18. Anomalies

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- In the classical theory, Noether's theorem ~~implies~~ states that a symmetry of the Lagrangian implies that there is a conserved quantity

- At the quantum level it's not ~~guaranteed~~ certained that there will be a conserved charge due to:

  i) quantization procedure

  ii) the need to deal with potentially divergent quantities.

- An anomaly is the failure of a classical symmetry to survive the process of quantization and regularization.

18.1 Linearly divergent integrals

For a linearly divergent integral we must be carefull with change of variables

Let's consider

$$\Delta(a) = \int_{-\infty}^{+\infty} dx \left(f(x+a) - f(x) \right) \qquad (1)$$

with f ~~[illegible]~~ such that $f(\pm\infty) =$ constants due to ~~the~~ the linear divergence of the integral.

If we are naive $x \to x-a$ in the first term leads to $\Delta = 0$. WRONG!

~~Exp~~ Taylor expanding $f(x+a)$:

$$\Delta(a) = \int_{-\infty}^{+\infty} dx \left[\cancel{f(x)} + a f'(x) + \frac{a^2}{2} f''(x) + \ldots - \cancel{f(x)} \right] = a\left[f(+\infty) - f(-\infty) \right] \qquad (2)$$

So $\Delta(a)$ has a linear dependence on the shift a!

In four dimension

$$\Delta^{\alpha}(a^{\mu}) = \int \frac{d^4k}{(2\pi)^4}\left[F^{\alpha}(k+a) - F^{\alpha}(k)\right] \quad (3)$$

with $\int d^4k\, F^{\alpha}(k)$ being linearly divergent. First we go to euclidean space

$$\Delta^{\alpha}(a^{\mu}) = i\int \frac{d^4k_E}{(2\pi)^4}\left(F^{\alpha}(k_E+a) - F^{\alpha}(k_E)\right) \quad (4)$$

and then we Taylor expand:

$$\Delta^{\alpha}(a^{\mu}) = i\int \frac{d^4k_E}{(2\pi)^4}\left[a^{\mu}\frac{\partial}{\partial k_E^{\mu}}(F^{\alpha}(k_E) + \frac{1}{2}a^{\mu}a^{\nu}\frac{\partial}{\partial k_E^{\mu}}\frac{\partial}{\partial k_E^{\nu}}F^{\alpha}(k_E) + \dots\right]$$

Since the integral is linearly divergent

$$F^{\alpha}(k_E) \xrightarrow[k_E\to\infty]{} A\frac{k_E^{\alpha}}{k_E^4} \quad (5)$$

so

$$\Delta^{\alpha}(a^{\mu}) = i a^{\mu}\int \frac{d^4k_E}{(2\pi)^4}\frac{\partial}{\partial k_E^{\mu}}(F^{\alpha}(k_E)) \overset{\text{stokes}}{=} i a^{\mu}\int \frac{d^3S_{\mu}}{(2\pi)^4}F^{\alpha}(k_E).$$

But $d^3S_{\mu} = k^2 k_{\mu}\, d\Omega_4 \Rightarrow \Delta^{\alpha}(a^{\mu}) = i a^{\mu} \lim_{k_E\to\infty}\int \frac{d\Omega_4}{(2\pi)^4} A \frac{k_E^{\alpha}k_{\mu}}{k^2}$

with $\Rightarrow k^{\alpha}k^{\mu} \to \frac{1}{4}k^2\delta^{\mu\alpha}$ by symmetry and $\Omega_4 = 2\pi^2$

$$\Rightarrow \Delta^{\alpha}(a^{\mu}) = \frac{i}{32\pi^2}Aa^{\alpha} \quad (6)$$

18.2 Symmetries of QED

Let's write

$$\mathcal{L} = \bar{\Psi}(i\partial\!\!\!/ - eA\!\!\!/ - m)\Psi = \bar{\Psi}_L(i\partial\!\!\!/ - A\!\!\!/)\Psi_L + \bar{\Psi}_R(i\partial\!\!\!/ - A\!\!\!/)\Psi_R - m\bar{\Psi}_L\Psi_R - m\bar{\Psi}_R\Psi_L \quad (7)$$

that has 2 global symmetries for $m\to 0$:

$$\psi \to \psi' = e^{i\alpha}\psi \quad \text{and} \quad \psi \to \psi' = e^{i\beta\gamma_5}\psi \quad (8)$$

The associated Noether currents are

$$J^\mu = \bar\Psi \gamma^\mu \Psi \qquad\qquad J_5^\mu = \bar\Psi \gamma^\mu \gamma^5 \Psi \tag{9}$$

Vector current (V^μ) Axial current (A^μ)

$$\text{The EOM} \Rightarrow \partial_\mu J^\mu = 0 \quad \text{and} \quad \partial_\mu J^{\mu 5} = 2 i m \bar\Psi \gamma^5 \Psi \tag{10}$$

We'll prove that after quantization and regularization

$$\partial_\mu J_5^\mu = 2 i m \bar\Psi \gamma^5 \Psi - \frac{e^2}{16\pi^2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \;!!! \tag{11}$$

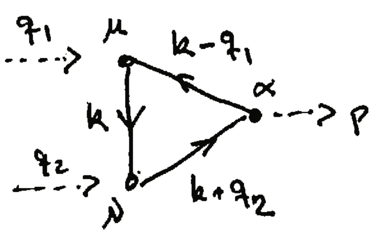
18.3 The anomaly

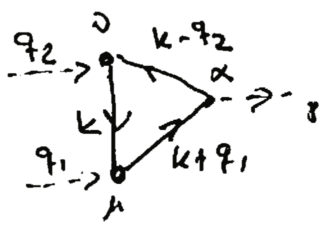
To simplify let's consider $m=0$ and evaluate the correlation function

$$M_5^{\alpha\mu\nu}(p,q_1,q_2)\,(2\pi)^4\,\delta(p-q_1-q_2) \equiv$$

$$\equiv \int d^4x\, d^4y\, d^4z \; e^{-ipx}\, e^{iq_1 y}\, e^{iq_2 z} \langle 0 | T J_5^\alpha(x)\, J^\mu(y)\, J^\nu(z) | 0 \rangle \tag{12}$$

that can be obtained from the diagrams





that lead to

$$M_5^{\alpha\mu\nu} = -\int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\left[\gamma^\mu \frac{i}{\not k} \gamma^\nu \frac{i}{\not k + \not q_2} \gamma^\alpha \gamma^5 \frac{i}{\not k - \not q_1} + \gamma^\nu \frac{i}{\not k} \gamma^\mu \frac{i}{\not k + \not q_1} \gamma^\alpha \gamma^5 \frac{i}{\not k - \not q_2} \right] \tag{13}$$

We must evaluate (13) such that

$$q_{1\mu} M_5^{\alpha\mu\nu} = q_{2\nu} M_5^{\alpha\mu\nu} = 0 \qquad (19)$$

to have ~~due to~~ $\partial_\mu J^\mu = 0$! Then, we can verify the result for $P_\alpha M_5^{\alpha\mu\nu}$!

Now,

$$q_{1\mu}^{\circ} M_5^{\alpha\mu\nu} = \int \frac{d^4k}{(2\pi)^4} \left[\frac{Tr[\not q_1 \not k \gamma^\nu (\not k + \not q_2) \gamma^\alpha \gamma^5 (\not k - \not q_1)]}{k^2 (k+q_2)^2 (k-q_1)^2} + \frac{Tr[\gamma^\nu \not k \not q_1 (\not k + \not q_1) \gamma^\alpha \gamma^5 (\not k - \not q_2)]}{k^2 (k+q_1)^2 (k-q_2^2)} \right.$$

Now $\not q_1 = \not k - (\not k - \not q_1)$ $\qquad$ $\not q_1 = (\not k + \not q_1) - \not k$

After evaluating the traces we are left with

$$q_{1\mu} M_5^{\alpha\mu\nu} = -4i\epsilon^{\alpha\nu\rho\sigma} \int \frac{d^4k}{(2\pi)^4} \left[\frac{(k-q_1)^\rho (k+q_2)^\sigma}{(k-q_1)^2 (k+q_2)^2} - \frac{(k-q_2)^\rho (k+q_1)^\sigma}{(k-q_2)^2 (k+q_1)^2} \right] (1$$

Notice: - the term quadratically divergent vanishes since $\epsilon^{\alpha\nu\rho\sigma} k_\rho k_\sigma \equiv 0$

- the integrals in (15) are linearly divergent.

~~- both integrals are equal if we change~~

~~$k^\mu \to k^\mu + a q_1^\mu + b q_2^\mu$ in the first integral~~

~~and~~ ~~$k^\mu \to k^\mu + b q_1^\mu + b q_2^\mu$ in the second integral~~

~~This shifting must be checked when evaluating $P_\alpha M_5^{\alpha\mu\nu}$!~~

~~Now using (8),~~

- Notice that both integrals in (15) can be obtained from

$$\int \frac{d^4k}{(2\pi)^4} \; \frac{k^\rho (k+q_1+q_2)^\sigma}{k^2 (k+q_1+q_2)^2} \tag{16}$$

with the changes

$$\begin{aligned} k &\longrightarrow k - q_1 \quad \text{for the 1st integral} \\ k &\longrightarrow k - q_2 \quad \text{for the 2nd integral.} \end{aligned} \tag{17}$$

Now using (6), (15), (16), and (17) we obtain

$$q_{1\mu} M_5^{\alpha\mu\nu} = \frac{i}{32\pi^2} (-4i\, \epsilon^{\alpha\nu\rho\sigma}) \left\{ -q_1^\rho (q_1+q_2)^\sigma - (-q_2^\rho)(q_1+q_2)^\sigma \right\}$$

$$= \frac{1}{8\pi^2} \epsilon^{\alpha\nu\rho\sigma} \left(-q_1^\rho q_2^\sigma + q_2^\rho q_1^\sigma \right) = -\frac{1}{4\pi^2} \epsilon^{\alpha\nu\rho\sigma} q_{1\rho} q_{2\sigma} \tag{18}$$

$$\neq 0 \;!!!$$

So it appears that the J^μ current is not conserved!! However, we do know that linearly divergent integrals are ill define! So, we can make a shift in k in (15) ~~such that~~ to force that $\partial_\mu J^\mu = 0$ at the 1-loop level. Moreover, after making this change for (15) we must use it to evaluate $p_\alpha M_5^{\alpha\mu\nu}$.

We make the change

$$k \to k + b_1 q_1 + b_2 q_2 \quad \text{in the } 1^{st} \text{ term of } (15)$$

$$k \to k + b_2 q_1 + b_1 q_2 \quad \text{" " } 2^{nd} \text{ " " "}$$

that corresponds to

2nd

$$\boxed{\begin{aligned} k &\to k - q_1 + b_1 q_1 + b_2 q_2 \\ k &\to k - q_2 + b_2 q_1 + b_1 q_2 \end{aligned}} \quad \text{in } (16) \qquad (19)$$

Now (06) + (15) + (16) + (19) $\Longrightarrow$

$$q_{\circ\mu} M_5^{\alpha\mu\nu} = \frac{i}{32\pi^2}\left(-4i\,\epsilon^{\alpha\nu\rho\sigma}\right)\Big[(q_1+q_2)^\sigma(-q_1+b_1q_1+b_2q_2)^\rho - (q_1+q_2)^\sigma(-q_2+b_1q_{\ldots} + b_2 q_{\ldots}$$

$$= \frac{1}{8\pi^2}\epsilon^{\alpha\nu\rho\sigma}\Big\{ q_1^\sigma q_2^\rho b_2 + q_2^\sigma q_1^\rho (b_1-1) - q_1^\sigma q_2^\rho (b_1-1) - q_2^\sigma q_1^\rho b_2$$

$$= \frac{1}{4\pi^2}\epsilon^{\alpha\nu\rho\sigma} q_1^\sigma q_2^\rho (b_2 - b_1 + 1) \qquad (20)$$

So we must have $b_1 - b_2 = 1$ (21)

Now let's evaluate $P_\alpha M_5^{\alpha\mu\nu}$:

$$P_\alpha M_5^{\alpha\mu\nu} = \int \frac{d^4k}{(2\pi)^4}\left[\frac{Tr\left(\gamma^\mu \not{k} \gamma^\nu (\not{k}+\not{q}_2)\not{P}\gamma^5(\not{k}-\not{q}_1)\right)}{k^2 (k+q_2)^2 (k-q_1)^2} + \begin{pmatrix} \mu \leftrightarrow \nu \\ 1 \leftrightarrow 2 \end{pmatrix}\right] \qquad (22)$$

Since $P = q_1 + q_2$ we can simplify (22) by writing $\boxed{18.7}$

$$\not{P}\gamma^5 = (\not{q}_1 + \not{q}_2)\gamma^5 = \gamma^5(\not{k} - \not{q}_1) + (\not{k} + \not{q}_2)\gamma^5$$

so

$$P_\alpha M_5^{\alpha\mu\nu} = \int \frac{d^4k}{(2\pi)^4}\left\{ \frac{\mathrm{Tr}[\gamma^\mu \not{k} \gamma^\nu (\not{k}+\not{q}_2)\gamma^5]}{k^2 (k+q_2)^2} + \frac{\mathrm{Tr}[\gamma^\mu \not{k} \gamma^\nu \gamma^5 (\not{k}-\not{q}_1)]}{k^2 (k-q_1)^2} + \begin{pmatrix}\mu \leftrightarrow \nu \\ 1 \leftrightarrow 2\end{pmatrix}\right.$$

$$= 4i\epsilon^{\mu\nu\rho\sigma} \int \frac{d^4k}{(2\pi)^4} \left\{ \frac{k^\rho q_2^\sigma}{k^2(k+q_2)^2} + \frac{k^\rho q_1^\sigma}{k^2(k-q_1)^2} + \begin{pmatrix}\mu \leftrightarrow \nu \\ 1 \leftrightarrow 2\end{pmatrix}\right\} \quad (23)$$

This appears to vanish since, by Lorentz invariance, the first and second terms are proportional to $q_2^\rho q_2^\sigma$ and $q_1^\rho q_1^\sigma$ respectively! However, we must remember that we have to shift the ~~...~~ ~~...~~ k variable. The shift (18) corresponds to

$$(24)\qquad \begin{aligned} k &\to k + b_1 q_1 + q_2 b_2 \quad \text{in the } 1^{st}\ 2 \text{ terms of } (23) \\ k &\to k + b_2 q_1 + b_1 q_2 \quad \text{in the last 2 terms of } (23) \end{aligned}$$

Now, (6) + (23) + (24) $\Rightarrow$

$$P_\alpha M_5^{\alpha\mu\nu} = 4i\epsilon^{\mu\nu\rho\sigma}\left(\frac{i}{32\pi^2}\right)\left\{(b_1 q_1 + b_2 q_2)^\rho q_2^\sigma + (b_1 q_1 + b_2 q_2)^\rho q_1^\sigma - (1 \leftrightarrow 2)\right\}$$

~~...~~

$$-\frac{1}{8\pi^2}\epsilon^{\mu\nu\rho\sigma}\left\{ b_1 q_1^\rho q_2^\sigma + b_2 q_1^\sigma q_2^\rho - (1\leftrightarrow 2)\right\} = -\frac{1}{4\pi^2}\epsilon^{\mu\nu\rho\sigma} q_1^\rho q_2^\sigma (b_1 - b_2) \quad (25)$$

In order to preserve $\partial_\mu J^\mu = 0$ we must have
$(b_1 - b_2) = 1$, so

$$P_\alpha M_5^{\alpha\mu\nu} = -\frac{1}{4\pi^2}\epsilon^{\mu\nu\rho\sigma} q_{1\rho} q_{2\sigma} \qquad (26)$$

So the ABJ anomaly can be summarized as

$$\partial_\mu J_5^\mu = -\frac{1}{16\pi^2}\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \qquad (27)$$

Notice, the choice $b_1 = b_2$ leads to

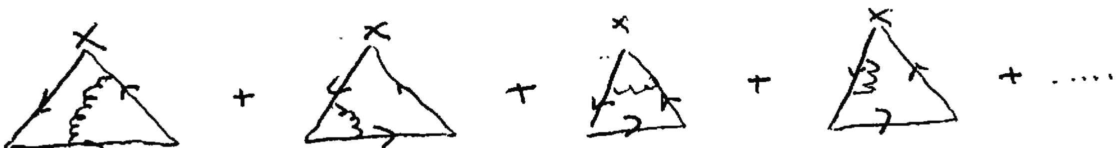
$$P_\alpha M_5^{\alpha\mu\nu} = 0 \qquad \text{and} \qquad q_{1\mu} M_5^{\alpha\mu\nu} = \frac{1}{4\pi^2}\epsilon^{\alpha\nu\rho\sigma} q_{1\sigma} q_{2\rho}$$

in which $\partial_\mu J_5^\mu = 0$ but $\partial_\mu J^\mu \neq 0$. It is impossible to have both currents conserved! References for 18.1-18.4:

- Schwartz sections 30.1 and 30.2
- Kaku section 12.4

18.4 Discussion

- Radiative corrections: Adler and Bardeen showed that their contribution to the axial anomaly vanishes:



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we will see that by an other method

- VVV triangle This triangle

$\langle 0|T J_\mu(x) J_\nu(y) J_\alpha(z)|0\rangle = M_{\mu\nu\alpha}$ vanishes due to charge conjugation (Furry's theorem)

– QED with Weyl fermion

Let's consider

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi}\,(i\not\partial - e\not A)\, P_L \Psi \qquad (28)$$

the conserved current is $J_L^\mu = \bar{\Psi}\gamma^\mu P_L \Psi$. We define

$$M_L^{\alpha\mu\nu} = \langle 0 | T\, J_L^\alpha\, J_L^\mu\, J_L^\nu | 0\rangle \equiv \langle J_L^\alpha J_L^\mu J_L^\nu \rangle$$

$$= \int \frac{d^4k}{(2\pi)^4} \left[\frac{\mathrm{Tr}\left[\gamma^\mu P_L \not k\, \gamma^\nu P_L (\not k + \not q_2)\gamma^\alpha P_L (\not k - \not q_1)\right]}{k^2 (k+q_2)^2 (k-q_1)^2} + \begin{pmatrix} \mu \leftrightarrow \nu \\ 1 \leftrightarrow 2 \end{pmatrix} \right.$$

i.e. each vertex has an additional P_L! Moving all P_L to be after γ^α we obtain:

$$M_L^{\alpha\mu\nu} = \frac{1}{2}\left(M_V^{\alpha\mu\nu} - M_5^{\alpha\mu\nu} \right) \qquad (29)$$

We know that $M_V \equiv 0$ and that it is impossible to have $q_{1\mu} M_5^{\alpha\mu\nu} = 0$ and $P_\alpha M_5^{\alpha\mu\nu} = 0$ $\Rightarrow$ The current J_L^μ is anomalous $\Rightarrow$ renormalizability is spoiled! So the model is not consistent!

Exercise Show that the model

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi}(i\not\partial + Q_L e \not A) P_L \Psi + \bar{\Psi}(i\not\partial + Q_R e \not A) P_R \Psi$$

is consistent only if $Q_L = Q_R$.

~~– Non-abelian symmetries~~

Later

~~In this case we evaluate~~

$$\cancel{M^{abc}_{\alpha\mu\nu} (2\pi)^4 \delta(p - q_1 - q_2) \equiv}$$

$$\cancel{\int d^4x\, d^4y\, d^4z\; e^{-ipx}\, e^{iq_1 y}\, e^{iq_2 z} \langle 0 | T\, J^a_{5\alpha}(x)\, J^b_{5\mu}(y)\, J^c_{5\nu}(z) | 0 \rangle} \qquad (30)$$

~~with $J^{a\mu} = \bar{\Psi}\gamma^\mu T^a \Psi$ and $J_5^{a\mu} = \bar{\Psi}\gamma^\mu \gamma^5 T^a \Psi$~~

18.5 Fujikawa's Method (1979)

Our starting point is the generating functional (focusing on the fermions!)

$$Z = \int D\psi D\bar\psi \, \exp\left[i \int d^4x \, \bar\psi i \not{D} \psi \right] \qquad (30)$$

Now, we make the change of variables:

$$\begin{aligned} \psi \to \psi'(x) &= (1 + i\alpha(x)\gamma^5)\psi(x) \\ \bar\psi \to \bar\psi'(x) &= \bar\psi(x)(1 + i\alpha(x)\gamma^5) \end{aligned} \qquad (31)$$

The exponent in (30) changes as

$$\begin{aligned} \int d^4x \, \bar\psi'(i\not{D})\psi' &= \int d^4x \left[\bar\psi i \not{D}\psi - (\partial_\mu \alpha)\, \bar\psi \gamma^\mu \gamma^5 \psi\right] \\ &\overset{\text{parts}}{=} \int d^4x \left[\bar\psi i\not{D}\psi + \alpha(x)\, \partial_\mu \underbrace{(\bar\psi\gamma^\mu\gamma^5\psi)}_{J_5^\mu}\right] \end{aligned} \qquad (32)$$

- Z is independent of α (it's a change of variables!) $\Rightarrow \frac{\delta Z}{\delta\alpha(y)} = 0$ ~~....~~ etc.
  So, if $D\psi D\bar\psi$ is invariant under (31) [wrong!] $\Longrightarrow \partial_\mu J_5^\mu = 0$.

- Although, (31) looks like a unitary transformation, it must be regularized since the Jacobian is divergent!

- Let's define the measure; for that, we evaluate the left and right eigenvectors

$$\begin{aligned} i\not{D}\phi_m &= \lambda_m \phi_m \\ \hat\phi_m (i\not{D}) &= -i\, \partial_\mu \hat\phi_m \gamma^\mu = \lambda_m \hat\phi_m \end{aligned} \qquad (33)$$

- For $A_\mu = 0 \Rightarrow \lambda_m = k^2$ (34).
- For $A_\mu \neq 0$ (34) still is the asymptotic form of λ_m for large k

Now we write

$$\psi(x) = \sum_m a_m \phi_m(x) \quad \text{and} \quad \bar\psi(x) = \sum_m \hat a_m \hat\phi_m(x) \qquad (35)$$

and we define

$$D\psi D\bar\psi = \prod_m da_m \, d\hat a_m \qquad (36)$$

- For the transformation (31), the coefficients of Ψ' are

$$a'_m = \sum_n \int d^4x\, \phi_m^\dagger(x)\,(1+i\alpha(x)\gamma^5)\,\phi_n(x)\, a_n = \sum_n \underbrace{(\delta_{mn} + C_{mn})}_{X_{mn}} a_n \qquad (37)$$

then

$$D\Psi' D\bar{\Psi}' = (\det X)^{-2}\, D\Psi D\bar{\Psi} \qquad (38)$$

α infinitesimal $\Rightarrow$ C infinitesimal

$$\det X = \exp\left[\mathrm{tr}(\log X)\right] \overset{\downarrow}{=} \exp\left[\mathrm{tr}(C_{mn})\right] = \exp\left(\sum_n C_{nn}\right) \qquad (39)$$

$$\Rightarrow \log(\det X) \overset{(37)}{=} i\int d^4x\, \alpha(x) \sum_n \phi_n^\dagger(x)\gamma^5\phi_n(x) \qquad (40)$$

- We have to regularize this sum in a gauge-invariant way. It is natural to choose

$$\sum_n \phi_n^\dagger(x)\gamma^5\phi_n(x) = \lim_{M\to\infty} \sum_n \phi_n^\dagger(x)\gamma^5\phi_n(x)\, e^{\lambda_n^2/M^2} \qquad (41)$$

Notice that going to euclidean space $\lambda_n^2 \to -\lambda_n^{E\,2}$! So

$$\sum_n \phi_n^\dagger(x)\gamma^5\phi_n(x) = \lim_{M\to\infty} \sum_n \phi_n^\dagger \gamma^5 e^{(i\not{D})^2/M^2}\phi_n(x)$$

$$= \lim_{M\to\infty} \langle x|\,\mathrm{tr}\left[\gamma_5\, e^{(i\not{D})^2/M^2}\right]|x\rangle \qquad (42)$$

over Dirac indices

- Notice

$$(i\not{D})^2 = -\gamma^\mu D_\mu \gamma^\nu D_\nu = -2g^{\mu\nu} D_\mu D_\nu + D_\mu D_\nu \gamma^\nu\gamma^\mu$$

$$= -2g^{\mu\nu} D_\mu D_\nu + D_\nu D_\mu \gamma^\nu\gamma^\mu + [D_\mu, D_\nu]\gamma^\nu\gamma^\mu$$

$$= -2D_\mu D^\mu + \underbrace{(\quad)}_{(\not{D})^2} + ie F_{\mu\nu}\gamma^\nu\gamma^\mu$$

$$\Rightarrow (i\not{D})^2 = -\underbrace{D_\mu D^\mu}_{\equiv D^2} + \frac{ie F_{\mu\nu}\sigma^{\mu\nu}}{2}$$

- Let's evaluate (42), finally

$$S \equiv \lim_{M\to\infty} \langle x | \mathrm{tr}\left[\gamma^5 e^{\frac{\left(-D^2 + \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}\right)}{M^2}}\right] | x \rangle$$

the first non zero after the trace is of order $\frac{1}{M^4}$. Using that

$$(\sigma_{\mu\nu}F^{\mu\nu})^2 = 2F_{\mu\nu}^2 \mathbb{1} + i\gamma^5 \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu}F_{\alpha\beta}$$

after Tr

so $$S = \lim_{M\to\infty} \left\{ \frac{ie^2}{2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu}F_{\alpha\beta} \frac{1}{M^4} \langle x | e^{(iD)^2/M^2} | x \rangle + \mathcal{O}\left(\frac{1}{M^5}\right) \right\}$$

then, $\mathbb{1} = \int \frac{d^4k}{(2\pi)^4} |k\rangle\langle k|$

~~At large momenta~~

At large momenta, what matters, $D \approx \partial$ and

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{M^4} \langle x | e^{\partial^2/M^2} | k\rangle\langle k | x \rangle = \frac{1}{M^4} \int \frac{d^4k}{(2\pi)^4} e^{k^2/M^2} = \frac{i}{M^4} \int \frac{d^4k_E}{(2\pi)^4} e^{-k_E^2/M^2} = \frac{i}{16\pi^2}$$

euclidian space

Putting all together,

$$S = \frac{-ie^2}{32\pi^2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu}F_{\alpha\beta} \overset{(40)}{\Longrightarrow} \det X = \exp\left\{-i\int d^4x\, \alpha(x) \frac{e^2}{32\pi^2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu}F_{\alpha\beta}\right. \quad (43)$$

Finally, we have that the change of variables (31) leads to

$$Z_{QED} = \int D\bar{\psi} D\psi DA \, \exp\left[i\int d^4x \left\{ \bar{\psi} i\not{D} \psi - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} \right]\right.$$

(32)(38)(40)(43)

$$= \int D\psi D\bar{\psi} \, DA \, \exp\left\{ i\int d^4x \left\{ \bar{\psi} i\not{D} \psi - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \alpha\, \partial_\mu J_5^\mu + \alpha \frac{e^2}{32\pi^2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu}F_{\alpha\beta} \right.\right. \quad (4$$

Consequently,

$$\frac{\delta \mathcal{Z}_{QED}}{\delta \alpha} = 0 \Longrightarrow \partial_\mu J_5^\mu = -\frac{e^2}{16\pi^2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \qquad (45)$$

This is an exact result since we did not use perturbation theory! The 1-loop result for the chiral anomaly is exact!

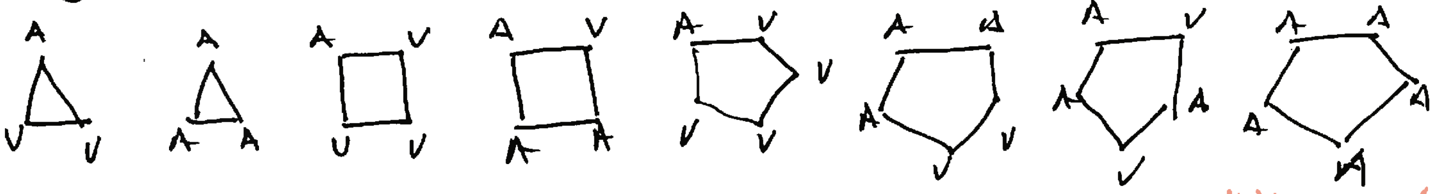
The same method applies to non-abelian gauge theories resulting in

$$\partial_\mu J_5^{\mu a} = \frac{g^2}{16\pi^2} \mathrm{Tr}\left(F_{\mu\nu} F_{\alpha\beta}\right) \epsilon^{\mu\nu\alpha\beta} \qquad (46)$$

with $F_{\mu\nu} = G^a_{\mu\nu} T^a$

18.6 Triangle contribution to ~~non-abelian~~ non-abelian anomalies

In general we can get a contribution to the anomaly from



(notice that these are the potentially divergent diagrams when we contract with the external momenta)

However, Bardeen showed that once the triangle anomaly is cancelled, then so are all the others. So we focus on the triangle anomaly to study the conditions to cancel the anomaly for a current coupling to gauge fields.

19/11/19

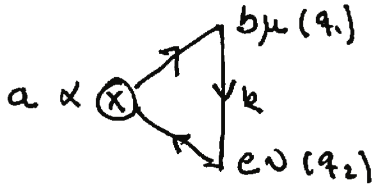
For a non-abelian current

$$J^{a\mu} = \bar{\Psi}_i \gamma^\mu T^a \Psi_j \qquad (47)$$

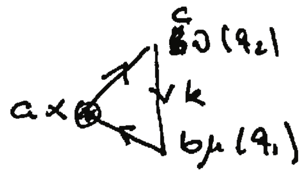
where Ψ_k might be chiral fermions (f_L or f_R). When we evaluate the loop diagrams we can always concentrate the $P_{L,R}$ in the vertex with T^a and obtain that, as in (29), each diagram can be written as:

$$M^{\alpha\mu\nu}_{L/R} = \frac{1}{2}\left(\underbrace{M_V^{\alpha\mu\nu}}_{0} \mp M_5^{\alpha\mu\nu} \right) \qquad (48a)$$

As before there are 2 contributions:



$+$



$\downarrow$ $Tr[T^aT^bT^c] \times 1^{st}$ term of (13)

$\downarrow$ $Tr[T^aT^cT^b] \times 2^{nd}$ term of (13)

Now,

$$Tr[T^aT^bT^c] = \frac{1}{2} Tr\left[[T^a,T^b]\,T^c\right] + \frac{1}{2} Tr\left[\{T^a,T^b\}\,T^c\right]$$

$$= \underbrace{\frac{i}{2} T_R\, f^{abc}}_{\text{anti-symmetric}} + \underbrace{\frac{1}{4} d_R^{abc}}_{\text{symmetric}} \qquad (48)$$

Due to the anti-symmetry of f^{abc}, the diagrams are subtracted, instead of summed and the divergence does not cancel. However, this can be renormalized by a counterterm in $f^{abc} A^a_\mu A^b_\nu \partial^\mu A^{\nu c}$!! Let's focus on the term d_R^{abc}.

In the case of SU(N) the totally symmetric three-index tensor is $\boxed{18.15}$ unique up to a constant (d^{abc}):

$$d_R^{abc} = 2\,\mathrm{tr}\left[T_R^a \{T_R^b, T_R^c\}\right] = 2A(R)\,\mathrm{Tr}\left[T^a\{T^b, T^c\}\right] = 2A(R)\,d^{abc} \quad (49)$$

anomaly coefficient.

thus, after summing the diagrams we have

$$\partial^\alpha J_\alpha^a = \Big(\sum_{\text{left}} A(R_\ell) - \sum_{\text{right}} A(R_R)\Big) d^{abc} \frac{g^2}{128\pi^2}\, \epsilon^{\mu\nu\alpha\beta} F^b_{\mu\nu} F^c_{\alpha\beta} \quad (5$$

Reference for 18.5: - Peskin- Schroeder section 19.2
- Schwartz section 30.3 and 30.4
- Pokorski chapter 13 to learn more about anomalies.

18.7 Gauge Anomalies in the SM

The SM group is $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$. We have to verify that

$$\partial_\mu \langle 0|T\, J^{j\mu}_a J^{k\alpha}_b J^{\ell\nu}|0\rangle = 0 \quad (51)$$

for j, k, ℓ standing for $SU(3)$, $SU(2)_L$ and U

The quantum numbers needed to evaluate (50) are: $\left(Q_{em} = T_3 + \frac{Y}{2}\right)$

Multiplet	$SU(3)_c$	$SU(2)_L$	$U(1)_Y$
$Q = \binom{u}{d}_L$	3	2	1/3
$L = \binom{\nu_e}{e}_L$	1	2	-1
u_R	3	1	4/3
d_R	3	1	-2/3
e_R	1	1	-2

(T1)

- $(U(1)_Y)^3$ anomaly, for which $j = k = l = U(1)_Y$. From (46) ⊕ (49a)

$$\partial_\mu J^\mu_Y = \underbrace{\left(\sum_{\text{left}} Y_L^3 - \sum_{\text{right}} Y_R^3 \right)}_{} \frac{g'^2}{32\pi^2} \epsilon^{\mu\nu\alpha\beta} B_{\mu\nu} B_{\alpha\beta} \quad (52)$$

$$\xrightarrow[\text{family}]{\text{1}} \left[2\times 3\times \left(\frac{1}{3}\right)^3 + 2\times(-1)^3 \right] - \left[3\times\left(\frac{4}{3}\right)^3 + 3\left(-\frac{2}{3}\right)^3 + (-2)^3 \right] = 0$$

- $(SU(2)_L)^3$ anomaly for which $j = k = l = SU(2)_L$. We have to evaluate

$$\sum_L \mathrm{tr}\left[T^a \{T^b, T^c\}\right] - \underbrace{\sum_R \mathrm{tr}\left[T^a \{T^b, T^c\}\right]}_{0}$$

since they are singlets

but $T^a = \frac{\sigma^a}{2} \Rightarrow \left\{ \frac{\sigma^a}{2}, \frac{\sigma^b}{2} \right\} = \frac{\delta^{ab}}{2} \mathbb{1}_{2\times 2} \Rightarrow \mathrm{Tr}\left[T^a \{T^b, T^c\}\right] \equiv 0 \Rightarrow$ <u>OK</u>

- $SU(3)_c^2\, U(1)_Y$ anomaly: $j = k = SU(3)_c$ and $l = U(1)_Y$

$$\mathrm{Tr}\left[Y \{T^a, T^b\}\right] = Y\, \mathrm{Tr}\left(2 T^a T^b\right) = \delta^{ab} Y \quad (53)$$

$$\Rightarrow \left(\sum_L Y_L - \sum_R Y_R \right) = 3 \times \left(\overset{\text{doublet}}{2\times\frac{1}{3}} - \frac{4}{3} + \left(-\frac{2}{3}\right) \right) = 0 \Rightarrow \underline{\underline{OK}}$$

- $SU(2)_L^2\, U(1)_Y$ anomaly $\quad (j=k=SU(2)_L$ and $l=U(1)_Y)$

Using (53) =>

$$\sum_L Y_L - \sum_R \cancel{Y_R}^{\,0} = 3\times 2\times \frac{1}{3} + 2\times(-1) = 0 \Rightarrow \text{OK}$$

Notice that there are no $(SU(3)_c)^3$ ~~and~~, $(SU(3)_c)^2 U(1)_{em}$ and $(U(1)_{em})^3$ anomalies since the L and R fermions couples in the same way to these gauge bosons; see (50).

- Reference Schwartz (30.4)

- Further reference: ~~the~~ it is worth reading chapter 3 of Donoghue, Golowich and Holstein, "Dynamics of the Standard Model".