Chiral Lagrangians

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1 Introduction

Symmetric models are usual in all quantum field theory, the standard model itself is a $SU(3)_C \times SU(2)_L \times U(1)_Y$ model. Spontaneous symmetry breaking, associated with many vacua in the theory, gives birth to Goldstone bosons, massless particles associated with the broken symmetries, but what happens when we add a small term to the Lagrangian that is not invariant under original symmetries? In these cases we have what we call pseudo-Goldstone bosons, with the main difference that they have mass.

We are going to discuss some theoretical aspects that enables us to evaluate these masses, and three models with non exact symmetries: the $SU(2) \times SU(2)$ two quark model (up and down quarks); $SU(2) \times SU(2)$ model that describes the interactions of pions among themselves and between pions and nucleons; and the $SU(3) \times SU(3)$ three quark model (up, down and strange). In all of these cases we break chiral symmetry and although in first two models we get only three pseudo-Goldstone bosons, identified as the pions, in the last one we get eight of them.

2 Exact Symmetries

To explore properties of Goldstone bosons, we are going to study what happens when we break the symmetry of an exactly symmetric Lagrangian and what are the properties of the field related with those broken symmetries.

When we have Lagrangian that is invariant under some continuous symmetry, whit the transformation given by

$$\phi_n(x) \longrightarrow \phi_n(x) + i\epsilon \sum_m t_{nm}\phi_m(x) \qquad (1)$$

with it_{nm} a real matrix. The effective action is also invariant under this field transformation, so we have

$$\delta\Gamma[\phi] = \int \frac{\delta\Gamma[\phi]}{\delta\phi_n(x)}\delta\phi_n(x)dx = \int \frac{\delta\Gamma[\phi]}{\delta\phi_n(x)}i\epsilon t_{nm}\phi_m(x)dx = 0 \qquad (2)$$

from now, when we have a duplicated index it means the we are summing over all values of it. For simplicity, we are going to treat only the constant field case, but our result is general. We can also relate this effective action with the effective potential by

$$\Gamma[\phi] = -Vol \cdot V_{eff}(\phi) \qquad (3)$$

where Vol is the volume of the spacetime. Substituting this in our previous relation, derivating it with respect to $\phi_a(x)$ and evaluating at the minimum of the effective potential, we have

$$\left. \frac{\partial^2 V(\phi)}{\partial\phi_n\partial\phi_a} \right|_{\phi=\bar{\phi}} t_{nm}\overline{\phi_m} = 0 \qquad (4)$$

with $\bar{\phi}$ the field that minimizes this potential. We can also relate the second derivative of the effective potential with the inverse of the propagator by

$$\Delta_{na}^{-1}(q) = \left(\frac{\partial^2 W[\phi]}{\partial J_n\partial J_a} \right)^{-1} = -\frac{\partial^2\Gamma[\phi]}{\partial\phi_n\partial\phi_a} = Vol\frac{\partial^2 V(\phi)}{\partial\phi_n\partial\phi_a} \qquad (5)$$

Using this in Eq. (4) we have that

$$\Delta_{na}^{-1}(0)t_{nm}\overline{\phi_m} = 0$$

So, if a symmetry associated with some transformation is broken, and by that i mean that this transformation does not leave the vacuum invariant ($\sum_m t_{nm}\langle\phi_m(x)\rangle_{VAC} \neq 0$), the term $t_{nm}\langle\phi_m(x)$ is an eigenvector of $\Delta_{na}^{-1}(0)$ with eigenvalue zero. If we have this situation, $\Delta_{na}(q)$ has a pole at $q=0$, what happens only if the associated particle has zero mass, so we must have one massless particle in our spectrum associated with our broken symmetry.

More general, we have that the Goldstone's theorem says that for each broken symmetry the spectrum of particles must contain one massless particle with zero spin, the same parity and internal quantum numbers as J^0, where J^0 is the 0th component of the current associated with the broken symmetry.

We can also calculate some amplitudes involving those bosons and the currents associated with them. By Lorentz invariance, considering that we have only one broken symmetry related to J^μ and one zero spin boson, we have that

$$\langle 0| J^\mu(x) |B\rangle = -i\frac{Fp_B^\mu e^{-ip_B\cdot x}}{(2\pi)^3\sqrt{2p_B^0}} \qquad (6)$$

and

$$\langle B| \phi_n(x) |0\rangle = \frac{Ze^{ip_B\cdot y}}{(2\pi)^3\sqrt{2p_B^0}}. \qquad (7)$$

More generally, when we have more than one broken symmetry and more than one boson associated with them, we have

$$\langle 0| J_a^\mu(x) |B_b\rangle = -i\frac{F_{ab}p_B^\mu e^{-ip_B\cdot x}}{(2\pi)^3\sqrt{2p_B^0}} \qquad (8)$$

and

$$\langle B_a| \phi_n(y) |0\rangle = \frac{Z_{an}e^{ip_B\cdot y}}{(2\pi)^3\sqrt{2p_B^0}} \qquad (9)$$

Now, using some relations that come from the general proof of Goldstone's theorem, we have that

$$i \sum_b F_{ab} Z_{bn} = -\sum_m [t_a]_{nm} \langle \phi_m(x) \rangle_{VAC} \qquad (10)$$

where the subscript VAC means that we are taking the expectation value with vacuum states, t_a's are related to the field transformations and Z and F were just defined above.

3 Approximate Symmetries

We just studied what happens when we break a symmetry that is exact in our Lagrangian, but one can ask what happens when we break a non-exact symmetry. We are going to see now that we do not have massless Goldstone bosons anymore, but Pseudo-Goldstone bosons that are massive.

To evaluate the mass of those Pseudo-Goldstone bosons, lets write the effective potential as

$$V(\phi) = V_0(\phi) + V_1(\phi) \qquad (11)$$

where V_0 is invariant under some transformation and V_1 is a small perturbation that is not invariant under this transformation. The invariance of V_0 can be written as

$$0 = \delta V_0(\phi) = \frac{\partial V_0(\phi)}{\partial \phi_n} (t_\alpha)_{nm} \phi_m \qquad (12)$$

with t_α the generator associated with the transformation that leaves V_0 invariant.

The minimum of the potential also changes due to this additional term in our potential, in a way that our minimum condition is given by

$$\left. \frac{\partial V(\phi)}{\partial \phi_n} \right|_{\phi=\phi_0+\phi_1} = 0 \qquad (13)$$

where ϕ_0 is the minimum of V_0 and ϕ_1 represents the small change of the minimum. As V_1 and ϕ_1 are small perturbations, we can expand this relation considering only first order terms and, using Eq.4 with V_0 as V and ϕ_0 as ϕ, we have

$$(t_\alpha)_{nm}(\phi_0)_m \left. \frac{\partial V_1(\phi)}{\partial \phi_n} \right|_{\phi=\phi_0} = 0 \qquad (14)$$

To evaluate the masses of our Pseudo-Goldstone bosons, at first order we have that them are related to the effective potential by

$$M_{ab}^2 = \sum_{nm} Z_{an} Z_{bm} \left. \frac{\partial^2 V(\phi)}{\partial \phi_n \partial \phi_m} \right|_{\phi=\phi_0+\phi_1} \qquad (15)$$

with Z's defined at Eq.9. Using Eq.10 and again doing expansion until first order in V, we have that

$$M_{cd}^2 = -\sum_{ab} F_{ca}^{-1} F_{db}^{-1} \left[\sum_{nm} (t_a \phi_0)_n (t_b \phi_0)_m \left. \frac{\partial^2 V_1(\phi)}{\partial \phi_n \partial \phi_m} \right|_{\phi=\phi_0} + \sum_n (t_a t_b \phi_0)_n \left. \frac{\partial V_1(\phi)}{\partial \phi_n} \right|_{\phi=\phi_0} \right] \qquad (16)$$

Consider now that the perturbation to our invariant part has the form

$$H_1 = u_n \Phi_n \qquad (17)$$

with Φ_n transforming like $[T_\alpha, \Phi_n] = -(t_\alpha)_{nm} \Phi_m$ with T_α the generators of our symmetry group. Equation 16 can be written as

$$M_{cd}^2 = -\sum_{ab} F_{ca}^{-1} F_{db}^{-1} \sum_n (t_a t_b \phi_0)_n u_n \qquad (18)$$

with $(\phi_0)_n$ the VEVs of the fields Φ_n. Using commutation rules, we can write the masses as

$$M_{cd}^2 = -\sum_{ab} F_{ca}^{-1} F_{db}^{-1} \langle [T_a, [T_b, H_1]] \rangle_{VAC} \qquad (19)$$

4 Two quark model

To apply those ideas of symmetry breaking, we can use a model that involves only two quark fields(u and d). This model involves the approximate chiral symmetry $SU(2) \times SU(2)$ of strong interactions. At first approximation, we are going to consider that the quarks are massless, so our Lagrangian can be written as

$$\mathcal{L} = \bar{u} i \slashed{D} u + \bar{d} i \slashed{D} d \qquad (20)$$

where D is the covariant derivative associated with QCD given by $(D_\mu \Psi(x))_l = \partial_\mu \Psi(x)_l - i A_\mu^\beta(x)(t_\beta)_l^m \Psi(x)_m$. The transformation that leaves the Lagrangian invariant is given by

$$\begin{pmatrix} u \\ d \end{pmatrix} \longrightarrow \exp\left\{ i\vec{\theta}^V \cdot \vec{t} + i\gamma_5 \vec{\theta}^A \cdot \vec{t} \right\} \begin{pmatrix} u \\ d \end{pmatrix} \qquad (21)$$

where the $\vec{t} = \dfrac{\vec{\sigma}}{2}$ and θ_A and θ_V are two real and independent three-vectors. Here we have two $SU(2)$ algebras related to the generators $\vec{x} = \gamma_5 \vec{t}$ and $\vec{t}$. Note that they are not two commuting algebras, but we can write them as $\vec{t_L} = \frac{1}{2}(1 - \gamma_5)$ and $\vec{t_R} = \frac{1}{2}(1 + \gamma_5)$ so we have two commuting algebras and identify our symmetry as $SU(2) \times SU(2)$. When we broke the symmetry related to the generators $\vec{x}$, as discussed in Section 2, we have now three massless particles in our spectrum that have the same quantum numbers as our broken generators: negative parity, zero spin, unit isospin and zero baryon number and strangeness. Therefore, we can associate those Goldstone's bosons as pions in our theory. Pions as massless particles is a good approximation since comparing their real masses with a typical scale of mass from QCD we get

$m_\pi^2/m_N^2 \approx 0,02$, where m_N is the mass of a nucleon. We can now evaluate the amplitude

$$\langle 0| A_i^\mu(x) |\pi_j\rangle = -i\frac{F_\pi \delta_{ij} p_\pi^\mu e^{-ip_\pi \cdot x}}{(2\pi)^3 \sqrt{2p_\pi^0}} \qquad (22)$$

where A^μ is the current related with the broken symmetry and F_π a constant. Using the relation of this current with the charged currents of electroweak interactions $A_\pm^\mu = A_1^\mu \pm A_2^\mu$, we can evaluate the decay rate of a pion into a muon and a neutrino as

$$\Gamma(\pi \longrightarrow \mu + \nu) = \frac{G_{wk}^2 F_\pi^2 m_\mu^2 (m_\pi^2 - m_\mu^2)^2}{16\pi m_\pi^3} \qquad (23)$$

Using our known values for G_{wk}, a constant of electroweak interactions, m_μ^2 and m_π^2, we get the value to our constant $F_\pi \approx 184 MeV$.

We can now study the effect of a perturbation in our Lagrangian that is not invariant under $SU(2) \times SU(2)$. As discussed before, we expect that this term generates mass for our pions. The term that explicitly breaks the symmetry is the one related to the mass of our quarks, given by

$$\mathcal{H}_1 = m_u \bar{u}u + m_d \bar{d}d = (m_u + m_d)\Phi_4^+ + (m_u - m_d)\Phi_3^-. \qquad (24)$$

Here, Φ_3^- and Φ_4^+ are the third and fourth components of

$$\Phi^+: \vec{\Phi}^+ = i\bar{q}\gamma_5 \vec{t}q; \ \ \Phi_4^+ = \frac{1}{2}\bar{q}q, \qquad (25)$$

$$\Phi^-: \vec{\Phi}^- = \bar{q}\vec{t}q; \ \ \Phi_4^- = \frac{1}{2}i\bar{q}\gamma_5 q. \qquad (26)$$

We can see that this term is not invariant since Φ^+ and Φ^- do not commute with the charges of our chiral symmetry. Explicitly, those commutation rules are given by

$$[\vec{X}, \Phi_n^\pm] = -\sum_m (\vec{\Lambda}_{nm})\Phi_m^\pm \qquad (27)$$

where Λ matrices are given by

$$(\Lambda_a)_{b4} = -(\Lambda_a)_{4b} = -i\delta_{ab}$$
$$(\Lambda_c)_{ab} = -(\Lambda_c)_{44} = 0$$

with $a, b, c = 1, 2, 3$. Just as an example, we can write one of those Λ matrices:

$$\Lambda_1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

As the chiral symmetry is broken, we expect that our Pseudo-Goldstone bosons, the pions, acquires mass as described in Eq. 16. To break this symmetry, we are going to take $\langle\Phi_4^+\rangle \neq 0$ and $\langle\Phi_3^-\rangle = 0$. So, using the form os Λ elements, we have

$$[X_b, \Phi_4^+] = -i\Phi_b^+ \implies [X_a, [X_b, \Phi_4^+]] = \delta_{ab}\Phi_4^+$$
$$[X_b, \Phi_3^-] = i\delta_{b3}\Phi_4^- \implies [X_a, [X_b, \Phi_3^-]] = \delta_{3b}\Phi_a^-$$

with $a, b = 1, 2, 3$. We can see that $\langle[X_a, [X_b, \Phi_4^+]]\rangle_{VAC} \neq 0$ and $\langle[X_a, [X_b, \Phi_3^-]]\rangle_{VAC} = 0$, so pions masses are given by

$$m_\pi^2 = 4(m_u + m_d)\langle\Phi_4^+\rangle_{VAC}/F_\pi^2. \qquad (28)$$

5 Pions and Nucleons interactions

We can also study how pions interact with themselves and with nucleons (protons and neutrons) using a $SO(4)$ model given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi_n^\mu \phi_n - \frac{M^2}{2}\phi_n \phi_n - \frac{\lambda}{4}(\phi_n \phi_n)^2 \qquad (29)$$

where $n = 1, 2, 3, 4$. We are considering that the energy of the Goldstone bosons (that we are going to identify as the pions again) are small, so we can calculate the amplitude of scattering involving those particles as powers of their energies. To do this, we want that the terms involving those fields appear with derivatives, so when we evaluate so scattering amplitude we get an energy dependence. To do this, we are going to write our original field $\phi_n(x)$ as

$$\phi_n = R(x) \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sigma(x) \end{pmatrix} \ \ with \ R^T(x)R(x) = 1 \qquad (30)$$

where $R(x)$ is just an orthogonal 4×4 matrix (the reason why we can do this is going to be clean in section 6) and the Goldstone's boson fields are going to be in $R(x)$ when we break the symmetry. The Lagrangian Eq. 29 has now the form of

$$\mathcal{L} = \frac{1}{2}\partial_\mu \sigma^\mu \sigma - \frac{\sigma^2}{2}\partial_\mu R_{n4}\partial^\mu R_{n4} - \frac{M^2}{2}\sigma^2 - \frac{\lambda}{4}\sigma^4. \quad (31)$$

To get a result in a simpler way to visualize, we can rewrite R matrix as

$$R_{a4} = -R_{4a} \equiv \frac{2\zeta_a}{1+\vec{\zeta}^2}; \ R_{44} = \frac{1-\vec{\zeta}^2}{1+\vec{\zeta}^2}; \ R_{ab} = \delta_{ab} - \frac{2\zeta_a \zeta_b}{1+\vec{\zeta}^2}$$

so we have the Lagrangian as

$$\mathcal{L} = \frac{1}{2}\partial_\mu \sigma^\mu \sigma - 2\sigma^2 \vec{D}_\mu \cdot \vec{D}^\mu - \frac{M^2}{2}\sigma^2 - \frac{\lambda}{4}\sigma^4 \qquad (32)$$

where this $\vec{D}_\mu$ is defined as $\vec{D}_\mu \equiv \frac{\partial_\mu \vec{\zeta}}{1+\vec{\zeta}^2}$. The isomorphism $SO(4) \cong SU(2) \times SU(2)$ allows us to recognize the $SO(4)$ transformation as an isospin and a chiral transformation. Although the ζ_a fields transform as expected for an isovector through isospin transformations, the way that they transform through chiral ones is different from what we expected for isovectors due to the non usual form of $\vec{D}_\mu$,

3

so we call this a non-linear realization of $SU(2) \times SU(2)$. Explicitly, under isospin rotations the $\sigma(x)$ field is invariant and $\vec{\zeta}$ transform like

$$\vec{\zeta} \to \vec{\zeta} + \delta\vec{\zeta} = \vec{\zeta} + \vec{\theta} \times \vec{\zeta} \qquad (33)$$

where θ is an infinitesimal parameter related with those rotations. Under chiral transformations, again $\sigma(x)$ is invariant and $\vec{\zeta}$ fields transform as

$$\vec{\zeta} \to \vec{\zeta} + \delta\vec{\zeta} = \vec{\zeta} + \vec{\alpha}(1 - \vec{\zeta}^2) + 2\vec{\zeta}(\vec{\alpha} \cdot \vec{\zeta}) \qquad (34)$$

where $\vec{\alpha}$ is an infinitesimal parametrization of chiral transformations. From the definition of $\vec{D}_\mu$ we see that it also just rotates under isospin transformations (since $\vec{\zeta}^2$ is invariant), but through chiral transformation we can show that it goes as

$$\vec{D}_\mu \to \vec{D}_\mu + \delta\vec{D}_\mu = \vec{D}_\mu + 2(\vec{\zeta} \times \vec{\alpha}) \times \vec{D}_\mu. \qquad (35)$$

When we break the chiral symmetry ($\langle\sigma\rangle = |M|/\sqrt{\lambda}$), the term with $\vec{D}_\mu$ gives rise to a kinetic term of ζ_a fields, and we also can recognize them as massless. Doing a redefinition of the fields as $\vec{\pi} \equiv F\vec{\zeta}$, with $F = 2\langle\sigma\rangle$, the kinetic term can be written as

$$\mathcal{L} = \frac{1}{2} \frac{\partial_\mu \vec{\pi} \cdot \partial^\mu \vec{\pi}}{(1 + \vec{\pi}^2/F^2)} \qquad (36)$$

where $\vec{\pi}$ are the usual pions of our theory, that in this exact symmetric model do not have mass. To study the interactions among pions, we can use effective theory with a bottom-up approach. We want that the interaction terms on pions have derivatives, so the more general Lagrangian involving Eq. 36 term and derivatives on $\vec{\pi}$ has the form

$$\mathcal{L} = \frac{F^2}{2} \vec{D}_\mu \cdot \vec{D}^\mu + \frac{c_4}{4}(\vec{D}_\mu \cdot \vec{D}^\mu)^2 + \frac{c_4'}{4}(\vec{D}_\mu \cdot \vec{D}_\nu)(\vec{D}^\mu \cdot \vec{D}^\nu) + \dots \qquad (37)$$

We can see that this is invariant under chiral transformations since

$$\vec{D}_\mu \cdot \delta\vec{D}^\mu = 2\epsilon_{ajk}\epsilon_{jlm}\zeta_l\alpha_m D_{\mu,k}D_{\mu,a} = 0$$
$$\vec{D}_\mu \cdot \delta\vec{D}^\nu = 2\epsilon_{ajk}\epsilon_{jlm}\zeta_l\alpha_m D_{\nu,k}D_{\mu,a} = -\delta\vec{D}_\mu \cdot \vec{D}^\nu$$

where the first line is equal to zero since it is a antisymmetric tensor ϵ_{ajk} on a, k contracted with a symmetric one ($\epsilon_{jlm}\zeta_l\alpha_m D_{\mu,k}D_{\mu,a}$); and the relation of the second line with invariance of $\mathcal{L}$ is that, when we do an infinitesimal transformation, the new terms originated from $\vec{D}_\mu \cdot \vec{D}_\nu$ cancel one another. The invariance under isospin transformations is direct since $\vec{D}_\mu$ just rotates, so $\vec{D}_\mu \cdot \vec{D}_\nu$ is invariant.

As discussed before, we are going to do perturbation theory with powers of Q, so if we want to evaluate our amplitude up to some order Q^ν, we must analyze which diagrams can contribute to it. To do this, we are going to write a relation between the order ν and the parts of the diagrams. Each internal line of our diagram associated with a propagator of pions contributes with Q^2; A vertex in our diagram associated with a term in the Lagrangian with d_i derivatives contributes with Q^{d_i}; each loop comes with an integral in four momentum directions, so it contributes with a Q^4. So, in general, a contribution of order ν involves diagrams that satisfy

$$\nu = \sum_i V_i d_i - 2I + 4L = \sum_i V_i(d_i - 2) + 2L + 2 \quad (38)$$

where V_i is the vertex related to the term in $\mathcal{L}$ with d_i derivatives, and the second equality comes from topological relations. Note that from Eq. 37 we have $d_i \geq 2$, so the lowest order diagrams are processes of order $\nu = 2$. Studying the scattering $\pi\pi \to \pi\pi$ at this order, we see that the only diagram that contributes is the one with $L = 0$, $V_i \neq 0$ for $d_i = 2$ and $V_i = 0$ for $d_i > 2$. So, the vertex term comes from the first term in Eq. 37, and the amplitude has the form of

$$M^{\nu=2} = \frac{4}{F^2}[\delta_{ab}\delta_{cd}(-p_A \cdot p_B - p_C \cdot p_D) + \\ + \delta_{ac}\delta_{bd}(p_A \cdot p_C + p_B \cdot p_D) + \delta_{ad}\delta_{cb}(p_A \cdot p_D + p_C \cdot p_B)]$$

where p_A, p_B, p_C and p_D are the moments associated with the pions. Note that each vertex term comes with two momenta, associated with the two derivatives that we have in our interaction term. If we want to study up to order $\nu = 4$, we have the contribution of diagrams with: one loop and interaction terms with two derivatives; no loops and interaction vertex with $d_i = 2$ or a single vertex diagram with $d_i = 4$. From the study of the current associated with the broken symmetry, we have that $F = F_\pi$.

Now we can study what are the corrections to the amplitudes that arise when we consider the mass of the pions. The mass term for pions can be identified as the term that comes with a m_π^2 as coefficient, and from Eq. 28 we see that the mass squared is proportional to $m_u + m_d$. In our two quark model, we saw that the term with the coefficient $m_u + m_d$ transforms as the fourth component of a chiral four vector, with the four component a scalar. In our $SU(2) \times SU(2)$ model, the chiral four vector is $\phi_n(x)$, with the fourth component $\sigma(x)$ a scalar (as discussed before), so the mass term for the pion is given by $\phi_4(x) = R_{44}\sigma(x)$, and apart from a constant and disconsidering σ since it is a scalar, we write it as

$$\mathcal{L} \supset -\frac{m_\pi^2}{2} \frac{\vec{\pi}^2}{(1 + \vec{\pi}^2/F_\pi^2)}. \qquad (39)$$

Note that this term is not invariant under chiral transformations, so this is not a symmetry of our Lagrangian anymore. We expected this since now our Goldstone bosons have mass. We can also note that this term does not have a derivative on pion field, but we are going to consider mass of the pion of the order of Q, in a way that this is

a term of order 2. Therefore, our relation that gives the order of Q^ν changes to

$$\nu = \sum_i V_i(d_i + 2m_i - 2) + 2L + 2 \qquad (40)$$

Now, reevaluating the amplitude of order two, we have that this additional interaction gives

$$M^{\nu=2} = \frac{4}{F_\pi^2}[\delta_{ab}\delta_{cd}(s - m_\pi^2) + \\ + \delta_{ac}\delta_{bd}(t - m_\pi^2) + \delta_{ad}\delta_{cb}(u - m_\pi^2)]$$

in particular, in the limit $s = 4m_\pi^2$ and $t = u = 0$, we have

$$M^{\nu=2}(s = 4m_\pi^2) = 32\pi m_\pi \left[\frac{7m_\pi}{8\pi F_\pi^2} M^{(0)} + \frac{-m_\pi}{4\pi F_\pi^2} M^{(2)} \right]$$

$$= 32\pi m_\pi \left[a_0 M^{(0)} + a_2 M^{(2)} \right]$$

where a_0 and a_2 are two coefficients that are measured and agree with experiments. Higher order amplitudes can also be evaluated by this method, where in those cases we are going to get corrections for this coefficients.

We can now try to describe the interactions among nucleons and pions by introducing a nucleon doublet N in our Lagrangian. To do this, we must write a term that is invariant under $SU(2) \times SU(2)$. Our first choice is

$$\mathcal{L}_N = -\overline{N}(\slashed{\partial} + g[\phi_4 + 2i\vec{t} \cdot \vec{\phi}\gamma_5])N \qquad (41)$$

where we can see explicitly that this is invariant under chiral transformations due to the way that those fields transform

$$\delta\vec{\phi} = 2\vec{\alpha}\phi_4 ; \quad \delta\phi_4 = -2\vec{\alpha} \cdot \vec{\phi}; \quad \delta N = -2i\gamma_5\vec{\alpha} \cdot \vec{t}N \quad (42)$$

with $\vec{t}$ the same as the two quark model. As discussed before, we do not want interaction terms of the pion fields without derivatives on them, so to solve this problem we are going to redefine our N field by

$$N \equiv \frac{(1 - 2i\gamma_5\vec{t} \cdot \vec{\zeta})}{\sqrt{1 + \vec{\zeta}^2}}\tilde{N}. \qquad (43)$$

With that definition, we see that this Lagrangian can be written as

$$\mathcal{L}_N = -\overline{\tilde{N}} \left[\slashed{\partial} + g\sigma + \frac{2i\vec{t} \cdot (\vec{\pi} \times \slashed{\partial}\vec{\pi})}{F_\pi^2[1 + \frac{\vec{\pi}^2}{F_\pi^2}]} + \frac{2i\gamma_5\vec{t} \cdot \slashed{\partial}\vec{\pi}}{F_\pi[1 + \frac{\vec{\pi}^2}{F_\pi^2}]} \right] \tilde{N}. \qquad (44)$$

We see that this is not the way we want since there is no coupling constant in the interaction terms, so we can try to study which terms are independently invariant in $\mathcal{L}_N$ and construct a Lagrangian from them. Combining Eq.

42 with Eq. 43, we can see that the $\tilde{N}$ field transform like

$$\tilde{N} \to \tilde{N} + 2i\vec{t} \cdot [\vec{\zeta} \times \vec{\alpha}]\tilde{N} \qquad (45)$$

so, based on our construction of the chiral invariant Lagrangian Eq. 41, knowing that the term $\overline{\tilde{N}}\sigma\tilde{N}$ comes from

$$\overline{\tilde{N}}\sigma\tilde{N} = \overline{N}(\phi_4 + 2i\vec{t} \cdot \vec{\phi}\gamma_5)N \qquad (46)$$

and that σ is a scalar under chiral transformations, we have that $\overline{\tilde{N}}\tilde{N}$ is a invariant term in our Lagrangian. We can also define

$$\mathcal{D}_\mu\tilde{N} \equiv \left[\partial_\mu + \frac{2i\vec{t} \cdot (\vec{\pi} \times \slashed{\partial}\vec{\pi})}{F_\pi^2[1 + \frac{\vec{\pi}^2}{F_\pi^2}]} \right] \tilde{N} \qquad (47)$$

and check that this field respects the same transformation rules as $\tilde{N}$. Then we have that, just like $\overline{\tilde{N}}\tilde{N}$, $\overline{\tilde{N}}\mathcal{D}_\mu\tilde{N}$ is invariant under chiral transformations. Being Eq. 44 invariant under those transformations, we also have that the term $\overline{\tilde{N}} \left[\frac{2i\gamma_5\vec{t} \cdot \slashed{\partial}\vec{\pi}}{F_\pi[1 + \frac{\vec{\pi}^2}{F_\pi^2}]} \right] \tilde{N}$ is invariant. Now we can construct the Lagrangian that describes the interactions among nucleons and pions by

$$\mathcal{L}_N = -\overline{\tilde{N}} \left[\slashed{\partial} + m_N + \frac{2ig_A\gamma_5\vec{t} \cdot \slashed{\partial}\vec{\pi}}{F_\pi[1 + \frac{\vec{\pi}^2}{F_\pi^2}]} \right] \tilde{N} \qquad (48)$$

where g_A is our coupling associated with the interaction term. In the context of effective field theory, we are considering $p_\pi \ll p_N$, so in this limit we have that the propagator associated with the nucleon can be written as

$$\frac{-i(\slashed{p} + \slashed{q}) + m_N}{(p + q)^2 + m_N^2} \longrightarrow \frac{-i\slashed{p} + m_N}{2p \cdot q} \qquad (49)$$

then the contribution of this propagator to the amplitude of some diagram is proportional to $1/Q$. We can then rewrite the relation to evaluate the power of Q^ν in some diagram by

$$\nu = \sum_i V_i(d_i + 2m_i) - 2I_\pi - I_N + 4L \\ = \sum_i V_i(d_i + 2m_i + \frac{n_i}{2} - 2) + 2L - E_N + 2 \qquad (50)$$

where the second equality comes from topological relations. Here, n_i is the number of nucleon fields in the interaction vertex V_i, E_N is the number of external nucleon lines and $I_{\pi,N}$ is the number of internal lines associated with each particle. With this, we can evaluate any scattering of nucleons with pions at any desired order. By example, if we want to evaluate $N + \pi \to N + \pi$ at lowest order we get a result that is compatible with experimental data. Before proceed with the three quark model, we are going to discuss some important group relations that are going to be useful to describe how our quarks and Goldstone boson fields transform under $SU(3) \times SU(3)$.

6   Groups and broken symmetries

Suppose that we have a theory that is invariant under some transformations of a compact group G, and this symmetry is broken to a subgroup $H \subset G$, and by that we mean that, given $h \in H$, we have

$$h_{nm}\langle\psi_m(x)\rangle = \langle\psi_n(x)\rangle \tag{51}$$

To exemplify, in our previous $SU(2) \times SU(2)$ model, $G = SO(4)$ and was broken to $H = SO(3)$. As we did in section 5, we want to write the field ψ as a rotation $\gamma \in G$ acting in a field $\tilde{\psi}$ where the Goldstone modes has been eliminated. We can represent this as

$$\psi_n(x) = \gamma_{nm}(x)\tilde{\psi}_m(x). \tag{52}$$

In sections 5, we had

$$\gamma(x) = R(x) \quad and \quad \tilde{\psi}(x) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sigma(x) \end{pmatrix}. \tag{53}$$

It is not clear what we mean by "Goldstone modes eliminated", so remembering from section 2 and generalizing our results, we have that some independent combinations of the vectors $[t^\alpha]_{nm}\langle\psi_m(0)\rangle_{VAC}$ are eigenvectors of the mass matrix associated with massless particles, where t^α are the generators of G. We can define $\tilde{\psi}(x)$ being Goldstone mode independent by

$$\tilde{\psi}_n(x)[t^\alpha]_{nm}\langle\psi_m(0)\rangle_{VAC} = 0. \tag{54}$$

To demonstrate that we always can write the relation given in Eq. 52, define

$$V_\psi(g) \equiv \psi_n(x)g_{nm}\langle\psi_m(0)\rangle_{VAC} \tag{55}$$

where $g \in G$ is in the real orthogonal representation of G. Since V_ψ is continuous and G is compact, so V_ψ is limited, to each x exists a $g = \gamma(x)$ associated with the maximum of $V_\psi(g)$. Under some infinitesimal transformation, a element of G goes as

$$g \to g(1 + i\epsilon^\alpha t^\alpha) = g + \delta g \tag{56}$$

where ϵ^α is just a transformation parameter. As $\gamma(x)$ is associated with the maximum at some fixed x, we have

$$\begin{aligned} \delta V_\psi(\gamma(x)) = 0 &= \psi_n(x)\delta g_{nm}\langle\psi_m(0)\rangle_{VAC} = \\ &= i\epsilon^\alpha \psi_n(x)\gamma_{nl}(x)[t^\alpha]_{lm}\langle\psi_m(0)\rangle_{VAC} \\ &= i\epsilon^\alpha(\gamma^{-1})_{ln}(x)\psi_n(x)[t^\alpha]_{lm}\langle\psi_m(0)\rangle_{VAC} \end{aligned}$$

where in the last equality we used that g is in an orthogonal representation. So, as ϵ^α is arbitrary, if we take $\tilde{\psi}_l(x) = (\gamma^{-1})_{ln}(x)\psi_n(x)$ with $\gamma(x)$ the matrix that maximizes $V_\psi(g)$ we have Eq. 54 satisfied, as we wanted.

If the broken Lagrangian is invariant under $h \in H$, by Eq. 51 and Eq. 55

$$V_\psi(g) = V_\psi(gh) \quad, h \in H \tag{57}$$

so if γ maximizes V_ψ, γh also does to any $h \in H$. Therefore, in Eq. 52 we can choose γ or γh, so there is an equivalent relation given by

$$\gamma_1 \sim \gamma_2 \Leftrightarrow \gamma_1 = \gamma_2 h \quad for \ some \ h \in H \tag{58}$$

it is easy to see that this is a real equivalent relation since H is a subgroup. The equivalent class associated with this equivalent relation is given by

$$[\gamma] = \{g \in G \mid g \sim \gamma\} \tag{59}$$

so we can divide G into disjoint equivalence classes, where those are known as the right cosets of G with respect to H. Let $\{t_i\}$ be the generators of H and $\{x_i\}$ the other independent generators of G, as they span Lie algebras we can write any element of G as

$$g = \exp\{i\xi_a x_a\}\exp\{i\theta_i t_i\} \tag{60}$$

but $\gamma(x)$ in Eq. 52 is defined up to a $h \in H$ multiplying in the right, so we can always take

$$\gamma(x) = \exp\{i\xi_a(x)x_a\} \tag{61}$$

where $\xi_a(x)$ are the Goldstone bosons of our theory. We can then label our γ's in different equivalence classes $[\gamma]$ by the fields $\xi(x)$, and relate the way that $\xi(x)$ transform with the way that our $\psi(x)$ fields transform. Note that, when we write our kinetic term as a function of the $\tilde{\psi}(x)$ field, we have

$$\partial_\mu\psi(x) = \gamma(x)\partial_\mu\tilde{\psi}(x) + \partial_\mu\gamma(x)\tilde{\psi}(x) \tag{62}$$

so our interaction terms between $\tilde{\psi}(x)$ and $\xi(x)$ involves derivative terms on $\xi(x)$.

To study how those fields transform under a general transformation of G, we get

$$\psi(x) \to \psi'(x) = g\psi(x) = g\gamma(\xi(x))\tilde{\psi}(x). \tag{63}$$

Also, as $g\gamma(\xi(x))$ is a element of G, we have that it belongs to some $[\gamma(\xi'(x))]$, so

$$g\gamma(\xi(x)) = \gamma(\xi'(x))h(\xi(x), g) \tag{64}$$

and putting everything together we have that $\psi'(x)$ and $\tilde{\psi}'(x)$ transform as

$$\psi'(x) = \gamma(\xi'(x))\tilde{\psi}'(x) \ with \ \tilde{\psi}'(x) = h(\xi(x), g)\tilde{\psi}(x) \tag{65}$$

and ξ fields by

$$h^{-1}(\xi(x), g) \, g \, \gamma(\xi(x)) = \gamma(\xi'(x)). \tag{66}$$

Note that by Eq. 65 the $\tilde{\psi}(x)$ field transformation depends only on the non broken subgroup H as we saw in the last section with $\tilde{N}$ fields in Eq. 45.

7 Three quark model

As we did for the $SU(2) \times SU(2)$ quark model, we are now going to construct a model that is involves three quarks and is invariant under the transformation

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix} \rightarrow \exp\left\{ i \sum_a (\theta_a^V \lambda_a + \theta_a^A \lambda_a \gamma_5) \right\} \begin{pmatrix} u \\ d \\ s \end{pmatrix} \quad (67)$$

where λ_a are the Gell-Mann matrices. In this model we have a $SU(3) \times SU(3)$ symmetry, where $SU(3)$ has 8 generator, so when we break the symmetry we get 8 Goldstone bosons the involves, besides the three pions, the eta meson η_0 and Kaons K^+, $\overline{K}^-$, K^0 and $\overline{K}^0$.

From the previous section, the quark fields that are independent on the Goldstone bosons can be obtained from

$$q(x) \equiv \exp\left\{ -i\gamma_5 \sum_a \xi_a(x)\lambda_a \right\} \tilde{q}(x). \quad (68)$$

where ξ_a are the eight Goldstone fields.

Using Eq. 64, the relation that dictates the way that $\xi(x)$ fields transform is

$$\begin{array}{c} \exp\left\{ i \sum_b (\theta_b^V \lambda_b + \theta_b^A \lambda_b \gamma_5) \right\} \exp\left\{ i\gamma_5 \sum_a \xi_a(x)\lambda_a \right\} = \\ \exp\left\{ i\gamma_5 \sum_a \xi_a'(x)\lambda_a \right\} \exp\left\{ i \sum_b \theta_b(x)\lambda_b \right\} \end{array}, \quad (69)$$

from where we have that the transformation of the Goldstone fields is given by

$$U(x) \rightarrow \exp\left\{ i \sum_a \lambda_a \theta_a^R \right\} U(x) \exp\left\{ -i \sum_a \lambda_a \theta_a^L \right\} \quad (70)$$

where $\theta^R = \theta^V + \theta^A$ and $\theta^L = \theta^V - \theta^A$, with θ^A, θ^V parameters of the transformations and $U(x)$ a unitary matrix given by

$$U(x) \equiv \exp\left\{ 2i \sum_a \lambda_a \xi_a(x) \right\}. \quad (71)$$

The only kinetic term involving Goldstone bosons and second derivatives as before is

$$\mathcal{L} = \frac{1}{16} F_\pi^2 Tr(\partial_\mu U \partial^\mu U^\dagger) \quad (72)$$

where the constant F_π is the same as the other models by the identification of pions as some Goldstone bosons. Explicitly, we can write the exponent of U in terms of the Goldstone fields as

$$\frac{\sqrt{2}B}{F} \equiv \sum_a \lambda_a \xi_a =$$

$$= \frac{\sqrt{2}}{F^2} \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta^0 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta^0 & K^0 \\ \overline{K}^- & \overline{K}^0 & -\sqrt{\frac{2}{3}}\eta^0 \end{pmatrix}.$$

Note that the kinetic term in 72 is obtained by the multiplication of the first terms in the expansion of $U(x)$, where using that U is unitary we get

$$Tr(\partial_\mu U \partial^\mu U^\dagger) = \frac{8}{F_\pi^2} Tr(\partial_\mu B \partial^\mu B) + ... =$$

$$= \frac{8}{F_\pi^2} \left[\partial_\mu \pi^0 \partial^\mu \pi^0 + \partial_\mu \eta^0 \partial^\mu \eta^0 + \partial_\mu K^0 \partial^\mu K^0 + ... \right] + ... \quad (73)$$

The choice of the multiplicative constant in Eq. 72 was made to have the kinematic terms as usual. At this point, we still have massless bosons, but by adding a quark mass term that breaks the $SU(3) \times SU(3)$ symmetry we also add mass to the bosons. So, the quark mass term in terms of the fields $\tilde{q}(x)$ is

$$\mathcal{L} \supset -\overline{q} M_q q = -\overline{\tilde{q}} e^{-i\sqrt{2}\gamma_5 B/F_\pi} M_q e^{-i\sqrt{2}\gamma_5 B/F_\pi} \tilde{q} \quad (74)$$

where M_q is the diagonal quark mass matrix. The mass terms of the Pseudo-Golstone bosons can be obtained from Eq. 74 by expanding both exponentials till second order as

$$-\overline{\tilde{q}} e^{-i\sqrt{2}\gamma_5 B/F_\pi} M_q e^{-i\sqrt{2}\gamma_5 B/F_\pi} \tilde{q} =$$

$$= -\overline{\tilde{q}} \left[M_q \frac{(-2(\gamma_5)^2 B^2)}{2F_\pi^2} + \frac{(-2(\gamma_5)^2 B^2)}{2F_\pi^2} M_q + \frac{(-2)\gamma_5 B M_q \gamma_5 B}{F_\pi^2} \right] \tilde{q}$$

$$= \frac{1}{F_\pi^2} \overline{\tilde{q}} (B(M_q B + B M_q) + (M_q B + B M_q)B) \tilde{q}$$

$$= \frac{1}{F_\pi^2} \overline{\tilde{q}} \{B, \{B, M_q\}\} \tilde{q} \quad (75)$$

and now, breaking the symmetry by taking

$$\langle \overline{\tilde{q}}_a \gamma_5 \tilde{q}_b \rangle_{VAC} = 0 \quad and \quad \langle \overline{\tilde{q}}_a \tilde{q}_b \rangle_{VAC} = -v\delta_{ab} \quad (76)$$

we have

$$\mathcal{L}_{mass} = -\frac{v}{F_\pi^2} Tr\{B, \{B, M_q\}\} \quad (77)$$

from where the mass terms for pions can be taken as

$$m_\pi^2 = \frac{4v}{F_\pi^2} [m_u + m_d]. \quad (78)$$

Note that we still have the same mass for different pions, the difference of their masses arises only when we consider electromagnetic effects. Also from Eq. 77, the other boson masses are

$$m_{K^+}^2 = \frac{4v}{F_\pi^2} [m_u + m_s]$$

$$m_{K^0}^2 = \frac{4v}{F_\pi^2} [m_d + m_s]$$

$$m_{\eta_0}^2 = \frac{4v}{\sqrt{3}F_\pi^2} \left[\frac{4m_s + m_d + m_u}{3} \right]$$

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