Principle of Virtual Work

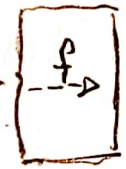
$\delta W_i + \delta W_e = 0$

$$\int_V \delta\varepsilon\,\sigma\,dV = \underbrace{\int_V \delta p\, f^B\, dV}_{\text{Body forces} = 0} + \underbrace{\int_V \delta p\, f^S\, dV}_{\text{Surface forces} = 0} + \underbrace{\sum_i \delta p_i R_i}_{\text{Concentr. Forces}}$$

$$\int_V \delta\varepsilon\,\sigma\,dV = \sum_i \delta p_i R_i$$

Truss - Linear case

FE from the governing equation

$\sigma \longleftarrow$  $\longrightarrow \sigma + \frac{\partial\sigma}{\partial x}dx$

dx

$$-\sigma A + \sigma A + \frac{\partial\sigma}{\partial x}dx\,A + \underbrace{f\,dx = 0}_{\text{static}}$$

$$\frac{\partial\sigma}{\partial x} + \frac{f}{A} = 0$$

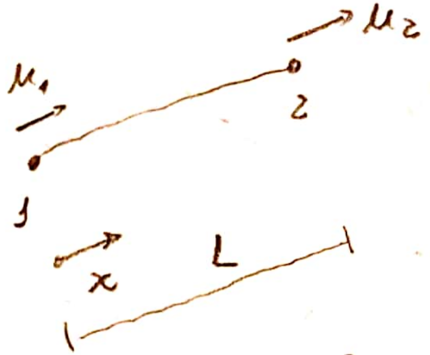
$$\sigma = E\varepsilon = E\frac{\partial u}{\partial x} \qquad \text{Hook, small strains}$$

$$E\frac{\partial^2 u}{\partial x^2} + \frac{f}{A} = 0 \qquad \times\ \delta u$$

$$\int E\frac{\partial^2 u}{\partial x^2}\cdot\delta u\,dx + \int \frac{f}{A}\cdot\delta u\,dx = 0$$

$$\{u\} = [N]\{u_e\} = [N_1, N_2]\begin{Bmatrix} u_{e_1} \\ u_{e_2} \end{Bmatrix}$$

Local coordinates



$$[N] = [N_1\ N_2] = \left[1 - \frac{x}{L} \quad \frac{x}{L}\right]$$

$$\int \left(E \frac{\partial^2 N}{\partial x^2} u_e\right) \cdot \left(N\, \delta u_e\right) dx + \int \left(\frac{f}{A}\right) \cdot \left(N \delta u_e\right) dx = 0$$

$$A \cdot B = B^T A \qquad (AB)^T = B^T A^T$$

$$\int (N \delta u_e)^T \left(E \frac{\partial^2 N}{\partial x^2} u_e\right) dx + \int (N \delta u_e)^T \frac{f}{A}\, dx = 0$$

$$\cancel{\delta u_e^T} \int \underbrace{N^T}_{u} E \underbrace{\frac{\partial^2 N}{\partial x^2} dx}_{dv}\, u_e + \cancel{\delta u_e^T} \int N^T \frac{f}{A}\, dx = 0$$

$$\frac{\partial^2 N}{\partial x^2} = 0 \quad \text{no integration by parts}$$

$$\int u\,dv = uv - \int v\,du$$

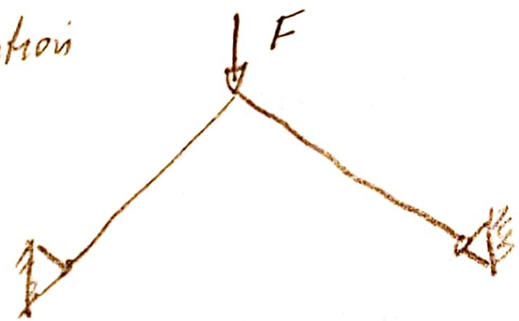
$$\left[E N^T \frac{\partial N}{\partial x} - \int E \frac{\partial N}{\partial x} \frac{\partial N^T}{\partial x} dx\right] u_e + \int N^T \frac{f}{A} dx = 0$$

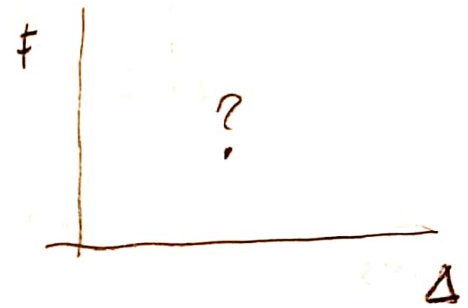
$$\underbrace{E \int \frac{\partial N}{\partial x} \frac{\partial N^T}{\partial x} dx}_{[K]}\, u_e = \underbrace{\int N^T \frac{f}{A} dx}_{\{F\}} + \underbrace{E N^T \frac{\partial N}{\partial x} u_e}_{0}$$

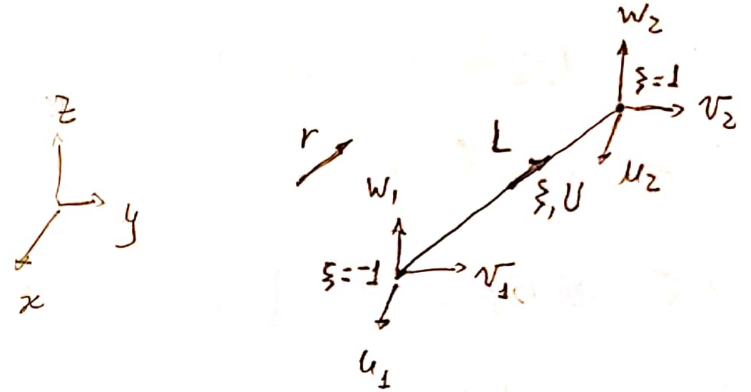
Exercises

1. Obtain an explicit expression for $[K]$
2. Is $E N^T N' = 0$?
3. Obtain the coordinate transformation matrix $T = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ $K_e' = T^T K_e T$
4. Write a FE program for spatial truss analysis

5. Using your program solve this structure and compare it with a theoretical solution







Displacement at nodes $\{p\} = [u_1\ v_1\ w_1\ u_2\ v_2\ w_2]^T$

Force at nodes $\{F\} = [R_1\ R_2\ R_3\ R_4\ R_5\ R_6]^T$

$$\delta W_e = \Sigma\, \delta p_i \cdot R_i = \underset{1\times 6}{F^T}\ \underset{6\times 1}{\delta p}$$

Displacement field within the element

$$U = Hp$$

$$\varepsilon = \frac{\partial U}{\partial r} = \frac{\partial}{\partial r} Hp = \frac{\partial H}{\partial r} p$$

$$\varepsilon = Bp$$

$$B = \frac{\partial H}{\partial r}$$

Linear Hooks law

$$\sigma = E\varepsilon$$

$$\sigma = E B p$$

$$\delta W_i = \int_V \delta\varepsilon \, \sigma \, dV$$

$$= \int_V B \delta p \, E B p \, dV$$

$$= \delta p \int_V B E B p \, dV$$

$$\delta W_i = \delta W_e$$

$$\delta p \int_V B E B p \, dV = \delta p \, F$$

$$F = \underbrace{\int B E B \, dV}_{K} \; p$$

Truss FE Linear : Geometry : Isoparametric

From $W_i = W_e$

$$h_1 = \frac{1}{2}(-\xi + 1) \qquad h_2 = \frac{1}{2}(\xi + 1)$$

$$h = \frac{1}{2}[1-\xi \quad 1+\xi]$$

$$x = \frac{1}{2}[1-\xi \quad 1+\xi]\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \sum h_i x_i$$

$$u = \frac{1}{2}[1-\xi \quad 1+\xi]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \sum h_i u_i$$

$$y = \sum h_i y_i \qquad z = \sum h_i z_i$$

$$v = \sum h_i v_i \qquad w = \sum h_i w_i$$

U → axial displacement

Define $x_{21} = x_2 - x_1$, $y_{21} = y_2 - y_1$, $z_{21} = z_2 - z_1$

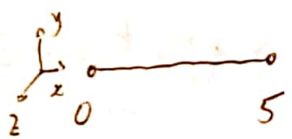
$$U = \frac{u x_{21} + v y_{21} + w z_{21}}{L}$$

Scanned by CamScanner

$$U = \frac{1}{2L}\begin{bmatrix} x_{21} & y_{21} & z_{21} \end{bmatrix} \underbrace{\begin{bmatrix} 1-\xi & 0 & 0 & 1+\xi & 0 & 0 \\ 0 & 1-\xi & 0 & 0 & 1+\xi & 0 \\ 0 & 0 & 1-\xi & 0 & 0 & 1+\xi \end{bmatrix}}_{H} \underbrace{\begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \end{Bmatrix}}_{\{p\}}$$

$U = Hp \qquad H =$ matriz de interpolação

Exemple



$$U = \frac{1}{2 \times 5}\begin{bmatrix} 5-0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1-\xi & 0 & 0 & 1+\xi & 0 & 0 \\ 0 & 1-\xi & 0 & 0 & 1+\xi & 0 \\ 0 & 0 & 1-\xi & 0 & 0 & 1+\xi \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 5 \\ 0 \\ 0 \end{Bmatrix}$$

$$U = \begin{bmatrix} \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} \; \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 5 \\ 0 \\ 0 \end{Bmatrix}$$

$$U = \begin{bmatrix} \frac{1-\xi}{2} & 0 & 0 & \frac{1+\xi}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 5 \\ 0 \\ 0 \end{Bmatrix}$$

$U = \frac{5}{2}(1+\xi) \qquad \xi = -1 \quad U = 0, \quad \xi = 1 \quad U = 5$

$$B = \frac{\partial H}{\partial r} = \frac{\partial H(\xi)}{\partial r} = \frac{\partial H}{\partial \xi}\frac{\partial \xi}{\partial r}$$

$r = 0 \quad \xi = -1 \qquad r = L \quad \xi = 1$

$$\xi = \frac{2r}{L} - 1$$

$$\frac{\partial \xi}{\partial r} = \frac{2}{L}$$

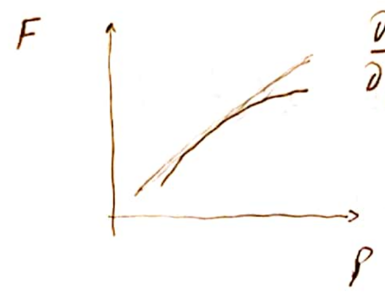
$$\frac{\partial H}{\partial \xi} = \frac{1}{2L}\underset{1\times 3}{\begin{bmatrix} x_{21} & y_{21} & z_{21} \end{bmatrix}} \underset{3\times 6}{\begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}}$$

$$B = \frac{\partial H}{\partial r} = \frac{\partial H}{\partial \xi}\frac{\partial \xi}{\partial r} = \frac{1}{L^2}\underbrace{\begin{bmatrix} -x_{21} & -y_{21} & -z_{21} & x_{21} & y_{21} & z_{21} \end{bmatrix}}_{b^T(x)}$$

$$K = \int B^T E B \, dV = EAL\, B^T B$$

$$K = EAL \frac{b}{L^2}\frac{b^T}{L^2} = \frac{EA}{L^3}\, b b^T$$

Non Linearity : Geometry



$$\frac{\partial F}{\partial p} \longrightarrow \begin{cases} \text{linear } K = \frac{\Delta F}{\Delta p} \\ \text{nonlinear } K_T = \frac{\partial F}{\partial p} \end{cases}$$

$$-F = g = F^{ext} - F^{int}$$

$$K_T = \frac{\partial F}{\partial p} = \frac{-\partial g}{\partial p} = \frac{-\partial F^{ext}}{\partial p} + \frac{\partial F^{int}}{\partial p}$$

$\downarrow$ does not vary with p

not follower force

$$K_T = \frac{\partial F^{int}}{\partial p} = \frac{\partial}{\partial p}(\sigma A L B^T)$$

linear case $\varepsilon = Bp$

Nonlinear case $\varepsilon = \ln \lambda$

$$\varepsilon = \ln \frac{L_n}{L}$$

$$\varepsilon = Bp$$

$$d\varepsilon = B\,dp$$

$$B^T = \frac{\partial \varepsilon}{\partial p} = \frac{\partial}{p}\left(\ln \frac{L_n}{L}\right) = \frac{\partial}{\partial p} \ln u = \frac{1}{u} \frac{\partial u}{\partial p}$$

$$B^T = \frac{L}{L_n} \frac{1}{L} \frac{\partial L_n}{\partial p} = \frac{1}{L_n} \frac{\partial L_n}{\partial p}$$

$$L_n^2 = [(x_2 - x_1) + (u_2 - u_1)]^2 + [(y_2 - y_1) + (v_2 - v_1)]^2 + [(z_2 - z_1) + (w_2 - w_1)]^2$$

$$L_n^2 = (x_{21} + p_{21})^T (x_{21} + p_{21})$$

$$L_n^2 = x_{21}^T x_{21} + x_{21}^T p_{21} + p_{21}^T x_{21} + p_{21}^T p_{21}$$

$$x_{21} = \begin{bmatrix} x_{21} \\ y_{21} \\ z_{21} \end{bmatrix} \qquad p_{21} = \begin{bmatrix} u_{21} \\ v_{21} \\ w_{21} \end{bmatrix}$$

$$x_{21}^T x_{21} = [x_{21}\ y_{21}\ z_{21}] \begin{bmatrix} x_{21} \\ y_{21} \\ z_{21} \end{bmatrix} = x_{21}^2 + y_{21}^2 + z_{21}^2$$

$$= (x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2$$

$$= x_2^2 + x_1^2 - 2x_1x_2 + y_2^2 + y_1^2 - 2y_1y_2 + z_2^2 + z_1^2 - 2z_1z_2$$

$$= x^T x - 2(x_2x_1 + y_2y_1 + z_2z_1)$$

Defining $x^T = [x_1\ y_1\ z_1\ x_2\ y_2\ z_2]$

$$x_{21}^T x_{21} = x^T x + [x_1\ x_2\ y_1\ y_2\ z_1\ z_2] \underbrace{\begin{bmatrix} 0&0&0&-1&0&0 \\ 0&0&0&0&-1&0 \\ 0&0&0&0&0&-1 \\ -1&0&0&0&0&0 \\ 0&-1&0&0&0&0 \\ 0&0&-1&0&0&0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ x_2 \\ y_2 \\ z_2 \end{bmatrix} =$$

$$= [x_1\ y_1\ z_1\ x_2\ y_2\ z_2] \begin{bmatrix} 1&0&0&-1&0&0 \\ 0&1&0&0&-1&0 \\ 0&0&1&0&0&-1 \\ -1&0&0&1&0&0 \\ 0&-1&0&0&0&0 \\ 0&0&-1&0&0&1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ x_2 \\ y_2 \\ z_2 \end{bmatrix} = x^T A x$$

$[A] \rightarrow$

$$p_{21}^T p_{21} = p^T A p$$

$$x_{21}^T p_{21} = [x_{21}\ y_{21}\ z_{21}] \begin{bmatrix} u_{21} \\ v_{21} \\ w_{21} \end{bmatrix}$$

$$= x_{21}(u_2-u_1) + y_{21}(v_2-v_1) + z_{21}(w_2-w_1)$$

$$= [-x_{21}\ -y_{21}\ -z_{21}\ x_{21}\ y_{21}\ z_{21}] \begin{bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \end{bmatrix} = b^T p$$

$$(x_{21}^T p_{21})^T = p_{21}^T x_{21} = (b^T p)^T = p^T b$$

$$b^T = x^T A \qquad \underbrace{b = Ax}$$

$$L_n^2 = x_{21}^T x_{21} + x_{21}^T p_{21} + p_{21}^T x_{21} + p_{21}^T p_{21}$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$x^T A x \qquad b^T p \qquad p^T b \qquad p^T A p$$

$$L_n^2 = (x+p)^T A (x+p)$$

$$B^T = \frac{1}{L_n}\frac{\partial L_n}{\partial p}$$

$$L_n^2 = x^T A x + p^T A p + x^T A p + p^T A x$$

$$\frac{\partial}{\partial p} L_n^2 = 2 L_n \frac{\partial L_n}{\partial p}$$

$$\frac{\partial}{\partial p} x^T A x = 0$$

$$\frac{\partial}{\partial p} \underset{1\times6}{x^T} \underset{6\times6}{A} \underset{6\times1}{p} = (x_2 - x_1)(u_2 - u_1) + (y_2 - y_1)(v_2 - v_1) + (z_2 - z_1)(w_2 - w_1)$$

$$\frac{\partial}{\partial p} x^T A p = \begin{matrix} \frac{\partial}{\partial u_1} \\ \frac{\partial}{\partial u_2} \\ \vdots \end{matrix} = \begin{bmatrix} -x_2 + x_1 \\ x_2 - x_1 \\ -y_2 + y_1 \\ y_2 + y_1 \\ -z_2 + z_1 \\ z_2 + z_1 \end{bmatrix} = b$$

$$\frac{\partial}{\partial p} p^T A x = \frac{\partial}{\partial p} p^T b = \frac{\partial}{\partial p} x^T A p = \frac{\partial}{\partial p} p^T A x$$
$$= b$$

$$\frac{\partial}{\partial p} p^T A p = 2 \begin{bmatrix} u_1 - u_2 \\ v_1 - v_2 \\ w_1 - w_2 \\ u_2 - u_1 \\ v_2 - v_1 \\ w_2 - w_1 \end{bmatrix} = 2Ap$$

$$2 L_n \frac{\partial L_n}{\partial p} = b + 2Ap + b$$

$$\frac{\partial L_n}{\partial p} = \frac{1}{L_n}(b + Ap) = \frac{1}{L_n}(Ax + Ap)$$

$$\frac{\partial L_n}{\partial p} = \frac{1}{L_n} A(x+p) = \frac{1}{L_n} A x'$$

$x' = x + p \longrightarrow$ update coordinates vector

$$B^T = \frac{1}{L_n}\frac{\partial L_n}{\partial p} = \frac{1}{L_n^2} A x' = \frac{Ax'}{(x+p)^T A (x+p)}$$

$$B^T = \frac{Ax'}{x'^T A x'}$$

Stiffness Matrix

$$K_T = \frac{\partial F^{int}}{\partial p} = \frac{\partial}{\partial p}\left[\sigma \frac{A_0}{L} \lambda^{-(1+2\nu)} A x'\right]$$

$\downarrow$ not area

$$K_T = \frac{\partial F_{int}}{\partial p} = \frac{\partial}{\partial p}\left(\sigma \frac{A_n}{L} \lambda^{-1} A x'\right)$$

$$K_T = \frac{\partial \sigma}{\partial p}\left(\frac{A_n}{L} \lambda^{-1} A x'\right) + \sigma \frac{\partial}{\partial p}\left(\frac{A_n}{L} \lambda^{-1} A x'\right)$$

$$K_T = \underbrace{\frac{\partial \sigma}{\partial p} \frac{A_n \lambda^{-1} A x'}{L}}_{K_{T_I}} + \sigma\left[\frac{\partial}{\partial p}\left(\frac{A_n}{L} \lambda^{-1}\right) A x' + \underbrace{\frac{A_n}{L} \lambda^{-1} A \frac{\partial x'}{\partial p}}_{K_{T_{\sigma_I}}/\sigma}\right]$$

$$K_T = K_{T_I} + K_{T_{\sigma_I}} + \sigma A x'\left[\frac{\partial}{\partial p}\left(\frac{A_n}{L}\right)\lambda^{-1} + \frac{A_n}{L}\frac{\partial}{\partial p}\lambda^{-1}\right]$$

$$K_T = K_{T_I} + K_{T_{\sigma_1}} + \sigma A x'\left[\frac{\partial}{\partial p}\left(\frac{A_0 \lambda^{-2\nu}}{L}\right)\lambda^{-1} + \frac{A_0 \lambda^{-2\nu}}{L}\frac{\partial}{\partial p}\lambda^{-1}\right]$$

$$K_T = K_{T_I} + K_{T_{\sigma_1}} + \sigma A x' \frac{A_0}{L}\left[\frac{\partial}{\partial p}\left(\lambda^{-2\nu}\right)\lambda^{-1} + \lambda^{-2\nu}\frac{\partial}{\partial p}\lambda^{-1}\right]$$

$$K_T = K_{T_I} + K_{T_{\sigma_1}} + \underbrace{\sigma A x' \frac{A_0}{L}\frac{\partial}{\partial p}\left(\lambda^{-1}\lambda^{-2\nu}\right)}_{K_{T_{\sigma_2}}}$$

$$K_T = K_{T_I} + K_{T_{\sigma_1}} + K_{T_{\sigma_2}}$$

$$\frac{\partial \sigma}{\partial p} = E\frac{\partial \varepsilon}{\partial p} = E B^T = E\frac{1}{L_n}\frac{\partial L_n}{\partial p} = E\frac{1}{L_n}\frac{1}{L_n} A x'$$

$$\frac{\partial \sigma}{\partial p} = \frac{E}{L_n^2} C(x')$$

$$C(x') = A x'$$

$$\frac{\partial x'}{\partial p} = \frac{\partial}{\partial p}(x + p) = \frac{\partial p}{\partial p} = I$$

$$\frac{\partial \lambda}{\partial p} = \frac{\partial}{\partial p}\left(\frac{L_n}{L}\right) = \frac{1}{L}\frac{\partial L_n}{\partial p} = \frac{1}{L}\frac{1}{L_n} C(x')$$

$$K_{T_I} = \frac{\partial \sigma}{\partial p} \frac{A_n \lambda^{-1} A x'}{L}$$

$$K_{T_I} = \frac{1}{L_n^2} \frac{C E A_n \lambda^{-1} A x'}{L} = \frac{1}{\lambda^2 L^2} \frac{C E A_n \lambda^{-1} A x'}{L}$$

$$K_{T_I} = \frac{1}{L^3 \lambda^3} C E A_0 \lambda^{-2\nu} A x'$$

$$K_{T_I} = \frac{E \lambda^{-(3+2\nu)} A_0}{L^3} C C^T$$

$$K_{T_{\sigma_1}} = \sigma \frac{A_n}{L} \lambda^{-1} A I = \frac{\sigma}{L} A_0 \lambda^{-2\nu} \lambda^{-1} A I = \frac{\sigma A_0}{L} \lambda^{-(1+2\nu)} A$$

$$K_{T_{\sigma_2}} = \sigma A x' \frac{A_0}{L} \frac{\partial}{\partial p} (\lambda^{-1} \lambda^{-2\nu})$$

$$K_{T_{\sigma_2}} = \sigma C \frac{A_0}{L} \cdot -(1+2\nu) \lambda^{-2(1+\nu)} \frac{\partial \lambda}{\partial p}$$

$$K_{T_{\sigma_2}} = -\sigma C \frac{A_0}{L} (1+2\nu) \lambda^{-2(1+\nu)} \frac{1}{L L_n} C$$

$$K_{T_{\sigma_2}} = -(1+2\nu) \lambda^{-2(1+\nu)} \sigma A_0 C C^T \frac{1}{L^2 L_n}$$

$$K_{T_{\sigma_2}} = -\frac{(1+2\nu) \lambda^{-(3+2\nu)}}{L^3} \sigma A_0 C C^T$$

$$K_T = K_{T_1} + K_{T_2} + K_{T_3}$$

$$K_T = \frac{E \lambda^{-(3+2\nu)}}{L^3} A_0 C C^T + \frac{\sigma A_0 \lambda^{-(1+2\nu)}}{L} A$$

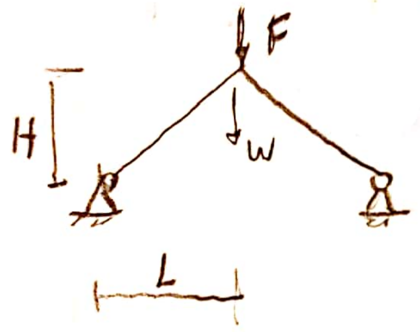
$$- \frac{(1+2\nu) \lambda^{-(3+2\nu)}}{L^3} \sigma A_0 C C^T$$

$$K_T = \frac{A_0}{L^3} \lambda^{-(3+2\nu)} \left[E - (1+2\nu) \sigma \right] C C^T + \frac{\sigma}{L} A_0 \lambda^{-(1+2\nu)} A$$

Non-linear elastic material

Newton-Raphson

deduzir, exercício



$$F = 2E(W-H)\left[1 - \frac{1}{L}\sqrt{\frac{L^2+H^2}{1+\left(\frac{W-H}{L}\right)^2}}\right]$$

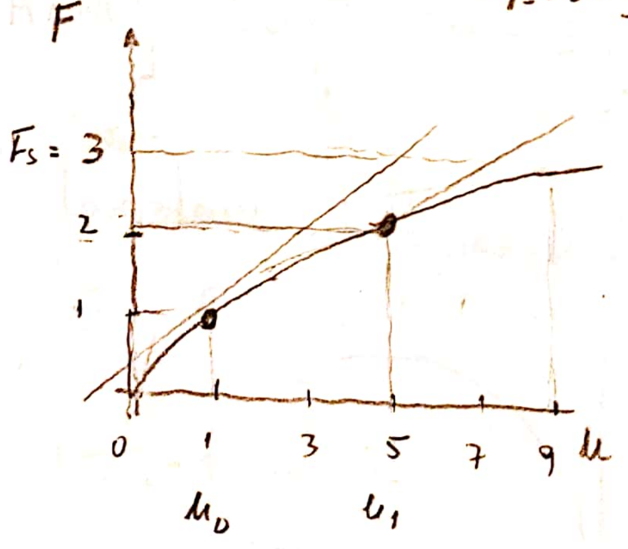
$F = \sqrt{u}$

guess $u \to F \to F'$

$1 \to 1 \to (F_s - F_1)/(u_1 - u_0) = \frac{1}{2\sqrt{u}} = \frac{1}{2}$

$$u_1 = u_0 + \frac{F_s - F_1}{k_T} = 5$$

no necessary to know and thuses K_T



$F_2 = \sqrt{u} = \sqrt{5}$

$$u_2 = u_1 + \frac{F_s - F_2}{F'(u_1)} = 5 + \frac{3-\sqrt{5}}{1/(2\sqrt{5})} = 8.42$$

General nonlinear equation

$$f(x_2) = f(x_1) + \frac{df(x_1)}{dx}(x_2 - x_1)$$

$$f(x_2) - f(x_1) = T(x_1)(x_2 - x_1)$$

$$x_2 = x_1 + \frac{f(x_2) - f(x_1)}{T(x_1)}$$

We target $f(x_2) = 0$ so

$$x_2 = x_1 - \frac{f(x_1)}{T(x_1)}$$

or

$$x_{r+1} = x_r - \frac{f(x_r)}{T(x_r)}$$

$$K(u)\, u = F$$

that is why we forget $f(z)=0$

$$r(u) = K(u)\, u - F$$

$$u_{r+1} = u_r - T^{-1}(u_r)\, r(u_r)$$

$$T(u_r) = \frac{\partial r(u_r)}{\partial u_r} = \frac{\partial K}{\partial u} u + K \frac{\partial u}{\partial u}$$

$$T(u_r) = \frac{\partial K(u_r)}{\partial u} u(r) + K(u_r)$$

Write a FE non linear truss element

- material: elastic non-linear
- geometry: non linear

Solve  + other example of your choice, comparing your program with abaqus / solome

Non Linear Beam Finite Element

Set 2019

$$\delta W_{int} = \delta W_{ext}$$

$$\int \delta \varepsilon_{ij} \sigma_{ij}\, dV = \underbrace{\int q\, \delta w\, dz}_{\text{transv}} + \underbrace{\int f\, \delta u\, dx}_{\text{axial}} + \underbrace{\sum Q_i \cdot \delta \Delta}_{\text{concent}}$$