

## ON STABLE CRACK GROWTH

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### SUMMARY

BEFORE catastrophic fracture occurs, certain changes in the distribution of the plastic strain near the crack-tip seem to take place. These changes make crack growth possible but contribute to a gradually increasing screening of the energy flow through the plastic region towards the crack-tip. Stable crack growth results. This phenomenon appears to be a feature common to almost all fractures. Therefore, a unified theory may be developed to describe fracture at both small-scale and large-scale yielding as well as stable crack growth which is accompanied by plastic collapse rather than being superseded by catastrophic fracture. In the present paper, an attempt is made to show how such a unified theory would work.

### 1. INTRODUCTION

AS HAS been discussed earlier by the writer (BROBERG, 1968, 1971), processes taking place in a certain region of material instability (or necking in cases of thin sheets) near a crack-tip play a dominant role in the initial growth of the crack. Under certain, rather general, conditions the state of this region at the critical load is a material property, and is thus independent of the crack length and the load distribution. Then, only one variable is needed to specify when the region of material instability (or necking), called the *end-region*, reaches the critical state. It has been shown by BROBERG (1971) that the *J*-integral (RICE, 1968) is a very suitable variable to effect this specification, but, naturally, other variables (such as some C.O.D.-measure) could in principle be used just as well. The advantages of using the *J*-integral and the limitations of the one-parameter approach have been recently discussed by BROBERG (1973).

However, as the *J*-integral criterion (or some other one-parameter criterion) is one of *initial crack growth* only, it cannot be used in a complete description of the fracture process when stable crack growth (SCG) occurs. From recent theoretical investigations (ANDERSSON, 1973b) as well as experimental ones (ROBINSON and TETELMAN, 1973) it has become apparent that SCG is a phenomenon that is much more common than has been previously understood to be the case. Thus, it has been expected that SCG is practically absent in cases where linear fracture mechanics is applicable. It is certainly true that, in these cases, the extent of SCG itself is much smaller than the original length of the crack, *but SCG may nevertheless start at a load well below the load at which catastrophic fracture occurs.*

Assuming that the end-region properties are conserved during SCG (i.e. that they are properties of the material, but not of crack length and load distribution) an energy criterion can be applied. Then, it is necessary to calculate the energy flow  $\Phi$  to the *end-region* per unit of crack growth. It appears (ANDERSSON, 1973b) that in

small-scale yielding,  $\Phi$  approximately equals the energy flow to the plastic region per unit of crack growth only at the very start of SCG. During the subsequent SCG the plastic region outside the end-region gives rise to a gradually increasing screening action, i.e.  $\Phi$  will then be smaller than the energy flow to the plastic region per unit of crack growth. The same kind of screening action takes place even if the plastic region is not very small compared to the crack length. The difference between this case and the case of small-scale yielding is mainly that  $\Phi$  will be noticeably smaller than the energy flow to the plastic region already from the start of SCG. This natural difference does not seem to be important. The similarity in the behaviour as regards the gradual increase of the screening action of the plastic region is more interesting and explains the mechanism of SCG both at small-scale and large-scale yielding.

As the complete fracture problem involves incipient crack growth, SCG and catastrophic fracture, a one-parameter approach is in principle not possible. Nevertheless, such an approach may be used in engineering applications in the case of small-scale yielding. In this case, SCG is negligible compared to the length of the crack, and therefore, when completely developed, the screening action of the plastic region is rather independent of crack length and load distribution, i.e. it is a material property. Then, simply, when catastrophic fracture occurs, the energy flow  $\Phi$  to the end-region equals a certain (material-dependent) fraction of the energy flow to the plastic region. Thus, the energy flow to the plastic region may be taken as the sole critical quantity in cases of small-scale yielding. This point of view provides an explanation for the success of methods used in so-called linear fracture mechanics.

## 2. THE END-REGION AT THE CRACK-TIP

The existence of a region of material instability, called the end-region, the size of which is given by the distance between inhomogeneities (e.g. atoms in a purely brittle fracture or inclusions in a dimple-type fracture) can be suggested from two different points of view. One argument (BROBERG, 1968) is based on simple dimensional considerations. It is known from tests that there exists a critical crack length in the case of a crack in a large body if the remote stresses are given. Therefore, a characteristic length parameter is needed, but there is no such parameter to be found for a homogeneous continuum and so the inhomogeneities have to be considered. The other argument (BROBERG, 1971) is based on the fact that a realistic solution to the crack problem is impossible if the material is regarded as a stable continuum ( $d\sigma_{ij} d\epsilon_{ij} \geq 0$ , where  $\sigma_{ij}$  and  $\epsilon_{ij}$  are the stress and the strain tensors) throughout, because this would imply "infinite" strains at the crack-tip. Therefore, material instability has to occur, the spatial extent of which is easily understood to be given by the distance between inhomogeneities. Both arguments can be used to suggest that the end-region may be characterized by necking, controlled by the sheet thickness in thin sheets.

## 3. THE ENERGY FLOW TO THE END-REGION

The only existing analytical solution to the problem of a (slowly) moving crack-tip in an elastic-plastic solid is restricted to the extensional mode, plane stress conditions, perfect plasticity, and the Tresca yield condition (DRUCKER and RICE, 1970). In

this case, a Dugdale zone develops (DUGDALE, 1960). The energy flow to the crack-tip per unit of crack growth is finite. For a Dugdale crack of finite length in a large sheet one finds from successive equilibrium solutions that the energy flow to the crack-tip per unit of crack growth increases with the crack growth at constant values of remote stresses. Thus, SCG is not possible when the whole plastic region is confined to a Dugdale zone. The opposite conclusion reached by CHEREPANOV (1969) seems to be based on an underestimation of the original crack length at a given energy flow to the crack-tip.

Besides the Dugdale treatment there exist some other attempts to obtain a solution to the problem of a steadily moving crack in a perfectly-plastic material. The results of these attempts show a logarithmic strain singularity at the crack-tip. This implies zero energy flow to the crack-tip. As the material properties near the end-region in a real case should be similar to those of a perfectly-plastic solid, this result would indicate that crack growth could hardly take place, a conclusion which, of course, contradicts experimental evidence. However, existing solutions showing logarithmic strain singularities seem to be in error. The most elaborate one is given for the anti-plane mode by CHITALEY and MCCLINTOCK (1971), but they did not consider the displacement condition, which is not satisfied by their solution. A detailed study of their problem (see Appendix I) suggests that the assumption of a shear stress field with centered fan characteristics has to be removed. The only possible alternative seems to be the inclusion of a *Dugdale-type zone inside a plastic region*. Then, the energy flow to the crack-tip will be non-zero†. The situation may be similar in the extensional mode under plane stress conditions even for a non Tresca material.

In the extensional mode under plane strain conditions, a Dugdale-type zone does not seem to be possible for kinematic reasons in a stable homogeneous continuum. However, a similar zone can be developed from the original end-region, the region of material instability. *This zone will be of the more general Barenblatt-type* (BARENBLATT, 1959). Unlike the Dugdale zone, it will not be characterized by the fulfilment of a plastic flow condition, but by the occurrence of separations in the material (such as the opening of holes and quasi-cleavage). In terms of engineering stress and strain across the zone it will be characterized by decreasing stresses at increasing strains. A progressing Barenblatt-zone implies energy flow to the end-region at crack growth.

In view of the circumstances discussed in this section, the calculations by ANDERSSON (1973b), briefly discussed in the Introduction, seem to provide the best basis for a further discussion.

At incipient crack growth the  $J$ -integral criterion may be used (BROBERG, 1971). Then,  $\Phi$  equals  $J$  at the start of SCG. This can be understood by comparing the real case with a hypothetical case where the material outside the end-region is non-linearly elastic, the stress-strain relations being the same as for the real material under loading conditions. The very first movement of the end-region (at incipient SCG) is then governed by the same boundary conditions in both cases. With progressing SCG, according to the investigations by ANDERSSON (1973b),  $\Phi$  will tend

† According to a recent study made by Dr. H. ANDERSSON (unpublished work, 1974) the energy flow to the crack-tip will be about one-fourth of the energy flow to the plastic region in this case.

towards a certain fraction of  $J$ , say

$$\Phi = \alpha J, \quad \alpha < 1. \quad (1)$$

The screening action of the plastic region ought to be fully developed after completion of the rearrangements inside the plastic zone which take place when Dugdale-type or Barenblatt-type zones are formed.

#### 4. APPROXIMATE PROCEDURE FOR PLANE STRESS

As it may be difficult to understand the mechanism of SCG from finite element calculations only, an approximate analytical method will be developed here. Possibly, also, this method could be used in engineering applications.

The results for plane stress given by ANDERSSON (1973b), discussed in Sections 1 and 3, can be approximately represented by

$$\Phi = J[\alpha + (1 - \alpha) \exp \{-\beta \sigma_Y^2 (a - a_0)/EJ\}], \quad (2)$$

where  $\beta$  is a numerical constant,  $\sigma_Y$  is yield stress,  $(a - a_0)$  is crack growth and  $E$  is Young's modulus. The exponent is given from dimensional considerations.

Equation (2) will now be used even in cases where the original crack length parameter  $a_0$  (usually the length or half-length of the crack) is not necessarily much larger than the linear extension of the plastic region. Then, we put

$$J = J(Q, a), \quad (3)$$

where  $Q$  is the applied load and  $J$  is calculated as for a crack with constant length parameter  $a$ .

Adopting an energy criterion for fracture, viz.

$$\Phi = J_c, \quad (4)$$

one finds the relation between the load  $Q$  and the crack growth  $(a - a_0)$  from (2) to (4).

Approximately, the displacement  $\delta$  under the load  $Q$  can be given by a relation

$$\delta = \delta(Q, a), \quad (5)$$

where  $\delta$  is calculated as for a crack with constant length parameter  $a$ . Then, it is possible to determine equilibrium values of  $Q$  and  $\delta$  as functions of  $a$ . Similarly, we write

$$Q = Q(\delta, a), \quad (6)$$

where  $Q$  is calculated as for a crack with constant length parameter  $a$ . Under *fixed grip conditions* SCG ceases when

$$\frac{d\delta}{da} = 0, \quad (7)$$

and under *fixed load conditions* it ceases when

$$\frac{dQ}{da} = 0. \quad (8)$$

Relation (5) can be derived from (3), or vice versa, by means of

$$\delta(Q, a) = \delta(Q, 0) + t \int_0^a \frac{\partial J(Q, a')}{\partial Q} da', \quad (9)$$

$$tJ(Q, a) = \int_0^a \frac{\partial \delta(Q', a)}{\partial a} dQ', \quad (10)$$

where  $t$  is the thickness of the body. For  $Q = Q(\delta, a)$ , one finds

$$Q(\delta, a) = Q(\delta, 0) - t \int_0^a \frac{\partial J(\delta, a')}{\partial \delta} da', \quad (11)$$

$$tJ(\delta, a) = - \int_0^\delta \frac{\partial Q(\delta', a)}{\partial a} d\delta'. \quad (12)$$

Relations (9)–(12) are variations of relations used in connection with  $K_c$ -calibration. They follow from the fact that in a (non-linear) elastic material the energy flow to the crack-tip during a crack growth from  $a$  to  $a + da$  must equal the difference in energy supplied during loading at constant crack length for the cases where the crack length is  $a$  and  $a + da$ .

It was noted by ANDERSSON (1973a) that a simple fracture criterion containing only one critical quantity (here given by (4)) is not obvious for a growing crack. However, combining (2) and (4), one finds that this difficulty may be met by a proper choice of  $\alpha$  in the present approximate procedure.

The approximate procedure for plane stress will be demonstrated in Section 5 by means of an example.

## 5. CENTRE-CRACKED SHEET

In this section a centre-cracked sheet will be considered (see Fig. 1).

The following formula is known to be rather accurate if the plastic zone is fairly small (FEDDERSEN, 1967):

$$J = (\pi Q^2 a / 4Eb^2 t^2) \sec(\pi a / 2b), \quad (13)$$

where  $b$  is the half-width and  $t$  the thickness of the sheet. The following non-dimen-

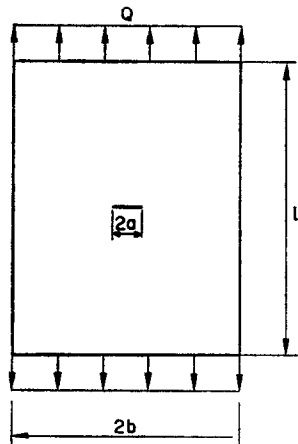


FIG. 1. Centre-cracked sheet.

sional variables and parameters are introduced:

$$EJ/\sigma_Y^2 b = S, \quad (a - a_0)/a_0 = \kappa, \quad Q/2\sigma_Y b t = r, \quad E\delta/\sigma_Y L = u,$$

$$EJ_c/\sigma_Y^2 b = S_c, \quad a_0/b = k, \quad 2b/L = w,$$

where  $L$  is the length of the sheet. With use of these non-dimensional quantities, equations (4), (7), (8) and (13) become

$$S[\alpha + (1 - \alpha) \exp(-\beta k \kappa / S)] = S_c, \quad (14)$$

$$\frac{du}{d\kappa} = 0, \quad (15)$$

$$\frac{dr}{d\kappa} = 0, \quad (16)$$

$$S = \pi r^2 k(\kappa + 1) \sec[\frac{1}{2}\pi k(\kappa + 1)]. \quad (17)$$

From (9) and (13),

$$u = r \left( 1 + \frac{2w}{\pi} \int_0^{\frac{1}{2}\pi k(\kappa + 1)} \zeta \sec \zeta \, d\zeta \right), \quad (18)$$

since

$$\delta(Q, 0) = QL/2Ebt.$$

Combination of (14), (17) and (18) gives the relation between non-dimensional load  $r$  and non-dimensional displacement under the load  $u$  during SCG. Naturally, before the start of SCG, the relation between  $r$  and  $u$  is given by

$$u = r \left( 1 + \frac{2w}{\pi} \int_0^{\frac{1}{2}\pi k} \zeta \sec \zeta \, d\zeta \right). \quad (19)$$

Very extensive experimental work on SCG has been carried out by D. Broek on aluminium plates. In order to make a comparison with certain of his experiments (BROEK, 1968a), the extent of SCG and the loads at the start and the end of SCG have been calculated as functions of the original crack length. With suitable choice of the parameters  $S_c$ ,  $\alpha$ ,  $\beta$ , satisfactory agreement is obtained as seen in Figs. 2 and 3. The scatter of the results, as regards the extent of SCG, seems to be caused by the weak condition of instability. The weakness depends on the fact that the maximum of  $u$  with regard to  $\kappa$  (which in the calculations is sought through (15)) is rather flat. Therefore, though  $u$ , and then also  $r$ , can be determined rather accurately,  $\kappa$  will be very sensitive to small changes in the experimental conditions.

BROEK (1973a) has also considered catastrophic failure after crack growth produced by sawing under a constant load, about equal to that giving incipient crack growth. He finds that catastrophic failure then occurs at a crack growth which is substantially larger than the one observed when crack growth is produced by increasing the load (i.e. SCG).

In the case of sawing,

$$S[\alpha + (1 - \alpha) \exp(-\beta k \kappa / S)] \leq S_c, \quad (20)$$

$$S = \pi r^2 k(\kappa + 1) \sec[\frac{1}{2}\pi k(\kappa + 1)], \quad (21)$$

where the equality sign in (20) is valid at the start and at the end of sawing and the

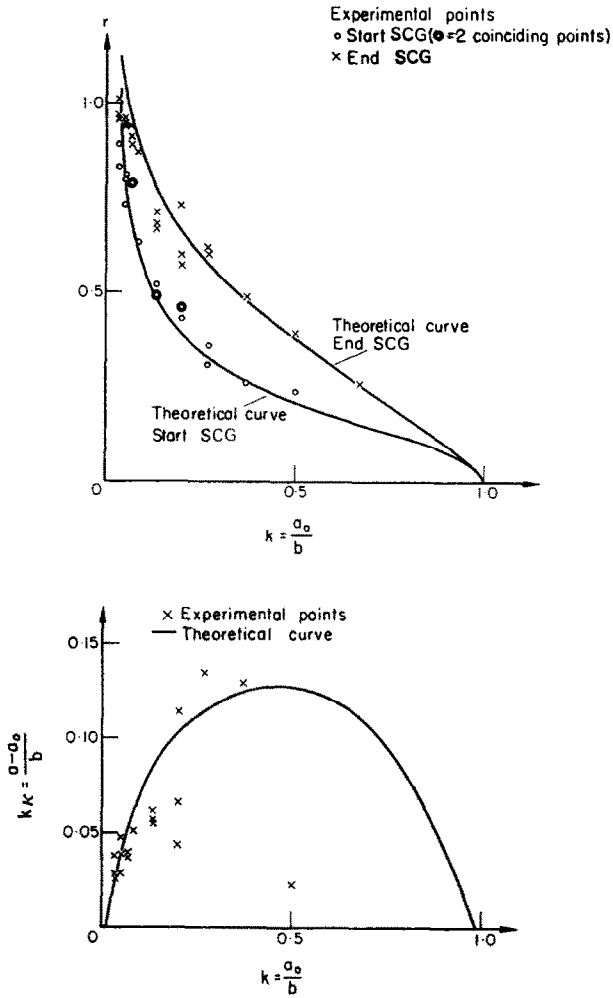


FIG. 2. SCG for 600 mm wide Al 2024-73 sheet. Experimental data are given by BROEK (1968a). Theoretical curves are obtained by choosing  $S_c = 0.1$ ,  $\alpha = 0.17$ ,  $\beta = 14$ . Upper diagram shows stresses at start and end of SCG; lower diagram shows extent of SCG.

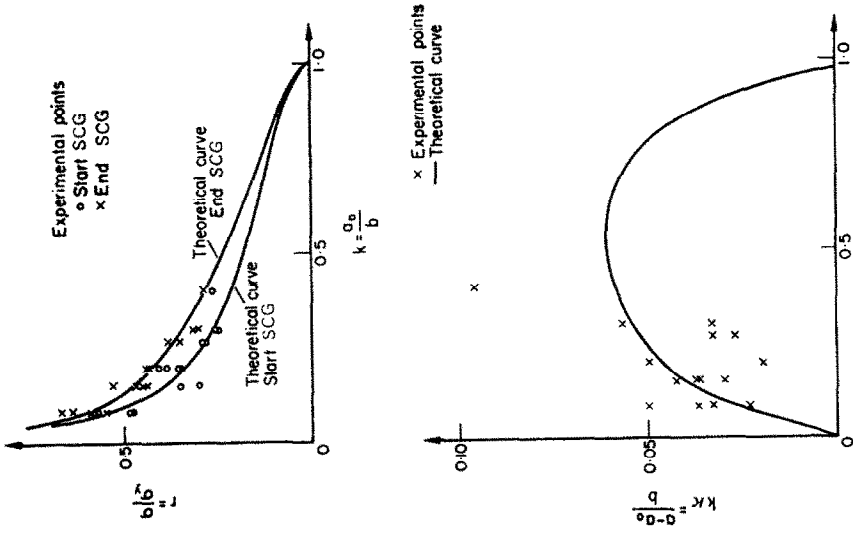


Fig. 3(b).

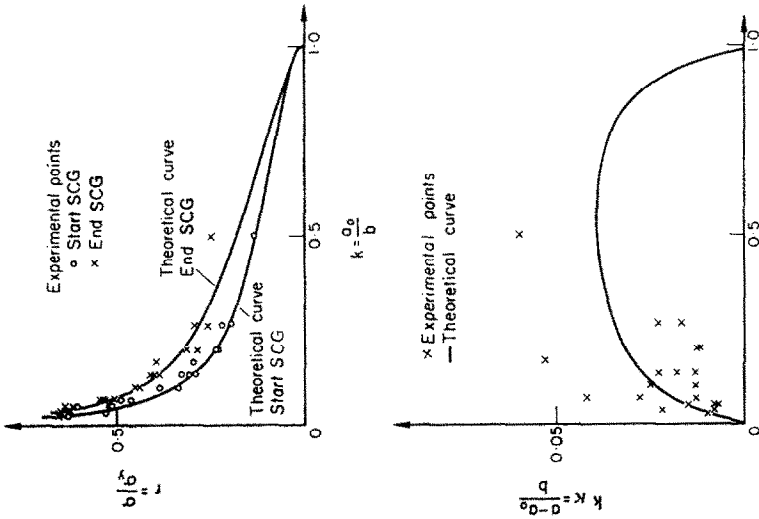


Fig. 3(a).



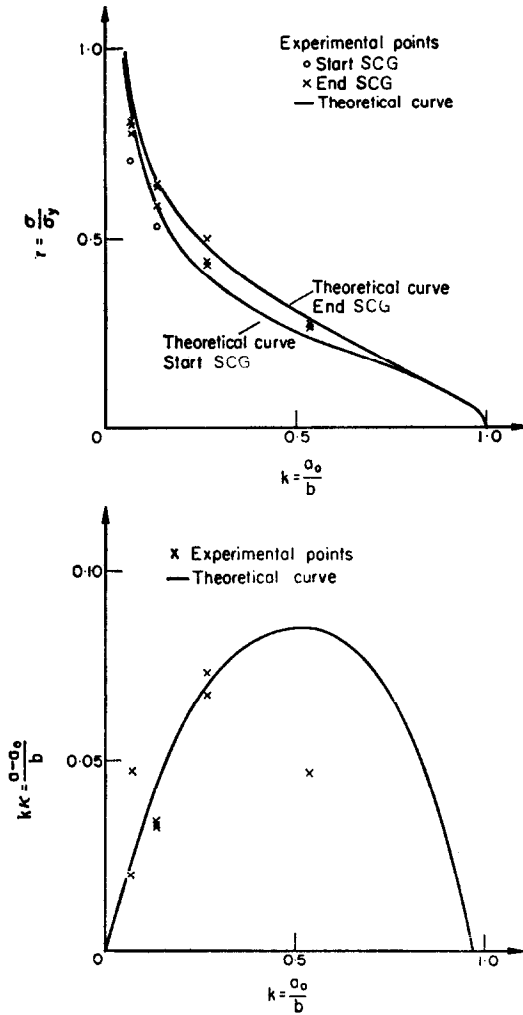


FIG. 3(c).

FIG. 3. SCG for Al 7075-T6 sheets. Experimental data are given by BROEK (1968a). Theoretical curves are obtained by choosing  $\alpha = 0.44$ ,  $\beta = 8$ ,  $S_c = 0.0375 \times 600/2b$  where  $2b$  is the sheet width in mm. Upper diagrams, stresses at start and end of SCG; lower diagrams, extent of SCG. Sheet width: (a) 600 mm, (b) 300 mm, (c) 150 mm.

non-dimensional load  $r$  is a constant given by setting  $\kappa = 0$  so that

$$\pi r^2 k \sec\left(\frac{1}{2}\pi k\right) = S_c. \quad (22)$$

The value of  $k\kappa$  at catastrophic failure has been calculated and compared with Broek's results (see Fig. 4). The plate material and dimensions are the same as in the case illustrated in Fig. 3(b). Therefore, the same values of the parameters  $S_c$ ,  $\alpha$ ,  $\beta$  have been used as when calculating the relations shown in Fig. 3(b).

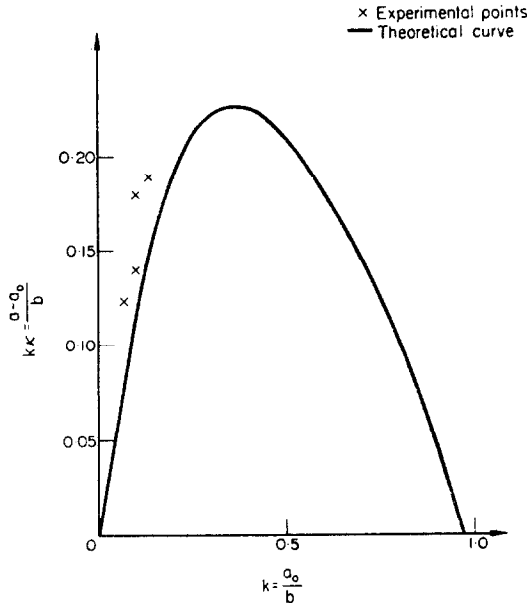


FIG. 4. Extent of crack growth produced by sawing. Experimental data by BROEK (1973a) for Al 7075-T6 sheet, width 300 mm. Theoretical curve obtained by choosing  $S_c = 0.075$ ,  $\alpha = 0.44$ ,  $\beta = 8$  as in the case shown in Fig. 3(b).

## 6. APPROXIMATE PROCEDURE. PLANE STRAIN

In the case of plane strain the idea of a Barenblatt-type zone rather than a Dugdale-type zone leads to a relation between  $J$ ,  $\alpha$ ,  $J_c$  that is somewhat different from that for the case of plane stress. As the size of the Barenblatt-type zone in plane strain is associated with microstructural properties, the relative extent of SCG ought to be expressed by comparing  $(a-a_0)$  with  $EJ_c/\sigma_Y^2$  rather than with  $EJ/\sigma_Y^2$  as in (2). Instead of (2) we put

$$J[\alpha + (1-\alpha) \exp\{-\beta\sigma_Y^2(a-a_0)/EJ_c\}] = J_c. \quad (23)$$

## 7. THREE-POINT BEND TEST SPECIMEN

In this section a three-point bend test specimen will be considered. The height is  $h$ , the length is  $8h$ , and the thickness is  $\frac{1}{2}h$ . If the original crack-length  $a$  is larger than a certain length, then unstable crack growth would never occur under fixed grip conditions (not even in the Griffith theory) because in the linear elastic case  $\partial J(\delta, a)/\partial a < 0$  if  $a$  is larger than a certain value, approximately  $\frac{1}{2}h$ , which happens

to be a value commonly used in fracture toughness tests (see Appendix II). One reason why unstable crack growth is known to supersede SCG in most fracture toughness tests with this specimen might be the presence of *springiness* in the loading system. This springiness should therefore be taken into account in an analysis of fracture toughness tests.

Here, we assume an extra springiness of about the same magnitude as the springiness of a test-piece without a crack. Then, for  $a_0 = \frac{1}{2}h$ , we may write approximately (see Appendix II)

$$S[\alpha + (1 - \alpha) \exp(-\beta\kappa/S_c)] = S_c, \quad (24)$$

$$S = 0.0025u^2(1 - 2\kappa)/[(1 - 2\kappa)^6 + (0.013u)^2]^{\frac{1}{2}}, \quad (25)$$

$$r = 0.00114u(1 - 2\kappa)^2/[(1 - 1.85\kappa)^2 + 0.0013u^2]^{\frac{1}{2}}, \quad (26)$$

in non-dimensional form, where

$$S = EJ/\sigma_Y^2 h, \quad S_c = EJ_c/\sigma_Y^2 h, \quad \kappa = (a - a_0)/h, \\ u = E\delta/\sigma_Y h, \quad r = Q/\sigma_Y h^2.$$

From (24) to (26) the relation between non-dimensional load  $r$  and non-dimensional displacement under the load  $u$  (including the displacement due to the springiness of the loading system) can be calculated. Under fixed grip conditions (referring the grips to the stiff parts of the loading system), SCG ceases when

$$\frac{du}{d\kappa} = 0. \quad (27)$$

Figure 5 shows examples of the calculations for certain values of the parameters  $S_c$ ,  $\alpha$ ,  $\beta$ . For comparison, experimental results are also shown in the Figure. When using the smallest test-piece the deviation from plane strain conditions is substantial. This deviation can be judged from Fig. 6 which shows the extension of SCG over the cross-section of the test-piece somewhat before catastrophic fracture occurs when the test was interrupted.

It should be observed that Fig. 5 covers a range not very far removed from the linear fracture mechanics range (Fig. 5(a)) to nearly plastic collapse without catastrophic fracture (Fig. 5(d)). The non-dimensional displacement  $u$  refers to displacement under load, not to clip-gauge measured displacements. As curves obtained by using clip-gauges are probably more familiar, the results of the calculations are also shown in Fig. 7 with clip-gauge measured displacement on the horizontal axis. The curves in Fig. 7 are derived from the curves in Fig. 5 by means of an estimate of the relation between clip-gauge measured displacements and displacement under load.

The parameters used for determining the curves shown in Fig. 5 are

$$J_c = 5860 \text{ N m}^{-1}, \quad \sigma_Y = 620 \text{ MN m}^{-2}, \quad E = 2.1 \times 10^5 \text{ MN m}^{-2}, \\ \alpha = 0.1, \quad \beta = 25.$$

With these values the linear fracture toughness (cf. equation (31)) is found to be

$$K_{Ic} \approx 110 \text{ MN m}^{-3/2}.$$

According to ASTM recommendations, for a test within the linear fracture mechanics range, a test-piece with minimum height  $h = 5K_{Ic}^2/\sigma_Y^2 \approx 160 \text{ mm}$  and the minimum width  $t = 2.5K_{Ic}^2/\sigma_Y^2 \approx 80 \text{ mm}$  would have been needed. Here, a fair determination of  $K_{Ic}$  is made with a test-piece with a height of only 10 mm.

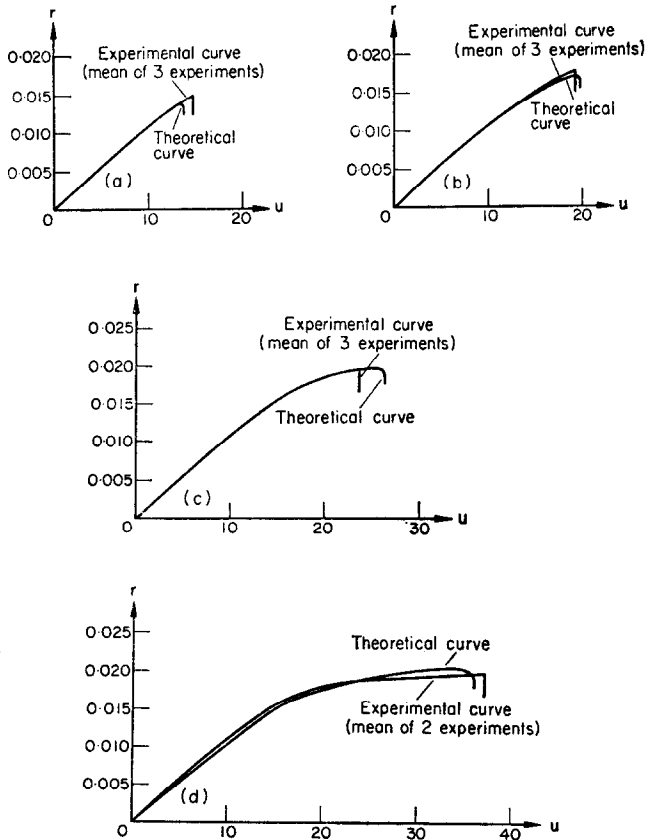


FIG. 5. Load-displacement curves for three-point bend test bars. Experimental data are obtained for steel DOMEX 450 at  $-100^{\circ}\text{C}$ . Displacement measured under the force. Choice of parameters is  $\alpha = 0.1$ ,  $\beta = 25$ ,  $S_c = 0.04 \times 80/h$  where  $h$  is beam height in mm. The very good agreement between experimental and theoretical curves is somewhat illusory because these experiments were used as a check when establishing the relation (26). (a)  $h = 80$  mm, (b)  $h = 40$  mm, (c)  $h = 20$  mm, (d)  $h = 10$  mm (test-piece width 10 mm).

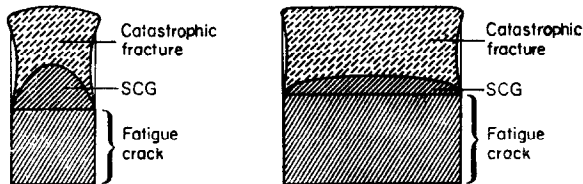


FIG. 6. SCG (determined by interrupting a test just before catastrophic fracture is expected and then placing the test-piece in a liquid with a dye in suspension) in test-pieces 10 mm high. The width 5 mm (left-hand diagram) is obviously too small to give fairly plane conditions, whereas width 10 mm (right-hand diagram) is acceptable.

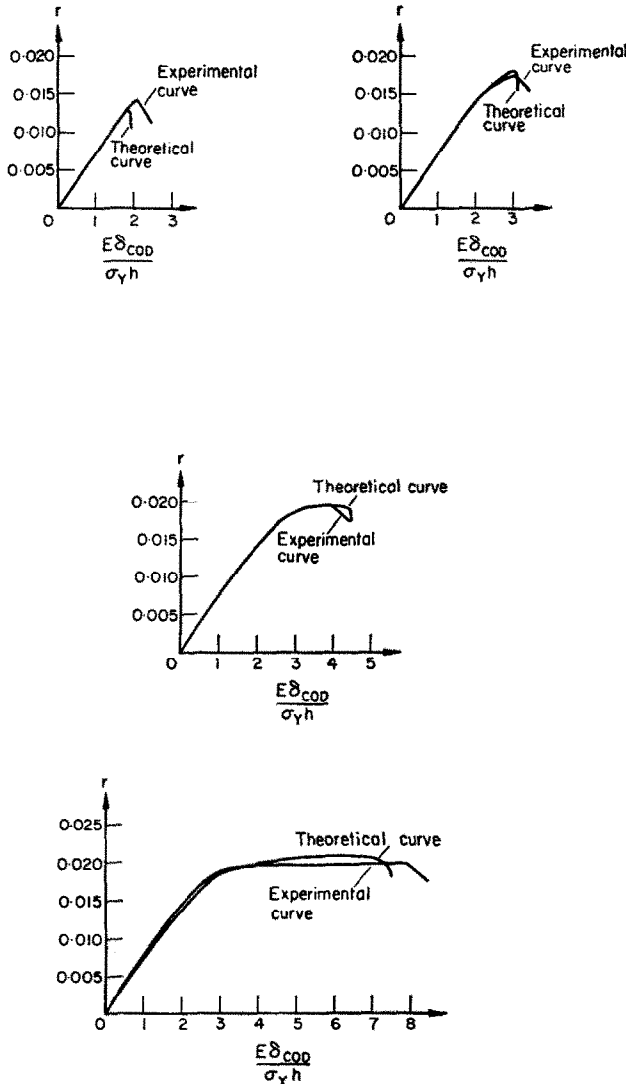


FIG. 7. Load-displacement curves as in Fig. 5, but with clip-gauge measured displacement.

It should be observed that a test within the linear fracture mechanics range according to ASTM specifications would have been impossible in the case studied. The plate material is rolled to 40 mm thickness which is therefore the maximum obtainable test-piece width. It cannot be expected that a plate rolled to about 80 mm thickness exhibits the same fracture toughness.

### 8. COMPARISON WITH OTHER RESULTS

BEGLEY and LANDES (1972) and LANDES and BEGLEY (1972) tried to verify experimentally a  $J$ -integral criterion of crack initiation. According to their investigations, crack growth was initiated when the load reached a maximum. This appears to be

erroneous. A sharp maximum of the load–displacement curve (as in tests within the linear fracture mechanics range) is related to catastrophic fracture. In such a case, SCG may have started at a substantially smaller load. A smooth maximum of the load–displacement curve (as in tests far within the non-linear fracture mechanics range) is a very strong indication of the occurrence of preceding SCG. Moreover, for some steels and specimen geometries the load-displacement curve shows a very flat maximum. A very small change in the original crack-length can considerably move the position of the maximum. The maximum of the load–displacement relation (or, more generally, a certain position on the load–displacement curve) seems to have little to do with the initiation of crack growth. The excellent agreement between theory and experimental results obtained by Begley and Landes is therefore quite astonishing and probably coincidental.

Methods using the  $R$ -curve concept are not directly comparable to the method presented here although they are energy methods. The  $R$ -curve (SRAWLEY and BROWN, 1965) gives the “resistance” to crack growth as an energy-rate quantity, varying with crack growth but not with load or body geometry. This “resistance” is compared with an energy “release” rate at fixed load. Instability is found to occur when these two energy-rates (viz. the “resistance” rate and the “release” rate) are equal in magnitude and in derivative with respect to crack-length. The  $R$ -curve concept does not recognize the difference between energy expended in the end-region and the energy which is dissipated in the plastic region outside the end-region. As the end-region ought to be fairly autonomous at critical states (i.e. during SCG), whereas the plastic region depends on specimen geometry, load distribution and crack-length, the  $R$ -curve concept does not seem to be well enough defined for general applications.

## 9. THE STRUCTURE OF THE LINEAR FRACTURE TOUGHNESS

The linear fracture mechanics range is characterized by an almost linear relation between load and displacement until catastrophic fracture occurs. The displacement may be measured under the load or between the crack surfaces by means of clip-gauges. From empirical results the following relation has been suggested (BROWN and SRAWLEY, 1967) to specify this range for plane strain:

$$\sigma_Y^2 a_0 / K_{Ic}^2 \geq 2.5. \quad (28)$$

Supposing the stress–strain relation for the material to be linear in the elastic region, the deviation from linearity is caused by plastic flow near the crack-tip and by SCG. Both the plastic region dimensions and the extent of SCG are small compared to the length of the crack when relation (28) is satisfied. Therefore, the deviation from linearity is hardly noticeable. The plastic flow starts at a very small load, whereas SCG starts at the load for which

$$J(Q, a_0) = J_c. \quad (29)$$

Catastrophic fracture occurs when the value of the exponential function in (23) is very small, i.e. when

$$J(Q, a_0) \sim J_c / \alpha. \quad (30)$$

Thus, the linear fracture toughness for plane strain is given by

$$K_{Ic} = [EJ/(1-\nu^2)]^{1/2} = [EJ_c/(1-\nu^2)\alpha]^{1/2}, \quad (31)$$

where  $\nu$  is Poisson's ratio.

The right-hand member of (31) shows the structure of the linear fracture toughness  $K_{Ic}$ . Each one of the three factors  $E/(1-\nu^2)$ ,  $\alpha$ ,  $J_c$  refers to the properties of a certain region of the body, viz, the first refers to the elastic region, the second refers to the plastic region outside the end-region, and the third refers to the end-region. The first two factors are properties of a homogeneous continuum for which  $d\sigma_{ij} d\epsilon_{ij} \geq 0$ . The third factor  $J_c$ , which is the only critical quantity, is a micro-structural property. It is associated with material instability, caused by separations occurring in the material. In terms of engineering stress and strain across the end-region, it is associated with decreasing stress at increasing strain.

From the results of ordinary tests within the linear fracture mechanics range,  $J_c$  and  $\alpha$  cannot be determined separately, but only their quotient  $J_c/\alpha$ , according to (30). From results of similar tests within the non-linear fracture mechanics range, a determination of  $J_c$  and  $\alpha$  separately is possible, although their quotient  $J_c/\alpha$  will be determined with better accuracy (cf. Fig. 5).

In tests made on thin plates, it is sometimes possible to observe directly the start of SCG (cf. BROEK, (1968a)). Then,  $J_c$  will be obtained, but not  $K_c$  which for plane stress would be written

$$K_c = (EJ_c/\alpha)^{1/2}. \quad (32)$$

Direct determination of  $K_c$  could be achieved only in a test where the extent of SCG is much smaller than the length of the crack. Very long cracks would then be needed in some cases (such as those with the thin plates used by Broek).

Direct observation of the start of SCG in cases of plane strain (where SCG generally is very small) may be possible by using advanced experimental techniques (cf. ROBINSON and TETELMAN, (1972)). Then,  $J_c$  may be determined separately.

Some attempts have been made to express linear fracture toughness in terms of the yield stress and the mean distance between inclusions (for dimple-type fractures) but without considering the factor  $\alpha$  in (31). These attempts then fail to explain the dependence of  $K_{Ic}$  on different degrees of strain-hardening.

## 10. COMMENTS ON SCG AT SMALL-SCALE YIELDING

It seems to be a common belief that SCG is not present at small-scale yielding. A simple argument, however, shows that the scale of yielding has very little to do with the phenomenon of SCG. For a given material very small scale yielding occurs (by implication of the concept itself) when the linear dimensions of the plastic region are much smaller than the length of the crack, whereas very large scale yielding occurs when the linear dimensions of the plastic region are at least of the same order of magnitude as the length of the crack†. At critical states, the plastic region is roughly the same in small-scale yielding as in not too large scale yielding. Thus, in a given material, the main difference between small-scale and large-scale yielding

† The division between small-scale and large-scale yielding may be assumed to occur at plastic zone sizes of 5 to 10% of the crack-length, as then there is just an obvious (though not very pronounced) deviation from linear elastic behaviour in the relation between applied load and clip-gauge measured displacement.

is not to be found in the plastic zones but simply in the crack-lengths (Fig. 8). (Taking, also, more large-scale yielding into consideration shows essentially the same result if one does not compare the entire plastic zones but only their inner parts.) This means that the fracture process, including SCG, is essentially the same in small-scale and large-scale yielding, whereas the response of the body is quite different. Naturally, Fig. 8 oversimplifies the matter as the plastic zones ought to be somewhat different, but this difference is rather insignificant and concerns mostly the outer parts of the plastic zones, where the plastic strain is smaller than the elastic strain.

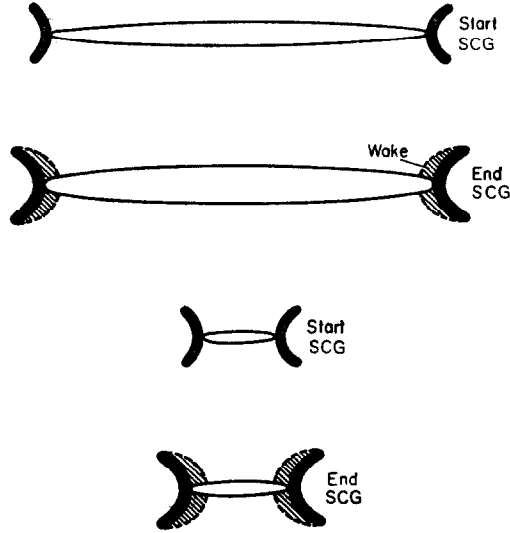


FIG. 8. Small-scale yielding (upper diagrams) and large-scale yielding (lower diagrams) at the start and at the end of SCG.

The crack-lengths used by Broek in the experiments on aluminium sheets, referred to in Section 5, are too small to comply with small-scale yielding. In fact, the sheet width and length and the crack-lengths ought to be increased by a factor of 10 or more if a test within the linear fracture mechanics range is desired. Such an increase would not appreciably change the absolute extent of SCG in corresponding cases, but in relation to the crack length this extent would be lowered in approximately inverse proportion to the linear scale. There would be a fairly linear relation, up to catastrophic failure, between the applied load and a clip-gauge measured C.O.D. SCG would start at a load well below the load at catastrophic fracture. It could hardly be detected by studying the load-displacement record. The  $K_c$ -value would be calculated from the load at catastrophic fracture, not from the load at incipient SCG. This would be quite in agreement with the requirements and concepts of linear fracture mechanics. Therefore, for "smaller" sheets, such as those actually used by Broek, the onset of SCG has nothing to do with the linear fracture toughness  $K_c$  for the sheet. The same discussion is applicable to plane strain fracture mechanics, the main difference being only that crack lengths less than 100 to 200 mm usually are sufficient for valid  $K_{Ic}$ -tests performed under the requirements of linear fracture mechanics.



## 11. COMMENTS ON THE PLASTIC FLOW AT SCG

The idea of a Dugdale-type or Barenblatt-type region inside the plastic region at SCG might explain experimental observations by BROEK (1968b) and Professor A. S. TETELMAN (unpublished work, 1973) that the crack profile after some extent of SCG does not seem to be a result of progressive blunting. Figure 9 shows schematically crack profiles at different stages of loading. Before SCG, rather extensive blunting occurs (cf. RICE and ROSENGREN (1970) and HUTCHINSON (1970)). During SCG, a rather sharp crack-tip is advancing. It is tempting to interpret these phenomena as the result of rearrangements of plastic strain after initiation of SCG. The occurrence of a Dugdale-type or a Barenblatt-type region during SCG should change the displacement field in the neighbourhood of the crack-tip in such a way that a "spearing" rather than a progressive blunting occurs. The basis for such a far-reaching conclusion is, however, at present rather uncertain.

The postulate of a Dugdale-type or a Barenblatt-type region at SCG might also shed some light on the mechanism of fatigue at higher load levels. Thus, it does not seem to be unreasonable to regard this mechanism as comprising cycles of blunting during loading before crack growth, "spearing" during continued loading when SCG occurs, and restoration of the stress-strain field during unloading.

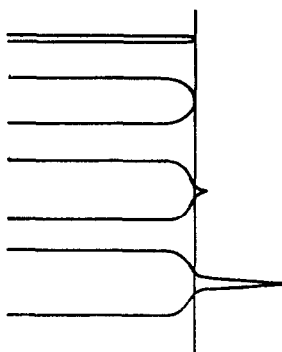


FIG. 9 Crack profiles, showing how a sharp fatigue crack at increased loading first becomes blunted, whereupon a sharp crack-tip grows.

## 12. CONCLUSIONS

Certain changes in the distribution of plastic strain near a crack-tip take place after incipient crack growth. As it appears, these changes favour "spearing", i.e. a fairly sharp crack-tip, rather than a progress of the blunting which occurs before crack growth. By such effects, energy flow to the end-region at crack extension is made possible, obviously a necessary requirement for crack growth to take place. On the other hand, with progressing crack growth, a gradually increasing screening of the energy flow through the plastic region occurs, almost always causing SCG, controlled by the loading system. The SCG is eventually superseded by catastrophic fracture. By consideration of the gradually increasing screening action of the plastic region a quantitative interpretation can be made, not only of observations of SCG in thin plates, but also of the fracture process in cases of plane strain, where the existence and importance of SCG has often been overlooked.

The screening action of the plastic region controls, to a certain degree, linear fracture toughness. As explained in the Introduction (last paragraph), this toughness can be directly determined from tests within the linear fracture mechanics range without separate determination or consideration of the screening action. This action must, however, be observed in interpretations of test results within the non-linear fracture mechanics range and in predictions of fracture properties in non-linear cases. In fact, analysis of tests even where catastrophic fracture does not occur before plastic collapse can give information both about the screening action and the fracture toughness. In this way, the whole range from fracture at very small scale yielding through fracture within the non-linear region up to plastic collapse can be covered.

Conditions of plane stress or plane strain have been assumed. Mixed cases (for instance, fractures in plates with substantial shear-lip formation) are not included. Neither are cases where the crack length is not much larger than the size of the end-region.

#### ACKNOWLEDGMENT

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APPENDIX I

*The anti-plane strain problem of steady crack growth*

CHITALEY and McCLINTOCK (1971) considered the anti-plane strain problem of steady crack growth in a perfectly-plastic solid. They found one primary and one secondary plastic zone. In the primary plastic zone (Fig. AI.1) the stresses  $\tau_{xz}$ ,  $\tau_{yz}$

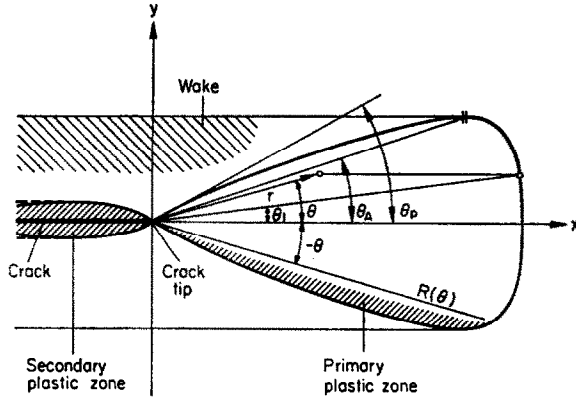


FIG. AI.1. Steady-state growth in anti-plane strain, according to CHITALEY and McCLINTOCK (1971).

are given by

$$\tau_{xz}^2 + \tau_{yz}^2 = k^2, \quad \tau_{xz} \cos \Theta + \tau_{yz} \sin \Theta = 0, \quad (AI.1)$$

where  $k$  is the yield stress in shear and  $\Theta$  is the angle between the radius vector from the crack-tip and the  $x$ -axis. The strains are given by

$$\gamma_{xz} = \gamma_{xz}^e + \gamma_{xz}^p, \quad \gamma_{yz} = \gamma_{yz}^e + \gamma_{yz}^p, \quad (AI.2)$$

$$\gamma_{xz}^e = \tau_{xz}/G, \quad \gamma_{yz}^e = \tau_{yz}/G, \quad (AI.3)$$

$$\gamma_{xz}^p = \frac{k}{G} \sin \Theta \ln (r/R(\Theta)) + \frac{k}{G} g(\Theta), \quad (AI.4)$$

$$\gamma_{yz}^p = -\frac{k}{G} (\cos \Theta + \ln (\tan \frac{1}{2}\Theta)) \ln (r/R(\Theta)) - \frac{k}{G} \int_r^R \ln (\tan \frac{1}{2}\Theta (s, \Theta)) \frac{ds}{s} + \frac{k}{G} h(\Theta), \quad (AI.5)$$

where  $G$  is the shear modulus,  $r$  is the length of the radius from the crack-tip,  $R(\Theta)$  is the distance from the crack-tip to the elastic-plastic boundary in the direction  $\Theta$ ,

$\Theta_1(r, \Theta)$  is an angle (see Fig. AI.1), and  $g(\Theta)$  and  $h(\Theta)$  are functions characterized by

$$g(\Theta) = h(\Theta) = 0, \quad \Theta \leq \Theta_A, \tag{AI.6}$$

$$g'(\Theta) = \cos \Theta + \sin \Theta R'(\Theta)/R(\Theta), \quad h'(\Theta) = -\cot \Theta g'(\Theta), \quad \Theta_A \leq \Theta \leq \Theta_p, \tag{AI.7}$$

where  $\Theta_A, \Theta_p$  are angles shown in Fig. AI.1.

CHITALEY and MCCLINTOCK (1971) found that the C.O.D. is  $\approx 0.15kR(0)/G$ . However, the logarithmic strain singularity at the crack-tip predicts zero C.O.D. This contradiction indicates that the displacement condition is not satisfied. In fact, as will be shown later, the displacement condition cannot be satisfied by centred fan characteristics in the shear field. The obvious alternative, and apparently the only possible one, is then a shear field where finite transverse slip is allowed along a finite segment of the  $x$ -axis, giving characteristic lines as shown in Fig. AI.2. The

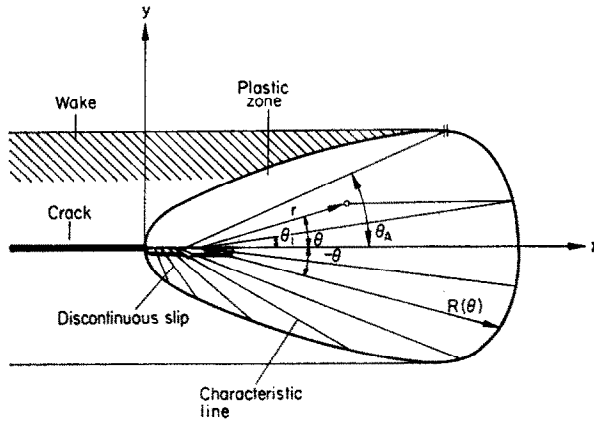


FIG. AI.2. Steady-state growth in anti-plane strain, taking a region of discontinuous slip into consideration.

plastic strains are then found to be

$$\gamma_{xz}^p = \frac{k}{G} \sin \Theta \ln \left[ \frac{r - (dx_0/d\Theta) \sin \Theta}{R(\Theta) - (dx_0/d\Theta) \sin \Theta} \right] + \frac{k}{G} g(\Theta), \tag{AI.8}$$

$$\begin{aligned} \gamma_{yz}^p = & -\frac{k}{G} \cos \Theta \ln \left[ \frac{r - (dx_0/d\Theta) \sin \Theta}{R(\Theta) - (dx_0/d\Theta) \sin \Theta} \right] + \\ & + \frac{k}{G} \sin \Theta \int_r^R \int_{\Theta_1(s, \Theta)}^{\Theta} \frac{d\Theta_v ds}{\sin \Theta_v (s \sin \Theta - (dx_0/d\Theta_v) \sin^2 \Theta_v)} + \frac{k}{G} h(\Theta), \end{aligned} \tag{AI.9}$$

where  $x_0(\Theta)$  is the  $x$ -coordinate of the intersection between the characteristic with slope  $\Theta$  and the  $x$ -axis,  $\Theta_1(r, \Theta)$  is an angle shown in Fig. AI.2, and

$$g(\Theta) = h(\Theta) = 0, \quad \Theta \leq \Theta_A, \tag{AI.10}$$

$$g'(\Theta) = \frac{R(\Theta) \cos \Theta + R'(\Theta) \sin \Theta}{R(\Theta) - (dx_0/d\Theta) \sin \Theta}, \quad h'(\Theta) = -\cot \Theta g'(\Theta), \quad \Theta_A \leq \Theta \leq \frac{1}{2}\pi, \tag{AI.11}$$

with  $\Theta_A$  an angle shown in Fig. AI.2. Further,

$$dx_0/d\Theta_v = d(x_0(\Theta_v))/d\Theta_v. \tag{AI.12}$$

The displacement condition can be found from

$$\oint (\gamma_{xz} dx + \gamma_{yz} dy) = 0, \tag{AI.13}$$

the integral being taken along the path shown in Fig. AI.3. The condition can be written

$$R(\Theta) + \sin \Theta (dx_0/d\Theta) \left[ 1 - \ln \left( \frac{-(dx_0/d\Theta) \sin \Theta}{R(\Theta) - (dx_0/d\Theta) \sin \Theta} \right) - g(\Theta) \right] - \frac{\partial}{\partial \Theta} \left[ \sin^2 \Theta \int_0^{R(\Theta)} \int_r^{R(\Theta)} \int_{\Theta_1(s, \Theta)}^{\Theta} \frac{d\Theta_v ds dr}{\sin \Theta_v (s \sin \Theta - (dx_0/d\Theta_v) \sin^2 \Theta_v)} \right] = 0. \quad (\text{AI.14})$$

Obviously, a complete solution to the problem requires the matching of the plastic zone to the outer field, which in turn is subject to conditions given by the remote load.

Integration along another path  $C = C_1 + C_2$  (Fig. AI.4) gives a special displacement condition

$$\frac{2k}{G} \int_{C_1} R(\Theta) d\Theta = - \frac{2k}{G} \int_{C_2} \sin \Theta (dx_0/d\Theta) \times \left[ 1 - \ln \left( \frac{-(dx_0/d\Theta) \sin \Theta}{R(\Theta) - (dx_0/d\Theta) \sin \Theta} \right) - g(\Theta) \right] d\Theta, \quad (\text{AI.15})$$

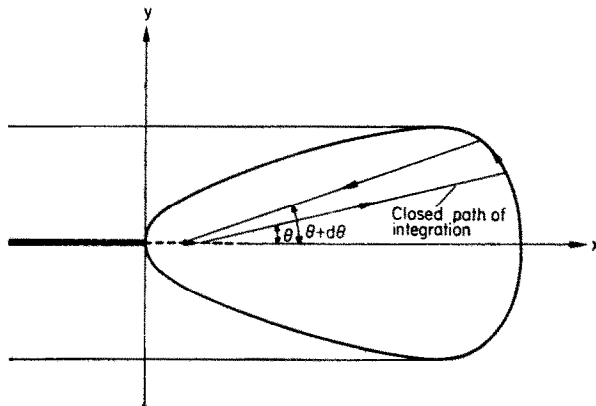


FIG. AI.3. Path of integration.

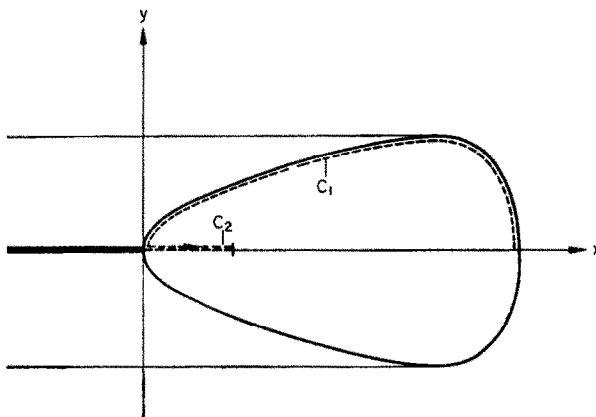


FIG. AI.4. Path of integration.

with  $C_1, C_2$  equal to  $\frac{1}{2}\pi \geq \Theta \geq 0, 0 \leq \Theta \leq \frac{1}{2}\pi$  respectively, where each member is equal to the C.O.D. From this relation it is seen that the displacement condition cannot be satisfied by  $x_0(\Theta) = 0$ , i.e. by a shear field with a centred fan of characteristics.

## APPENDIX II

### *Approximate expressions of some properties of a three-point bend specimen*

The dimensions of the specimen considered are shown in Fig. AII.1. For this specimen, BROWN and SRAWLEY (1967) give the following expression for the calculation of the stress intensity factor  $K$ :

$$K = 24(Qa^{1/2}/h^2)(1.96 - 2.75a/h + 13.66a^2/h^2 - 23.98a^3/h^3 + 25.22a^4/h^4). \quad (\text{AII.1})$$

By means of (9) to (12) it is possible to calculate  $J$  as a function of the displacement  $\delta$  under the load and the crack length  $a$ . The result is shown in Fig. AII.2.

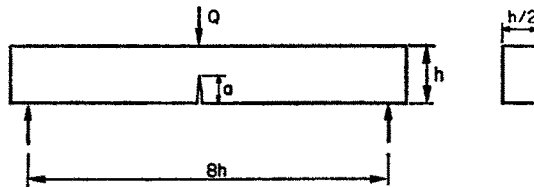


FIG. AII.1. Dimensions of three-point bend specimen.

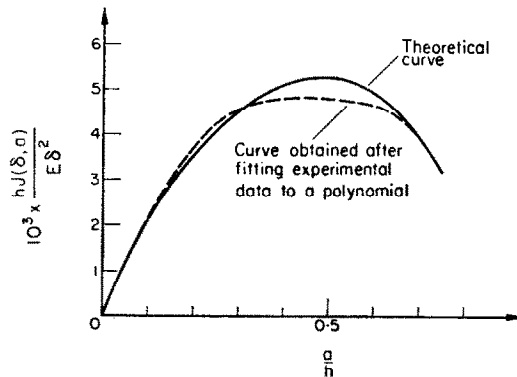


FIG. AII.2.  $J(\delta, a)$  as a function of  $a$ .

It is seen that  $J(\delta, a)/\delta^2$  decreases with increasing  $a$  if  $a > 0.49h$ . As (AII.1) assumes plane conditions and the real case should be a mixed case (plane strain near the crack-tip and plane stress far from the crack-tip), a  $K$ -calibration was performed in order to test the reliability of (AII.1). Although the accuracy of such a calibration (performed on a specimen with  $h = 30$  mm and milled slits 0.3 mm wide) is much poorer than the accuracy of numerical calculations in plane cases, it seems definitely to be good enough to confirm the existence of a maximum of  $J(\delta, a)/\delta^2$  for

$a < 0.55h$ . The result of a numerical calculation of the relation  $J(\delta, a)/\delta^2$  based on the experimental data is shown in Fig. AII.2.

It is obvious that unstable crack growth under fixed grip conditions cannot take place at very small scale yielding if the crack length is larger than that for which  $J(\delta, a)/\delta^2$  is a maximum.

The difficulty of obtaining unstable crack growth under fixed grip conditions is reduced if the loading system exhibits a springiness between the grips. It is here assumed that such a springiness exists, equal to the springiness of a test-piece with  $a = 0$ . The relations  $J(\delta, a)$  and  $Q(\delta, a)$  are then calculated. In this calculation, (AII.1) is used for the small-scale yielding part and Dr. H. Andersson's (unpublished work, 1973) finite element calculations for  $a = \frac{1}{2}h$  for non-linear parts. As regards the dependence on  $a$ , some consideration of very large scale yielding (through-plasticity) as well as experimental results has been used. The final result is given by (25) and (26).