Álgebra de Boole – caso geral (não restrito apenas a 0 e 1)
Extraído de Gajski, D. D. "Principles of Digital Design". Prentice Hall, 1997.

3.2 AXIOMATIC DEFINITION OF BOOLEAN ALGEBRA

In 1854, George Boole introduced the formalism that we use for the systematic treatment of logic, which is now called **Boolean algebra**. In 1938, C. E. Shannon applied this particular algebra to demonstrate that the properties of electrical switching circuits can be represented by a two-valued Boolean algebra, which is called **switching algebra**. The formal definition of Boolean algebra that is given below was first formulated in 1904 by E. V. Huntington.

At the most general level, Boolean algebra is an algebraic structure that is defined on a set of elements B with two binary operators, $+$ and $\cdot$, which satisfies the following axioms.

Axiom 1 (Closure Property). (a) B is closed with respect to the operator $+$; (b) B is also closed with respect to the operator $\cdot$.

Axiom 2 (Identity Element). (a) B has an identity element with respect to $\cdot$, designated by 1; (b) B also has an identity element with respect to $+$, designated by 0.

Axiom 3 (Commutativity Property). (a) B is commutative with respect to $+$; (b) B is also commutative with respect to $\cdot$.

Axiom 4 (Distributivity Property). (a) The operator $\cdot$ is distributive over $+$; (b) similarly, the operator $+$ is distributive over $\cdot$.

Axiom 5 (Complement Element). For every $x \in B$, there exists an element $x' \in B$ such that (a) $x + x' = 1$ and (b) $x \cdot x' = 0$. This second element, x', is called the complement of x.

Axiom 6 (Cardinality Bound). There are at least two elements $x, y \in B$ such that $x \neq y$.

Observações:

1. In ordinary algebra, $+$ is not distributive over $\cdot$.
2. Boolean algebra does not have inverses with respect to $+$ and $\cdot$; therefore, there are no subtraction or division operations in Boolean algebra.
3. Complements are available in Boolean algebra but not in ordinary algebra.
4. Boolean algebra applies to a finite set of elements, whereas ordinary algebra applies to the infinite set of real numbers.
5. Huntington's definition of Boolean algebra does not include associativity, since it can be derived from the other axioms.

Basic Theorems of Boolean Algebra

Theorem 1	(a)	$x + x$	=	x
(idempotency)	(b)	xx	=	x
Theorem 2	(a)	$x + 1$	=	1
	(b)	$x \cdot 0$	=	0
Theorem 3	(a)	$yx + x$	=	x
(absorption)	(b)	$(y + x)x$	=	x
Theorem 4		$(x')'$	=	x
(involution)				
Theorem 5	(a)	$(x + y) + z$	=	$x + (y + z)$
(associativity)	(b)	$x(yz)$	=	$(xy)z$
Theorem 6	(a)	$(x + y)'$	=	$x'y'$
(De Morgan's law)	(b)	$(xy)'$	=	$x' + y'$

T7 (absorção): (a) a.(a' + b) = a.b
(b) a + a'.b = a + b

T8 (absorção): (a) a.b + a'.c + b.c = a.b + a'.c
(b) (a+b).(a'+c).(b+c) = (a+b).(a'+c)

T9 (dualidade): "+" e "." são duais.
"0" e "1" são duais.

T10 (dualidade): Sejam $E(b_0, b_1, \ldots b_{n-1})$ uma expressão boolena de n variáveis, e $E_D(b_0, b_1, \ldots b_{n-1})$ a sua expressão *dual*, obtida a partir de E inter-cambiando-se: "+" por ".", e "0" por "1" . Tem-se que:

$$E'(b_0, b_1, \ldots b_{n-1}) = E_D(b_0', b_1', \ldots b_{n-1}')$$

Exemplo de demostração

THEOREM 1(a) (Idempotency). $x + x = x$.

PROOF:

$x + x = (x + x) \cdot 1$	by identity [Axiom 2(b)]
$= (x + x)(x + x')$	by complement [Axiom 5(a)]
$= x + xx'$	by distributivity [Axiom 4(b)]
$= x + 0$	by identity [Axiom 5(b)]
$= x$	by complement [Axiom 2(a)]