ESCOLA POLITÉCNICA DA UNIVERSIDADE DE SÃO PAULO

Equações de Laplace e Poisson Solução por Integração (1D)

Determinação da Função Potencial

Campo de Correntes Estacionárias

$$\nabla \cdot (-\boldsymbol{\sigma} \nabla \varphi) = 0$$

$$\nabla^2 \varphi = -\frac{\nabla \boldsymbol{\sigma}}{\boldsymbol{\sigma}} \nabla \varphi$$

Campo Eletrostático

$$\nabla \cdot (-\boldsymbol{\varepsilon} \nabla \varphi) = \boldsymbol{\rho_v}$$

$$\nabla^2 \varphi = -\frac{\boldsymbol{\rho_v}}{\boldsymbol{\varepsilon}} - \frac{\nabla \boldsymbol{\varepsilon}}{\boldsymbol{\varepsilon}} \nabla \varphi$$

Em meios homogêneos

$$\boldsymbol{\nabla^2 \varphi = 0}$$

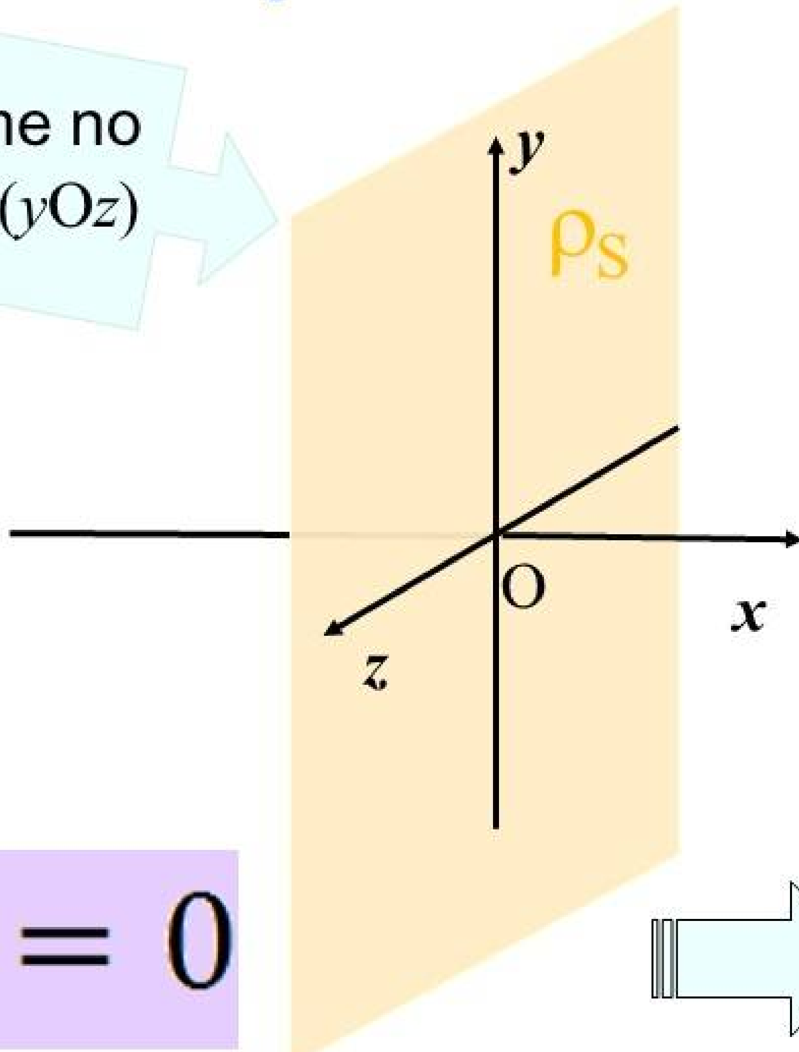
Equação de Laplace

$$\boldsymbol{\nabla^2 \varphi = -\frac{\rho_v}{\varepsilon}}$$

Equação de Poisson

Campo Unidimensional (1D)

ρ_S uniforme no plano $\boldsymbol{x}=\mathbf{0}$ (yOz)



Potencial $\varphi = \varphi(x)$

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0$$

$$\nabla^2 \varphi = 0$$

$$\frac{\partial^2 \varphi}{\partial x^2} = 0$$

Solução da Equação de Laplace em 1D

$$\frac{\partial^2 \varphi}{\partial x^2} = 0 \quad \Rightarrow \quad \varphi(x) = k_1 x + k_2$$

Devido à simetria do problema, $\varphi(x) = \varphi(-x)$

$$\varphi(x) = k_1 |x| + k_2$$

k_1, k_2 ?? → **Condições de Contorno**

Solução da Equação de Laplace em 1D

Condições de contorno sobre os campos

$$\vec{E} = -\nabla\varphi = \begin{cases} -k_1\hat{u}_x\,, & x > 0 \\ k_1\hat{u}_x\,, & x < 0 \end{cases}$$

$$\vec{D} = \varepsilon\vec{E} = \begin{cases} -\varepsilon k_1\hat{u}_x\,, & x > 0 \\ \varepsilon k_1\hat{u}_x\,, & x < 0 \end{cases}$$

$$D_{n_1} - D_{n_2} = \rho_S$$

$$D_x(0^+) - D_x(0^-) = \rho_S$$

$$-\varepsilon k_1 - \varepsilon k_1 = \rho_S$$

Solução da Equação de Laplace em 1D

$$-2\varepsilon k_1 = \rho_S \quad \rightarrow \quad k_1 = -\rho_S/2\varepsilon$$

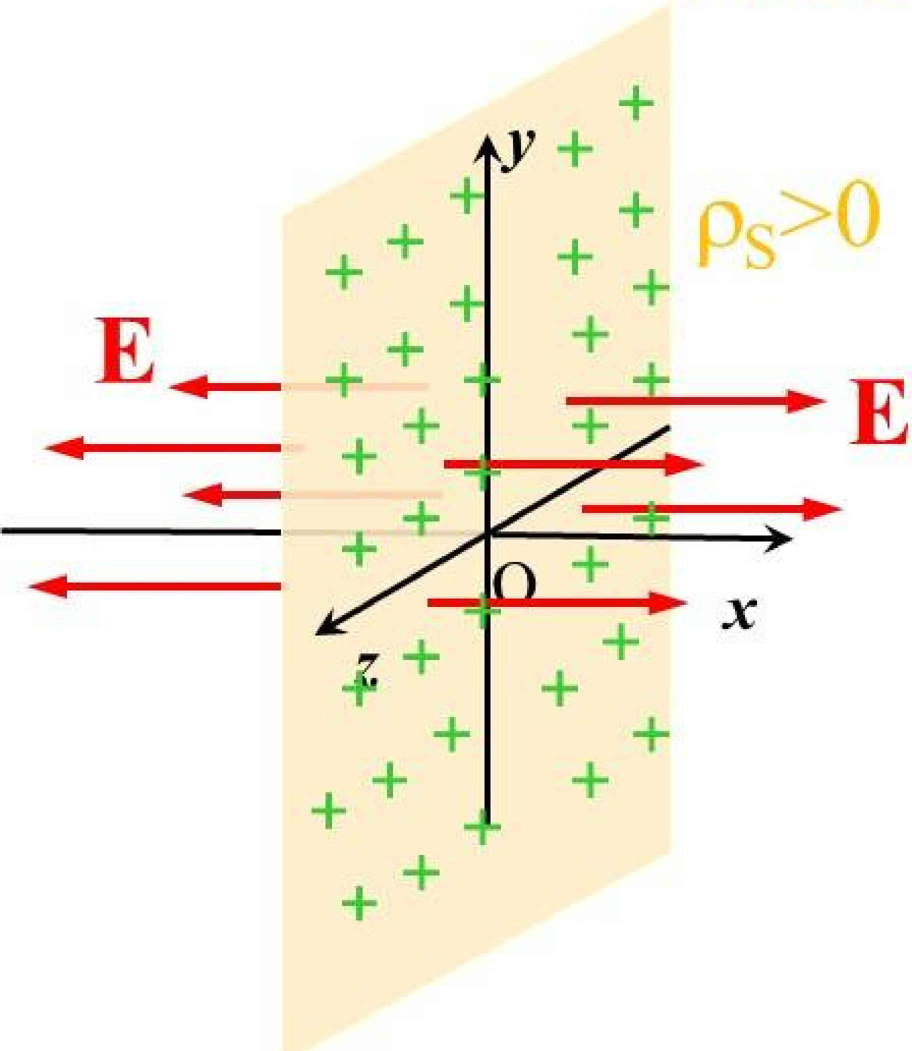
$$\vec{E} = -\nabla\varphi = \begin{cases} \dfrac{\rho_S}{2\varepsilon}\hat{u}_x\,,\ x > 0 \\ \dfrac{-\rho_S}{2\varepsilon}\hat{u}_x\,,\ x < 0 \end{cases}$$

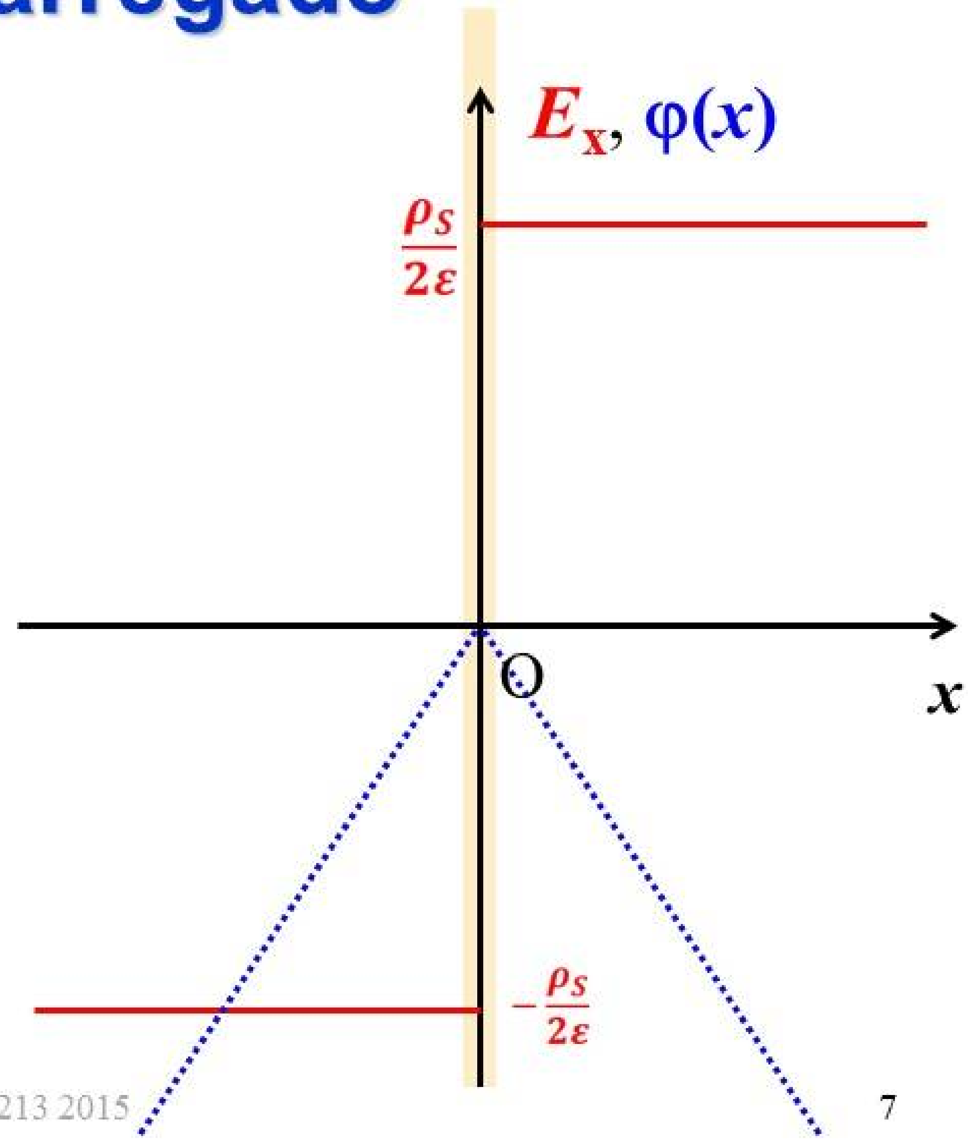
Faz-se arbitrariamente

$$k_2 = 0$$

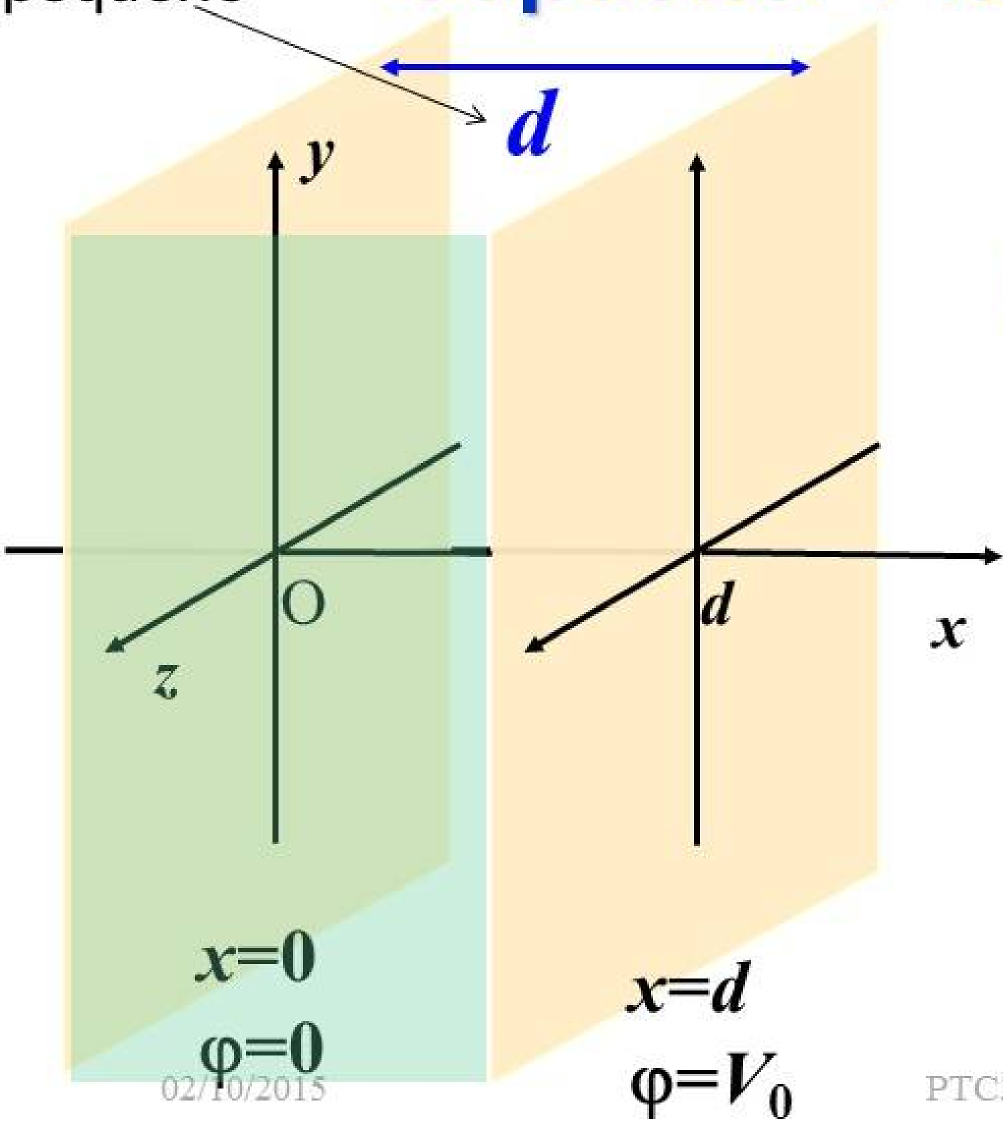
$$\varphi(x) = -\frac{\rho_S}{2\varepsilon}|x|$$

Plano carregado





Capacitor Plano de Placas //



$$\varphi(x) = k_1 x + k_2$$

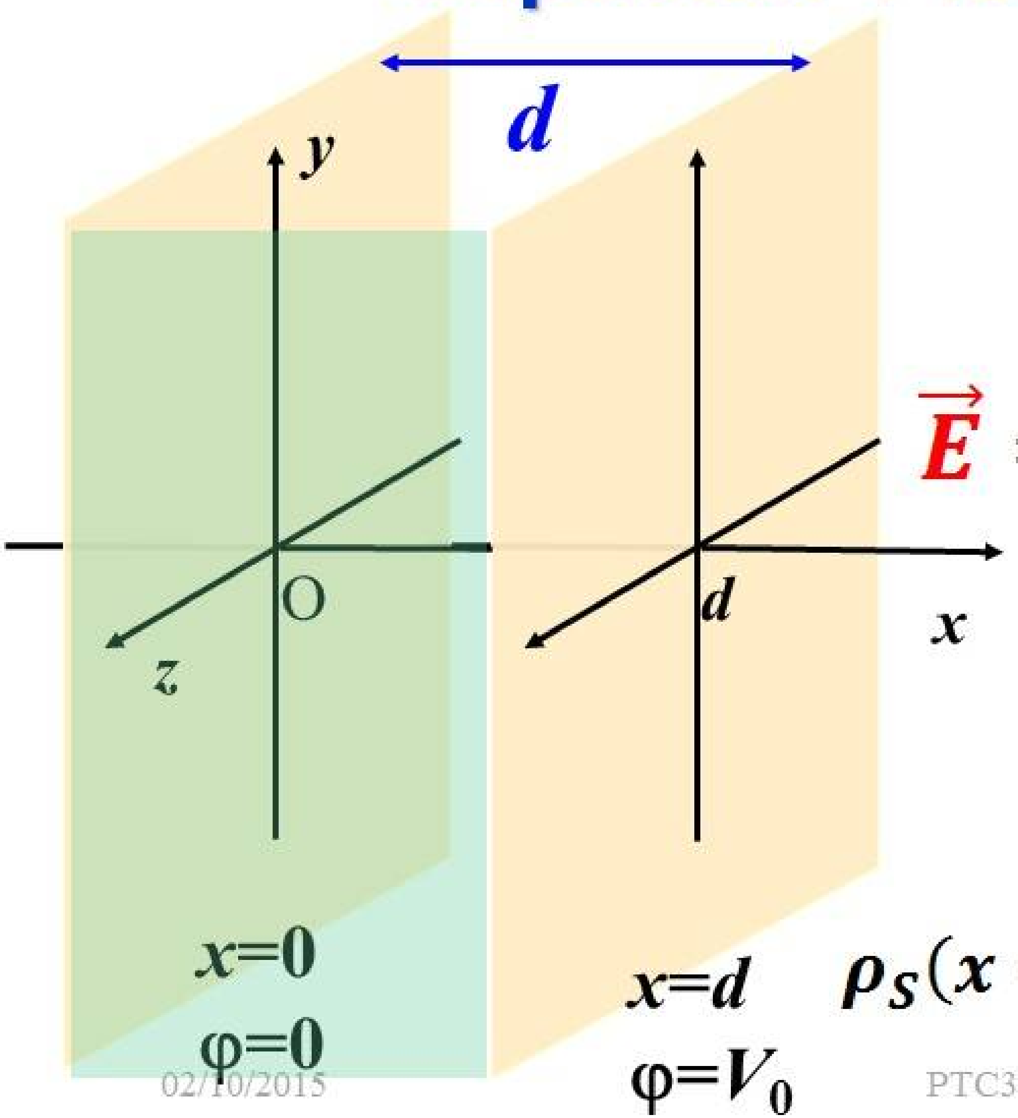
$$V_0 = \varphi(d) - \varphi(0) = k_1 d$$

$$k_1 = \frac{V_0}{d}$$

$$\varphi(0) = 0 \Rightarrow k_2 = 0$$

$$\varphi(x) = \frac{V_0}{d} x$$

Capacitor Plano de Placas //



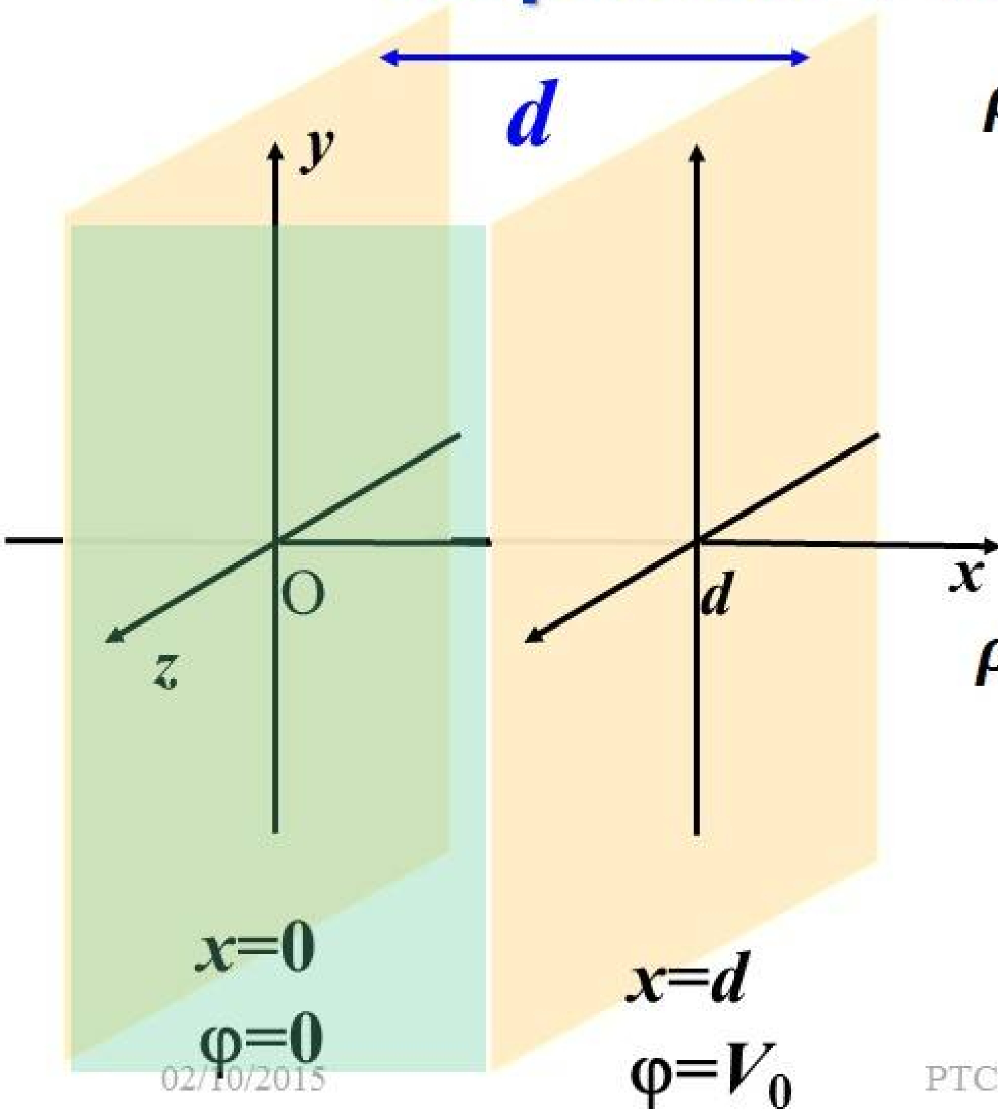
$$\varphi(x) = \frac{V_0}{d}x$$

$$\vec{E} = -\nabla\varphi \quad \Rightarrow \quad E_x = -\frac{\partial \varphi}{\partial x}$$

$$\vec{E} = -\frac{V_0}{d}\hat{u}_x$$

$$\rho_S(x = 0) = -\frac{\varepsilon V_0}{d} - 0 = -\frac{\varepsilon V_0}{d}$$

Capacitor Plano de Placas //



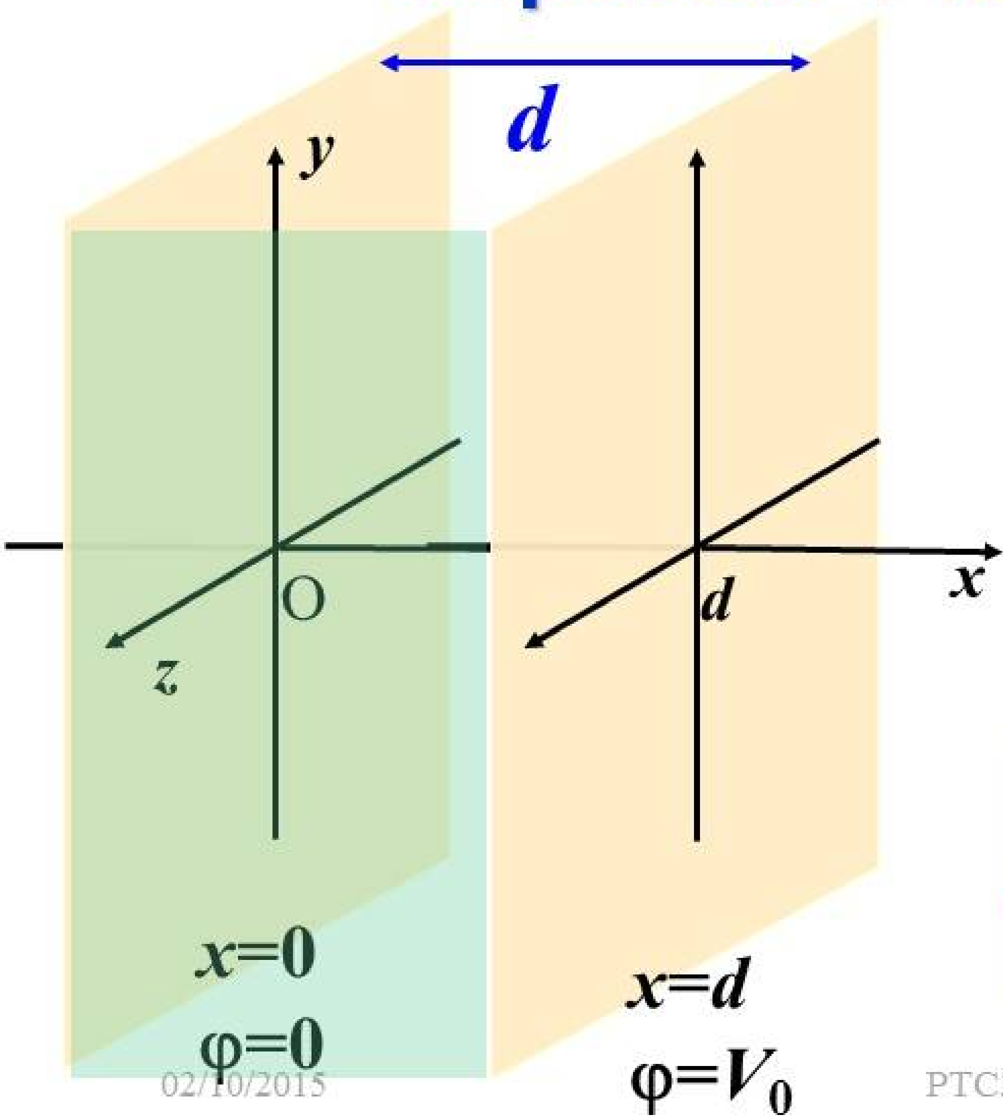
$$\boldsymbol{\rho_S}(\boldsymbol{x} = \mathbf{0}) = \boldsymbol{D}_x(\boldsymbol{x} = \mathbf{0}^+) - \boldsymbol{D}_x(\boldsymbol{x} = \mathbf{0}^-)$$

$$\boldsymbol{\rho_S}(\boldsymbol{x} = \mathbf{0}) = -\frac{\varepsilon V_0}{d} - 0 = -\frac{\varepsilon V_0}{d}$$

$$\boldsymbol{\rho_S}(\boldsymbol{x} = \boldsymbol{d}) = \boldsymbol{D}_x(\boldsymbol{x} = \boldsymbol{d}^+) - \boldsymbol{D}_x(\boldsymbol{x} = \boldsymbol{d}^-)$$

$$\boldsymbol{\rho_S}(\boldsymbol{x} = \boldsymbol{d}) = \mathbf{0} - \left(-\frac{\varepsilon V_0}{d}\right) = \frac{\varepsilon V_0}{d}$$

Capacitor Plano de Placas //



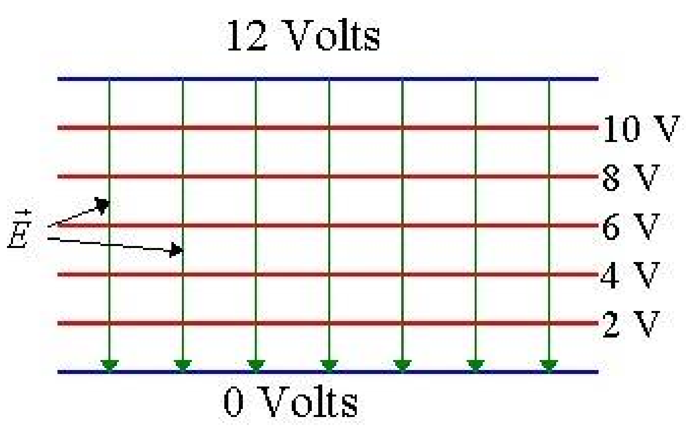
$$C = \frac{Q}{V_0} = \frac{\iint_S \vec{D} \cdot d\vec{S}}{V_0}$$

$$\iint_S \vec{D} \cdot d\vec{S} = \boldsymbol{D}.S = \frac{\varepsilon V_0}{d}.S$$

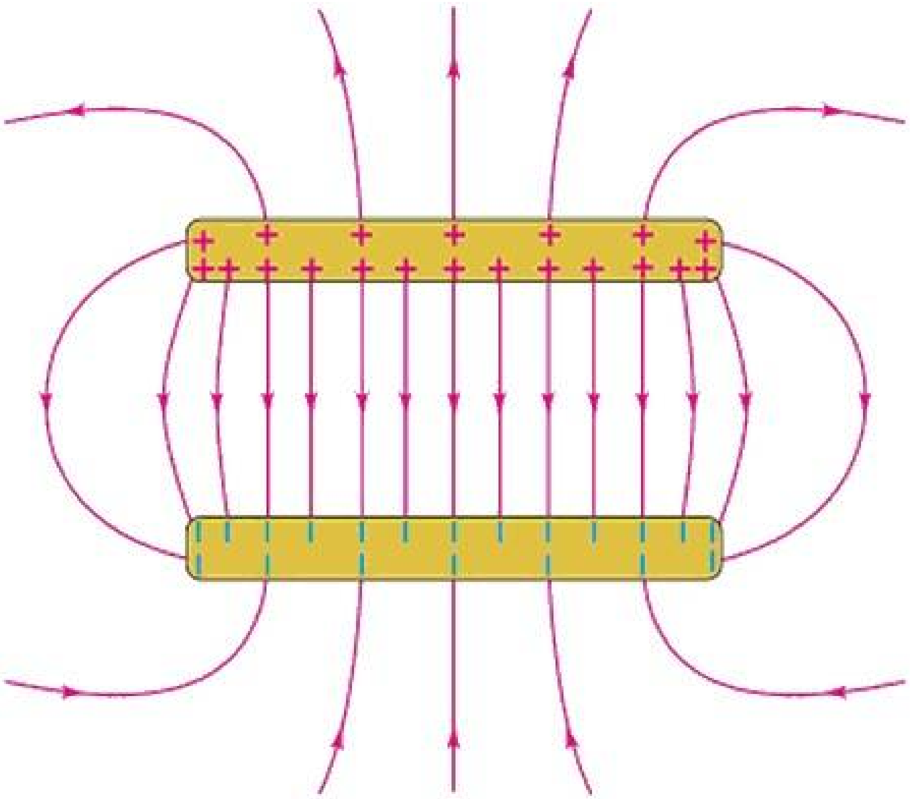
$$C = \frac{\varepsilon S}{d}$$

$$RC = \frac{\varepsilon}{\sigma} \Rightarrow R = \frac{d}{\sigma S}$$

Capacitor Plano de Placas //

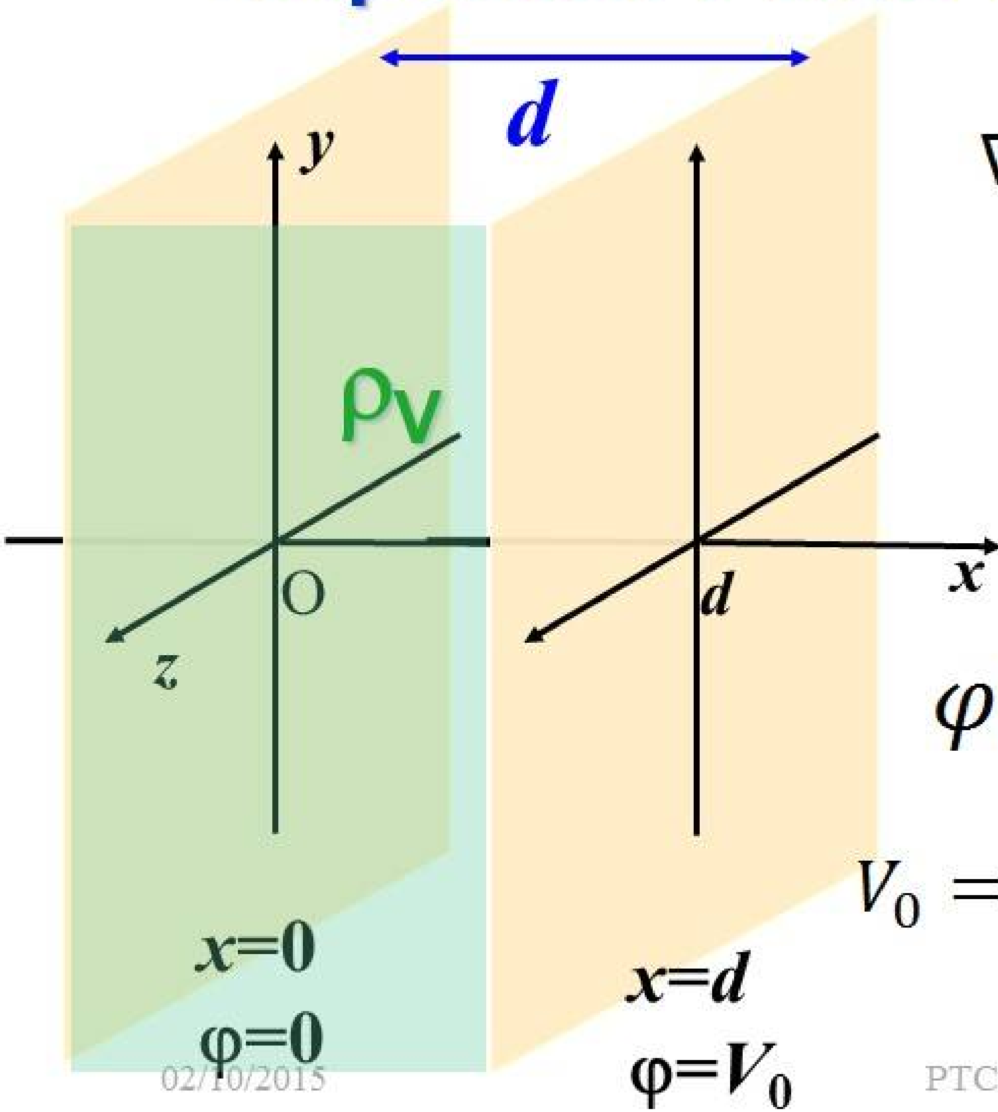


Ideal → sem efeito de bordas



Real → com efeito de bordas

Capacitor Plano de Placas // com $\rho_V \neq 0$



$$\nabla^2 \varphi = -\frac{\rho_v}{\varepsilon} \quad \rightarrow \quad \frac{\partial^2 \varphi}{\partial x^2} = -\frac{\rho_v}{\varepsilon}$$

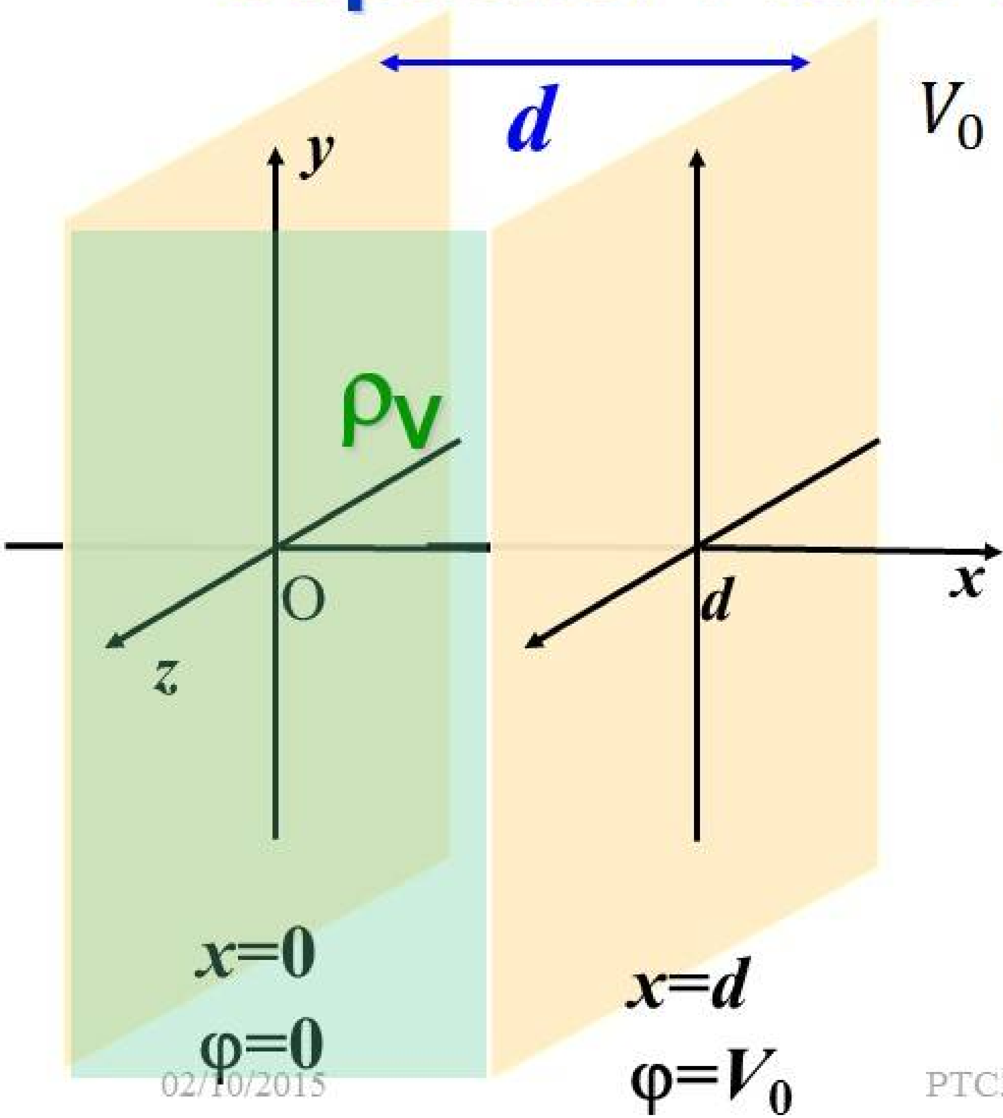
$$\frac{\partial \varphi}{\partial x} = -\frac{\rho_v}{\varepsilon} x + k_1$$

$$\varphi(x) = -\frac{\rho_v}{2\varepsilon} x^2 + k_1 x + k_2$$

$$V_0 = \varphi(d) - \varphi(0) = -\frac{\rho_v}{2\varepsilon} d^2 + k_1 d$$

$$\varphi(0) = 0 \Rightarrow \ k_2 = 0$$

Capacitor Plano de Placas // com ρV



$$V_0 = \varphi(d) - \varphi(0) = -\frac{\rho_v}{2\varepsilon} d^2 + k_1 d$$

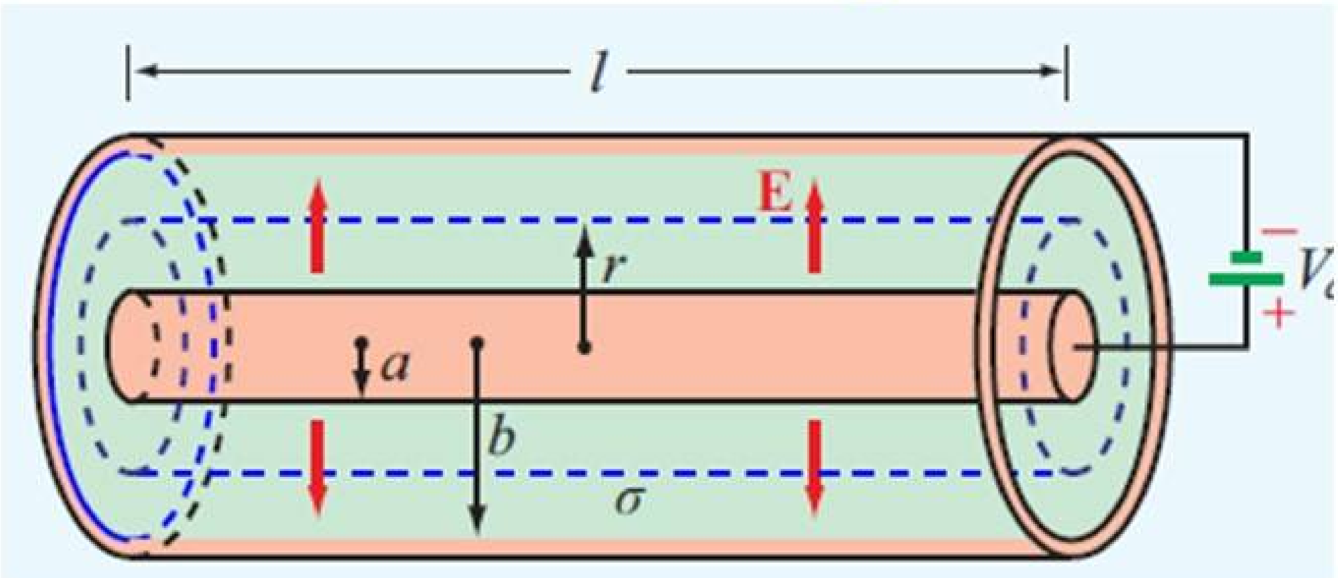
$$k_1 = \frac{V_0}{d} + \frac{\rho_v}{2\varepsilon} d$$

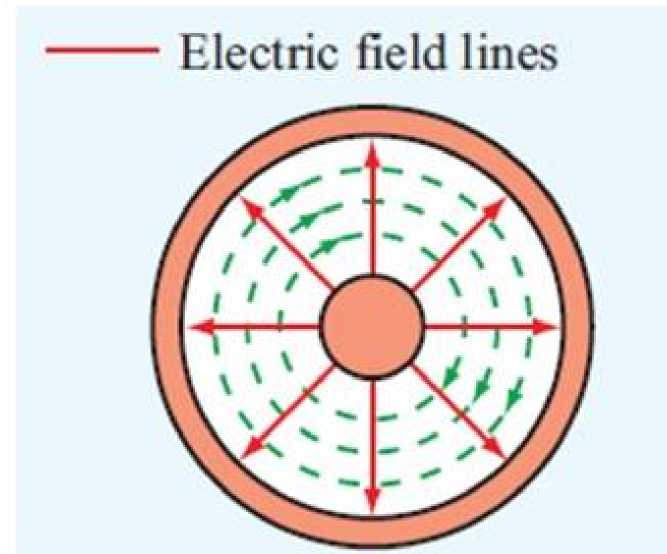
$$\varphi(x) = -\frac{\rho_v}{2\varepsilon} x^2 + \left(\frac{V_0}{d} + \frac{\rho_v d}{2\varepsilon}\right) x$$

$$\vec{E} = -\nabla\varphi \quad \Rightarrow \quad E_x = -\frac{\partial\varphi}{\partial x}$$

$$\vec{E} = \frac{\rho_v}{2\varepsilon} x - \left(\frac{V_0}{d} + \frac{\rho_v d}{2\varepsilon}\right)$$

Campo Cilíndrico

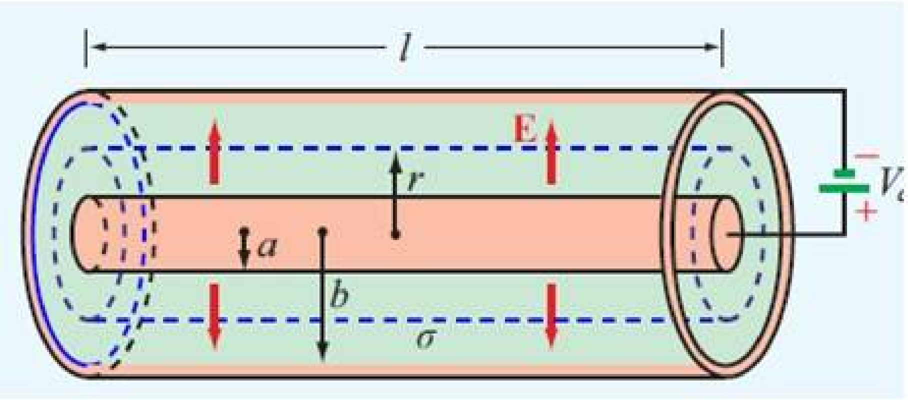


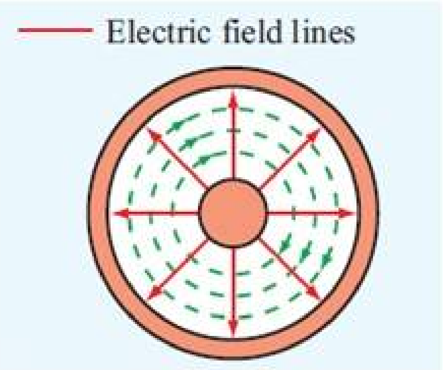


$$\varphi(a) = V_0 \qquad \varphi(b) = 0$$

$$\varphi = \varphi(r) \rightarrow \quad \frac{\partial \varphi}{\partial \phi} = \frac{\partial \varphi}{\partial z} = 0$$

Laplaciano em Coordenadas Cilíndricas



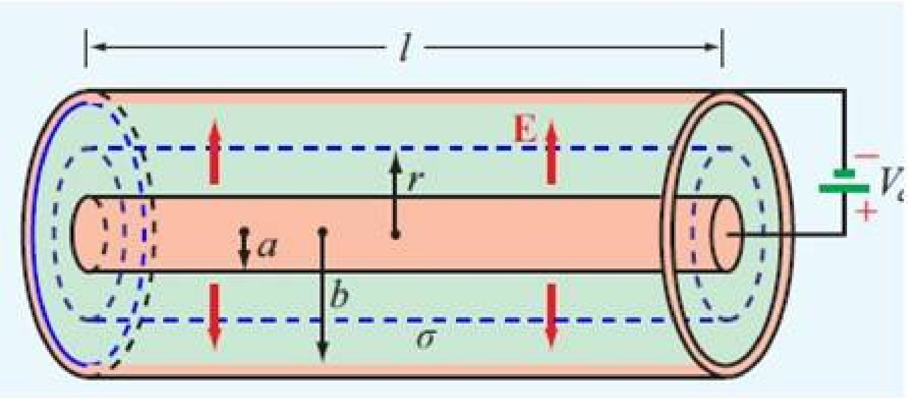


$$\varphi(a) = V_0 \qquad \varphi(b) = 0$$

$$\varphi = \varphi(r) \rightarrow \quad \frac{\partial \varphi}{\partial \phi} = \frac{\partial \varphi}{\partial z} = 0$$

$$\nabla^2 \varphi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \varphi}{\partial r}\right) = 0$$

Laplaciano em Coordenadas Cilíndricas



$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right)=0 \;\rightarrow \frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right)=0$$

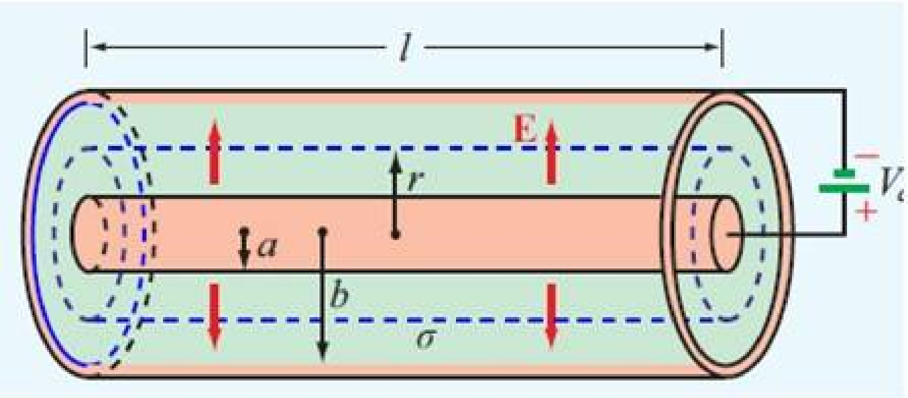
$$r\frac{\partial\varphi}{\partial r}=k_1$$

$$\varphi(r)=k_1\ln r+k_2$$

$$V_0=\varphi(a)-\varphi(b)=k_1\ln a-k_1\ln b \qquad \varphi(b)=0$$

$$k_1=\frac{V_0}{\ln{a/b}}=-\frac{V_0}{\ln{b/a}}$$

Laplaciano em Coordenadas Cilíndricas



$$\varphi(b) = 0 \Rightarrow k_1 \ln b - k_2 = 0$$

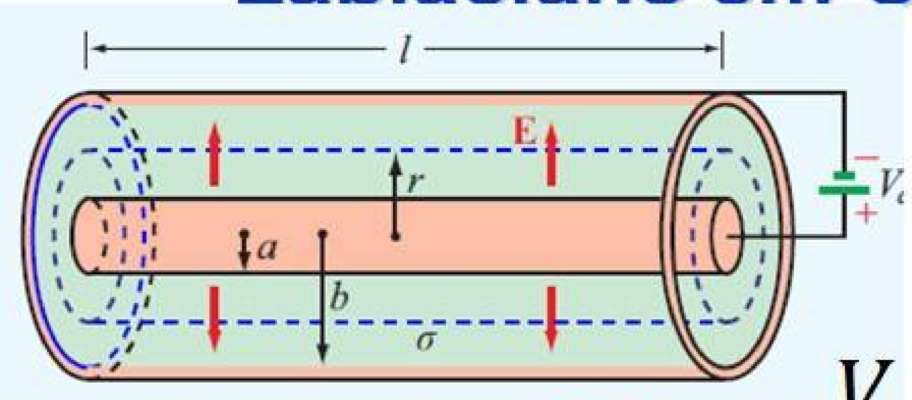
$$k_1 = \frac{V_0}{\ln {}^{a}/_{b}} = -\frac{V_0}{\ln {}^{b}/_{a}}$$

$$k_2 = -\frac{V_0}{\ln {}^{b}/_{a}} \ln b \Rightarrow \varphi(r) = k_1 \ln r + k_2$$

$$\varphi(r) = -\frac{V_0}{\ln {}^{b}/_{a}} \ln r + \frac{V_0}{\ln {}^{b}/_{a}} \ln b \Rightarrow \varphi(r) = -\frac{V_0}{\ln {}^{b}/_{a}} \ln {}^{b}/_{r}$$

ESCOLA POLITÉCNICA DA UNIVERSIDADE DE SÃO PAULO

Laplaciano em Coordenadas Cilíndricas



$$\vec{E} = -\nabla\varphi \quad \Rightarrow \quad E_r = -\frac{\partial\varphi}{\partial r}$$

$$\vec{E} = \frac{V_0}{\ln {}^{b}/_{a}}\frac{1}{r}\hat{u}_r \quad \Rightarrow \quad \vec{D} = \frac{\varepsilon V_0}{\ln {}^{b}/_{a}}\frac{1}{r}\hat{u}_r$$

$$\iint_S \vec{D}\cdot d\vec{S} = D.S = \frac{\varepsilon V_0}{\ln {}^{b}/_{a}}\frac{2\pi r l}{r}. \quad \Rightarrow \quad C = \frac{2\pi\varepsilon l}{\ln {}^{b}/_{a}}$$

$$R = \frac{\ln {}^{b}/_{a}}{2\pi\sigma l}$$

Campo Esférico

$$\varphi = \varphi(r) \rightarrow \quad \frac{\partial \varphi}{\partial \phi} = \frac{\partial \varphi}{\partial \theta} = 0$$

$$\nabla^2 \varphi = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial \varphi}{\partial r}\right) = 0 \quad \Rightarrow \quad \varphi(r) = -\frac{k_1}{r} + k_2$$

$$V_0 = \varphi(a) - \varphi(b) = \frac{k_1}{b} - \frac{k_1}{a} = k_1\left(\frac{1}{b} - \frac{1}{a}\right)$$

$$\varphi(a) = V_0 \qquad \varphi(b) = 0$$

$$k_2 = \frac{k_1}{b} = -\frac{V_0}{b\left(\frac{1}{a} - \frac{1}{b}\right)}$$

Campo Esférico

$$\varphi(r) = \frac{V_0}{\left(\frac{1}{a} - \frac{1}{b}\right)}\frac{1}{r} - \frac{V_0}{b\left(\frac{1}{a} - \frac{1}{b}\right)}$$

$$\vec{\boldsymbol{E}} = -\boldsymbol{\nabla}\boldsymbol{\varphi} \quad \Rightarrow \quad \vec{\boldsymbol{E}} = -\frac{\boldsymbol{\partial\varphi}}{\boldsymbol{\partial r}}\hat{\boldsymbol{u}}_r$$

$$\vec{\boldsymbol{E}} = \frac{V_0}{\left(\frac{1}{a} - \frac{1}{b}\right)}\frac{1}{r^2}\hat{\boldsymbol{u}}_r$$

$$C = \frac{Q}{V_0} = \frac{\iint_{\mathbf{S}} \boldsymbol{\varepsilon}\vec{\boldsymbol{E}} \cdot \boldsymbol{d}\vec{\boldsymbol{S}}}{V_0}$$

$$C = \frac{4\pi\varepsilon}{\left(\frac{1}{a} - \frac{1}{b}\right)}$$

$$R = \frac{\left(\frac{1}{a} - \frac{1}{b}\right)}{4\pi\sigma}$$