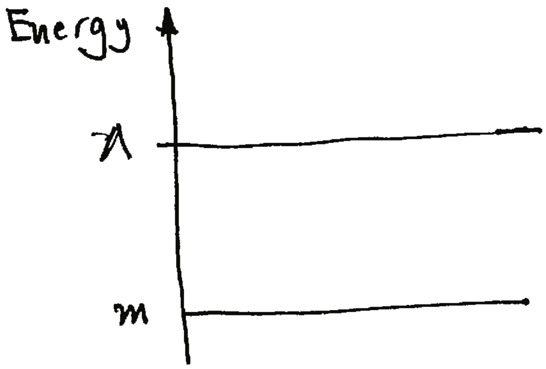
19. Effective Theories

There are many scales in Physics. However, we don't need to know what is going at all scales to understand physics at a particular one. Let's assume that we have 2 scales separated by a mass gap:



An effective field theory (EFT) describe the physics at the scale m taking into account virtual contributions from state at the scale Λ.

- Basic idea: non-analytic parts of scattering amplitudes are due to intermediate processes where physical particles can exist on shell. For instance $\frac{1}{p^2-m^2+i\epsilon}$ with $p^2=m^2$. The contribution of heavy particles can be expressed as powers series of p^2/Λ^2 since they can not be on shell at low energies!

- All our quantum field theories are EFT since they have a finite range of validity!

- In the 70's, Wilson and others changed the way we think about renormalization since we incorporate into the Lagrangian operators whose dimensions are larger than 4.

- Main uses of EFT

  i) Top-down approach: we know the high energy theory by we construct an EFT with the light degrees of freedom to describe its effect at low energy. For example, the SM and the Fermi 4-fermion interaction.

ii) Bottom-up approach: we don't know the high energy theory, so we introduce high dimension operators to probe what might be going on at shorter distances.

19.1 Effective Action from Matching

For simplicity, let's consider a toy model with two real scalar fields l and H with masses m and Λ with $m \ll \Lambda$. We are going to obtain the low energy effective action requiring that the Green's functions of the light fields are the same as the ones in the full theory, order by order in perturbation theory:

$$\langle 0 | T\, llll | 0 \rangle_{l_{eff}} = \langle 0 | T\, llll | 0 \rangle_{l+H} \qquad (1)$$

with $\mathcal{L}_{eff} = \sum_i C_i \mathcal{O}_i$

Wilson coefficient ↑ ↳ operators containing just l fields

Let's consider

$$\mathcal{L}(l,H) = \frac{1}{2}\partial_\mu l\, \partial^\mu l + \frac{1}{2}\partial_\mu H\, \partial^\mu H - \frac{1}{2}m^2 l^2 - \frac{1}{2}\Lambda^2 H^2 - \frac{g_l}{4!}\, l^4 - \frac{g_h}{4!}\, H^4$$

$$- \frac{g_{lH}}{4}\, l^2 H^2 - \frac{\tilde{m}}{2}\, l^2 H - \frac{\tilde{g}_h \Lambda}{3!}\, H^3. \qquad (2)$$

the scattering amplitude at tree level for $ll \to ll$ is given by

$$M(p_1\, p_2\, p_3\, p_4) = -g_\ell\, \tilde{m}^2\left[\frac{1}{(p_1-p_3)^2-M^2} + \frac{1}{(p_1-p_4)^2-M^2} + \frac{1}{(p_1+p_2)^2-M^2}\right] \quad (3)$$

At low energies, $(p_1-p_3)^2 \ll \Lambda^2$ $\quad (p_1-p_4)^2 \ll \Lambda^2$ $\quad (p_1+p_2)^2 \ll M^2$ we have that

$$M = -g_\ell + \frac{3\tilde{m}^2}{\Lambda^2} + \frac{\tilde{m}^2}{\Lambda^4}\underbrace{\left[(p_1-p_3)^2 + (p_1-p_4)^2 + (p_1+p_2)^2\right]}_{4m^2} + \mathcal{O}(\Lambda^{-6})$$

So we can describe this scattering with

$$\mathcal{L}_{eff} = \frac{1}{2}\partial_\mu \ell\, \partial^\mu \ell + \frac{1}{2} m^2 \ell^2 - \frac{\ell^4}{4!}\left(g_\ell - \frac{3\tilde{m}^2}{\Lambda^2} - \frac{4\tilde{m}^2 m^2}{\Lambda^4}\right) \quad (4) ✓$$

to order $\mathcal{O}(\Lambda^{-4})$. Analogously, the scattering $\ell\ell \to \ell\ell\ell\ell$ leads to a contribution to $\mathcal{L}_{eff}$:

$$-\frac{\tilde{m}^2}{\Lambda^4}\left(\frac{g_{eh}}{16} - \frac{g_\ell}{36}\right)\ell^6 \quad (5) ✓$$

36? it must be! And it is!

Obs.: This is not a matching using just 1LPI diagrams ??

19.2 General Formulation

A few comments:

i) To describe physics at scale E with accuracy ϵ we need just a ~~[illegible]~~ finite set of parameters.

ii) At each order in $1/\Lambda^k$ there are a finite number of operator with dimension $k-4$.

iii) The contribution to a process of an operator of dimension $k-4$ at energy E is, by dimensional analysis,

$$(E/\Lambda)^k$$

iv) To obtain an accuracy $\epsilon_k \Rightarrow$ we need ~~operators~~ to including to k_ϵ: $\left(\frac{E}{M}\right)^{k_\epsilon} \approx \epsilon \Rightarrow k_\epsilon \cong \frac{\ln(1/\epsilon)}{\ln(M/E)}$ (finite)

Formally, in the case of two real scalar discussed above

$$\int \mathcal{D}\ell \; e^{iS_{eff}} = \int \mathcal{D}\ell\,\mathcal{D}H \; e^{i\,S(\ell,H)} \qquad (6)$$

with $S_{eff} = \int d^4x \; \mathcal{L}_{eff}$.

H particles are not produced

Comments: i) S_{eff} contains the exact same physics at low energy than $S(\ell, H)$.

ii) We do not use a wilsonian approach that integrate the high frequency modes of the light field.

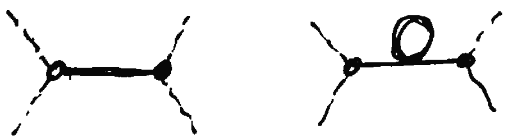
iii) To regularize the calculations we use dimensional regularization (DR). This is followed by minimal substraction (MS).

We can obtain the effective action for ℓ by the standard procedure:

$$e^{iW[j]} = \int \mathcal{D}\ell \; e^{iS_{eff} + i\int d^4x\, j\ell} = \int \mathcal{D}\ell\,\mathcal{D}H \; e^{i\,S(\ell,H) + \int d^4x\, j\ell} \qquad (7)$$

$$\ell_c(x) = \frac{\delta W[j]}{\delta j(x)} \quad \text{and} \quad \Gamma[\ell_c] = W[j] - \int d^4x \; j(x)\,\ell_c(x) \qquad (8)$$

Notice that the procedure (8) eliminates the 1-particle-reducible diagrams when we cut a ℓ line. However, due to the absence of a source for H also generates 1PI reducible diagrams with respect to cutting internal H lines! The graphs generated by $\Gamma[\ell_c]$ are called one-light-particle-irreducible (1LPI) diagrams. For instance, in our two-scalars model the following graphs are (1LPI):



Let's work (6) up to one-loop level. We can use the saddle-point method to perform the DH integration:

$$\left.\frac{\delta S[\ell,H]}{\delta H}\right|_{H_c} = 0 \Rightarrow H_c[\ell] \qquad (9)$$

↳ considered a background field.

sol'n to the EOM

Then, we expand

$$S[\ell, H_c+\eta] = S[\ell,H_c] + \frac{1}{2}\int dx\,dy \left.\frac{\delta^2 S}{\delta H^2}\right|_{H_c} \eta(x)\eta(y) + \mathcal{O}(\eta^3) \qquad (10)$$

Obtaining

$$e^{iS_{eff}} \simeq \int DH\, e^{iS[\ell,H_c]}\, e^{i\int dx\,dy\, \eta(y) \left.\frac{\delta^2 S}{\delta H^2}\right|_{H_c} \eta(x)} = e^{iS[\ell,H_c]} \left[\det\left(-\left.\frac{\delta^2 S}{\delta H^2}\right|_{H_c}\right)\right]^{-1/2} \quad (11)$$

tree level

1-loop level

$$\Rightarrow S_{eff}[\ell] \simeq S[\ell,H_c] + \frac{i}{2}\,\mathrm{Tr}\left[\log\left(-\left.\frac{\delta^2 S}{\delta H^2}\right|_{H_c}\right)\right] \quad (12).$$

For the model in Eq. (2), (9) leads to

$$\Box H_c + \Lambda^2 H_c + \frac{g_{eH}}{2}\ell^2 H_c + \frac{\tilde{m}\,\ell^2}{2} + \mathcal{O}(H_c^2) = 0 \qquad (13)$$

$$\Rightarrow H_c = \frac{1}{\left(-\partial_\mu\partial^\mu - \Lambda^2 - \frac{g_{eH}\ell^2}{2}\right)}\left(\frac{\tilde{m}\,\ell^2}{2} + \mathcal{O}(H_c^2)\right) \qquad (14)$$

In order to have an approximation that is manageable we expand in powers of Λ, as suggested by the initial comments of section 19.2:

$$H_c = -\frac{1}{\Lambda^2}\left(\frac{\tilde{m}\,\ell^2}{2}\right) + \frac{1}{\Lambda^4}\left(+\partial_\mu\partial^\mu + \frac{g_{eH}\ell^2}{2}\right)\frac{\tilde{m}\,\ell^2}{2} + \ldots\ldots \qquad (15)$$

Substituting (15) into (2) we obtain:

$$\mathcal{O}\left(\frac{1}{\Lambda^0}\right) \Rightarrow \mathcal{L}^{(0)}_{eff} = \frac{1}{2}\partial_\mu \ell\, \partial^\mu \ell - \frac{m^2}{2}\ell^2 - \frac{g_e}{4!}\ell^4 \qquad (16.a) ✓$$

$$\mathcal{O}\left(\frac{1}{\Lambda^2}\right) \Rightarrow \mathcal{L}^{(2)}_{eff} = -\frac{\tilde{m}}{4}\ell^2 H_c = -\frac{1}{\Lambda^2}\frac{\tilde{m}^2\ell^2}{8} ✓ \qquad (16.b) ✓$$

$$\cancel{\mathcal{O}\left(\frac{1}{\Lambda^4}\right) \Rightarrow \mathcal{L}^{(4)}_{eff} = -\frac{\tilde{m}\ell^2}{2}\frac{1}{\Lambda^2}\left(-\partial_\mu\partial^\mu - \frac{g_{eH}\ell^2}{2}\right)\frac{\tilde{m}\ell^2}{2}}$$

$$\mathcal{O}\left(\frac{1}{\Lambda^4}\right) \Rightarrow \mathcal{L}^{(4)}_{eff} = -\frac{\tilde{m}\ell^2}{4}\frac{1}{\Lambda^4}\left(+\partial_\mu\partial^\mu + \frac{g_{eH}\ell^2}{2}\right)\frac{\tilde{m}\ell^2}{2}$$

$$= -\frac{\tilde{m}^2}{8\Lambda^4}\ell^2\partial_\mu\partial^\mu\ell^2 - \frac{g_{eH}\tilde{m}^2}{16\Lambda^4}\ell^6 ✓$$

$$= \frac{\tilde{m}^2}{2\Lambda^4}\ell^2\partial_\mu\ell\,\partial^\mu\ell - \frac{g_{eH}\tilde{m}^2}{16\Lambda^4}\ell^6. \qquad (16.c) ✓$$

Comments:

i) Notice that the induced operators vanish for $M \to \infty$. When this happens we say that there is decoupling. This depends on the structure of the couplings. ⇒ see page 19.6a

ii) The operator in (14)

$$\frac{1}{-\partial_\mu\partial^\mu - \Lambda^2} \text{ is non-local: } \langle x|\frac{1}{-\partial_\mu\partial^\mu-\Lambda^2}|x'\rangle = \int\frac{d^4p}{(2\pi)^4}\frac{e^{ip\cdot(x'-x)}}{p^2-\Lambda^2} = G \qquad (17.a)$$

but after the large M expansions, the terms in the series are local

$$G = \left(\frac{1}{M^2} - \frac{\Box}{M^4} + \dots\right)\delta(x-x') \qquad (17.b)$$

This is a consequence of the uncertainty principle!

Decoupling Theorem (Appelquist and Carrezone 1995)

If the remaining low energy theory is renormalizable, then all effects of the heavy particle appear either as a renormalization of the coupling constants in the theory or else are suppressed by powers of the heavy particle mass.

iii) Again, at low energy H can not be on shell. $\boxed{19.7}$

Therefore, there is no non-analicity associated to it so we can expand in an $1/H$ power series!

19.3 Redundant Interactions

Notice that the effective action obtained above by two different ethods are apparently different; compare (9.5) to (16a-c)!

Theorem: (Haag 1958; Coleman; Wess and Zumino 69, Callan, Coleman, Wess and Zumino 69)

If two fields are related non-linearly $\varphi = \chi F(\chi)$ with $F(0) = 1$ then the same experimental observables results if one calculates with the field φ and $\mathcal{L}(\varphi)$ or instead with χ and $\mathcal{L}(\chi F(\chi))$.

Justification: $F(0)=1 \Rightarrow$ the free fields χ and φ have same mass and charge, so experiments can not distinguish isolated χ from isolated φ. When the scattering experiment is performed, the results can not depend whether calculations are done with $\mathcal{L}(\varphi)$ or $\mathcal{L}(\chi F(\chi))$!

From this theorem it follows that if two $\mathcal{L}_{eff}$ differ by he use of EOM, then they lead to the same physical results! To see this, let's write the effective action as

$$S_{eff}[\Phi] = S_0[\Phi] + \varepsilon S_1[\Phi] + \varepsilon^2 S_2[\Phi] + \cdots \tag{18}$$

where ε is the expansion parameter ($\frac{1}{\varepsilon}$, number of loops..)

Now let's assume that there is a term that vanishes when we use the EOM, i e,

$$S_u^R[\phi] = \int d^4_x\, f(x) \frac{\delta S_0}{\delta \phi(x)} \qquad (19)$$

function of the fields and its derivatives

This term can be removed to order ε^n if we perform the field redefinition

$$\phi(x) \longrightarrow \tilde{\phi}(x) = \phi(x) + \varepsilon^n f(x) \qquad (20)$$

To order ε^n the only term that changes is S_0

$$S_0[\phi] = S_0[\tilde{\phi} - \varepsilon^n f(x)] = S_0[\tilde{\phi}] - \varepsilon^n \int d^4_x\, f(x) \frac{\delta S_0}{\delta \phi(x)} + O(\varepsilon^{n+1})$$

Therefore,

$$S[\phi] \to S[\tilde{\phi}] = S[\phi] - \varepsilon^n \int d^4_x\, f(x) \frac{\delta S_0}{\delta \phi} \qquad (21)$$

$\Rightarrow$ the only effect is to cancel (19)! The above theorem guarantees that the scattering amplitudes are the same at this order!

In the case of our example; (16.c) contains,

$$\int d^4_x\, \ell^2 \partial_\mu \ell \partial^\mu \ell = -\frac{1}{3}\int d^4_x \ell^3 \Box \ell \overset{\text{EOM}}{=} -\frac{1}{3}\int d^4_x\, \ell^3\left(-m^2 \ell - \frac{g_\ell}{3!}\ell^3\right)$$

$$= -\frac{1}{3}\int d^4_x\left(-m^2 \ell^4 - \frac{g_\ell}{3!}\ell^6\right)$$

with this change to (16.c) we obtain

$$(16.6) = -\frac{g_{eH}\tilde{m}^2}{16\Lambda^4}\ell^6 + \frac{\tilde{m}^2}{2\Lambda^4}\left(\frac{m^2\ell^4}{3} + \frac{g_\ell\,\ell^6}{18}\right)$$

$$= -\frac{g_{\ell H}\tilde{m}^2\ell^6}{16\Lambda^4} + \frac{m^2\tilde{m}^2}{\Lambda^4\,6}\ell^4 + \frac{g_\ell\tilde{m}^2}{\Lambda^4}\frac{\ell^6}{36}$$

19.4 Matching & running

In the ~~bottom~~ top-down approach we follow the procedure:

i) Match at scale Λ:
   - integration or equal Green's functions
   - followed by a Taylor expansion

ii) Renormalization group running of the Wilson coefficients of the effective operators from the scale Λ to the m one.

iii) evaluation of the physical observables at the scale m.

19.5 Power Counting

26/11/19 19.10

We need an estimate of the power of ε ($=1/M$) in a given process to identify which terms in $\mathcal{L}_{eff}$ contribute to ~~a given~~ i~~...~~ To make the discussion general we ~~...~~ consider

$$\mathcal{L}_{eff} = f^4 \sum_k \frac{c_k}{M^{d_k}} \mathcal{O}_k\left(\frac{\varphi}{v}\right) \qquad (22)$$

where we considered a scalar field φ. It's trivial to extend to other fields. Explanation:

- $f, M, v \longrightarrow$ dimension of mass
- k runs over the effective operators
- $\mathcal{O}_k$ has mass dimension (mass)^{d_k}
- ϕ/v dimensionless $\Rightarrow d_k$ counts the number of derivatives in $\mathcal{O}_k$!
- Here it's assumed that c_k is dimensionless. If we have to keep track of the powers of $\wedge$ associated to it.

Let's estimate $M(q)$ of a diagram with

- E external lines with momenta q
- I internal lines
- V_{ik} vertices $\Rightarrow$ $i \equiv$ number of lines in k
  $k \equiv$ power of momentum in the vertex

We know that:

number of lines in the vertices $\quad \sum_{ik} i V_{ik} = 2I + E \qquad (23)$

14.11

known number of loops

$$L = 1 + I - \sum_{jk} V_{jk} \qquad (24)$$

The vertices contribute with

$$\prod_{jk}\left[\left(\frac{p}{M}\right)^{k}\left(\frac{f^4}{\vartheta^j}\right)\; i\,(2\pi)^4\delta^4(p)\right]^{V_{jk}} \qquad (25)$$

Internal lines lead to

$$\left[\int\frac{d^4p}{(2\pi)^4}\left(\frac{M^2\vartheta^2}{f^4}\right)\frac{i}{p^2-m^2}\right]^{I}$$

↳ due to the propagator being the inverse of ~~...~~ the quadratic term

The whole graph amplitude is

$$(2\pi)^4\,\delta(q)\;M(q) \qquad (26)$$

The only missing element is an estimate of integrals like

$$\int\cdots\int\left(\frac{d^4p}{(2\pi)^4}\right)^{A}\frac{p^{B}}{(p^2-m^2)^{C}} \sim \left(\frac{1}{4\pi}\right)^{2A} m^{nA+B-2C} \qquad (27)$$

with $n\to 4$. If $m=0$ or $q \gg m$ we exchange ~~...~~ $m\to q$ in (27)

Finally, $M(q)$ contains an integral

$$\int\cdots\int\left(\frac{d^4p}{(2\pi)^4}\right)^{L}\frac{p^{\sum_{jk}kV_{jk}}}{(p^2-q^2)^{I}} \sim \left(\frac{1}{4\pi}\right)^{2L} q^{4L-2I+\sum_{jk}kV_{jk}} \qquad (28)$$

Now, (23-28) lead to

$$M(q) \sim f^4 \left(\frac{1}{\theta}\right)^E \left(\frac{Mq}{4\pi f^2}\right)^{2L} \left(\frac{q}{M}\right)^{2+\sum_k (k-2) V_{ik}} \tag{29}$$

Notice that:

- large number of loops suppresses the contribution
- large number of vertices also suppress the contribution

~~Example: The model defined by eq'n (2) ... in [4] and [5]~~

How to use effective lagrangians:

0. Sequence 19.12A
1. Choose accuracy that $M(q)$ is to be computed.
2. ~~Determine the order of which~~ Use the power couting eq'n (29) to determine which terms of the effective lagrangian contribute to the process given $\frac{q}{\Lambda}$ or $\frac{m}{\Lambda}$.
3. For top-down approach, compute the wilson coefficients of the required operators. If this is impossible follow the recipe for the bottom-up approach. ~~This is complemented with~~ use section 19.4

   For bottom-up approach, treat the wilson coefficients of the required operators as unknown parameters, which must be determined from data!

Important comments: 4

1. the bottom-up approach broadens the utility of EFT since we do not need to know the underlying theory.

0. The first thing that we need is a list of the light degrees of freedom and the symmetries that the EFT obeys! Then, we can construct the effective operators used in the EFT.

The symmetries may be realized:

i) linearly, like we did ~~xxxx~~ for gauge theories:

$M_R \to$ multiplet $\Longrightarrow$ $M_R' = U_R M_R$

where U_R is an unitary matrix representation of the group.

ii) non-linearly, sometimes a member of a multiplet M is much heavier and we integrate it out! In this case the EFT is still symmetric but we need to use a non-linear representation of it. See Luigi's (chiral models) talk!

2. Due to (29), there are a finite number of terms in $\mathcal{L}_{eff}$. This restrict the number of loops with divergences which can to a given precision. This restrict the number of loops with divergences which can be absorbed by renormalizing the ~~parameters~~ Wilson coefficients of the operators being considered!

Renormalizable theories are a special case for which it's enough to renormalize the order zero Lagrangian!

3. Non-renormalizable theories have an infinite number of parameters. However, for a given accuracy we need just a finite subset of them, making the EFT predictive!

19.6 Application to low energy QED

Let's consider QED only with electrons

$$\mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\not{\partial} - m_e)\psi \qquad (30)$$

and focus on energies $E \ll m_e$. In this case, it is enough to consider macroscopic external currents J^μ_{em} such that $\partial_\mu J^\mu_{em} = 0$.

In a bottom-up approach, we include the effective interaction

$$\mathcal{L}_J = -eA_\mu J^\mu_{em} \qquad (31)$$

The most general effective theory containing just photons (and J^μ_{em}) i

$$\mathcal{L}_{eff} = \mathcal{L}_4 + \mathcal{L}_6 + \mathcal{L}_8 + \dots - eA_\mu J^\mu_{em} \qquad (32)$$

with

$$\mathcal{L}_4 = -\frac{Z}{4} F_{\mu\nu} F^{\mu\nu} \qquad (33)$$

$$\mathcal{L}_6 = \frac{a}{m_e^2} F_{\mu\nu} \Box F^{\mu\nu} + \frac{a'}{m_e^2} \partial_\nu F^{\mu\nu} \partial^\lambda F_{\lambda\nu} \qquad (34)$$

$$\mathcal{L}_8 = \frac{b}{m_e^4}\left(F_{\mu\nu}F^{\mu\nu}\right)^2 + \frac{c}{m_e^4}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)^2 + \left(\partial^4 F^2 \text{ terms}\right) \qquad (35)$$

↳ include 3em using EOM!

where we imposed the symmetries of (30), i.e., P and C. Notice that C forbids terms containing just an odd number of A_μ's. $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$. In this case the free parameters are

$Z, a, a', b, c, \ldots$.

Using the EOM at lowest order, (34) reads

$$\mathcal{L}_6 = \frac{e^2}{m_e^2}(a' - 2a)\, J^\mu_{em} J_{em\,\nu}$$

so it's not relevant for photon propagation and scattering!

Let's focus on light by light scattering. We start by applying (29) with $f = M = v = m_e$, obtaining

$$M(q) \sim q^2 m_e^2 \left(\frac{1}{m_e}\right)^E \left(\frac{q}{4\pi m_e}\right)^{2L} \left(\frac{q}{m_e}\right)^{\sum_{ik}(k-2)V_{ik}} \qquad (36)$$

Notice that gauge invariance requires using $F_{\mu\nu}$, so A_μ is always accompanied by a derivative, so

$$V_{ik} = 0 \quad \text{if} \quad i > k.$$

For $k = 2 \Rightarrow i \leq 2 \Rightarrow$ the only term is $F_{\mu\nu}F^{\mu\nu}$ that is not an interactio[n]

So

$$V_{ik} = 0 \quad \text{for } k \leq 2$$

Therefore, the new interactions satisfy

$$\sum_{ik} (k-2) V_{ik} \geq 2$$

This sum is 2 only for $V_{ik}=0$ for all $k>4$ and $V_{i4}=1$.

The leading term comes from

$$L=0 \qquad V_{ik}=0 \text{ for } k>4 \quad \text{and} \quad V_{i4}=1 \qquad (37)$$

being of $\mathcal{O}\left(\frac{q^4}{m_e^4}\right)$ and containing b and c.

This leads to

prediction

$$\frac{d\sigma_{\gamma\gamma\to\gamma\gamma}}{d\Omega} = \frac{278}{65\pi^2}\left[(b+c)^2+(b-c)^2\right]\left(\frac{E_{cm}^6}{m_e^8}\right)(3+\cos^2\theta)^2 \qquad (38)$$

Corrections to this result come from

$L=1$ and $\sum_{ik}(k-2)V_{ik}=2$ $\Longrightarrow$ not possible!

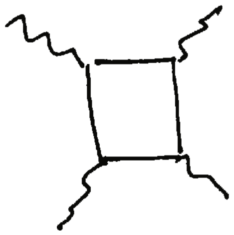
or

$L=0$ and $\sum_{ik}(k-2)V_{i4}=4$ $\Longrightarrow$ not possible since the interactions satisfy $k\geq i\geq 4$

So the first corrections to M are of $\mathcal{O}\left(\frac{q^4}{m_e^4}\right) \Longrightarrow$

the corrections to (37) are $\mathcal{O}\left(\frac{E_{cm}^4}{m_e^4}\right)$ (the interference term with the leading order one)

Had we used a top-down approach evaluating



+ crossed diagrams $\Rightarrow$ $b = \frac{4}{7} c = \frac{\alpha^2}{90}$

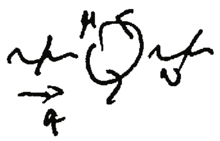
19.7 Largeology or RGE or decoupling!

The RGE is an efficient way to deal with potentially large logarithmics, as we saw previously. In QED with one fermion, at one-loop level,

$$\beta_{QED}^{(\alpha)} = \frac{2\alpha^2}{3\pi} \Rightarrow \bar{\alpha}(t) = \frac{\alpha_R}{1 - \frac{2\alpha_R}{3\pi} t} \quad \text{with } t = -\ln\frac{\mu}{\mu_0} \qquad (39)$$

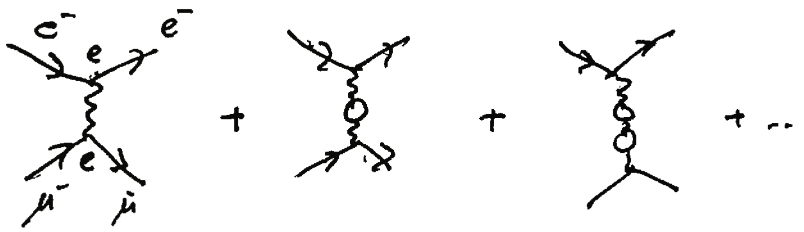
this result was obtained using the MS subtraction.

We also evaluated the vacuum polarization

$i\Pi^{\mu\nu} =$  $= i\Pi(q^2)\left[q^2 g^{\mu\nu} - q^\mu q^\nu\right]$

gauge invariance

By summing up the series



we obtained that

$$e_{eff}^2(q) = \frac{e^2}{1 - \Pi(q^2)}$$

In the On-mass-shell scheme we know, and just one charged fermion

$$\Pi^{ON}(q^2) \xrightarrow[-q^2/m^2 \to \infty]{} \frac{\alpha}{\pi}\left[\frac{1}{3}\ln\frac{|q^2|}{m^2} - \frac{5}{9} + \mathcal{O}\left(\frac{m^2}{q^2}\right)\right] \quad (40)$$

and

$$\Pi^{ON}(q^2) \xrightarrow[q^2 \to 0]{} -\frac{\alpha}{\pi}\frac{q^2}{m^2}\frac{1}{15} + \mathcal{O}\left(\frac{q^4}{m^4}\right) \ldots \quad (41)$$

Comparing (39) with (40) and (41), clearly the RGE describes the correctly the limit $|q^2| \gg m^2$. However, it fails for $|q|^2 \ll M^2$!

The way to reconcile these results is to consider QED as an effective field theory. ~~Above~~ For energy scales above $\underline{m}$ we keep the fermion and use (30). Below m, we integrate out the fermion which leads to $\beta_{QED}=0$. ~~the~~ The effect of (41) is kept as ~~an~~ ~~effect~~ through the first operator in (34)! This procedure ~~is~~ is known as decoupling of the fermion!

To further illustrate the point, let's consider 2 fermions, i.e., electron and muon. For μ^2 larger than M_μ

$$\beta = 2 \times \frac{2\alpha^2}{3\pi} \quad \text{and} \quad \mu\frac{d\bar{\alpha}}{d\mu} = \frac{4\bar{\alpha}^2}{3\pi} \quad (42)$$

For $m_e < \mu < m_\mu$

$$\beta = \frac{2\alpha^2}{3\pi^2} \quad \text{and} \quad \mu\frac{d\bar{\alpha}}{d\mu} = \frac{2\bar{\alpha}^2}{3\pi} \quad (43)$$

Finally, for $\mu < m_e$

$$\beta = 0 \quad \text{and} \quad \mu \frac{d\bar{\alpha}}{d\mu} = 0 \qquad (44)$$

As boundary condition, it is required that $\bar{\alpha}$ should be continuous at $\mu = m_\mu$ and $\mu = m_e$. Note that for $\mu < m_e$ $\bar{\alpha}$ does not run and it is equal to the physical value α_{phys}.

References: – C.P. Borgess, Anno. Rev. Nucl. Part. Sci (2007), 57, 32

– A. Pich, hep-ph/9806303

– H. Georgi, Anno. Rev. Nucl. Part. Sci, volume 43

– For details on how to integrate the heavy fields, see Schwartz chapter 33.