Solitons, Lumps, Kinks and all that

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1 Introduction

The soliton phenomenon was first observed in 1834, by John Scott Russell, while he was observing the motion of a boat through a narrow canal. He observed that after the boat had stopped a “large solitary elevation, a rounded, smooth and well-defined heap of water”[1] continued moving along the canal without changing its form or speed. The phenomenon described by J. S. Russel relates to the focus of this text, the idea of a non-spreading wave of constant energy; that is what lays behind the idea of *solitons*.

Most of the simple field theories that are around have the property that all their non-singular solutions of finite energy are dissipative. By that we mean that given the energy density component of the energy-momentum tensor $\mathbf{\Theta}_{00}$ we have

$$\lim_{t\to\infty} \max_{\boldsymbol{x}} \mathbf{\Theta}_{00}(\boldsymbol{x}, t) = 0 \tag{1}$$

However, there are some theories that possess non-singular non-dissipative solutions of finite energy. Even in simple theories, those solution maybe be time-independent, *lumps* energy holding themselves together by means of a self-interaction. We'll deal with some simple cases of classical scalar field theories in order to understand this phenomenon, and then study how topological properties of our soliton-like solutions hold important physical information.

2 Lumps in 1+1 dimensions

A quick commentary about the nomenclature seems necessary. Although there seems to be no consensus around the nomenclature of *lumps*, both of our main references [2, 5] agree in not calling them *solitons*. In Coleman's book he states “I avoid the word ‘soliton’ here because it has a very precise and narrow definition in applied mathematics(...)”. To Rajaraman the difference between solitons and *kinks* as stated above is that solitons are unperturbed by collisions with other solitons while kinks are not. We will be somewhat clumsy and pay little attention to this in the text since the spirit is to give a general view of the properties common to both.

[1] His words, as described in [6], p. 13.

We will start our discussion by studying the simplest case, theories given by a single scalar field in 1+1 dimensions, given by the following Lagrangian

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - U(\phi) \tag{2}$$

where $\mu = 0, 1$ and the potential $U(\phi)$ depends only on the field ϕ. The energy of any configuration of the field ϕ is given by the integral

$$E = \int dx \left[\frac{1}{2}(\partial_0\phi)^2 + \frac{1}{2}(\partial_x\phi)^2 + U\right] \tag{3}$$

It is easily seen that for the energy to be bounded below we must require that U be bounded below. And in this case, we can always sum a constant to the energy in order to make its minimum value equals zero. From now on that is always assumed unless stated otherwise. As one can see from (3), for ϕ to be a minimum (ground state) its time and spatial derivative must vanish, as well as the potential U.

As we are concerned in finding time-independent solutions, meaning $\partial_0\phi = 0$, the variation of E give us

$$\delta E = \delta\int dx \left[\frac{1}{2}(\partial_x\phi)^2 + U(\phi)\right] \tag{4}$$

Looking at expression (4) we see that it is analogous to the variational principle applied to a particle of mass $m = 1$ under the effect of a potential $-U$

$$\delta S = \delta\int dt\mathscr{L}(x,t) = \delta\int dt\left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + U(x)\right] \tag{5}$$

Since we assumed $\partial_0\phi = 0$, solutions for the equation of motion defined by (4) are time-independent. And since we want finite energy solutions the integral (3) must converge when $x \to \pm\infty$. For the integral to converge it is required that ϕ approaches the zeroes of the potential U. In our analogous problem that means that the particle must approach the *hills* in the graph of Fig. 1 as $t \to \pm\infty$. We get a trivial solution when the particle stays forever at one of the hills. Something more interesting happens in a less trivial case; when the particle goes from one hill to another one. The solution still has zero energy but we are allowed to change vacua!

We summarize the discussion above in two cases:

1. When the potential U has only one zero[2], we don't have a non-trivial time-independent solution.

2. If the potential U has more than one zero: there's always a non-trivial time-independent solution of finite energy.

[2]Equivalently: if the ground-state of the theory is unique.

The equation of motion determined by (5) is[3]

$$\frac{\partial \mathscr{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{x}}\right) = 0 \Rightarrow \frac{d^2 x}{dt^2} - \frac{dU(x)}{dx} = 0 \tag{6}$$

We multiply (6) by $\dot{x}$ and integrate over t to get

$$\int dt \frac{dx}{dt}\frac{d^2 x}{dt^2} = \int dt \frac{dx}{dt}\frac{dU(x)}{dx} \tag{7}$$

Substituting $u = dx/dt$ on the l.h.s. and noting that the r.h.s. is equal to $U(x)$ after integration, we get the following result

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 = U(x) \tag{8}$$

Now, the analogy that was made in (5) was $\phi(x) \rightarrow x(t)$ and $x \rightarrow t$, if we translate it back to the field ϕ, the equation (8) becomes

$$\frac{1}{2}\left(\partial_x \phi\right)^2 = U(\phi) \tag{9}$$

Which we solve simply by integrating over both sides

$$\int_{x_0}^{x} dx = \pm \int_{\phi_0}^{\phi(x)} \frac{d\phi}{\sqrt{2U(\phi)}} \tag{10}$$

Where x_0 is the point where $\phi(x_0) = \phi_0$. And we remember that followed from our discussion that $\phi(x)$ approaches the zeroes of $U(\phi)$ as $x \rightarrow \pm\infty$. And we can go back to (3) and substitute (7) to get various expressions for the energy

$$E = \int dx (\partial_x \phi)^2 = 2\int U(\phi(x)) dx = \int \sqrt{2U(\phi)} d\phi = \int d\phi \partial_x \phi \tag{11}$$

2.1 The ϕ^4 and sine-Gordon theories

We will now delve into two simple theories, the ϕ^4 with two ground state[4] and the sine-Gordon theory. First lets start with an old friend of ours, the ϕ^4 interaction. In general

$$U = \frac{\lambda}{4}\left(\phi^2 - a^2\right)^2 \tag{12}$$

We see that U has two vacua, $\phi = \pm a$, thus it has a non-trivial time-independent solution of finite energy. In the case we deal with the traditional Lagrangian of the ϕ^4 interaction given by

$$\mathscr{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4}\phi^4 \tag{13}$$

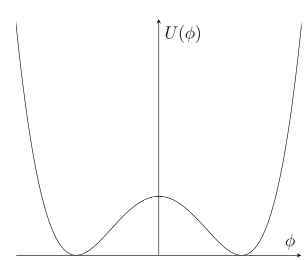


Figure 1: Potential for eq. (12).

[3]We use the notation $\dot{x} \equiv \dfrac{dx}{dt}$

[4]Where symmetry is spontaneously broken.

we readily see that a in (12) must be $a = \sqrt{\frac{\mu^2}{\lambda}}$. If we substitute (12) into (10) and solve for $\phi(x)$, with $\phi_0 = 0$, we get that the *lump* has the form

$$\phi(x) = \sqrt{\frac{\mu^2}{\lambda}} \tanh\left(\pm\frac{\mu}{\sqrt{2}}(x - x_0)\right) \tag{14}$$

The solution with + sign is called a *lump* (or *kink*), and with − sign is called *anti-lump* (or *anti-kink*). Equivalently, we can do the same analysis for the sine-Gordon potential given by

$$U(\phi) = \frac{\alpha}{\beta^2}\left(1 - \cos(\beta\phi)\right) \tag{15}$$

As we see, the potential in Fig.2 has zeroes at $\phi = \frac{2n\pi}{\beta}$, with $n \in \mathbb{Z}$, meaning that we can find non-trivial solutions. If we substitute the potential again in (10) and solve for $\phi(x)$ we get

$$\phi(x) = \frac{4}{\beta}\arctan\left(\mathrm{e}^{\pm\sqrt{\alpha}(x-x_0)}\right) \tag{16}$$

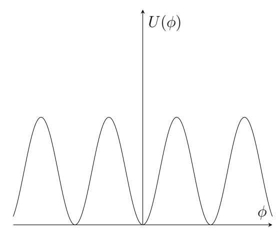


Figure 2: Potential for eq. (15).

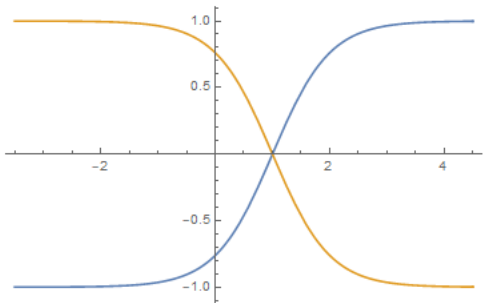
In Fig. 3 we chose for simplicity $a = \lambda = x_0 = \beta = \alpha = 1$. We clearly see the expected behavior of $\phi \xrightarrow{x\to\pm\infty} \pm 1$, for the solution depicted in Fig. 3a). And for the solution (16) we see that it goes from $\phi = 0$ to $\phi = 2\pi$, also the expected behavior.

We now calculate the total energy of the lumps, often called the *mass* of the lump, by means of equation (11)

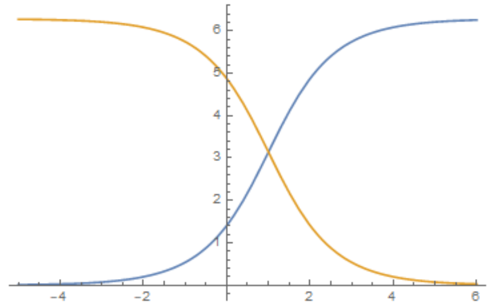
$$E_{\phi^4} = 2\int_{-\infty}^{\infty} U(\phi(x))dx = \frac{\lambda}{2}\int_{-\infty}^{\infty} dx\left\{\left[a\tanh\left(\frac{\mu}{\sqrt{2}}(x-x_0)\right)\right]^2 - a^2\right\}^2 = \frac{2\sqrt{2}\mu^3}{3\lambda} \tag{17}$$

$$E_{sG} = \int_{-\infty}^{\infty} dx(\partial_x\phi(x))^2 = \int_{-\infty}^{\infty}\left(\frac{4}{\beta^2}\,\frac{\sqrt{\alpha}e^{\sqrt{\alpha}(x-x_0)}}{1+e^{2\sqrt{\alpha}(x-x_0)}}\right)^2 dx = \frac{8\sqrt{\alpha}}{\beta^2} \tag{18}$$

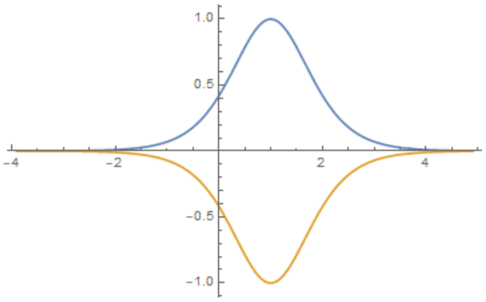
As we expected, both of the energies are finite. From graphs 3c) and 3d), we see easily that their energy densities are localized in space, and as they are time-independent they remain localized.



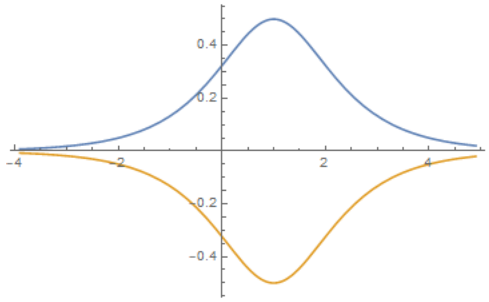
(a) Graph of $\phi(x)$ given by (14).



(b) Graph of $\phi(x)$ given by (16).



(c) Graph of $\partial_x \phi(x)$ given by (14).



(d) Graph of $\partial_x \phi(x)$ given by (16).

Figure 3: The color blue graphs represents the lumps and the orange represents the anti-lumps

2.2 Stability of the Lumps

We now ask ourselves if the lumps are stable under small perturbations. We expand the $\phi(x)$ into $\phi(x,t) = f(x) + \delta(x,t)$ where $f(x)$ is a time-independent solution and $\delta(x,t)$ is a small perturbation (that may depend on time). From (6) we find the equation of motion to be

$$\partial_\mu \partial^\mu \phi - \frac{\partial U(\phi)}{\partial \phi} = 0 \tag{19}$$

We expand $U(\phi)$ around one of its minimum ϕ_{min}

$$U(\phi) = U(\phi_{\text{min}}) + \frac{\partial U(\phi_{\text{min}})}{\partial \phi}(\phi - \phi_{\text{min}}) + \frac{\partial^2 U(\phi_{\text{min}})}{\partial \phi^2} \frac{(\phi - \phi_{\text{min}})^2}{2!} + \mathcal{O}(\phi^3) \tag{20}$$

Since ϕ_{min} is a minimum of $U(\phi)$ we have $U(\phi_{\text{min}}) = 0 = \frac{\partial U(\phi_{\text{min}})}{\partial \phi}$. Our equation of motion in first order is then

$$\partial_\mu \partial^\mu \delta(x,t) + \frac{\partial^2 U}{\partial \phi^2}(\phi_{\text{min}})\, \delta(x,t) = 0 \tag{21}$$

From the fact that the above equation of motion is invariant under a time translation operation we expand the solution as

$$\delta(x,t) = \text{Re} \sum_n a_n e^{i\omega_n t} \psi_n(x) \tag{22}$$

Where ω_n is an eigenfrequency associated with the n-th function of the basis $\psi_n(x)$. Since there is a complex exponential phase in (22), for the solution be stable[5] under small perturbations we require that $\omega_n \in \mathbb{R}$, otherwise we would have a real exponential that either goes to 0 or to infinity as $t \to \infty$.

2.3 Time dependence for non-trivial solution

We find a time-dependent solution by applying a Lorentz transformation to

$$x \to x' = \gamma(x - vt) \tag{23}$$

with $\gamma = \dfrac{1}{\sqrt{1-v^2}}$ and $c = 1$, thus our solutions given by (14) and (16) become

$$\phi_{\phi^4}(x) = a \tanh\left(\pm\frac{\mu}{\sqrt{2}}\frac{(x-x_0)-vt}{\sqrt{1-v^2}}\right) \tag{24}$$

$$\phi_{sG}(x) = \frac{4}{\beta}\arctan\left(\exp\left(\pm\sqrt{\alpha}\frac{(x-x_0)-vt}{\sqrt{1-v^2}}\right)\right) \tag{25}$$

If we calculate again the energy of the lumps –or its mass– we see that they transform as

$$E \to E' = \frac{E}{\sqrt{1-v^2}} \tag{26}$$

In this sense lumps resemble particles. They have definite rest mass and definite location, the location of its center-of-mass. By the Lorentz invariance they obey the relativistic energy relation $E^2 = p^2 + m^4$, but they are not quite particles. Also noteworthy is the fact that for any time t_0 the limit $\lim_{x\to\pm\infty}\phi(x,t_0) = \phi(\pm\infty, t_0)$ is still a zero of the potential $U(\phi)$.

2.4 Derrick's Theorem

We now proceed to ask one important question about possible generalization to higher dimensions and for more fields. We want to know which general theories possess non-trivial time-independent solution. The answer is quite discouraging.

Theorem 2.1 (Derrick's Theorem)**.** *Let ϕ be a set of scalar fields in d+1 dimensions that follows the dynamics determined by the Lagrangian density*

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - U(\phi) \tag{27}$$

Where the potential U is non-negative and zero for the ground states. Then for $d \geq 2$ the only non-singular time-independent solutions of finite energy are the ground states.

[5]By stable we mean a solution that does not goes to $\pm\infty$ nor to 0.

Since the proof is simple and straightforward we won't bother to reproduce it here, instead we refer to [2, p. 194], where you can find it.

We discuss briefly what this theorem means. Derrick's theorem is a powerful tool since already with small effort, we can decide for certain field theories whether *solitons* can exist in this theory or not. The theorem shows us that for $d \geq 2$ the only time-independent solution are the ground states; it does not state anything about non-dissipative time-dependent solutions.

3 Topological Conservation Laws

We want a way to study and classify the properties of theories which possess non-trivial time-independent solutions. It is often possible to make topological classifications of solutions of a given system of equations. We will focus on non-trivial finite-energy solutions, of which lumps and solitons are special cases.

We saw in the last section that the field ϕ, whether static or time-dependent, must tend at any instant t, to a minimum of $U(\phi)$ at spatial infinity, in order to the energy be finite. The set of zeroes of U in both of cases above was discrete. In one space-dimension, spatial infinity consists of two points, $x = \pm\infty$; for any instant t_0 the limit below must be a zero of $U(\phi)$

$$\lim_{x \to \pm\infty} \phi(x, t_0) = \phi(\pm\infty, t_0) \equiv \phi_\pm \tag{28}$$

Where $\phi_\pm$ are zeroes of U. We note that the non-dissipative solutions are continuous functions of time t, and that since the energy of the solution is finite, $\phi(\infty, t)$ must be a zero of $U(\phi)$ for any t. For U having a discrete sets of zeroes, the continuity in t asserts that $\phi(\infty, t)$ cannot *jump* from ϕ_+ to ϕ_- or vice-versa; it must remain stationary at one of the zeroes of U.

A slightly more abstract way of expressing that is to say that we have divided the space of non-singular finite-energy solutions at a fixed time into subspaces, labeled by a ordered pair $(\phi_\pm, \phi_\pm)$. These subspaces are disconnected components of the whole space in the topological sense; it is not possible to continuously change a solution in one component into a solution in another component. Since time evolution is continuous, this implies that if a solution is in one component at any one time, it is in the same component at any other time.

To exemplify we return to the ϕ^4 theory where we can divide the space of all non-singular finite-energy solution into 4 sector: $(-a, -a)$, $(a, -a)$, $(-a, a)$ and (a, a); where $a = \sqrt{\frac{\mu^2}{\lambda}}$. We see that the lump is in the 3^{rd} sector, the anti-lump in the 2^{nd}, and the constant solution in the 1^{st} and 4^{th}

We can generalize this thought by introducing the concept of homotopy, homotopy classes and more. We will follow some of the exposition in [4].

3.1 A glimpse of Homotopy

We will define some important mathematical notions that will be helpful in the future. We will first define what is a homotopy.

Definition 3.1 (Homotopy)**.** Given two continuous function $g, h : X \to Y$, they are said to be *homotopic* if there is a continuous function $F : X \times [0,1] \to Y$ such that $F(s,0) = g(s)$ and $F(s,1) = h(s)$, $\forall s$. The function F is called a homotopy between g and h.

We can imagine that if g and h are homotopic we can in a way, continuously deform g into h. We note that a homotopy defines a equivalence relation between functions. Given a function g the set of all functions which are homotopic to g is called a *homotopy class* and is usually denoted by $[g]$.

Definition 3.2 (Homotopy Group)**.** The set of all homotopy classes in the space Y is denoted by $\pi_1(Y)$ and is called the *fundamental group* of Y. It is a group with the product $[g] * [h] = [g * h]$, for $f, g : [0,1] \to Y$, which is defined by

$$g * h(s) = \begin{cases} g(2s) & 0 \leq s \leq \frac{1}{2} \\ h(2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases} \tag{29}$$

We can generalize the homotopy group of a function $f : X^n \to Y$, for $n \geq 1$, which is denoted by $\pi_n(Y)$ and is called the *n-th homotopy group*.

We can now formulate our problem in terms of this language. The idea is given a theory in d-dimensions, and the set of zeroes of U, $(\mathcal{Z})$, we impose the following condition on the field ϕ at the spatial infinity

$$\phi(x) \in \mathcal{Z}, \forall x \in S^{d-1}_{\infty} \tag{30}$$

where S^{d-1}_{∞} is the sphere surface manifold in d-dimension at $r = \infty$. The boundary condition (30) defines us a map $\phi^{\infty} : S^{d-1}_{\infty} \to \mathcal{Z}$ from the sphere at spatial infinity (in S^{d-1}_{∞}) to the vacuum space $\mathcal{Z}$. Those maps can be classified by an element of the homotopy group $\pi_{d-1}(\mathcal{Z})$. From the fact that two field configurations ϕ_1, ϕ_2 −with distinct asymptotic data ϕ_1^{∞}, ϕ_2^{∞}− are homotopic to each other if ϕ_1^{∞} is homotopic to ϕ_2^{∞}, we can also classify the field configurations $\phi(x)$ with the same element of the homotopy group $\pi_{d-1}(\mathcal{Z})$.

4 Topological Charges

A topological charge is a quantity originated only from topological considerations of our problem, that allows us to quantify properties of our system. It means that just from a topological point of view there is a rich set of information available to us.

4.1 The case for $d = 1$

The case of one-dimensional field theories is very simple but not less instructive. If we want to classify the solitons in this theory we consider ϕ^{∞} which take values on $S^0_{\infty} = \{-\infty, \infty\}$. We see then that $\phi^{\infty}(x)$ is a function from two points to the set $\mathcal{Z}$ of zeroes of U. From our earlier discussion we can classify the solutions by pairs of the form

$$(\phi_1, \phi_2) \in \pi_0(\mathcal{Z}) \times \pi_0(\mathcal{Z}) \tag{31}$$

Where $\pi_0(\mathcal{Z}) = \{\phi_-, \phi_+\}$. Note that we have two cases to analyze here. First if $\phi_1 = \phi_2 = \phi_{\pm}$ the field is in the homotopy class of the vacuum configuration $\phi_{\pm}$ that means that we can continuously deform the solution $\phi(x)$ to the constant solution $\phi_\pm$ (in the lump example it would correspond to the case where the lump was in either $(-a,-a)$ or (a,a)). If we had otherwise, that $\phi_1 \neq \phi_2$, than the solution will interpolate between the vacua in $\mathcal{Z}$.

Lets remember our ϕ^4 case and calculate the topological charge for this theory. We will first define the topological current k^μ. It is defined by

$$k^\mu = \left(\frac{1}{2a}\right)\varepsilon^{\mu\nu}\partial_\nu\phi \tag{32}$$

where $\varepsilon^{01} = 1$ and $\varepsilon^{\mu\nu} = -\varepsilon^{\nu\mu}$, and a is the same as in (12). We note that the current thus defined satisfies $\partial_\mu k^\mu = 0$. The topological charge is the integral given by

$$Q = \int_{-\infty}^{\infty} dx k^0 \tag{33}$$

$$Q = \int_{-\infty}^{\infty} dx k^0 = \frac{1}{a}\int_{-\infty}^{\infty} dx(\partial_x\phi(x)) = \frac{1}{2a}[\phi(\infty) - \phi(-\infty)] \tag{34}$$

Calculating it for the the solution in (14) we have then for the lump and anti-lump $Q_{lump} = 1$ and $Q_{anti-lump} = -1$. It is also straightforward to calculate the topological charges for the potential (15).

4.2 Vortices in $d = 2$

For the case in $d = 2$ we turn our attention to the case where the vacuum manifold is $\mathcal{Z} = S^1$ (think for example the Mexican hat potential). For this case the homotopy group is the fundamental group $\pi_1(\mathcal{Z})$, and it is well-known that the $\pi_1(S^1) = \mathbb{Z}$, this means that we can characterize the solitons by integers. Since in two dimension we have

$$\lim_{r\to\infty}\phi(r,\varphi) = \phi^\infty(\varphi) \equiv ae^{ig(\varphi)} \tag{35}$$

where $g(\varphi)$ is a function of the angle φ that determines the value of the solution at the spatial circle at infinity. We call those solutions *vortices* and we compute their conserved charge[6] as

$$Q = \frac{1}{2\pi}\int_0^{2\pi}\frac{dg}{d\varphi}d\varphi = \frac{1}{2\pi}(g(2\pi) - g(0)) \tag{36}$$

4.3 Monopoles in $d = 3$

Since we discussed the cases of $d = 1$ and $d = 2$, we complete by discussing briefly the case $d = 3$. In 3 dimensions the homotopy group of interest is the group $\pi_2(\mathcal{Z})$. To begin our analysis we assume our vacuum manifold ($\mathcal{Z}$) is S^{n-1}. If we have $n \neq 3$ the homotopy group $\pi_2(S^{n-1})$ is trivial and the solution is in the same homotopy class as the vacuum configuration. Now if $d = 3$ we have $\pi_2(S^2) = \mathbb{Z}$, and as before, this means we can classify

[6] Also called winding number because it is related to how many times your field winds around when you complete a circle in position space.

the solitons (now called monopoles) by integer topological labels. The monopole charge is given then

$$N = -\frac{1}{4\pi}\int_{S^2_\infty} \mathrm{Tr}(\phi d\phi \wedge d\phi) \tag{37}$$

5 Energy of vortices and monopoles

Lets analyze the energy of those solutions for spatial dimensions $d \geq 2$. We assume that our field at spatial infinity is topologically non-trivial. This is justified because if the field configuration were constant at spatial infinity, we would find that this field lies in the vacuum homotopy since the homotopy group of this configuration is trivial. Lets begin by studying the vortex energy. It is given by

$$E = \int_{\mathbb{R}^2}\left[\frac{1}{2}|\nabla\phi|^2 + U(\phi)\right] d^2x = \int_0^{2\pi} d\theta \int_0^\infty r dr \left[\frac{1}{2}\partial_r\bar{\phi}\partial_r\phi + \frac{1}{2r^2}\partial_\theta\bar{\phi}\partial_\theta\phi + U(\phi)\right] \tag{38}$$

Since the field is non-trivial at the spatial infinity, the angular derivative does not vanish. From that we see that the angular term is proportional to $1/r$ which is logarithmically divergent. The case is even worse in $d = 3$ where the divergence that appears is linear. This is expected since from Derrick's theorem there is no non-trivial time-independent solution with finite energy for $d \geq 2$.

There is a neat trick to try and fix those divergences in higher dimensions. The idea is to couple the scalar field ϕ to a gauge field A_μ, note that this does not change the topological classification of the solitons. We proceed to change the derivative ∂_μ for the covariant derivative

$$\partial_\mu \mapsto D_\mu = \partial_\mu - iqA_\mu \tag{39}$$

Now we have this new freedom of choice that comes from the gauge field, then if we choose wisely A_θ such that as $r \to \infty$

$$iqA_\theta\phi = \partial_\theta\phi \tag{40}$$

it is easy to see that the divergences in the vortex and the monopole cases, vanishes since the covariant derivative goes to zero no matter how complicated the field ϕ might be. This means that the integral for the vortex

$$\int_0^{2\pi} d\theta \int_0^\infty r dr \left[\frac{1}{2}\overline{D_r\phi}D_r\phi + \frac{1}{2r^2}\overline{D_\theta\phi}D_\theta\phi + U(\phi)\right] \tag{41}$$

become convergent since we can always ask D_r to fall faster than $1/r$. The same analysis is possible to $d = 3$, if we carefully choose the gauge field we can absorb the divergence present in the energy into A_μ and make the energy finite, just as we have done for $d = 2$.

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