Entanglement Entropy in QFT.

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Abstract. Quantum information measures are becoming increasingly important in quantum field theory, particularly in proposals for its extension to curved spacetimes and quantum gravity. The Unruh effect, Hawking radiation, and the holographic principles stemming from black hole physics are shaped under the proposal that entanglement is in part responsible for the thermal character of these phenomena and the area law of black hole entropy. I show in these notes some first steps towards understanding entanglement in a continuous spacetime, and mention how spacetime seems to be intimatelly related to information.

Contents

1 Introduction

Definition 1. Let ρ be a state. Its von Neumann entropy is

$$S(\rho) = -\operatorname{tr}(\rho \log \rho). \tag{1}$$

The von Neumann entropy has an interpretation of a measure of lack of information one has about the given system at state ρ. This absense of information can originate from many sources, but we are mainly interested here in the entropy that comes from entanglement.

Take for instance a pure state $\rho = |\psi\rangle\langle\psi|$. In a basis spanned by the vector $|\psi\rangle$ and the remaining linearly independent ones that together span the Hilbert space, this state is an orthogonal projector onto span $|\psi\rangle$.

The trace that defines the von Neumann entropy is a sum over the spectrum of ρ. By defining the functional calculus that goes into the trace as a continuous function

$$S(p) = \begin{cases} 0 & \text{if} \quad p = 0 \\ p \ln p & \text{if} \quad p > 0 \end{cases}, \tag{2}$$

we see that the von Neumann entropy of the given state is

$$S(|\psi\rangle\langle\psi|) = -1 \ln 1 = 0. \tag{3}$$

We conclude that the von Neumann entropy of pure states vanishes.

Now compare this to the case where $|\psi\rangle = 2^{-1/2}(|{\uparrow_1}{\uparrow_2}\rangle + |{\downarrow_1}{\downarrow_2}\rangle)$, i.e. that we have a Bell state of two spins. Consider now the reduced density matrix $\rho = \mathrm{tr}_2 |\psi\rangle\langle\psi|$:

$$\begin{aligned} \rho = \mathrm{tr}_2 |\psi\rangle\langle\psi| &= \langle{\uparrow_2}||\psi\rangle\langle\psi||{\uparrow_2}\rangle + \langle{\downarrow_2}||\psi\rangle\langle\psi||{\downarrow_2}\rangle \\ &= \frac{1}{2}(|{\uparrow_1}\rangle\langle{\uparrow_1}| + |{\downarrow_1}\rangle\langle{\downarrow_1}|). \end{aligned} \tag{4}$$

The von Neumann entropy resulting from this reduced state is

$$S(\rho) = -\frac{1}{2} 2 \log 2^{-1} = \log 2. \tag{5}$$

We see that the entropy went up, i.e. we have less information about ρ than we had about the full, pure Bell state $|\psi\rangle\langle\psi|$. By tracing out the second spin, we are ignoring the correlations it had with the first one. In this case, this correlation is entanglement. This is what will be the main topic in this text, as it applies to field theory in a continuum.

Other entropic functions also play a role in quantum information. We will particularly be interested in the von Neumann entropy as a limit of the *Rényi entropy*:

$$S(\rho) = -\lim_{n\to 1} \frac{\mathrm{d}}{\mathrm{d}n} \log \mathrm{tr}\, \rho^n, \tag{6}$$

for the *replica parameter* n. The expression inside the limit is known as the Rényi entropy S_n, defined for integer n. In order to compute the v.N. entropy, however, an analytic continuation to real values is used. This is the expression that we will look for when computing the v.N. in field theory.

2 Path integrals and wavefunctionals

Recall from ordinary quantum mechanics the expression for the transition amplitude between position eigenstates:

$$\langle x|\mathrm{e}^{-\mathrm{i}H(t_2-t_1)}|y\rangle\,. \tag{7}$$

Consider setting $t_2 = 0$ and $t_1 = t$ and a Wick rotation $t = -\mathrm{i}\tau$. Inserting energy eigenbasis completeness relations on each side of this expression yields

$$\langle x|\mathrm{e}^{-\mathrm{i}H(t_2-t_1)}|y\rangle = \sum_n \psi_n(x)\psi_n^*(y)\mathrm{e}^{-E_n\tau}. \tag{8}$$

By further considering a particle coming from a long time in the past, i.e. $-\tau \ll 0$, the energy eigenvalues above the ground state are exponentially suppressed, and the transition amplitude reduces to

$$\langle x|\mathrm{e}^{-\mathrm{i}H(t_2-t_1)}|y\rangle \approx \psi_0(x)\psi_0^*(y)\mathrm{e}^{-E_0\tau}. \tag{9}$$

In terms of path integrals, we had

$$\langle x|\mathrm{e}^{-\mathrm{i}H(t_2-t_1)}|y\rangle = \int_y^x \mathrm{D}x(\tau)\,\mathrm{e}^{-I_\mathrm{E}}, \tag{10}$$

where I put the boundary condition for the paths as the upper and lower limit of the integral.

If we now take the inner product of these expression with the normalised wavefunction $\psi_0(y)$, we get

$$\begin{aligned} \psi_0(x)\mathrm{e}^{-E_0\tau}\int \mathrm{d}y\,\psi_0(y)\psi_0^*(y) &\approx \int \mathrm{d}y \int_y^x \mathrm{D}x(\tau)\,\mathrm{e}^{-I_\mathrm{E}}\psi_0(y) \\ &\Updownarrow \\ \psi_0(x)\mathrm{e}^{-E_0\tau} &\approx \int \mathrm{d}y \int_y^x \mathrm{D}x(\tau)\,\mathrm{e}^{-I_\mathrm{E}}\psi_0(y). \end{aligned} \tag{11}$$

In what follows we will no longer specify the boundary condition for amplitude distributions at the asymptotic past times.

The interpretation we are looking for when we see this expression is that to find the amplitude distribution of a particle at $t = 0$, we need to consider a sum over all possible paths from all possible positions the particle could have in the far past, weighted by its initial amplitude distribution. The extra factor can be ignored, especially if we set, as conventional, $E_0 = 0$.

Field theory. We switch gears to field theory. The degrees of freedom now are no longer positions but field amplitudes. The configuration space is therefore some function space, e.g. smooth functions on spacetime $\mathcal{C} \doteq \mathrm{C}^\infty(\mathcal{M})$. We work with eigenstates of the field configuration operator $\phi(x)$,

$$\phi(x)\,|\varphi\rangle = \varphi(x)\,|\varphi\rangle\,. \tag{12}$$

These states are analogous to coherent states in quantum mechanics, where one sets them as eigenstates of the annihilation operator. In our case, the annihilation operator is the negative frequency part of the field operator.[1] For simplicity we will not delve into these details.

Since we are now talking about amplitude distributions for functions, we have instead a *wavefunctional*:

$$\langle\varphi_0|0\rangle \doteq \Psi[\varphi_0] = \int_{\varphi(\boldsymbol{x},t=-\infty)}^{\varphi(\boldsymbol{x},t=0)=\varphi_0} \mathrm{D}\varphi\,\mathrm{e}^{-I_\mathrm{E}}, \tag{13}$$

that is a sum over field configurations for all times from the asymptotic past to $t = 0$.

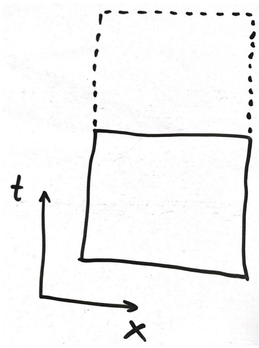
It is not difficult to be convinced that the complex conjugate is

$$\langle 0|\varphi_0\rangle \doteq \Psi^*[\varphi_0] = \int_{\varphi(\boldsymbol{x},t=0)=\varphi_0}^{\varphi(\boldsymbol{x},t=\infty)} \mathrm{D}\varphi\,\mathrm{e}^{-I_\mathrm{E}}. \tag{14}$$

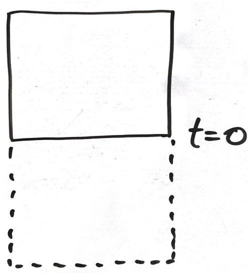
[1] $\phi = \phi_- + \phi_+$

With a sum over configuration from $t = 0$ to the asymptotic future.

With these computations we can also introduce an auxiliary pictorial representation. Imagine spacetime as the sheets in the figure below. The part with the full line indicates the region whereupon the field configurations have support. This diagram represents the wavefunctional $\Psi[\varphi_0]$.



By flipping it upside down, we get $\Psi^*[\varphi_0]$.



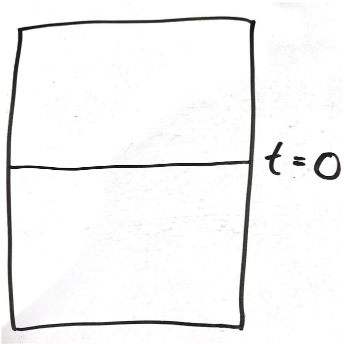
In these expression we have left the infinite past time configuration unspecified. If one wishes to really compute the wavefunction at the present time $t = 0$, it is enough to plug in the boundary condition. On the other hand, we are making explicit the amplitude of probability of finding the system with field configuration φ_0 at the present time slice.

In these terms, the familiar example of the generating functional is a sum over the diagonal of the vacuum state, at some fixed time slice:

$$\begin{aligned} Z = \operatorname{tr} |0\rangle\langle 0| &= \int_{\varphi_0 \in \mathcal{C}} \mathrm{D}\varphi_0 \; \langle \varphi_0 | 0 \rangle \, \langle 0 | \varphi_0 \rangle \\ &= \int_{\varphi_0 \in \mathcal{C}} \mathrm{D}\varphi_0 \, \Psi[\varphi_0] \Psi^*[\varphi_0] \\ &= \int_{\varphi_0 \in \mathcal{C}} \mathrm{D}\varphi_0 \int^{\varphi_0} \mathrm{D}\varphi_{<0} \, \mathrm{e}^{-I_\mathrm{E}[\varphi_{<0}]} \int_{\varphi_0} \mathrm{D}\varphi_{>0} \, \mathrm{e}^{-I_\mathrm{E}[\varphi_{>0}]} \\ &= \int_{\varphi(t) \in \mathcal{C}} \mathrm{D}\varphi(t) \, \mathrm{e}^{-I_\mathrm{E}} \end{aligned} \tag{15}$$

with an integration over all of configuration space, for all times. In what follows we will set $Z = 1$ for simplicity, and recover it at the end.

Going back to the pictorial representation, since we are now integrating over all spacetime, the generating functional is given by the following picture.



Entanglement in field theory. If we are interested in studying entanglement in field theory as it regards to spatial degrees of freedom, the treatment above is necessary. Using the Fock state basis (i.e. working with particles) is not sufficient, as they are delocalised. By having control over regions of spacetime as we treat the field configurations, we are able to judge whether the field fluctuations at distinct regions are correlated.

As in the first section, we analysed a system and the von Neumann entropy of a choice of subsystem (the first spin in that case). Since the full system is in a pure state, the only origin of the entropy measured by the v.N. entropy is entanglement. We will find something similar happening in field theory, for the vacuum state.

Let us choose our subsystem of interest as $\mathcal{R} \subset \mathcal{M}_0$, a finite region of space at the zero time slice. This means that $\mathcal{R}^{\complement} \doteq \mathcal{M}_0 \smallsetminus \mathcal{R}$, the complement of our chosen region inside the zero time slice, has d.o.f. to be traced out.

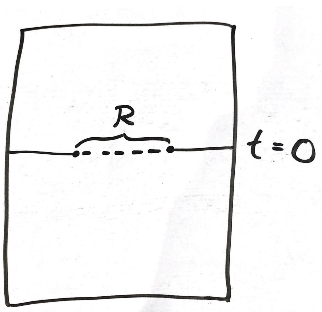
The reduced state, for region $\mathcal{R}$, is

$$\rho_{\mathcal{R}} = \mathrm{tr}_{\mathcal{R}^{\complement}}\, \rho_0. \tag{16}$$

The procedure to take this partial trace is very similar to what has been done to find the generating function, but now we are restricting the sum over diagonal terms of configurations only on the complement region, i.e. we letting the configurations fixed at $\mathcal{R}$, and averaging over the rest:

$$\rho_{\mathcal{R}} = \int_{\mathcal{C}(\mathcal{R}^{\complement})} \mathrm{D}\varphi_0 \, \langle \varphi_0 | 0 \rangle \, \langle 0 | \varphi_0 \rangle \,, \tag{17}$$

for which $\mathcal{C}(\mathcal{R}^{\complement}) \doteq \{\varphi_0\big|_{\mathcal{R}^{\complement}}\}$. The corresponding diagram is found below.



Recall that we are interested in computing

$$S(\rho) = -\lim_{n\to 1} \frac{\mathrm{d}}{\mathrm{d}n} \log \operatorname{tr} \rho^n. \tag{18}$$

We will first compute the nth power of the density matrix, and then the trace. The matrix elements come into play.

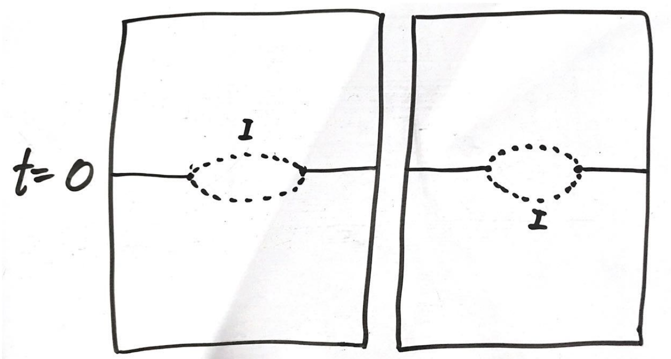
Let $\rho_0 = |0\rangle\langle 0|$ be the vacuum state of our field theory. We have the results above telling us how to express it in terms of path integrals. Before taking the trace over configuration space to find the generating function, we compute the matrix elements of the vacuum density matrix at $t = 0$:

$$\langle \tilde{\varphi}_0 | \rho_0 | \varphi_0 \rangle = \Psi[\tilde{\varphi}_0] \Psi^*[\varphi_0]. \tag{19}$$

Take now, for instance, $\rho_{\mathcal{R}}^2$. This has a straightforward expression in terms of these matrix elements:

$$\langle \tilde{\varphi}_0 | \rho_{\mathcal{R}}^2 | \varphi_0 \rangle = \int_{\mathcal{C}(\mathcal{R}^{\complement})} \mathrm{D}\hat{\varphi}_0 \, \langle \tilde{\varphi}_0 | \, \rho_{\mathcal{R}} \, | \hat{\varphi}_0 \rangle \langle \hat{\varphi}_0 | \, \rho_{\mathcal{R}} \, | \varphi_0 \rangle \, . \tag{20}$$

For n powers of the density matrix, therefore, one inserts $n - 1$ functional integrals inbetween the factors, and whose diagrammatics is the following:

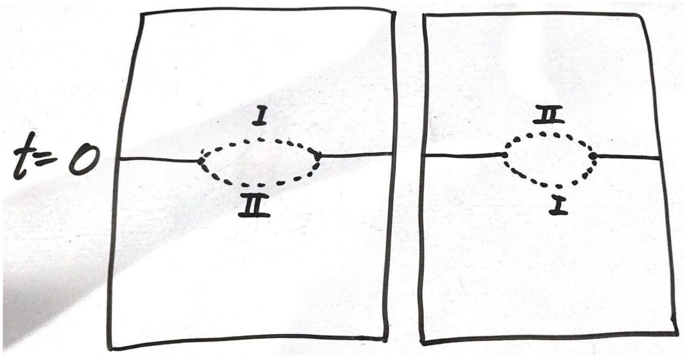


Where now we are depicting the two free field configuration indices as the two dashed lines for the region $\mathcal{R}$. By summing over two of them in the definition of the matrix multiplication, we identify the corresponding dashed lines. In this case, denoted by I.

The trace is again the sum over the diagonal, now of the nth power of the density matrix:

$$\operatorname{tr}_{\mathcal{R}} \rho_{\mathcal{R}}^2 = \int_{\mathcal{C}(\mathcal{R})} \mathrm{D}\tilde{\varphi}_0 \, \mathrm{D}\hat{\varphi}_0 \, \langle \tilde{\varphi}_0 | \, \rho_{\mathcal{R}} \, | \hat{\varphi}_0 \rangle \langle \hat{\varphi}_0 | \, \rho_{\mathcal{R}} \, | \tilde{\varphi}_0 \rangle \, . \tag{21}$$

Correspondingly, the picture just identifies the remaining two dashed lines. Below we denote this as II.



Ultimately, we are using this pictorial representation to create an intuition over what the computation of the Rényi entropy leads to. By starting with a spacetime $\mathcal{M}$, gluing these replicas of spacetime creates a new geometry, the *n-fold cover* $\mathcal{M}_n$. The computation of $\operatorname{tr}\rho^n$ in the end yields the partition function of the field theory on this new geometry,

$$\operatorname{tr}\rho^n = \frac{Z_n}{Z^n}, \tag{22}$$

where we are now making the partition function in the original spacetime explicit again.

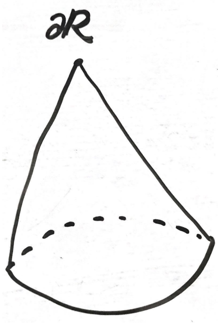
Finally, the v.N. entropy in these terms is

$$S(\rho_{\mathcal{R}}) = -\lim_{n\to 1}\frac{\mathrm{d}}{\mathrm{d}n}\log Z_n + \log Z. \tag{23}$$

This configures the *replica trick*.

3 Sketching an example

In general determining the resulting geometry of the n-fold cover and solving for the v.N. entropy is difficult. An explicit example exists, which we sketch below.



Set $\mathcal{M} = \mathbb{R}^d$ to be Minkowski spacetime, and choose the subregion to be the causally complete right Rindler wedge, $\mathcal{R} = \{x \in \mathcal{M} \mid t = 0, x^1 > 0\}$, that is, a half-space. The n-fold cover can be determined to be the direct product $\mathcal{M}_n = C_n \times \partial\mathcal{R}$, between a cone C_n and the boundary between the two half-spaces, $\partial\mathcal{R} = \partial\mathcal{R}^{\complement} = \mathbb{R}^{d-2}$. Let the theory to be a scalar free field theory $I_{\mathrm{E}} = \nabla\varphi\nabla\varphi/2 + m^2\varphi^2/2$.

We have seen from the lectures that $\log Z_n$ is the one-loop determinant:

$$\begin{aligned}\log Z_n &= \frac{-1}{2}\log\det\big(-\nabla^2+m^2\big)\\ &= \frac{-1}{2}\operatorname{tr}\log\big(-\nabla^2+m^2\big)\\ &= \frac{1}{2}\int_{\epsilon^2}^{\infty}\frac{\mathrm{d}s}{s}\operatorname{tr}\Big(\mathrm{e}^{-s(-\nabla^2+m^2)}-\mathrm{e}^{-s}\Big),\end{aligned} \tag{24}$$

where the last line comes from using Schwinger parameters, and where we regularised by introducing ϵ^2.

The cone is flat (with closed boundaries), and therefore solving for the eigenvalues of the laplacian is possible. Ultimately, one can compute these expression and find

$$S(\rho_{\mathcal{R}}) = \frac{\pi\,\mathrm{vol}(\partial\mathcal{R})}{3}\int_{\epsilon^2}^{\infty}\mathrm{d}s\,\frac{\mathrm{e}^{-sm^2}}{(4\pi s)^{d/2}}. \tag{25}$$

At $\epsilon \approx 0$ we can expand this expression and find

$$S(\rho_{\mathcal{R}}) = \frac{\mathrm{vol}(\partial\mathcal{R})}{6(4\pi)^{d-1}}\left(\frac{1}{(d-2)\epsilon^{d-2}}+\dots\right). \tag{26}$$

It is believed that this last expression has far-reaching consequences. The first term in the expression particularly is known as the *area law for entanglement entropy*. It is at the root of the most accepted intepretations of the black hole entropy, which is also proportional to the area of a region, namely the event horizon.

As mentioned previously, the non-zero value of this v.N. entropy is interpreted as coming from entanglement which is being ignored by tracing out a region of spacetime. This is what happens effectively during the Unruh effect: the uniformly accelerating observer in the region $\mathcal{R}$ is causally disconnected from the complementary region, with a light-cone separating these two parts. Since he has only access two one half-space, the vacuum to him appears as $\rho_{\mathcal{R}}$, whose entropy is shown above. He sees this entropy as a thermal entropy, originating from the thermal character of the reduced density matrix.

References

[1] Nishioka, T. (2018). Entanglement entropy: holography and renormalization group. Reviews of Modern Physics, 90(3), 035007.