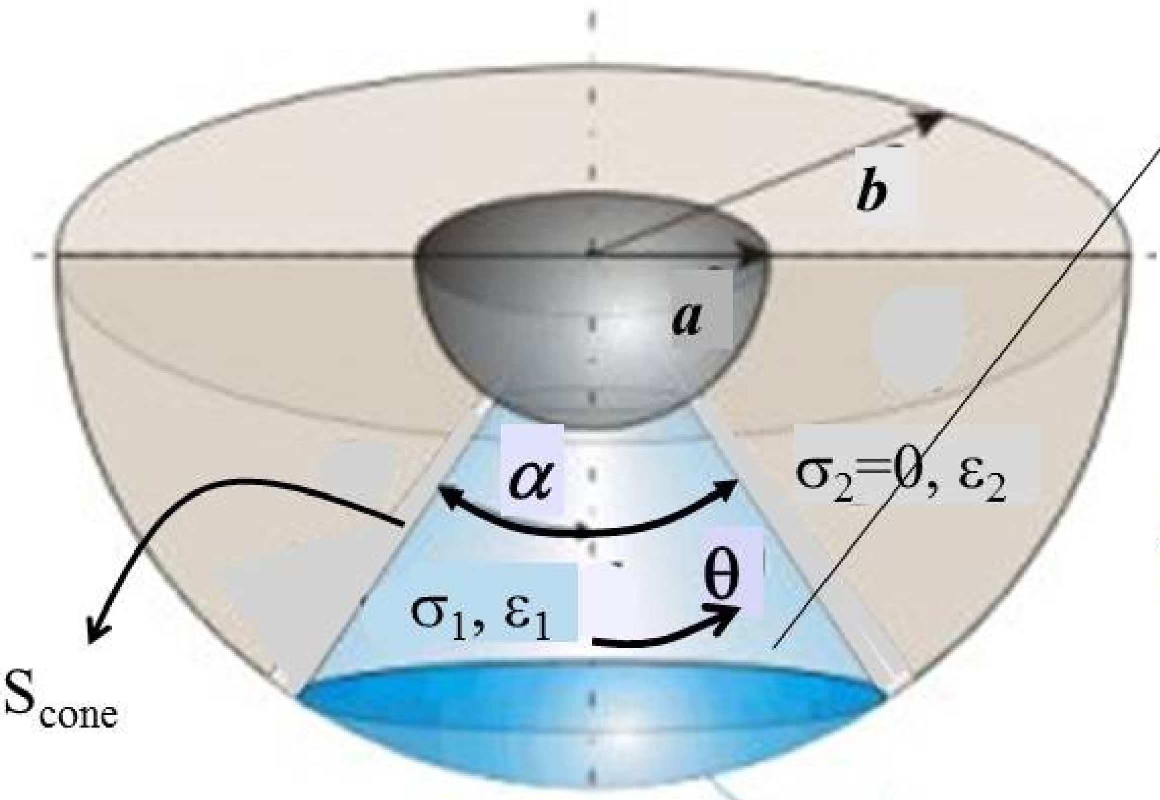

$$C = \frac{\varepsilon S}{d} + \frac{\iint_{S_{ext}} \vec{D} \cdot d\vec{S}}{V_0} > \frac{\varepsilon S}{d}$$

# Teorema da Unicidade - Capacitor




$$C = \frac{\varepsilon S}{d} + \frac{\iint_{S_{ext}} \vec{\boldsymbol{D}} \cdot d\vec{S}}{V_0} \quad > \frac{\varepsilon S}{d}$$

# Teorema da Unicidade – Capacitor Esférico sem Simetria




**1**: Condutor homogêneo

$$\nabla^2 \varphi = 0$$

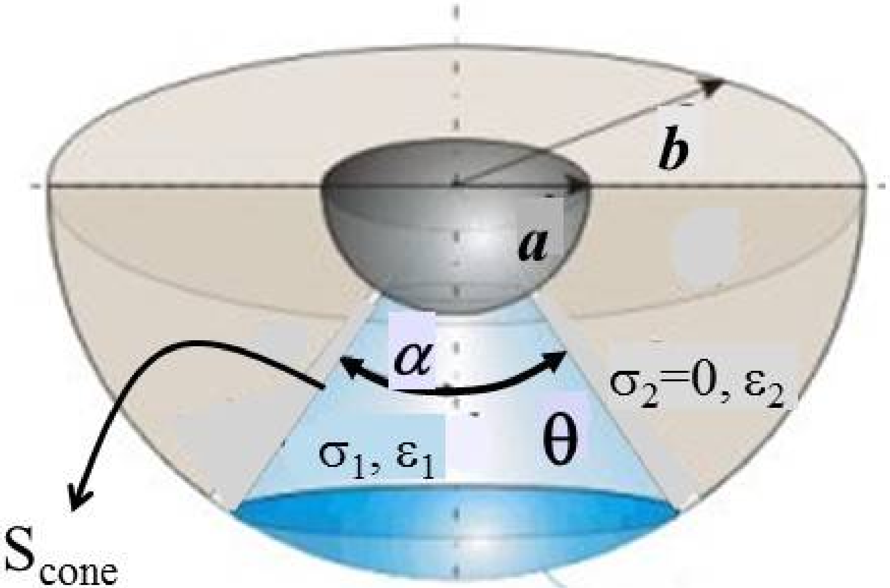
Condições de Contorno

$$\varphi(r = b, 0 \leq \theta \leq \alpha/2) = 0$$

$$\varphi(r = a, 0 \leq \theta \leq \alpha/2) = V_0$$

$$\left.\frac{\partial \varphi}{\partial n}\right|_{S_{cone}} = 0$$

# Capacitor Esférico sem Simetria – Solução em 1




$$\varphi(r) = \frac{V_0}{\left(\frac{1}{a} - \frac{1}{b}\right)}\left(\frac{1}{r} - \frac{1}{b}\right)$$

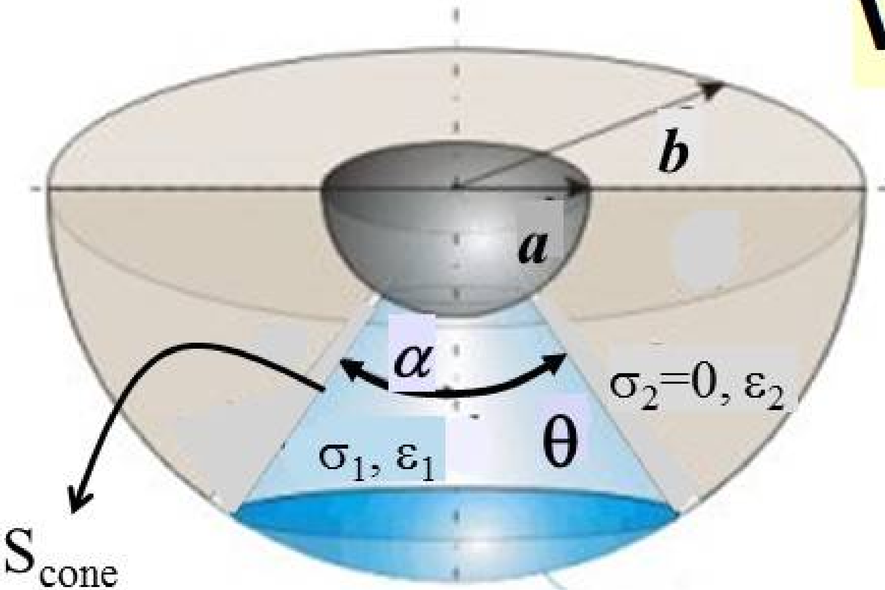
$$\varphi(r = \boldsymbol{b}) = \frac{V_0}{\left(\frac{1}{a} - \frac{1}{b}\right)}\left(\frac{1}{b} - \frac{1}{b}\right) = \mathbf{0}$$

$$\varphi(r = \boldsymbol{a}) = \frac{V_0}{\left(\frac{1}{a} - \frac{1}{b}\right)}\left(\frac{1}{a} - \frac{1}{b}\right) = \boldsymbol{V_0}$$

$$\left.\frac{\partial\varphi}{\partial n}\right|_{S_{cone}} = \frac{1}{r}\left.\frac{\partial\varphi(r)}{\partial\theta}\right|_{\theta=\alpha/2} = 0$$

# Capacitor Esférico sem Simetria – Solução em 2

$$\nabla^2\varphi = 0$$




$$\varphi(r=b)=0$$

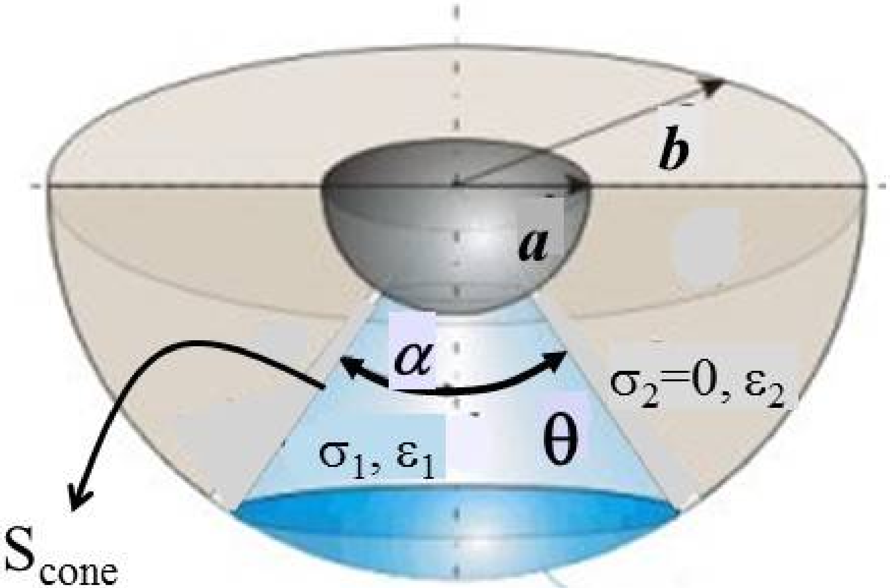
$$\varphi(r=a, \alpha/2 \le \theta \le \pi)=V_0$$

$$\varphi(r, \theta=\alpha/2) = \frac{V_0}{\left(\frac{1}{a}-\frac{1}{b}\right)}\left(\frac{1}{r}-\frac{1}{b}\right)$$

Solução Geral

$$\varphi(r) = \frac{V_0}{\left(\frac{1}{a}-\frac{1}{b}\right)}\left(\frac{1}{r}-\frac{1}{b}\right)$$

# Capacitor Esférico sem Simetria – Resistência de 1




$$\varphi(r) = \frac{V_0}{\left(\frac{1}{a} - \frac{1}{b}\right)}\left(\frac{1}{r} - \frac{1}{b}\right)$$

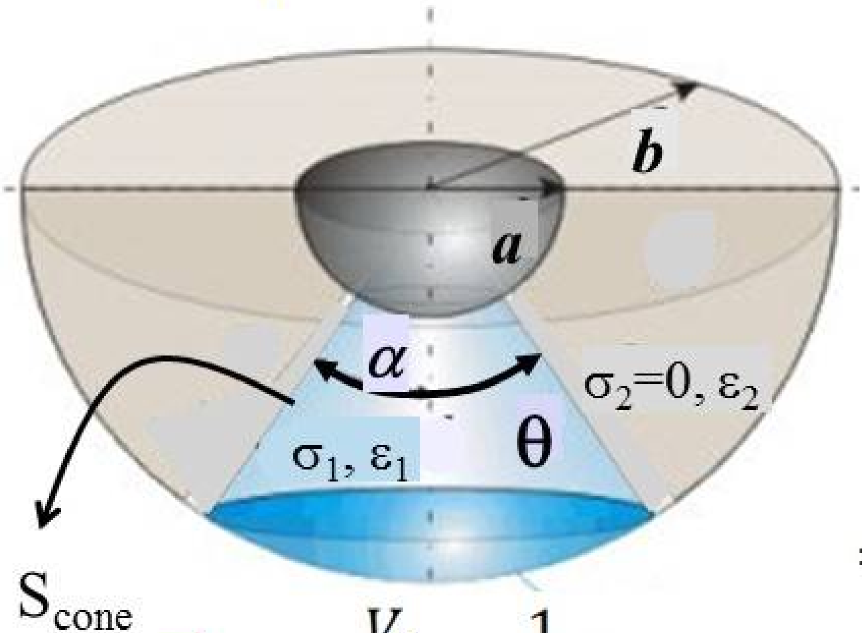
$$\vec{E} = \frac{V_0}{\left(\frac{1}{a} - \frac{1}{b}\right)}\frac{1}{r^2}\hat{u}_r$$

$$I = \iint\limits_{\text{sup esf } r} \vec{J} \cdot d\vec{S} = \int_0^{\alpha/2} \sigma_1 \vec{E} \cdot \hat{u}_r 2\pi r^2 \sin\theta \; d\theta$$

$$= \frac{2\pi\sigma_1 V_0}{\left(\frac{1}{a} - \frac{1}{b}\right)}\left(1 - \cos\frac{\alpha}{2}\right)$$

$$R = \frac{(b-a)}{2\pi\sigma_1 ab\left(1 - \cos\frac{\alpha}{2}\right)}$$

# Capacitor Esférico sem Simetria – Capacitância Total




$$\vec{E} = \frac{V_0}{\left(\frac{1}{a} - \frac{1}{b}\right)} \frac{1}{r^2} \hat{\boldsymbol{u}}_r$$

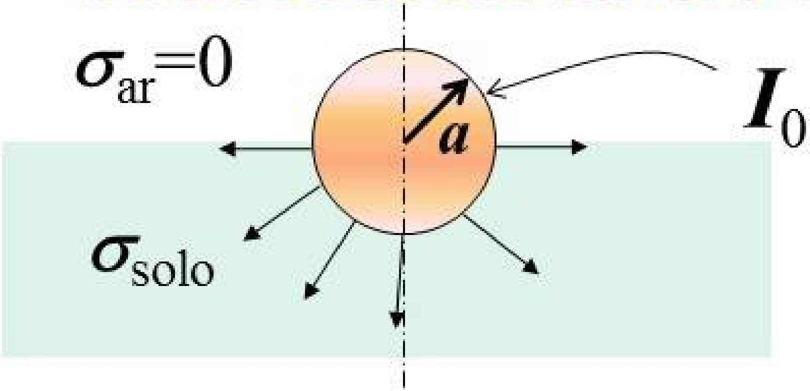
$$Q_a = \iint_{\text{sup esf } r} \vec{\boldsymbol{D}} \cdot d\vec{S} = \int_0^{\frac{\alpha}{2}} \varepsilon_1 \vec{\boldsymbol{E}} \cdot \hat{u}_r 2\pi r^2 \sin\theta \; d\theta + \int_{\alpha/2}^{\pi} \varepsilon_2 \vec{\boldsymbol{E}} \cdot \hat{u}_r 2\pi r^2 \sin\theta \; d\theta$$

$$= \frac{2\pi\varepsilon_1 V_0}{\left(\frac{1}{a} - \frac{1}{b}\right)}\left(1 - \cos\frac{\alpha}{2}\right) + \frac{2\pi\varepsilon_2 V_0}{\left(\frac{1}{a} - \frac{1}{b}\right)}\left(1 + \cos\frac{\alpha}{2}\right)$$

$$= \frac{2\pi V_0 ab}{(b-a)}\left[(\varepsilon_1 + \varepsilon_2) + \cos\frac{\alpha}{2}(\varepsilon_1 - \varepsilon_2)\right]$$

$$C = \frac{2\pi ab}{(b-a)}\left[(\varepsilon_1 + \varepsilon_2) + \cos\frac{\alpha}{2}(\varepsilon_1 - \varepsilon_2)\right]$$

# Resistência de Aterramento - Esfera




$\alpha=\pi$ e $b\to\infty$

$$\varphi(r)=\frac{V_0}{\left(\frac{1}{a}-\frac{1}{b}\right)}\left(\frac{1}{r}-\frac{1}{b}\right)$$

$$\varphi(r)=\frac{aV_0}{r}$$

$$\vec{E}=\frac{V_0}{\left(\frac{1}{a}-\frac{1}{b}\right)}\frac{1}{r^2}\hat{u}_r \Rightarrow \vec{E}=\frac{aV_0}{r^2}\hat{u}_r$$

$$\vec{J}=\frac{\sigma_{solo}aV_0}{r^2}\hat{u}_r$$

$$I=2\pi a\sigma_{solo}V_0$$

$$R=\frac{1}{2\pi a\sigma_{solo}}$$

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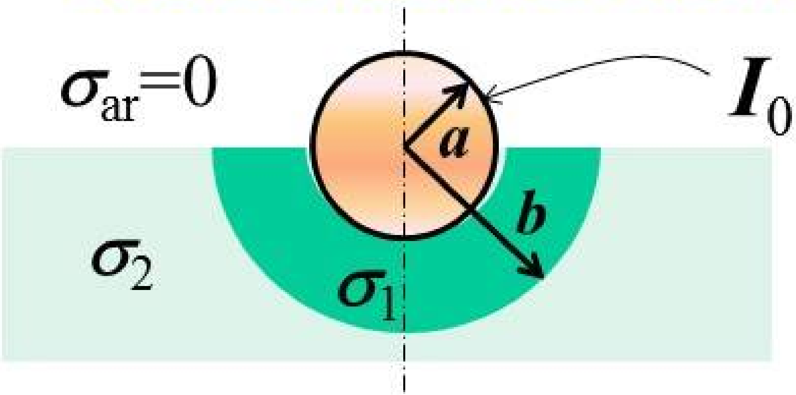
# Resistência de Aterramento – Solo Estratificado




$$\varphi_2(r=\infty, 0 \le \theta \le \pi/2) = 0$$

$$\varphi_1(r=a, 0 \le \theta \le \pi/2) = V_0$$

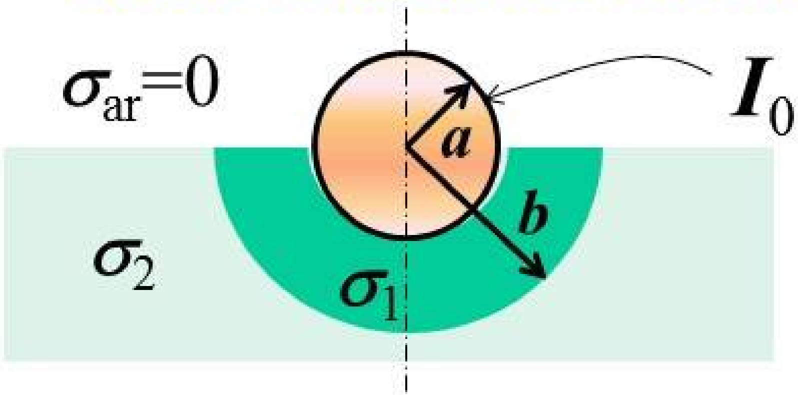
$$\left.\frac{\partial \varphi_{1,2}}{\partial n}\right|_{\substack{\text{interface}\\ \text{solo-ar}}} = 0$$

$$\left.\frac{-\sigma_1 \partial \varphi_1}{\partial r}\right|_{r=b} = \left.\frac{-\sigma_2 \partial \varphi_2}{\partial r}\right|_{r=b}$$

$$\varphi_1(r) = \frac{V_0 - V_b}{\left(\frac{1}{a} - \frac{1}{b}\right)}\left(\frac{1}{r} - \frac{1}{b}\right) + V_b$$

$$\varphi_2(r) = b V_b \frac{1}{r}$$

# Resistência de Aterramento – Solo Estratificado




$$\left.\frac{-\sigma_1 \partial\varphi_1}{\partial r}\right|_{r=b} = \left.\frac{-\sigma_2 \partial\varphi_2}{\partial r}\right|_{r=b}$$

$$\sigma_1 \frac{V_0 - V_b}{\left(\frac{1}{a} - \frac{1}{b}\right)} \frac{1}{b^2} = \sigma_2 b V_b \frac{1}{b^2}$$

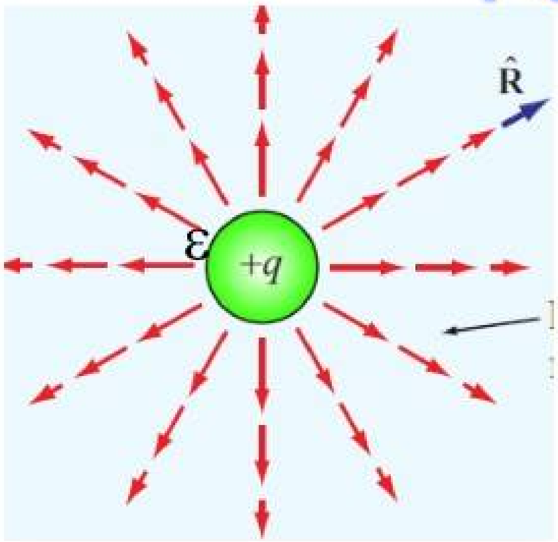
$$V_b = \frac{\sigma_1 a}{\sigma_1 a + \sigma_2 (b-a)} V_0$$

$$I = \iint_{S_{r>b}} \vec{J} \cdot d\vec{S} = \frac{\sigma_2 b V_b}{r^2} 2\pi r^2 = \frac{2\pi\sigma_1\sigma_2 ab}{\sigma_1 a + \sigma_2 (b-a)} V_0$$

$$R = \frac{1}{2\pi\sigma_2 b} + \frac{(b-a)}{2\pi\sigma_1 ab}$$

$a$=1 m, $b$=3 m, $\sigma_1$= 0,01 S/m e $\sigma_2$=0,1 S/m ⇨ $\boldsymbol{R = 5{,}6\ \Omega}$

# Potencial a partir de Q, ρ




$$\nabla^2 \boldsymbol{\varphi} = 0 \;\Rightarrow\; \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \varphi}{\partial r}\right) = 0$$

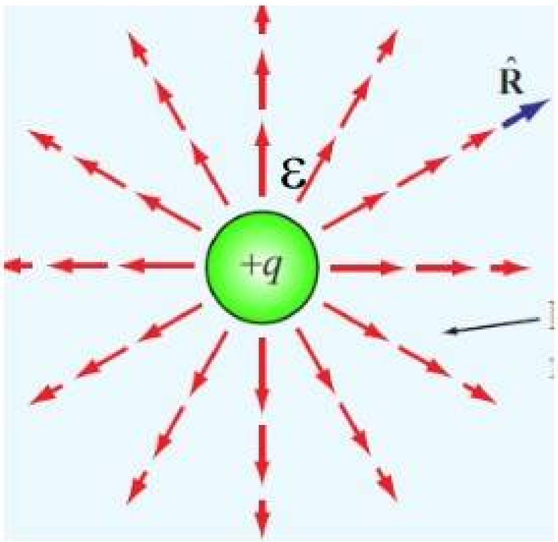
Simetria Esférica → 1D

$$r^2 \frac{\partial \varphi}{\partial r} = k_1 \;\Rightarrow\; \varphi(r) = -\frac{k_1}{r} + k_2$$

$$\varphi(\infty) = \lim_{r\to\infty}\left(-\frac{k_1}{r} + k_2\right) = 0 \quad \Rightarrow k_2 = 0$$

$$\varphi(r) = -\frac{k_1}{r}$$

# Potencial a partir de *q*, ρ




$$\vec{E} = -\nabla\varphi \quad \Rightarrow \quad \vec{E} = -\frac{\partial \varphi}{\partial r}\hat{u}_r$$

$$\vec{E} = -\frac{k_1}{r^2}\hat{u}_r \quad \Rightarrow \quad \vec{D} = \varepsilon\vec{E} = -\varepsilon\frac{k_1}{r^2}\hat{u}_r$$

$$q = \oiint_S \vec{D}\cdot d\vec{S} = -\varepsilon\frac{k_1}{r^2}\oiint_S dS = -4\pi\varepsilon k_1 \quad \Rightarrow \quad k_1 = -\frac{q}{4\pi\varepsilon}$$

$$\varphi(r) = \frac{q}{4\pi\varepsilon r}$$

$$\vec{E} = \frac{q}{4\pi\varepsilon r^2}\hat{u}_r$$

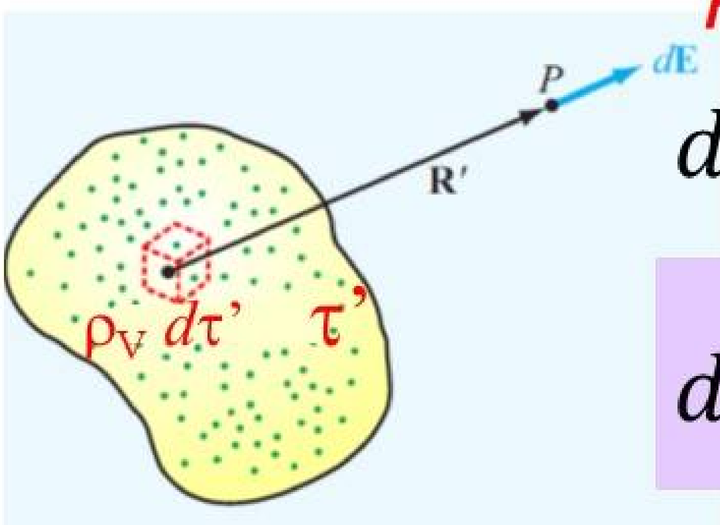
$$\vec{D} = \frac{q}{4\pi r^2}\hat{u}_r$$

# Potencial a partir de *q*, ρ

$$\varphi(P) = \sum_{i=1}^{n} \frac{q_i}{4\pi\varepsilon r_i}$$

$$\rho_V(x', y', z') \qquad d\tau' = dx'dy'dz'$$

$$dq = \rho_V(x', y', z')\, d\tau' \qquad P(x, y, z)$$

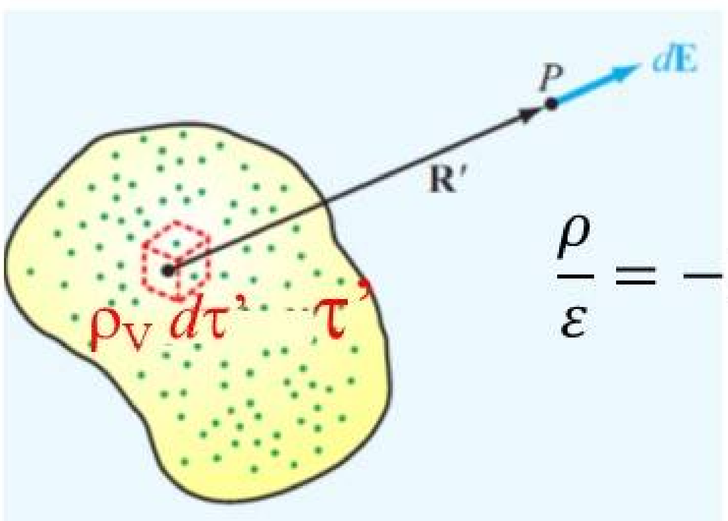



$$d\varphi(x, y, z) = \frac{\rho_V(x', y', z')\, d\tau'}{4\pi\varepsilon R}$$

# Potencial a partir de *q*, ρ

$$d\varphi(x,y,z) = \frac{\rho_V(x',y',z')\,d\tau'}{4\pi\varepsilon R}$$

$$R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$




$$\frac{\rho}{\varepsilon} = -\nabla^2\varphi$$

$$\varphi(x,y,z) = \iiint_{\tau} \frac{\rho_V(x',y',z')\,d\tau'}{4\pi\varepsilon R}$$

$$\varphi(x,y,z) = -\iiint_{\tau} \frac{\nabla^2\varphi\,d\tau'}{4\pi R}$$