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Cumulative fatigue damage and life prediction theories: a survey of the state of the art for homogeneous materials

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Fatigue damage increases with applied load cycles in a cumulative manner. Cumulative fatigue damage analysis plays a key role in life prediction of components and structures subjected to field load histories. Since the introduction of damage accumulation concept by Palmgren about 70 years ago and 'linear damage rule' by Miner about 50 years ago, the treatment of cumulative fatigue damage has received increasingly more attention. As a result, many damage models have been developed. Even though early theories on cumulative fatigue damage have been reviewed by several researchers, no comprehensive report has appeared recently to review the considerable efforts made since the late 1970s. This article provides a comprehensive review of cumulative fatigue damage theories for metals and their alloys, emphasizing the approaches developed between the early 1970s to the early 1990s. These theories are grouped into six categories: linear damage rules; nonlinear damage curve and two-stage linearization approaches; life curve modification methods; approaches based on crack growth concepts; continuum damage mechanics models; and energy-based theories. © 1998 Elsevier Science Ltd.

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INTRODUCTION

Fatigue damage increases with applied cycles in a cumulative manner which may lead to fracture. Cumulative fatigue damage is an old, but not yet resolved problem. More than seventy years ago, Palmgren[1] suggested the concept which is now known as the 'linear rule'. In 1945, Miner[2] first expressed this concept in a mathematical form as: $D = \Sigma(n_i/N_{fi})$, where D denotes the damage, and n_i and N_{fi} are the applied cycles and the total cycles to failure under ith constant-amplitude loading level, respectively. Since then, the treatment of cumulative fatigue damage has received increasingly more attention. As a result, many related research papers are published every year and many different fatigue damage models have been developed.

Some of the progress on the subject of cumulative fatigue damage has been summarized in several review papers. Newmark[3] in a comprehensive early review discussed several issues relating to cumulative damage in fatigue such as damage cumulation process, damage vs cycle ratio curve, and influence of prestressing on cumulative cycle ratios. Socie and Morrow[4] presented a review of contemporary approaches for fatigue damage analysis employing smooth specimen material data for predicting service life of components and structures subjected to variable loading. The early theories on cumulative fatigue damage have also been reviewed by Kaechele[5], Manson[6], Leve[7], O'Neill[8], Schive[9], Laflen and Cook[10] and Golos and Ellyin[11]. However, as pointed out by Manson and Halford[12] in 1986, no comprehensive report has appeared recently to review the considerable effort made since Schive's publication. In addition, no such review has been published since the late 1980s.

This review paper provides a comprehensive overview of cumulative fatigue damage theories for metals and their alloys. Damage models developed before 1970s were mainly phenomenological, while those after 1970s have gradually become semi-analytical or analytical. Several researchers[4–9] have reviewed the theories developed before 1970s. These damage rules are first reviewed in this paper. Then a more detailed discussion on the selected approaches developed after 1970s is presented. Even though some of the continuum damage mechanics (CDM) models are also mentioned, these approaches are not reviewed in this paper. An important application of these models has been in

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damage assessment of inhomogeneous materials. It should also be noted that this review paper deals with damage rules and life prediction aspects of cumulative fatigue damage. Another review paper[13] provides a comprehensive overview of cumulative fatigue damage mechanisms and quantifying parameters.

WORK BEFORE 1970s

The phenomenologically-based damage theories developed before 1970s were originated from three early concepts (discussed below) and attempted to improve the linear damage rule (LDR). These theories can be categorized into five groups: the damage curve approach (DCA); endurance limit-based approach; *S–N* curve modification approach; two-stage damage approach; and crack growth-based approach.

Three early concepts

The history of fatigue damage modeling can be dated back to 1920s and 1930s. It was Palmgren[1] who first introduced the concept of linear summation of fatigue damage in 1924. French[14] first reported the significant investigation of the overstress effect on endurance limit in 1933. In 1938, Kommers[15] suggested using the change in the endurance limit as a damage measure. In 1937, Langer[16] first proposed to separate the fatigue damage process into two stages of crack initiation and crack propagation. The linear rule was proposed for each stage. These three early concepts (linear summation, change in endurance limit and two-stage damage process) laid the foundation for phenomenological cumulative fatigue damage models.

Linear damage rules

Miner[2] first represented the Palmgren linear damage concept in mathematical form as the LDR presented by:

$$D = \Sigma r_i = \Sigma n_i/N_{fi} \qquad (1)$$

In the LDR, the measure of damage is simply the cycle ratio with basic assumptions of constant work absorption per cycle, and characteristic amount of work absorbed at failure. The energy accumulation, therefore, leads to a linear summation of cycle ratio or damage. Failure is deemed to occur when $\Sigma r_i = 1$, where r_i is the cycle ratio corresponding to the ith load level, or $r_i = (n/N_f)_i$. Damage vs cycle ratio plot (the damage curve or *D–r* curve as it is usually called) for this rule is simply a diagonal straight line, independent of loading levels. In a *S–N* diagram, the residual life curves corresponding to different life fractions are essentially parallel to the original *S–N* curve at failure. The main deficiencies with LDR are its load-level independence, load-sequence independence and lack of load-interaction accountability. In 1949, Machlin[17] proposed a metallurgically based cumulative damage theory, which is basically another form of LDR. In 1950s, Coffin and co-workers[18,19] expressed the LDR in terms of plastic strain range, which is related to fatigue life through the Coffin–Manson relation. In a later study, Topper and Biggs[20] used the strain-based LDR to correlate their experimental results. A review on the applications of the LDR to strain-controlled fatigue damage analysis was given by Miller[21] in 1970. However, due to the inherent deficiencies of the LDR, no matter which version is used, life prediction based on this rule is often unsatisfactory. Experimental evidence under completely reversed loading condition often indicates that $\Sigma r_i > 1$ for a low-to-high (L–H) loading sequence, and $\Sigma r_i < 1$ for a high-to-low (H–L) loading sequence.

Marco–Starkey theory

To remedy the deficiencies associated with the LDR, Richart and Newmark[22] introduced the concept of damage curve (or *D–r* diagram) in 1948 and speculated that the *D–r* curves ought to be different at different stress-levels. Upon this concept and the results of load sequence experiments, Marco and Starkey[23] proposed the first nonlinear load-dependent damage theory in 1954, represented by a power relationship, $D = \Sigma r_i^{x_i}$, where x_i is a variable quantity related to the *i*th loading level. The *D–r* plots representing this relationship are shown in *Figure 1*. In this figure, a diagonal straight line represents the Miner rule, which is a special case of the above equation with $x_i = 1$. As illustrated by *Figure 1*, life calculations based on Marco–Starkey theory would result in $\Sigma r_i > 1$ for L–H load sequence, and in $\Sigma r_i < 1$ for H–L load sequence.

Damage theories based on endurance limit reduction

On the other hand, the concept of change in endurance limit due to prestress exerted an important influence on subsequent cumulative fatigue damage research. Kommers[24] and Bennett[25] further investigated the effect of fatigue prestressing on endurance properties using a two-level step loading method. Their experimental results suggested that the reduction in endurance strength could be used as a damage measure, but they did not correlate this damage parameter to the life fraction. This kind of correlation was first deduced by Henry[26] in 1955 and later by Gatts[27,28], and Bluhm[29]. All of these damage models based on endurance limit reduction are nonlinear and able to

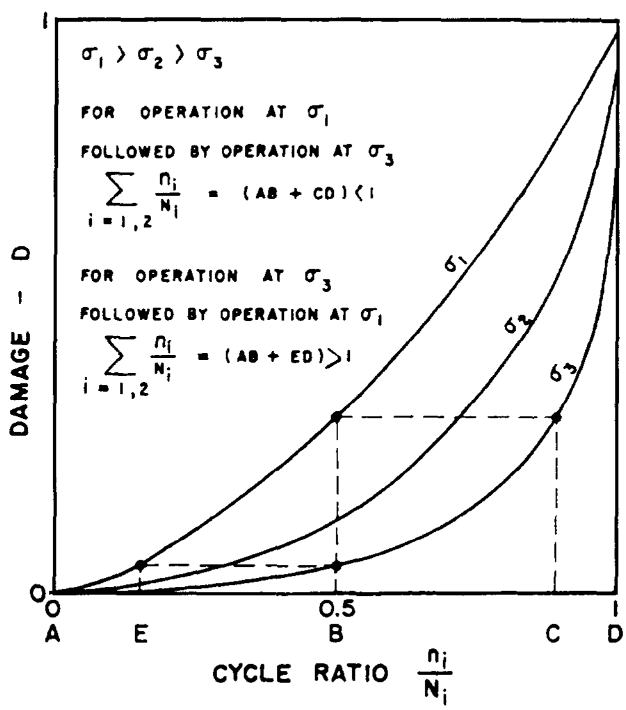


Figure 1 Schematic representation of damage vs cycle ratio for the Marco–Starkey theory[23]

account for the load sequence effect. Some of these models can also be used for predicting the instantaneous endurance limit of a material, if the loading history is known. None of these models, however, take into account load interaction effects.

Early theories accounting for load interaction effects

These theories include Corten–Dolon model[30] and Freudenthal–Heller[31,32] approach. Both theories are based on the modification of the *S–N* diagram, which is simply a clockwise rotation of the original *S–N* line around a reference point on the line. In the Corten–Dolon model, a point corresponding to the highest level in the load history is selected as the reference point, while in the Freudenthal-Heller approach, this reference is chosen at the stress level corresponding to a fatigue life of 10^3–10^4 cycles. Later, Spitzer and Corten[33] attempted to further improve the Corten–Dolon approach. They suggested to obtain the slope of the modified *S–N* line from the average result of a few repeated two-step block tests. With rotating bending specimens of SAE 4130 steel, Manson *et al.*[34,35], also examined the approach based on the *S–N* line rotation and convergence concept. They suggested that a point corresponding to a fatigue life between 10^2 and 10^3 cycles on the original *S–N* line can be selected as the convergence point. Their approach also provides a method for predicting the reduction in endurance limit due to precycling damage, and is therefore able to account not only for the load interaction effect, but also for small cycle damage. *Figure 2* shows a schematic representation for two-level L–H and H–L stressing. In these figures, the Miner rule is represented by the solid lines which are parallel to the original *S–N* curves. It can be seen that the LDR and the *S–N* line rotation approaches differ in their abilities to account for the load interaction effects.

Two-Stage linear damage theories

The two-stage linear damage approach improves on the LDR shortcomings, while still retains its simplicity in form. Following Langer's concept[16], Grover[36] considered cycle ratios for two separate stages in the fatigue damage process of constant amplitude stressing:

1. damage due to crack initiation, $N_I = \alpha N_f$; and
2. damage due to crack propagation, $N_{II} = (1 - \alpha)N_f$, where α is a life fraction factor for the initiation stage.

In either stage, the LDR is then applied. Manson[37] reverted to Grover's work and proposed the double linear damage rule (DLDR) in 1966. This damage model and its applications were further examined and discussed in Ref. 38. In the original version of DLDR, the two stages were separated by equations of: $N_I = N_f - PN_f^{0.6}$ and $N_{II} = PN_f^{0.6}$, where P is a coefficient of the second stage fatigue life. A graphical representation of DLDR applied to a H–L two-level step load sequence is shown in *Figure 3*. Recently, Bilir[39] carried out an experimental investigation with two-level cycling on notched 1100 Al specimens. A reasonable agreement between predictions by the DLDR and the experimental data was obtained.

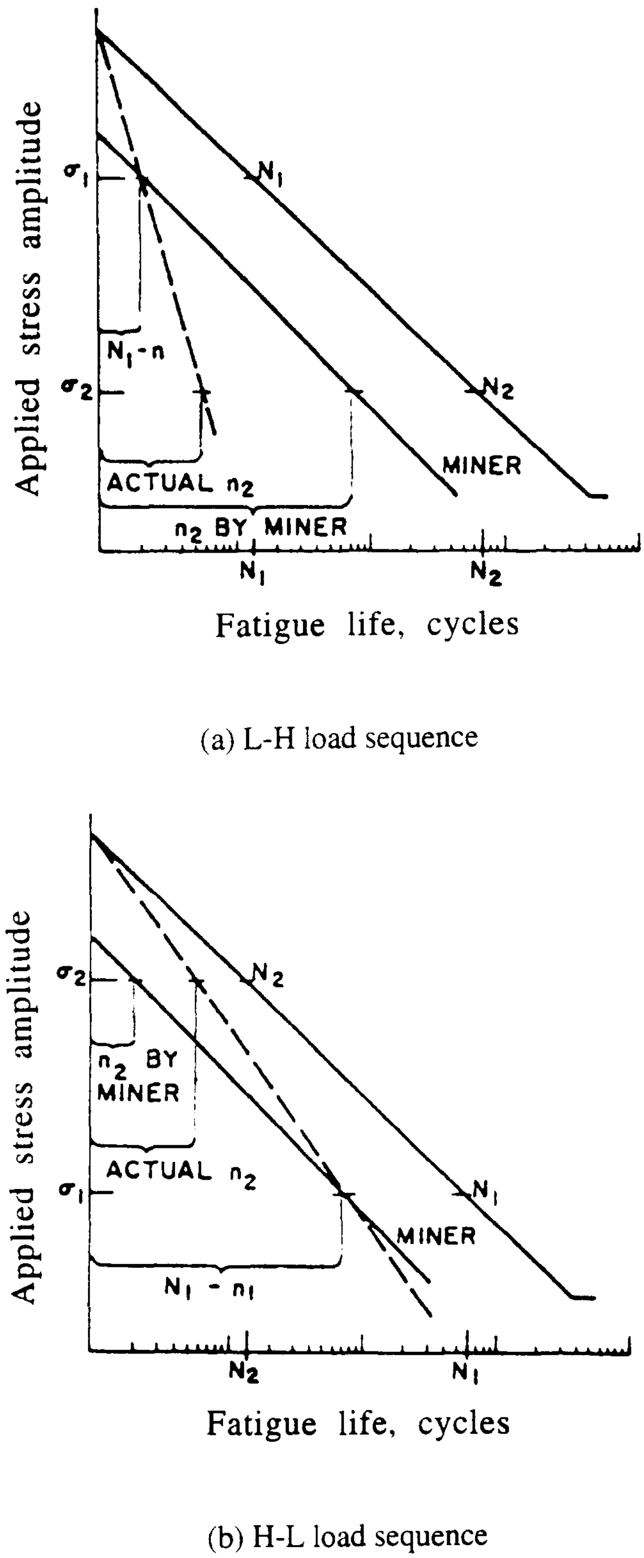


Figure 2 Schematic representation of fatigue behavior by the rotation method and by the Miner rule for (a) L–H, and (b) H–L load sequences[35]

Damage theories based on crack growth concept

Another approach in cumulative fatigue damage analysis is the crack growth concept. On the basis of the mechanism of progressive unbounding of atoms as a result of reversed slip induced by stress cycling, Shanley[40] introduced a damage theory by defining crack length as a damage measure in 1952. It was suggested that the crack growth rate varies with the

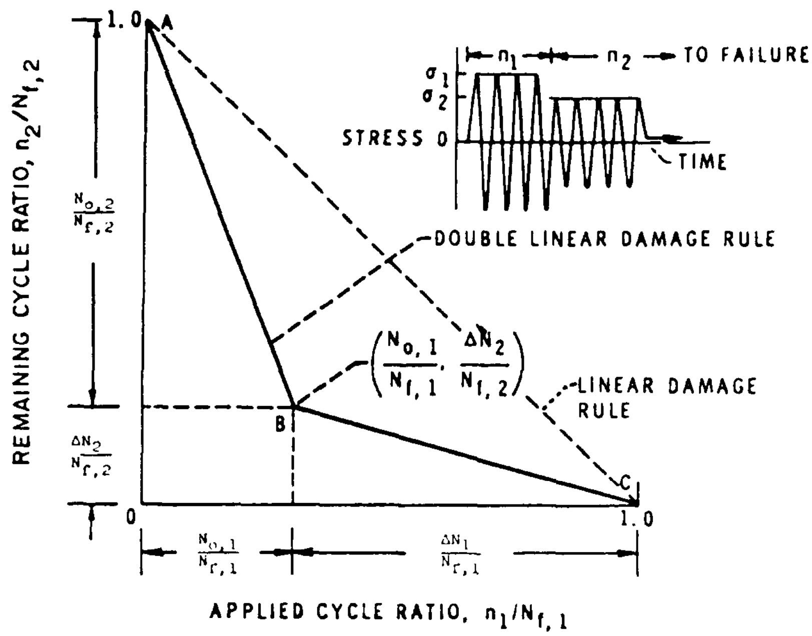


Figure 3 Illustration of the double linear damage rule for H–L two-level load cycling[38]

applied stress level in either a linear or an exponential manner. Valluri[41,42] presented a crack growth damage model in a differential form in 1961. The quantitative development of the theory is based on concepts derived from dislocation theory and a synthesis of the macroscopic elasto-plastic fracture theory. The equation formulated is in a form similar to that expressed by linear elastic fracture mechanics (LEFM): $da/dN = Cf(\sigma)a$, where a is the crack length, C is a constant and $f(\sigma)$ is a function which depends on the material and loading configuration. Another damage theory using crack growth concept was formulated by Scharton and Crandall[43] in 1966. Its mathematical expression is represented by: $da/dN = a^{m+1}f(\sigma_{ij})$, where m is a material constant.

DAMAGE CURVE APPROACH, REFINED DOUBLE LINEAR DAMAGE RULE AND DOUBLE DAMAGE CURVE APPROACH

The DCA, refined DLDR, and double damage curve approach (DDCA) were developed by Manson, Halford, and their associates[12,44] and have many common features.

Damage curve approach

This approach was developed to refine the original DLDR through a reliable physical basis. It is recognized that the major manifestation of damage is crack growth which involves many complicated processes such as dislocation agglomeration, subcell formation, multiple micro-crack formation and the independent growth of these cracks until they link and form a dominant crack. Based on this phenomenological recognition, Manson and Halford[44] empirically formulated the 'effective crack growth' model that accounts for the effects of these processes, but without a specific identification. This model is represented by:

$$a = a_o + (a_f - a_o)r^q \tag{2}$$

where a_o, a and a_f are initial ($r = 0$), instantaneous, and final ($r = 1$) crack lengths, respectively; and q is a function of N in the form $q = BN^{\beta}$ (B and β are two material constants). Damage is then defined as the ratio of instantaneous to final crack length, $D = a/a_f$. In most cases, $a_o = 0$, and the damage function of the DCA simply becomes:

$$D = r^q \tag{3}$$

Obviously, this form is similar to the Marco–Starkey theory[23]. Through a series of two-level tests, the constant β can be determined from the slope of the regression line of the experimental data: that is, $\log[\log(1 - r_2)/\log r_1]$ vs $\log(N_1/N_2)$. A value of $\beta = 0.4$ was determined in Ref. 44. Furthermore, if a reference level, N_r, is selected, the other constant, B, can then be expressed as $N_r^{\ \beta}$. Therefore, the exponent q in Equation (3) can be written as $q = (N/N_r)^{\beta}$, which is load level dependent.

Refined double linear damage rule

The original DLDR can be refined by linearization of damage curves defined by DCA model. In the refined DLDR, the knee points in a damage vs cycle-ratio (D–r) plot, which divide the damage process into two phases, are determined by:

$$D_{knee} = A(N_r/N)^{\alpha} \text{ and } r_{knee} = 1 - (1 - A)(N_r/N)^{\alpha} \tag{4}$$

where A and α are two constants determined from regression analysis of the experimental data. The empirical values of these two constants were found to be $A = 0.35$ and $\alpha = 0.25$ for high strength steels[12,44]. Shi *et al.*[45], have recently used a similar approach to define the knee points. They proposed a knee point coordinate formula based on the two-stage damage rule.

Double-Damage curve approach

This approach is developed by adding a linear term to the DCA equation with some mathematical manipulation and can be presented as:

$$D = [(pr)^k + (1 - p^k)r^{kq}]^{1/k} \tag{5}$$

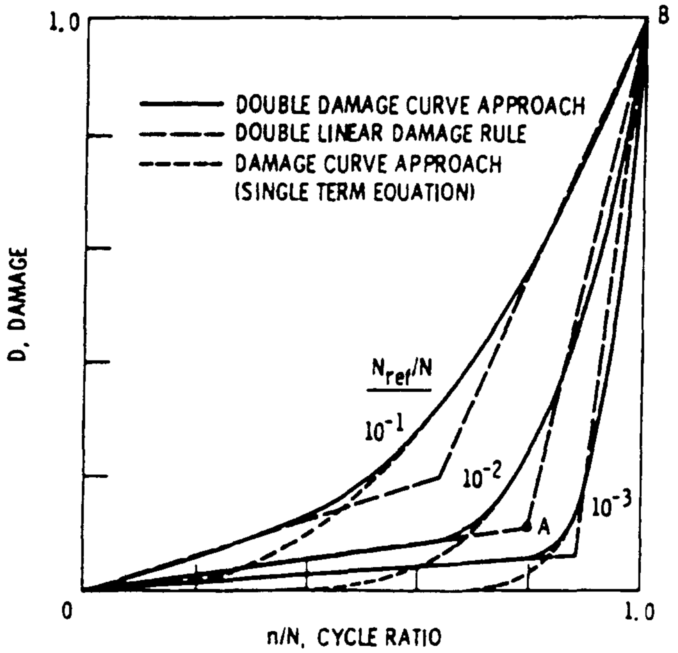


Figure 4 Comparison of the DDCA with DLDR and DCA[12]

where k is a mathematical exponent to give a close fit to the double linear damage line, and p is a constant measured from the slope of the first damage accumulation line in DLDR:

$$p = \frac{D_{\text{knee}}}{r_{\text{knee}}} = \frac{A(N_r/N)^{\alpha}}{1 - (1 - A)(N_r/N)^{\alpha}} \tag{6}$$

As can be seen from *Figure 4*, the DDCA represents a continuous damage curve which conforms to the DLDR line in the early portion of the Phase I regime, but blends into the DCA curve which is also close to the DLDR in Phase II. To evaluate the effectiveness of the developed DDCA, Manson and Halford and co-workers[46–48] conducted cumulative damage experiments on both 316 stainless steel and Haynes Alloy 188. A comparison of the experimental results with the DDCA predictions indicate good agreements. *Figure 5* shows

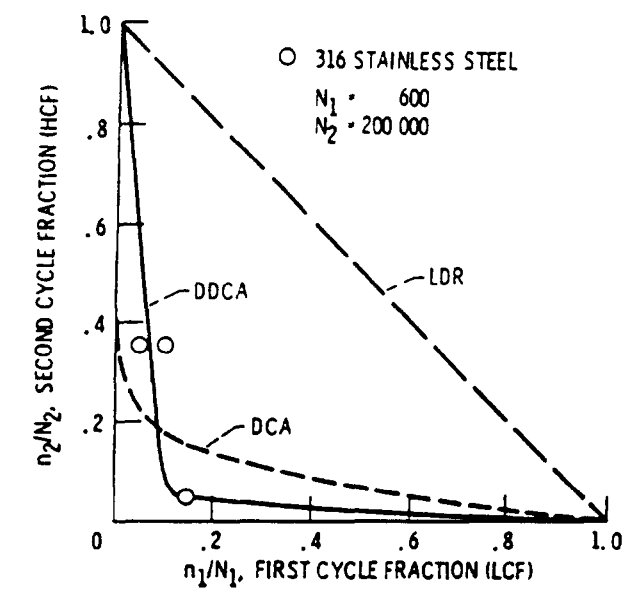


Figure 5 Improved representation of H–L load interaction tests of 316 stainless steel using DDCA as compared with LDR and DCA[12]

an improved representation of data by DDCA over LDR and DCA for 316 stainless steel tested under high-to-low loading. The DDCA has also been applied to two other materials used in the turbo pump blade of the main engines of the space shuttle[49].

The three aforementioned models possess similar characteristics. They are all load-level dependent, but do not account for the load interaction effect and small-amplitude cycle damage. With some modification in procedure, the mean stress equation by Heidmann[50] can be incorporated into these damage models. The details of this incorporation can be found in Ref. 12.

HYBRID THEORY

Bui-Quoc and colleagues presented their work dealing with cumulative fatigue damage under stress-controlled[51] and strain-controlled conditions[52] in 1971. The theory for stress-controlled fatigue was first developed from the hybridization of four prior damage models by Henry[26], Gatts[27], Shanley[40] and Valluri[41]. It was later adopted to strain-controlled cycling fatigue. Both theories were then combined into a 'unified theory'[53]. Noting the interaction effect under cyclic loading involving several stress levels[54], Bui-Quoc and co-workers.[51,55–58] modified their damage models to account for this sequence effect. These damage models had already been extended to include high temperature fatigue[59], creep[60–63] and creep–fatigue damage[64–71] conditions. They were further modified to take into account not only the effects of mean stress/strain[72], but also the effects of temperature and strain rates[73] on fatigue damage accumulation.

Stress-Controlled version

The main hypotheses in the development of this damage theory is that cracks growing in a material subjected to cyclic loading lead to a continuous reduction in fatigue strength and endurance limit. For convenience, all the parameters in this model were expressed by dimensionless ratios with respect to the original endurance limit, σ_{eo}. These include the instantaneous endurance limit ratio, $\gamma_e = \sigma_e/\sigma_{eo}$, the applied stress ratio, $\gamma = \sigma/\sigma_{eo}$, and the critical endurance limit ratio, $\gamma_{ec} = \sigma_{ec}/\sigma_{eo}$, which corresponds to failure. A differential equation for strength evaluation rate was obtained by combining three fundamental damage theories:

1. Shanley's power rule of crack growth rate in terms of the maximum cyclic stress;
2. Valluri's relation between crack growth and cyclic stress range; and
3. Gatts' damage function described by the second power of the stresses in excess of the instantaneous value of the endurance limit.

An integration of this differential equation with some mathematical manipulations results in the damage function for the stress-controlled condition as:

$$D = \frac{1 - \gamma_e}{1 - \gamma_{ec}} = \frac{r}{r + (1 - r)\dfrac{\gamma - (\gamma/\gamma_u)^m}{\gamma - 1}} \tag{7a}$$

where $\gamma_u = \sigma_u/\sigma_{eo}$, and m is a material constant. The characteristic of this equation is shown in *Figure 6* as

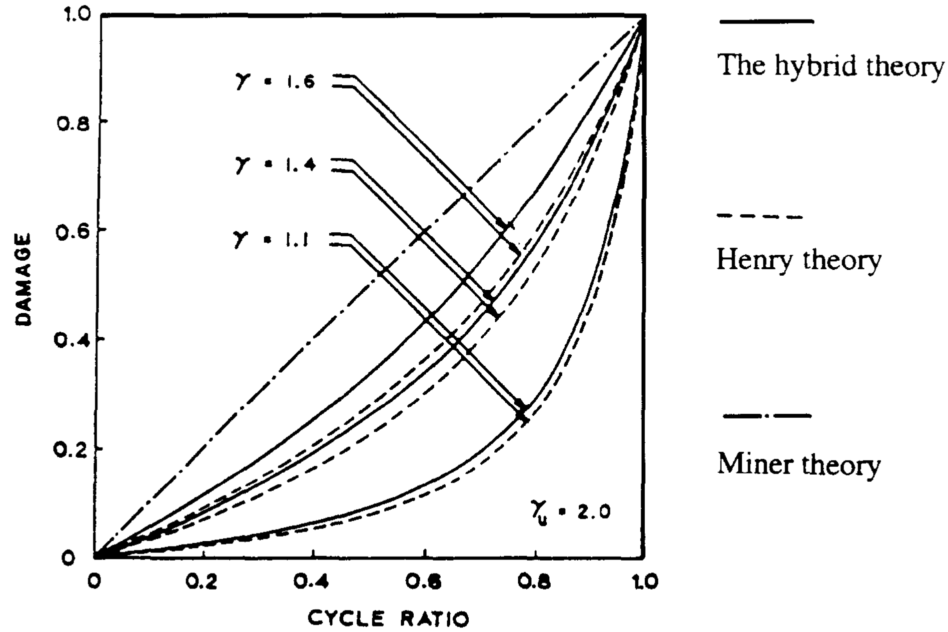


Figure 6 Characteristic of the hybrid damage function[209] and comparison with the Miner rule and the Henry theory

compared with the Henry theory and the LDR. For large γ, it is clear that the difference between the two models becomes appreciable.

Strain-Controlled version

The conversion of the stress-controlled theory to strain-controlled version was made simply by replacing the stress parameters γ_x in Equation (7)a, with the corresponding strain parameters, λ_x, which are defined as: $\lambda_x = 1 + \ln(\epsilon_x/\epsilon_{eo})$. The symbol '$x$' stands for different subscripts. Therefore, the strain-controlled version of hybrid theory can be mathematically presented as:

$$D = \frac{\ln(\epsilon_e/\epsilon_{eo})}{\ln(\epsilon_{ec}/\epsilon_{eo})} = \frac{r}{r + (1 - r)\dfrac{\lambda - (\lambda/\lambda_f)^m}{\lambda - 1}} \tag{7b}$$

where $\lambda = 1 + ln(\epsilon/\epsilon_{eo})$ and $\lambda_f = 1 + ln(\epsilon_f/\epsilon_{eo})$, in which ϵ_e, ϵ_{eo} and ϵ_{ec} are instantaneous, initial and critical strain endurance limit, ϵ is the applied maximum cyclic strain, and ϵ_f is fracture ductility or the true strain at fracture. The D–r plot of Equation (7)b is similar to *Figure 6* described by Equation (7)a.

Both Equations (7)a and (7)b give a nonlinear, load level dependent damage assessment. They also account for the effect of reduction in strain endurance limit resulting from prior strain cycling. These models improve life predictions compared to the LDR, but deviations from experimental results are still found[74], mainly due to the load interaction effects which are not accounted for by this model.

Modified version to account for load interaction effects

To account for load interaction effects, Bui-Quoc developed two approaches to improve the model. One is the fictitious load approach[51,55,57] and another is the cycle ratio modification approach[55,56]. The fictitious load approach was developed only for two-step load cycling. In this approach, there is no modification of the load parameter for the first level, λ_1. For the second load level, however, the load parameter, λ_2, is replaced by an imaginary strain, λ_2', which is, therefore, called 'fictitious load'. To determine the fictitious value, λ_2', a parameter Y used in regression analysis is proposed:

$$Y = 1 + B_1\left(\frac{|\Delta\lambda|}{\Delta\lambda^*}\right)^{B_2} r_1^{B_3} \tag{8}$$

where B_1, B_2 and B_3 are constants to be determined experimentally; $\Delta\lambda$ is the difference between strain levels: $\Delta\lambda = \lambda_2 - \lambda_1$; Y and $\Delta\lambda^*$ are sequence-related parameters defined as follows for the L–H increasing step:

$$Y = \frac{\lambda_f^* - \lambda_2}{\lambda_f^* - \lambda_2'} \text{ and } \Delta\lambda^* = \lambda_f^* - \lambda_1 \tag{9a}$$

where $\lambda_f^* = \lambda_f^{8/7}$; and for the H–L decreasing step:

$$Y = \frac{\lambda_2 - 1}{\lambda_2' - 1} \text{ and } \Delta\lambda^* = \lambda_1 - 1 \tag{9b}$$

In the cycle ratio modification approach, the damage function in Equation (7)a, (7)b is modified by introducing an exponent, v, to the cycle ratio, r^v. Therefore, v is called a load-interaction parameter. For two-step cycling, v is related to another parameter, α, by the empirical equation:

$$v = \left[1 - \left(\frac{|\Delta\lambda|}{\lambda_f - 1}\right)^{\alpha}\right]^{\Delta\lambda/|\Delta\lambda|} \tag{10}$$

where $\Delta\lambda = \lambda_2 - \lambda_1$. The value of the material constant α is in the range of 0–1. It can be experimentally determined from two-step fatigue tests, or empirically estimated by taking $\alpha = 0.5$ as a reasonable approximation[58]. This approach can be extended to multi-step loading by defining the interaction parameter v_k' ($k = 2, 3, \ldots, i$) between any two successive strain levels $k - 1$ and k in the same form as Equation (10), but with $\Delta\lambda = \lambda_k - \lambda_{k-1}$. Under the assumption that a multi-step fatigue process accumulates interaction effect as well as damage, the interaction parameter appropriate for the ith load level becomes:

$$v_i = 1 \times v_2' \times v_3' \times ... v_k' \times ... \times v_{i-1}' \times v_i' \; (k = 2, 3, ..., i) \quad (11)$$

Iterative calculations from $i = 2$ to $i = i$ ($i \geq 2$) following a similar procedure presented for two-step cycling would provide a prediction of the remaining life fraction for the ith level loading.

THEORIES USING THE CRACK GROWTH CONCEPT

The crack growth concepts developed in 1950s and 1960s have enjoyed wide acceptance since cracks are directly related to damage, and since modern technology has provided sophisticated tools and techniques which enable measurement of very small cracks in the order of 1 μm. Several macro fatigue crack growth models based on LEFM concepts were developed in the early 1970s to account for load interaction effects in the crack propagation phase (stage II) of the cumulative fatigue damage process. These models attempt to explain macrocrack growth retardation resulting from overloads under variable amplitude loading conditions. After the early 1970s, several new fatigue damage theories have been developed based on the microcrack growth concept. Though some are still phenomenological, most of these newer models better explain the physics of the damage than those developed before 1970s.

Macro fatigue crack growth models

A popular macro fatigue crack growth retardation model is the Wheeler model[75]. This model assumes the crack growth rate to be related to the interaction of crack-tip plastic zones under residual compressive stresses created by overloads. This model modifies the constant amplitude growth rate equation, $da/dN = A(\Delta K)^n$, by an empirical retardation factor, C_i:

$$da/dN = C_i[A(\Delta K)^n] \text{ where: } C_i = (r_{pi}/r_{max})^p \quad (12)$$

Here r_{pi} is the plastic zone size associated with the ith loading cycle, r_{max} is the distance from the current crack tip to the largest prior elastic–plastic zone created by the overload, and p is an empirical shaping parameter depending on material properties and load spectrum. A similar retardation model based on crack tip plasticity is the Willenborg model[76]. This model uses an effective stress intensity factor at the crack tip, $(\Delta K_{eff})_i$, to reduce the applied crack tip stress intensity factor, ΔK_i, due to the increased crack tip residual compressive stress induced by the overloads. The reduction in the applied ΔK is a function of the instantaneous plastic zone size at the ith load cycle and of the maximum plastic zone size caused by the overload. Unlike the Wheeler model however, the Willenborg model does not require an empirical shaping parameter.

Based on his experimental observations, Elber[77,78] suggested that a fatigue crack can close at a remotely applied tensile stress due to a zone of compressive residual stresses left in the crack tip wake. This results in a reduced driving force for fatigue crack growth. The crack tip stress intensity factor driving the crack is then an effective stress intensity factor based on the effective stress range, $\Delta S_{eff} = S_{max} - S_{op}$, where S_{op} is the crack tip opening stress. Other crack closure models have also been developed which include those by Newman[79,80], Dill *et al.*[81,82], Fuhring and Seeger[83] and de Koning[84]. The difficulty in using crack closure models is in determining the opening stress, S_{op}. Newman's model[79] predicts the crack opening stress by an iterative solution procedure for a cycle-by-cycle closure calculation using detailed finite element programs. In addition to the plasticity induced crack closure, other forms of fatigue crack closure can arise from corrosion (oxide-induced closure), fracture surface roughness (roughness-induced closure), and other microstructural and environmental factors as categorized by Ritchie and Suresh[85–88].

Statistical macrocrack growth models have also been proposed[89,90] in which crack growth rate is related to an effective stress intensity factor range based on probability-density curve characteristics of the load spectrum. The effective stress intensity factor range described in terms of the root-mean-square value of stress intensity factor range, ΔK_{rms}, proposed by Barsom[90] is given by:

$$\Delta K_{rms} = \sqrt{\left(\sum_{i=1}^{n} \Delta K_i^2\right)/n} \quad (13)$$

where ΔK_i is the stress intensity factor in the ith cycle for a load sequence consisting of n cycles. These models are empirical and do not account for load sequence effects such as crack growth rate retardation.

Double exponential law

For the accumulation of fatigue damage in crack initiation and stage I growth, Miller and Zachariah[91] introduced an exponential relation between the crack length and elapsed life for each phase. The approach is thus termed double exponential law. In this model damage is normalized as: $D = a/a_f$, where a and a_f are instantaneous and final crack lengths, respectively. Later, Ibrahim and Miller[92] significantly modified this model. Based on the growth mechanism of very small cracks, crack propagation behavior in stage I was then mathematically described in a manner similar to that expressed by LEFM for stage II growth as:

$$\frac{da}{dN} = \phi(\Delta\gamma_p)^\alpha a \quad (14)$$

where ϕ and α are material constants, and $\Delta\gamma_p$ is the plastic shear strain range. From this equation, a linear relationship between the initial cycle ratio, r_1, and the final cycle ratio, r_2, in two level cycling can be found for r_1 in excess of the initiation boundary $r_{I,1} = N_{I,1}/N_{f,1}$. To determine the phase boundary between initiation and stage I propagation, data from a series of two level strain-controlled tests are then collected and plotted in the $r_1 - r_2$ frame. An example of this type of plot and its comparison with the linear rule is shown in *Figure 7*. In a further study by Miller and Ibrahim[93], N_i and a_i data were correlated with the corresponding values of plastic shear strain range, $\Delta\gamma_p$, through a power function. The phase boundary in the D–r frame is then also defined through $\Delta\gamma_p$. The damage equation for stage I propagation can, therefore, be described as:

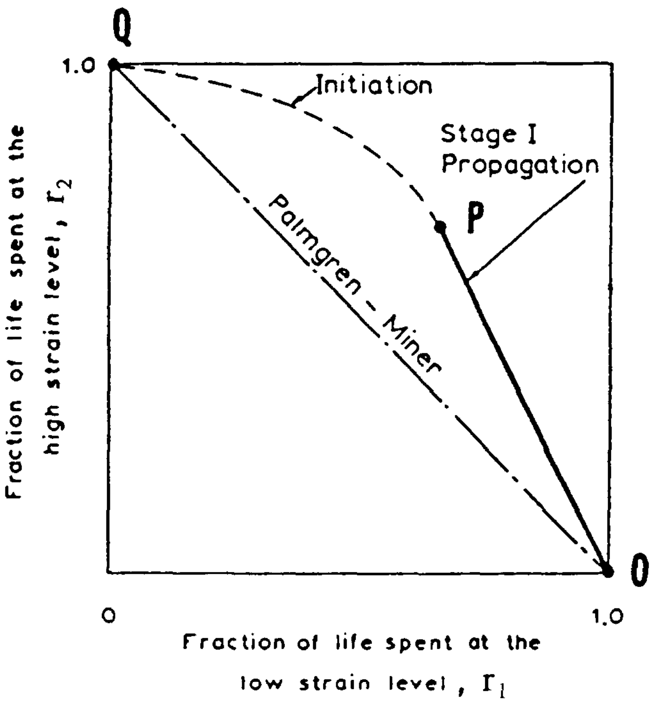


Figure 7 Schematic representation of the cumulative damage curve based on the modified Ibrahim–Miller model in a L–H two-level step test[93]

$$D = \frac{a}{a_f} = \left(\frac{a_1}{a_f}\right)^{(1-r)/(1-r_f)} \tag{15}$$

As summarized in *Figure 8* for damage lines at various strain range levels, the above equation represents a bundle of line segments radiated from point (1.0, 1.0) and terminated at the phase boundary defined by (N_I/N_f, a_I/a_f). However, the predictive model for damage

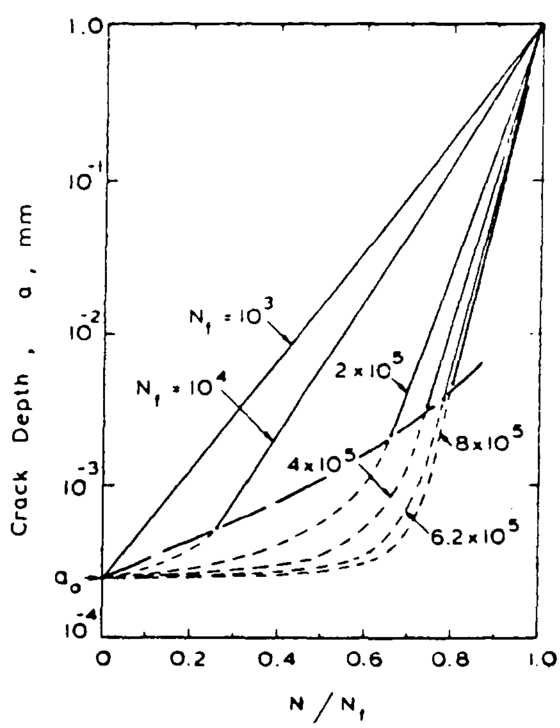


Figure 8 A summary of the accumulation of fatigue damage at various load levels based on the double exponential rule proposed by Ibrahim and Miller[93]

accumulation in the initiation phase is not yet established, but only schematically indicated by dashed curves. Difficulties of modeling this damage phase can hardly be overcome, unless the damage mechanisms of this regime are well understood.

Short crack theory

Miller and co-workers[94–101] investigated the behavior of very short cracks and proposed that crack initiation occurs immediately in metal fatigue, and that the fatigue lifetime is composed entirely of crack propagation from an initial defect size, a_o. The early two phases were renamed as microstructurally short crack (MSC) growth and physically small crack (PSC) growth, rather than as initiation and stage I propagation. Both MSCs and PSCs are elasto-plastic fracture mechanics (EPFM) type cracks. The growth behavior of MSC cracks is, however, significantly influenced by the microstructure in addition to the loading condition. The phase boundary between MSCs and PSCs, and that between EPFM and LEFM cracks are schematically represented in *Figure 9*. This is a modification of Kitagawa–Takahashi $\Delta\sigma$–a diagram by Miller[97,98]. Based on experimental observation and data analysis, crack growth models for MSCs and PSCs were established and mathematically described as[96–98,100]:

$$\frac{da}{dN} = A(\Delta\gamma)^{\alpha}(d-a) \text{ for MSCs: } a_o \le a \le a_t \tag{16a}$$

$$\frac{da}{dN} = B(\Delta\gamma)^{\beta}a - C \text{ for PSCs: } a_t \le a \le a_f \tag{16b}$$

where A, B, α and β are constants obtained by fitting of the experimental data; $\Delta\gamma$ is the shear strain range, a_t is the crack length corresponding to phase transition from MSC growth to PSC propagation, d represents the barrier size, and C is the crack growth rate at the threshold condition. Equation (16)b was also derived in Ref. 102 for high strain torsional fatigue damage accumulation. The mathematical forms of Equation (16)a, (16)b seem convenient for application to the analysis of fatigue damage accumulation. However, the physics and validity of the short crack theory still needs further experimental evidence.

Ma–Laird model

Ma and Laird[103] found that in the short crack regime, similar to the MSC region defined by Miller, crack population, P, is linearly related to the applied strain amplitude and used life, and can therefore act as a damage indicator. Based on this concept, Ma and Laird proposed a new approach to summing cumulative damage and predicting fatigue life, which is formulated as:

$$D = \Sigma(P_i/P_{crit}) = K\Sigma n_i[(\Delta\gamma_p/2)_i\alpha_i - (\Delta\gamma_p/2)_{limit}] \tag{17}$$

where $(\Delta\gamma_p/2)_{limit}$ is the fatigue limit strain, $K = C/P_{crit}$ (C is a constant in the strain-life equation), P_{crit} is the critical crack population at which failure is deemed, and α_i is the loading history factor corresponding to the ith load level. Based on the experimental findings in Ref. 104, Ma and Laird defined α_i as the ratio of

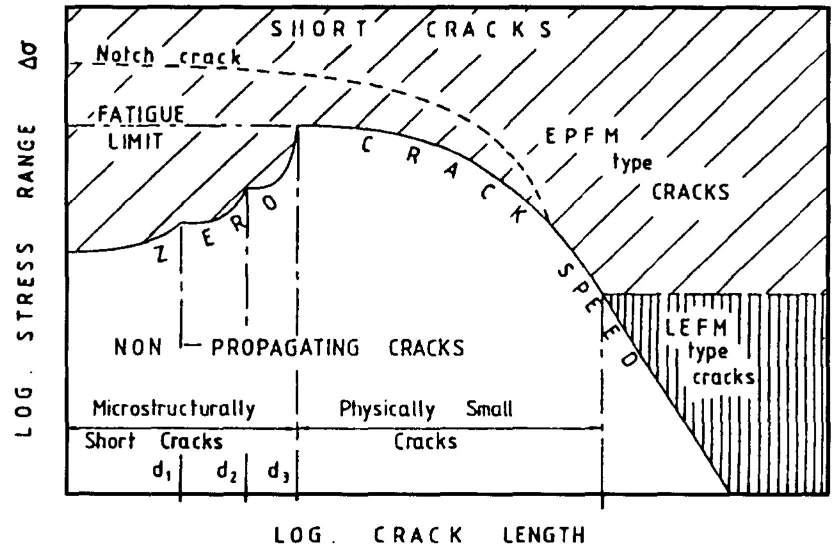


Figure 9 A modified Kitagawa–Takahashi $\Delta\sigma$–a diagram showing boundaries between MSCs and PSCs, and between EPFM cracks and LEFM cracks[98]

the currently applied strain amplitude to the maximum strain amplitude in the pre-loading history including the current cycle:

$$\alpha_i = \left(\frac{\Delta\gamma_p}{2}\right)_i \bigg/ \left(\frac{\Delta\gamma_p}{2}\right)_{\text{max. in pre-loading history}} \quad (\alpha_i \leq 1) \tag{18}$$

The model represented by Equation (17) has the ability to account for the load interaction effects. However, it should be pointed out that this model predicts a longer life for H–L strain sequence where $\alpha < 1$, than for L–H strain sequence where α is always equal to 1. This is in contradiction with the common experimental observations in completely reversed loading.

Vasek–Polak approach

Based on their experimental observations and interpretation, Vasek and Polak[105] identified two damage regimes. In the crack initiation regime, a constant crack growth rate was proposed, described by:

$$\frac{da}{dN} = \nu_i \text{ for } a_o \leq a \leq a_c \tag{19a}$$

and in the crack propagation regime, the dependence of da/dN on the crack length was approximated by a linear relation:

$$\frac{da}{dN} = \nu_i + k(a - a_c) \text{ for } a_c \leq a \leq a_f \tag{19b}$$

where ν_i is the crack growth rate independent of applied cycles, k is a coefficient, and a_o, a_c and a_f are the initial, critical, and final crack lengths, respectively. The critical crack length, a_c, defines the transition from initiation phase to propagation phase. Also, the magnitudes of ν_i, k and a_c are load level dependent. In their experiments, Vasek and Polak found the values of these three quantities to increase with increasing the loading level, and a_c was reached approximately at half-life ($r = n/N_f = 1/2$) under constant amplitude cycling. Subsequently, integrating Equation (19)a, (19)b, damage evolution functions can be explicitly expressed as:

$$D = 2D_c r \text{ for initiation: } D \leq r \leq 1/2 \tag{20a}$$

and

$$D = D_c + \frac{D_c}{m}\left[e^{m(2r-1)} - 1\right] \text{ for propagation: } 1/2 \leq r \leq 1 \tag{20b}$$

where $D_c = a_c/a_f$, and $m = kN_f/2$. This is essentially a linear-exponential model. *Figure 10* schematically represents this approach in the D–r frame.

MORE RECENT THEORIES BASED ON LIFE CURVE MODIFICATIONS

The life curve modification approaches introduced before the 1970s possess attractive features of relative simplicity in form, and effectiveness in implementation. Since 1970s several other damage rules have been developed based on life curve modifications. These

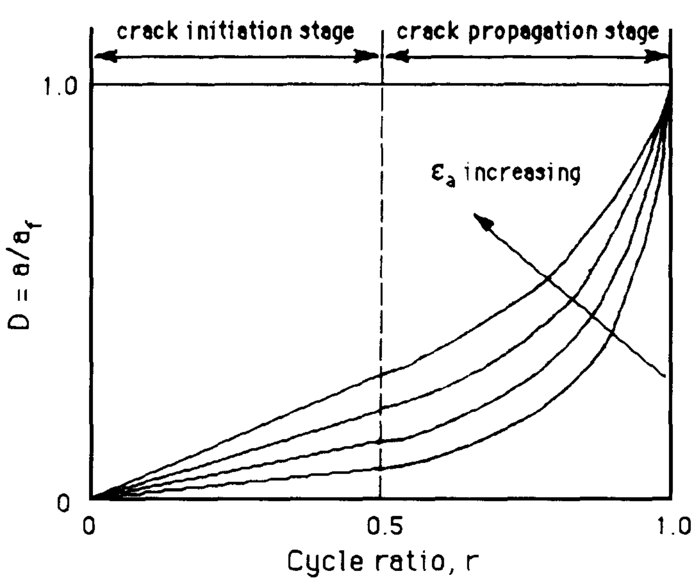


Figure 10 Schematic representation of damage functions proposed by Vasek and Polak

models are load-level dependent and can account for the load sequence effects.

Subramanyan's knee point approach

A knee point-based approach was introduced by Subramanyan[106] based on observations of experimental results. In his study, a set of isodamage lines were introduced which were postulated to converge to the endurance knee point of the S–N curve. The damage is then defined as the ratio of the slope of an isodamage line to that of the original S–N curve. This implies an assumption that the endurance limit of a material remains constant at all stages of the damage process. Mathematical expression for any isodamage line can easily be obtained from this postulation, provided the original S–N curve is linearized and has a knee point. For a loading sequence including i ($i \geq 1$) steps, a mathematical form for the residual cycle ratio at the ith level can be found:

$$r_i = 1 - \{r_{i-1} + [r_{i-2} + \ldots + (r_2 + r_1^{\alpha_1})^{\alpha_2} \ldots]^{\alpha_{i-2}}\}^{\alpha_{i-1}} \qquad (21)$$

where $\alpha_k = \log(N_{k+1}/N_e)/\log(N_k/N_e)$ for $k = 1, 2, \ldots, i - 1$. However, it should be noted that this approach is not valid at stress levels near the fatigue limit of the material. There are two reasons for this limitation. One reason is the singularity at the knee point since all the isodamage lines pass through this point. The second reason is the nonlinearity that a log–log S–N plot usually exhibits in the vicinity of the fatigue limit.

Hashin–Rotem model

Hashin and Rotem[107] presented a discussion of the S–N line convergence and rotation approaches in the framework they have devised for cumulative damage analysis. Two types of convergence were speculated. In the first model, all damage lines pass through the intersection of the original S–N line with the S-axis (called static ultimate). This approach avoids determination of the convergence point as in the earlier S–N curve modification models[30–35]. In the second model, the convergence point is at the endurance limit. Essentially, this is Subramanyan's concept, which has already been discussed. Based on the proposed approaches, Hashin and Rotem[107] performed analytical calculations on two-stage, three-stage, periodic two-stage and amplitude continuously changed cyclic loadings. Experiments were carried out by Hashin and Laird[108] with two-stage cycling and the data were used to test the effectiveness of this predictive model characterized by the endurance point convergence. Predicted results were found to be in good agreement with test data, as well as with those predicted from the double exponential damage rule.

Ben-Amoz's bound theory

Fatigue damage is a statistical phenomenon in nature and test data are inevitably scattered. Based on this argument, Ben-Amoz[109] introduced a concept of bands on residual fatigue life instead of seeking a definite form for a damage rule. This theory states that a residual life line obtained from the rotation of the original life line would fall in the upper and lower bounds. For the first approximation, Ben-Amoz represented the two bounds by Miner LDR (a parallel translation of S–N line) and Subramanyan's theory (a rotation around the endurance limit). He proved that these bounds are also applicable to nonlinear life curves. Bounds are narrowed by the inclusion of additional information from the fatigue damage process of crack initiation and propagation. This is simply a replacement of LDR with DLDR. The initiation life fractions were determined from the empirical relation given in Ref. 91. Further improvement[110], of this model was made by considering all parameters to be functions of the random variables N_1, N_2 and N_e. For these parameters in the bounds, if extreme values associated with any desired number of standard deviations are used, statistically optimized bounds will result and data scatter can be bracketed. The extent to which the bounds can bracket the data scatter depends on the choice of the number of standard deviations. Based on the mathematical analogy between the fatigue and creep cumulative damage problems, the bound theory was modified to predict creep residual time in a two-stage exposure to stresses σ_1 and σ_2 at a fixed elevated temperature. The theory was further extended to creep–fatigue interactions. A full presentation of the derivation can be found in Ref. 111.

Leipholz's approach

In agreement with Freudenthal's and Heller's opinion that the errors in life predictions based on LDR are due not to its linear summation but to the assumption of damage-rate independence of loading levels, Leipholz[112–114] resumed the concept of replacing the original S–N curve with a modified curve, S–N', which accounts for load interaction effect. Leipholz's model is represented as:

$$N_\Sigma = 1/\Sigma(\beta_i/N_i') \qquad (22)$$

where N_Σ is the total accumulated life, and β_i and N_i' are the frequency of cycles (n_i/N_Σ) and the modified life with loading level σ_i, respectively. *Figure 11* describes the typical manner in which the modified S–N curve converges to the original curve at a high loading level, and deviates from it at low loading levels. The S–N' curve is determined from multi-level repeated block tests along with Equation (22). Details of the method for obtaining the modified S–N curve are referred to Refs 112,113,115. The experimental verifications of this modified life theory were given in Refs 113 and 115. Results show that this model can provide accurate predictions of fatigue lives under

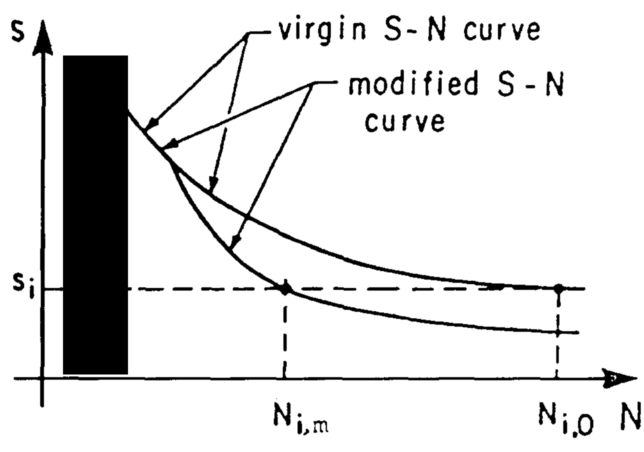


Figure 11 Schematic representation of the modified S–N curve according to the Leipholz approach[113]

repeated block loading. This predictive theory is also expanded to stochastic loading histories [113,115].

ENERGY BASED DAMAGE THEORIES

Since the report of connection between hysteresis energy and fatigue behavior by Inglis[116], many studies have been carried out on energy methods. Several failure criteria based on strain energy were established by Morrow[117] and Halford[118] in the 1960s. However, cumulative damage theories based on strain energy were mainly developed in the last two decades. Some energy-based damage parameters have been proposed such as those by Zuchowski[119] and Budiansky and O'Connell[120]. It has been realised that an energy-based damage parameter can unify the damage caused by different types of loading such as thermal cycling, creep, and fatigue. In conjunction with Glinka's rule[121], it is possible to analyze the damage accumulation of notched specimen or components with the energy approach. Energy-based damage models can also include mean stress and multiaxial loads since multiaxial fatigue parameters based on strain energy have been developed[122,123].

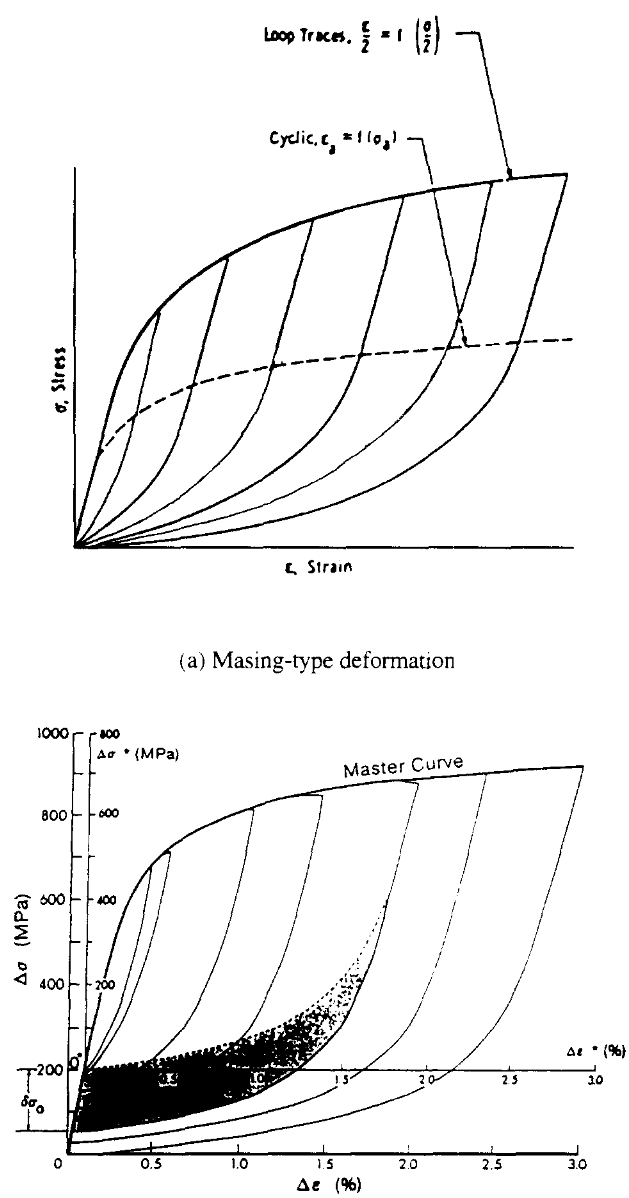
Models proposed by Ellyin and co-workers

Kujawski and Ellyin[124] developed a preliminary damage model by using plastic strain energy density as a parameter. Theoretically, plastic strain energy absorbed in a complete cycle can be obtained by integrating the area included in a hysteresis loop. It is, therefore, also referred to as the hysteresis energy and denoted by Δw^p. Another alternative is the master curve technique. It has been found[125] that there are two types of materials, Masing type and nonMasing type, as shown in *Figure 12*. For a Masing material, the master curve can directly be constructed from the cyclic stress–strain curve. For a nonMasing material, however, this is not straight-forward. Ellyin and co-workers[11,126–129] employed the Jhansale–Topper technique[130] to construct the master curve. Once the master curve is constructed, the calculation of Δw^p can be formulated.

It was later found that some inefficiencies were associated with the plastic strain energy approach. For example, the effect of mean stress cannot be directly incorporated in the determination of the hysteresis energy. Also, for the low strain high-cycle fatigue, the plastic strain energy density is very small. In some cases, though the macroscopic (bulk) response of the material is elastic or quasi-elastic, microscopic (local) plastic deformation may still exist in the material due to the nonuniformity of local strain distribution and/or due to the strain concentration by high prestraining. To overcome these shortcomings, Golos and Ellyin[11,126,127] modified the plastic strain energy-based model by using total strain energy density, Δw^t. The total strain energy density combines both plastic (Δw^p) and elastic (Δw^e) portions. The elastic portion is thought to be associated with the tensile mode and can facilitate crack growth. The calculation of Δw^e is obtained from:

$$\Delta W^e = \frac{1}{2E}\left(\frac{\Delta\sigma}{2} + \sigma_m\right)^2 \qquad (23)$$

where σ_m is the mean stress.



(a) Masing-type deformation

(b) non-Masing type deformation

Figure 12 Materials exhibiting hysteresis loops with (a) Masing-type deformation, and (b) nonMasing type deformation[126]

Regardless of the type of energy model, the concept used in damage modeling is the same. Both energy models are essentially similar to Subramanyan's convergence approach[106]. A power function analogous to an *S–N* relation was employed to describe the energy-life relation, which is a straight line in a log frame. As illustrated in *Figure 13* for a two-level load test, isodamage lines intersect the extension of the original energy-life line at the point (N_e*, Δw_e*), rather than at (N_e, Δw_e) which is the original endurance limit. The point (N_e*, Δw_e*) is, therefore, called the 'apparent' fatigue limit. There are several methods to determine the coordinates N_e* and Δw_e*. One method is based on the predictive equation of change in endurance limit such as Bui-Quoc's hypothesis[55]. Another method[131] is based on the use of the relation between the threshold stress intensity factor, ΔK_{th}, and the apparent fatigue stress limit in conjunction with the cyclic stress–strain equation. In later modifications, Ellyin *et al.*[11,126,127]

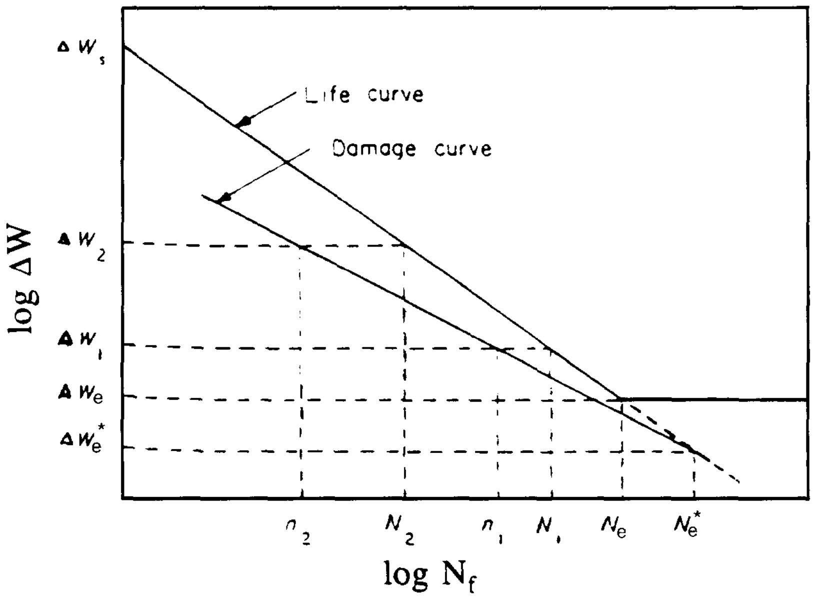


Figure 13 Damage line through 'apparent' fatigue limit defined by point (N_e*, ΔW_e*)[124]

fixed the point (N_e*, Δw_e*) at the intersection of the original energy-life curve extension with a critical damage curve which delineates the boundary between fatigue initiation and propagation phases. This critical curve can be experimentally determined, similar to the determination of the Frence curve[132]. Once the convergence point is determined, damage lines corresponding to different degrees of damage will converge to or radiate from it. The energy-based damage models proposed by Ellyin and co-workers possess features similar to Subramanyan's model. However, the energy approaches have no singularity problem at endurance limit, since the convergence point is selected at the 'apparent' fatigue limit below the original knee point.

Leis theory

Leis[133] proposed an energy-based nonlinear history-dependent damage model which links the damage parameter to fatigue life in a manner similar to the Smith–Watson–Topper parameter[134]:

$$D = \frac{4\sigma_f'}{E}(2N_f)^{2b_1} + 4\sigma_f'\epsilon_f'(2N_f)^{b_1+c_1} \tag{24}$$

where σ_f' and ϵ_f' are the fatigue strength and ductility coefficients, respectively. However, the exponents b_1 and c_1 are analogous to but different from the fatigue strength exponent, b, and fatigue ductility exponent, c. In this model, b_1 and c_1 are two variables related to the instantaneous strain-hardening exponent, n_1, through:

$$c_1 = \frac{-1}{1+5n_1} \text{ and } b_1 = \frac{-n_1}{1+5n_1} \tag{25}$$

which are asserted by an analogy to the Morrow's correlations[117]. From experimental observations, Leis speculated that the parameter n_1 can be characterized as a function of the accumulation of plastic strain, $\Sigma\Delta\epsilon_p$. The model represented by Equation (24) is therefore an analytical formulation in terms of the deformation history. Clearly, properly defining the function $n_1 = n_1(\Sigma\Delta\epsilon_p)$ is crucial in the application of this damage model.

Model developed by Niu et al.

By examining constant strain amplitude test data, Niu *et al.*[135,136] found the cyclic strain hardening coefficient to change during the cycling process, while the cyclic strain hardening exponent had a negligible change. Therefore, a new cyclic stress–strain relation was proposed as:

$$\frac{\Delta\sigma}{2} = K^*\left(\frac{\Delta\epsilon_p}{2}\right)^{n^*} r^{\beta} \tag{26}$$

where K^* and n^* are cyclic strain hardening coefficient and exponent determined near failure ($r = n/N_f = 1$), and β is the cyclic hardening rate. The expression for β was given as:

$$\beta = a\left(\frac{\Delta\sigma}{2}\frac{\Delta\epsilon_p}{2}\right)^{b} \tag{27}$$

where a and b are two constants. The incremental rate of plastic strain energy was then derived as:

$$\frac{dW}{dN} = 4\frac{1-n^*}{1+n^*}r^{\beta}K^*\left(\frac{\Delta\epsilon_p}{2}\right)^{1+n^*} \tag{28}$$

and the energy accumulation is defined by introducing a parameter called the fraction of plastic strain energy, $\Phi = W/W_f = r^{1+\beta}$. Finally, the fatigue damage function was constructed as:

$$D = \Phi^{1/[(n'+\alpha)(1+\beta)]} = r^{1/(n'+\alpha)} \tag{29}$$

where $\alpha = (\Delta\sigma\Delta\epsilon_p/4)^{2b}\sqrt{a}$, and n' is the cyclic strain hardening exponent. The model represented by Equation (29) is a nonlinear, load-dependent damage accumulation model. It accounts for the load interaction effect and the change in strain hardening through the stress response. This damage model is specially suitable for materials which exhibit cyclic hardening. Experimental verification of this model can be found in Ref. 135.

In addition to the energy approaches reviewed above, several other energy-based models also exist. In the early 1970s, Bui-Quoc[137] conducted an experimental investigation on fatigue damage with five-step loading tests. Both step-up (increasing loads) and step-down (decreasing loads) experiments were conducted on two steels. The number of strain levels in multiple-step tests did not have an influence on the cyclic strain-hardening coefficient, and the strain ratio had little effect on the cyclic stress–strain curves and a negligible effect on the total plastic energy at fracture, W_f. Based on these experimental observations, a model for calculating the value of W_f accumulated during a fatigue damage process was proposed, given by:

$$W_f = \Sigma n_i \Delta W_i = \frac{2K'M^{n'+1}}{n'+1} \Sigma r_i N_{fi}^{1-C(n'+1)} \tag{30}$$

where Δw_i is the hysteresis energy for the ith loading level, and M and c are material constants in the relation: $\Delta\epsilon_p N_f^c = M$. However, since it is found[117,118] that the total plastic energy at failure is not constant for most materials, application of this model to the cumulative fatigue damage problems is questionable.

From the viewpoint of crack growth, Radhakrishnan[138,139] postulated that the crack growth rate is proportional to the plastic strain energy density which is linearly accumulated to failure. For a m-level load variation, an expression for predicting the remaining life fraction at the last load step was, therefore, formulated as:

$$r_m = 1 - \sum_{i=1}^{m-1} \frac{W_{fi}}{W_{fm}} r_i \tag{31}$$

where W_{fi} and W_{fm} are total plastic strain energy at failure for the ith and the last (mth) levels under constant amplitude cycling, respectively. This formulation implies that failure occurs when the accumulated plastic energy reaches the value of W_{fm} in the last stage. This implication excludes the influence of load interaction effect on either W_{fi} or W_{fm}.

A concept similar to Radhakrishnan's was also proposed by Kliman[140] and applied to repeated block tests with harmonic loading cycles. A linear energy accumulation was applied. The hysteresis energy of each loading block was calculated as the sum of the multiplication of frequency, n_{bi}, by the corresponding plastic strain energy density, Δw_i, as $W_b = \Sigma \Delta w_i n_{bi}$. Apparently, this hypothesis does not consider the load sequence effects. In reality, however, it has been experimentally shown[141] that the value of W_b changes with the spectrum sequence pattern in a block. Disregarding loading sequence effects, Kliman defined the damage fraction per block as:

$$D_b = \frac{W_b}{W_{fR}} = \frac{1}{W_{fR}} \Sigma \Delta W_i n_{bi} \tag{32}$$

where W_{fR} is the total energy at fracture for a given sequence. Failure is deemed to occur when $D = D_b B_f = 1$. Based on this damage accumulation model for block loading, one can calculate the accumulated energy following a successive procedure, cycle by cycle and block by block.

CONTINUUM DAMAGE MECHANICS APPROACHES

Continuum damage mechanics is a relatively new subject in engineering mechanics and deals with the mechanical behavior of a deteriorating medium at the continuum scale. This approach is developed based on the original concepts of Kachanov[142] and Rabotnov[143] in treating creep damage problems. The general concepts and fundamental aspects of this subject can be found in Refs 144–151. Hult[146,147] and Chaboche[152] argued the importance of CDM in damage analysis. The success of CDM application in modeling the creep damage process has encouraged many researchers to extend this approach to ductile plastic damage, creep–fatigue interaction, brittle fracture and fatigue damage. In addition to metallic materials, CDM can also be applied to composites[153] and concrete materials[154]. Krajcinovic[155] and Chaboche[156,157] have reviewed the main features of the CDM approach and its applications.

For the one-dimensional case, Chaboche[158] postulated that fatigue damage evolution per cycle can be generalized by a function of the load condition and damage state. Tests conducted under completely reversed strain-controlled conditions provided supportive information. By measuring the changes in tensile load-carrying capacity and using the effective stress concept, he formulated a nonlinear damage evolution equation as[158,159]:

$$D = 1 - [1 - r^{1/(1-\alpha)}]^{1/(1+\beta)} \tag{33}$$

where β is a material constant and α is a function of the stress state. This damage model is highly nonlinear in damage evolution and is able to account for the mean stress effect. It is, therefore, called a nonlinear-continuous-damage (NLCD) model[160]. This model has three main advantages. First, it allows for the growth of damage below the initial fatigue limit, when the material is subjected to prior cycling above the fatigue limit[160]. Second, the model is able to take into account the influence of initial hardening effect by introducing a new internal variable which keeps memory of the largest plastic strain range in the prior loading history[160,161]. Third, mean stress effect is directly incorporated in the model. However, since a scalar damage variable is employed and the model is written in its uniaxial form involving the maximum and mean stresses, difficulties will inevitably be present when the model is extended to multiaxial loading conditions. The main features, advantages and some deficiencies of the NLCD model are summarized in Ref. 160.

Based on the CDM concept, many other forms of fatigue damage equations have been developed after Chaboche's work[158]. Such models include those proposed by Lemaitre and Chaboche[145,162], Lemaitre and Plumtree[163], Wang[164], Wang and Lou[165] and Li *et al.*[166] Basically, all these CDM-based approaches are very similar to Chaboche NLCD model in both form and nature. The main differences lie in the number and the characteristics of the parameters used in the model, in the requirements for additional experiments, and in their applicability.

Socie and co-workers[167,168] applied the Lemaitre–Plumtree model to the fatigue damage analysis of cast iron to account for the influence of defects on fatigue life. They reported improved life predictions as compared to the Miner rule[167]. Plumtree and O'Connor[169] attempted to analyze damage accumulation and fatigue crack propagation in 6066-T6 aluminum using a modified Lemaitre–Plumtree Model. Hua and Socie[170] also evaluated the Chaboche and the Lemaitre–Plumtree models in their investigations of biaxial fatigue. They found the Chaboche model to be better than the Lemaitre–Plumtree model for fatigue damage assessment.

The CDM models aforementioned were mainly developed for uniaxial fatigue loading. Some difficulties arise when these models are extended to multiaxial loading[160]. To overcome these difficulties, Chow and Wei[171] have recently attempted a generalized three-dimensional isotropic CDM model by introducing a damage effect tensor. However, due to the complexity of nonproportional multiaxial fatigue problems, a three-dimensional anisotropic CDM model does not yet exist. Though the framework was already proposed by Chaboche in Ref. 148, great efforts are still needed to obtain an appropriate generalized prediction model for cumulative fatigue damage.

OTHER DAMAGE THEORIES

Kramer's surface layer stress model

Recognizing that information from the surface of a fatigued material usually plays an important role in damage analysis, Kramer[172] introduced the concept of surface layer stress to characterize fatigue damage. It was postulated that during fatigue cycling, the specimen surface layer hardens due to a higher dislocation density than the interior. Consequently, to attain a given plastic strain, more stress must be imposed than would otherwise be required if the hardened layer were not present. Kramer defined this additional stress as the surface layer stress, σ_s. Under constant amplitude cycling, this stress was found to linearly increase with applied cycles, n, as: $\sigma_s = Sn$, where the proportionality coefficient S is the increase rate of σ_s and is load amplitude dependent. This coefficient can be described as: $S = K\sigma_a^p$, where K and p are material constants. As the fatigue process continues, the surface layer stress would reach a critical value, σ_s^*, when a fatal crack forms. This critical stress is found to be independent of the stress amplitude and can be expressed as: $\sigma_s^* = SN_f$. A stress-life equation was, therefore, derived:

$$\sigma_a = \frac{2\sigma_s^*}{K}(2N_f)^{-1/p} \tag{34}$$

In this equation, $2\sigma_s^*/K$ is equal to the fatigue strength coefficient σ_f' and $-1/p$ to the fatigue strength exponent, b. Considering accumulation of the surface layer stress directly to quantify the damage process, Kramer defined the damage rule simply as:

$$D = \Sigma(\sigma_{si}/\sigma_s^*) \tag{35}$$

Failure is deemed to occur when $D = 1$. Since σ_{si} is linearly developed with applied cycles as indicated by $\sigma_{si} = Sn_i$, Equation (35) is actually another version of the LDR. Based on experimental observation of the surface layer stress evolution under two-level cycling, Kramer modified the previous model to:

$$D = r_1 + r_2\left(\frac{\sigma_{a,1}}{\sigma_{a,2}}\right)^{-r_1/b} + r_3\left(\frac{\sigma_{a,2}}{\sigma_{a,3}}\right)^{r_2/b}\left(\frac{\sigma_{a,1}}{\sigma_{a,2}}\right)^{-r_1r_2/b} + \dots \tag{36}$$

According to this equation, damage evolves linearly at a load level, but sums up nonlinearly from level to level through a modification factor which keeps the memory of loading histories. Therefore, Equation (36) represents a load dependent damage model with linear evolution, nonlinear accumulation, and accounts for load interaction effect. Kramer believed the model could also be extended to corrosion–fatigue damage analysis, since corrosive attack promotes the surface layer stress. Applications of the model represented by Equation (36) to two and three step load level fatigue tests were reported by Kramer with a 2014-T6 aluminum[173] and by Jeelani *et al.* with a titanium 6AL–4V alloy[174] and a AISI 4130 steel[175].

A model based on internal and effective stresses

The concept of internal and effective stresses was generated from the discovery that the average dislocation velocity and thus the plastic strain rate is proportionally related to the effective (resolved) stress acting on a dislocation[176–179]. This effective stress, a concept different from the effective stress defined in CDM approach, is equivalent to the difference between the applied and internal (back) stresses. Ikai and coworkers[180–183] introduced this concept for the analysis of cumulative fatigue damage. As illustrated in *Figure 14*, the highest level of an elastic range (HG) is defined as the internal stress, σ_i, and the difference (HC), $\sigma_a - \sigma_i$, gives the effective stress, σ_{eff} (where σ_a is the applied stress amplitude). Based on the 'stress-dip' technique[184,185], σ_i and σ_{eff} for a given applied stress level, σ_a, can experimentally be determined.

Based on their experimental observations under both constant and variable amplitude stress cycling, Ikai *et al.* concluded that internal stress (as a result of elastic interaction of dislocations) is representative of the fatigue resistance of a material and that the effective stress above a critical value is responsible for the fatigue deformation, and hence fatigue damage in the material. It was speculated that a material under cycling

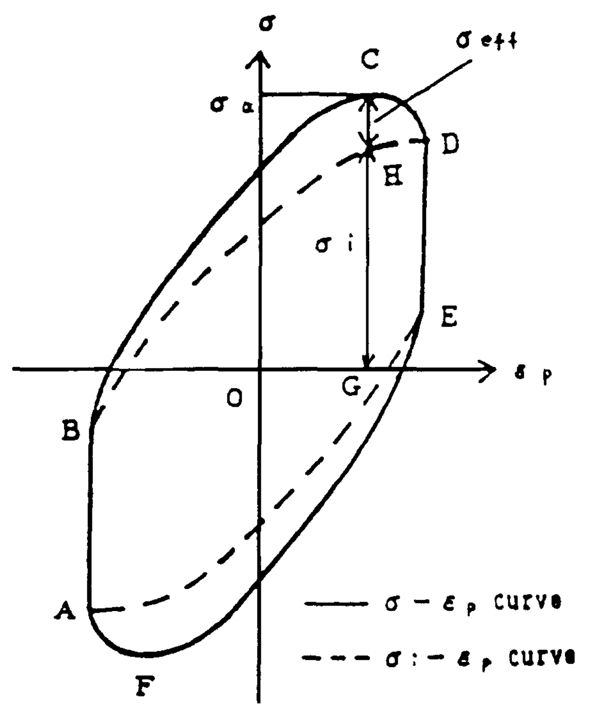


Figure 14 Illustration of internal stress and effective stress on a stress–plastic strain hysteresis loop[180]

reacts in two contradictory manners: suffering damage as a consequence of the effective stress; and being strengthened in proportion to the internal stress to endure further stressing. The internal stress was found to increase through dislocation reaction or strain aging of the material. This mechanism can probably be used to explain many phenomena in cumulative fatigue damage processes such as the coaxing effect[22,25], fatigue limit and its load history dependency, and fatigue failure caused by small cycles. In addition, the internal stress concept may provide physical interpretations for cyclic creep and cyclic stress relaxation[179,186,187], barriers for growth of MSCs[97–99,188,189], load interaction effects, cyclic hardening or softening and possibly other phenomena. However, this method can not be applied to damage assessment for fatigue crack propagation process, where internal stress measurement becomes meaningless.

On the basis of the behavior of internal stress and effective stress, Ikai and co-workers[180,183] proposed a new approach to fatigue damage accumulation. This model is simply an effective stress version of the Miner rule. In the so called ES-Miner rule, a σ_{eff}–$\log N_f$ plot is used as the baseline in damage calculation, instead of the conventional *S–N* diagram, and the Miner linear summation is then applied.

An overload damage model

Both tensile and compressive overloads are frequently encountered in real loading spectra. Brose *et al.*[190] conducted a systematic study on the effects of overloads on the fatigue behavior. They performed completely reversed strain-controlled fatigue tests on small smooth specimens with either 10 completely reversed initial overstrain cycles, or one fully reversed periodic overstrain cycle at intervals of 10^5 cycles. Recently, Topper and co-workers[191,192] carried out intensive studies on this subject by subjecting small smooth specimens to uniaxial stress histories consisting of repeated blocks, where each block contained one underload or overload and a fixed number of small fully reversed cycles. A damage model which accounts for the overload effect was then established to predict the experimental results. Their work has been reviewed by DuQuesnay[193]. In this model, damage summation was expressed by:

$$D = \Sigma D_{ol} + \Sigma D_{ss} + \Sigma D_{int} \text{ with } D = 1 \text{ at failure} \quad (37)$$

where D_{ol} is the damage due to the overloads, D_{ss} is the damage due to the smaller amplitude cycles in the absence of overloads (at a steady-state condition), and D_{int} is the additional 'interactive' damage due to the smaller cycles succeeding the overload. For a periodic overload history, the interactive damage per block is determined through:

$$(D_{int})_b = \begin{cases} D_1 \sum_{N=1}^{\eta+1} N^{\alpha} & \text{(for compressive overload)} \\ D_1 \sum_{N=1}^{\eta} N^{\alpha} & \text{(for tensile overload)} \end{cases} \quad (38)$$

where D_1 is the interactive damage due to the first smaller cycle ($N = 1$) after the overload, and the exponent α is a material constant. For an accurate prediction of fatigue life, quantities D_1 and α must be determined by experiments. This damage model accounts for the interactive damage resulting from either tensile and/or compressive overloads. It also takes into account the damage from post-overload small-amplitude cycles below the constant amplitude fatigue limit, which is normally ignored in other damage models. The results in Refs 191,192 indicate this damage model to be promising in applications.

A plastic strain evolution model

In order to represent the relationship between damage evolution and changes in mechanical properties, Azari *et al.*[194] postulated a general form of damage function, that is, $D = f(Y, X)$, where Y denotes the damage and X is the material's property. It follows that the proper selection of property and evolution of its change enable an accurate evaluation of fatigue damage and prediction of fatigue life. In their study, total strain range, $\Delta\epsilon$, was controlled and plastic strain range, $\Delta\epsilon_p$, was then chosen as an evaluation property. Therefore, damage accumulation was expressed as:

$$D = \Sigma\left(\frac{\Delta\epsilon_p - \Delta\epsilon_{po}}{\Delta\epsilon_{pf} - \Delta\epsilon_{po}}\right)_i^{-1/C} \quad (39)$$

where C is a constant, and $\Delta\epsilon_{po}$, $\Delta\epsilon_p$ and $\Delta\epsilon_{pf}$ are initial, present and final values of the plastic strain range, respectively. $\Delta\epsilon_p$ is usually a function of the applied strain, $\Delta\epsilon$, and elapsed cycles, n. For a given $\Delta\epsilon$ of constant amplitude, damage can be plotted with respect to n, which results in a damage evolution curve. The area under the curve, A, can be calculated by integration. Experiments[194] showed that the quantity A divided by the value of N_f is a constant (about 0.55). Therefore, Azari *et al.*[194] proposed a criterion for fatigue life prediction under complex loading expressed as:

$$\Sigma(A_i/N_{fi}) = \text{Constant} \quad (40)$$

The form of this model is simple and its application only requires information from constant amplitude tests. Moreover, in the constant amplitude tests, only the plastic strain response needs to be monitored. This damage model can be applied to multi-level complex loading situations. However, this model cannot account for load interaction effects and small cycle damage.

Additional approaches

In addition to the aforementioned models, there still exist other approaches for cumulative fatigue damage analysis. In order to cover as many models as possible while avoiding being lengthy, this subsection reviews the remaining approaches, but only very briefly.

Fong[195] postulated that fatigue damage rate, as a first approximation can be linearly related with damage itself. With the initial and final conditions ($D = 0$ at $r = 0$ and $D = 1$ at $r = 1$ at failure), the integration of the damage rate equation gives a damage function in the form of:

$$D = \Sigma \frac{e^{kr_i} - 1}{e^k - 1} \text{ for } k \neq 0 \quad (41)$$

The validity of this model was experimentally evaluated and compared with three other models in Ref. [170].

Its nonlinearity matches the experimental data fairly well.

To analyze complex strain histories, Landgraf[196] derived a damage equation based on the linear summation hypothesis from the strain-life equation. Damage per reversal was defined as reciprocal reversals to failure. Combined with Morrow's mean stress modification[197], the damage rate equation was derived as:

$$D/\text{reversal} = \frac{1}{2N_f} = \left[\frac{\sigma_f' - \sigma_m}{\epsilon_f' E}\left(\frac{\Delta\epsilon_p}{\Delta\epsilon_e} + \frac{\sigma_m}{\sigma_f'}\right)\right]^{1/(b-c)} \quad (42)$$

where σ_f', ϵ_f', b and c are the low cycle fatigue properties. The ratio of plastic to elastic strain range provides the experimental input. $\Delta\epsilon_p$ and $\Delta\epsilon_e$ can be determined from a block of steady-state stress–strain responses. Using $\Delta\epsilon_p/\Delta\epsilon_e$ as a damage parameter, this method entails a reversal-by-reversal damage analysis of a complex history.

Also considering variable amplitude histories representative of service load situations, Kurath *et al.*[198] examined both the effect of selected sub-cycle sequences on fatigue damage, as well as the applicability of plastic work model and *J*-integral for damage summation. Plastic work was employed to account for interaction effects. By introducing an interactive factor into each summation term in linear damage hypothesis, the damage for a block sequence with k levels was defined as:

$$D_b = \sum_{i=1}^{k} \frac{2n_i}{(2N_f)_i}\left(\frac{\Delta\sigma_i}{\Delta\sigma_h}\right)^{1/d} \quad (43)$$

where d is an interactive exponent derived from baseline data ($d = b/[b + c + 1]$ in which b and c are fatigue strength and ductility exponents), and $\Delta\sigma_h$ is the highest range of stress response in the sequence. For complex loading histories, an event is considered to be a cycle identified based on the rainflow counting technique[199,200]. The model suggests that the factor $(\Delta\sigma_i/\Delta\sigma_h)^{1/d}$ modifies the slope of the stress–life curve in a way similar to that in Corten–Dolon Theory[30]. As to the *J*-integral damage model, it characterizes crack growth rate in terms of elastic–plastic work required to open and extend the crack.

Recently, Pasic[201] attempted to combine the fracture mechanics-based damage model with the CDM approach. In his 'unified' approach, damage in stress-controlled fatigue is defined as the product of the longest or 'equivalent' surface crack length and the accumulated strain range, both in exponential form. Employing fracture mechanics, a relation between cycle ratio and evolution of normalized crack length was first derived. The normalized crack length was then related to normalized damage through the effective stress concept in CDM. Therefore, the damage evolution rule is implied by these two relations. This model, however, lacks experimental verification.

Rather than using macro damage parameters, some investigators focused their efforts to search for appropriate micro-variables which best describe fatigue initiation damage mechanisms and process. Cordero *et al.*[202] formulated cumulative fatigue damage for persistent slip band (PSB) type materials through PSB density, which is defined as the ratio of PSB length to its spacing. Then a linear summation of the normalized PSB density enables life prediction. In another study, Inoue *et al.*[203] developed a multiaxial micro-damage approach. The damage accumulation was presumed to be a function of the applied cycles and the intensity of slip bands, $\Psi_{(\vec{N})}^*$, occurring in all grains with common $\vec{N}$ (unit vector normal to the slip plane). Normalized by the intensity at the termination of crack initiation life where the maximum value of $\Psi_{(\vec{N})}^*$ is reached, a scalar damage variable was then defined by:

$$D(\vec{N}, n) = \frac{\Psi_{(\vec{N})}^*}{\Psi_{(\vec{N})_{max}}^*}\left(\frac{n}{N_I}\right)^{ck/m} \quad (44)$$

where m is the hardening exponent, and c and k are the exponents in relations: $\Delta\epsilon = g(N_i)^c$ and $\Delta\sigma = h(\Delta\epsilon)^k$, respectively. In multiaxial fatigue, $\Delta\sigma$ and $\Delta\epsilon$ are replaced by the corresponding equivalent quantities. This model assumes that crack initiation occurs at the neighborhood of the grain with slip plane $\vec{N}$, when the value of $D(\vec{N}, n)$ reaches unity.

Different from others, Abuelfoutouh and Halford[204] introduced 'resistance to flow (X),' instead of 'damage (D)', as a scalar state variable. The rate of evolution of X was defined by a state variable constitutive relation as:

$$\frac{dX}{dN} = \pm J_2^{1+b}\exp(cX + d) \quad (45)$$

where b, c and d are material constants; J_2 is the second invariant of the deviatory stress tensor; and the sign is positive for a hardening material and negative for a softening or damaged material. Integration of Equation (45) gives the evolution rule of X with respect to the applied cycles or life fraction r, which determines a resistance–life (X–r) curve somewhat similar to an upside down damage curve. For complex fatigue block loading, a concept of local iso-resistance curve was introduced into the X–r diagram. This enables a level-by-level successive fatigue life prediction. However, since these distribution curves were just conceptually described in Ref. 204, this approach cannot be applied unless these curves can practically be determined.

A correction method, called the relative Miner rule, is suggested to be a possible solution for improving fatigue life prediction accuracy for variable amplitude or service loading spectra[205–207]. The basic assumption of this method is that for sufficiently similar loading histories, the deviations from the Miner linear damage summation have the same trend and similar relative magnitudes. Under this hypothesis it is possible from the experimental results of a reference loading spectrum, to predict the fatigue life, N'', of other loading histories with sufficiently similar loading patterns to the reference pattern. If the life ratio of experimental to calculated values of the reference loading spectrum is N_{exp}'/N_{cal}', then the corrected prediction of life becomes:

$$N'' = N_{cal}''\frac{N_{exp}'}{N_{cal}'} = N_{cal}''C \quad (46)$$

where N_{cal}'' is the calculated life for the present loading condition, and C is a correction factor ($= N_{exp}'/N_{cal}'$). For improved life estimation with the application of this relative method, the considered load spectra should have basic loading parameters identical or similar to those in the reference spectrum such as maximum/minimum amplitudes, mean loads and event frequency distribution.

SUMMARY

More than 50 fatigue damage models have been proposed since the Palmgren damage accumulation concept and the Miner LDR were introduced. Most of these models are summarized in *Tables 1–8*. The physical basis, damage expression and the main characteristics of each model are listed in these tables. The abbreviations used for describing the physical basis and characteristics of each model are defined at the bottom of the tables. In general, damage theories developed before 1970s are mainly phenomenological, while those after 1970s are semi-analytical because, to some extent, they involve damage mechanism(s).

As a whole, six major categories in cumulative fatigue damage modeling exist:

1. linear damage evolution and linear summation;
2. nonlinear damage curve and two-stage linearization approaches;
3. life curve modifications to account for load interactions;
4. approaches based on crack growth concept;
5. models based on CDM; and
6. energy-based methods.

No clear boundaries exist among some of these approaches. LDRs can not account for load sequence and interaction effects due to their linear nature. The first nonlinear load-dependent damage theory represented by the power relationship, $D = \Sigma r_i^{x_i}$, was proposed by Marco and Starkey in 1954. In two-stage linearization approaches, the damage process is divided into two stages of crack initiation and crack propagation and the LDR is then applied to each stage. Life curve modification approaches are based on modifying the material *S–N* curve, are load-level dependent, and can account for the load sequence effects. Approaches based on the crack growth concept including macro crack growth retardation models have enjoyed a wide degree of acceptance since crack growth can directly be related to the physics of the damage process. CDM approaches are relatively new approaches, modeling the material damage process at the continuum scale. These approaches were originally developed to model creep damage and later extended to include the fatigue damage process. Cumulative damage theories based on energy have mainly been developed since the late 1970s and have the potential to unify the damage caused by different types of loads such as thermal cycling, creep and fatigue.

Though many damage models have been developed, unfortunately, none of them enjoys universal acceptance. Each damage model can only account for one or several phenomenological factors, such as load dependence, multiple damage stages, nonlinear damage evolution, load sequence and interaction effects, overload effects, small amplitude cycles below fatigue limit and mean stress. Due to the complexity of the problem, none of the existing predictive models can encompass all of these factors. The applicability of each model varies from case to case. Consequently, the Palmgren–Miner LDR is still dominantly used in design, in spite of its major shortcomings. Also, the most common method for cumulative damage assessment using LEFM has been based on integration of a Paris-type crack growth rate equation, with modifications to account for load ratio and interaction effects. More efforts in the study of cumulative fatigue damage are needed in order to provide design engineers with a general and reliable fatigue damage analysis and life prediction model.

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Table 1 Summary of cumulative fatigue damage theories: work before the 1970s

Model	Model developer	Year	Physical basis[a]	Expression	Characteristics[b]	Ref.
Miner LDR	Miner	1945	Constant energy absorption per cycle (CON)	$D = \Sigma n_i/N_{fi} = \Sigma r_i$	LDE, nLLD, nLIA, nSC, many Appl (popular), S	2
Machlin theory (metallurgic LDR)	Machlin	1949	Constant dislocation generation per cycle (CON)	$D = \Sigma n_i \int_0^{P/2} R_{gi}\,\mathrm{d}t$	LDE, nLLD, nLIA, nSC, nAppl, C	208
Strain version of LDR	Coffin	1956	Directly converted from stress version	$D = \frac{\Sigma n_i(\Delta\epsilon_p)_i^{1/\alpha}}{C^{1/\alpha}}$	LDE, nLLD, nLIA, nSC, some Appl, G	18
Marco–Starkey theory	Marco and Starkey	1954	Conceptual (CON)	$D = \Sigma r_{ix_i}, x_i > 1$	nLDE, LLD, nLIA, nAppl, S	23
Henry theory	Henry	1955	Endurance limit change (PHE)	$D = \frac{\sigma_{eo} - \sigma_e}{\sigma_{eo}} = \Sigma \frac{r_i}{1 + (1 - r_i)/\lambda_i}$	nLDE, LLD, nLIA, SC, few Appl, G	22
Gatts theory	Gatts	1961	Endurance limit change (PHE)	$\gamma_e = \frac{\sigma_e}{\sigma_{eo}} = \gamma_a\left[1 - \frac{1}{\frac{\gamma_a}{\gamma_{a-1}} r + \frac{\gamma_a}{\gamma_a - 1}(1 - r)}\right]$	nLDE, LLD, nLIA, SC, few Appl, C	27
Bluhm's hypothesis	Bluhm	1961	Endurance limit change (CON)	$d_n = \lvert\gamma_{(n)}\rvert - \lvert\gamma_{e(n-1)}\rvert$	nLDE, LLD, nLIA, SC, nAppl, C	29
Corten–Dolon model	Corten and Dolon	1956	Number of damage nuclei (CON)	$D = \Sigma m_i p_i n_{n_i}$	nLDE, LLD, LIA, SC, few Appl, G	30
Frudenthal–Heller theory	Frudenthal and Heller	1959	Fictitious life curve, probabilistic analysis (sANA)	$D = \Sigma\left(\frac{n_i\omega_i}{N_i}\right)$ where ω_i is an interaction factor	nLDE, LLD, LIA, nSC, some Appl, G	31
Grover's two-stage damage theory	Grover	1960	Crack initiation and crack propagation, two-stage linear evolution (CON)	$\Sigma \frac{n_i}{\alpha_i N_i} = 1$ for initiation stage $\Sigma \frac{m_i}{(1 - \alpha_i)N_i} = 1$ for propagation stage	Two-stage LDE, LLD, nLIA, nSC, few Appl, S	36
Double linear damage rule (DLDR)	Manson *et al.*	1966 1967	Crack initiation and crack propagation, two-stage linear evolution (EMP)	$\Sigma \frac{n_i}{N_{I,i}} = \Sigma \frac{n_i}{N_f - PN_f^{0.6}} = 1$ for phase I $\Sigma \frac{m_i}{N_{II,i}} = \Sigma \frac{m_i}{PN_f^{0.6}} = 1$ for phase II	Two-stage LDE, LLD, nLIA, SC, some Appl, S	37 38
Shanley theory	Shanley	1952	Crack growth, crack length as damage measure (PHE)	$D = \Sigma(\frac{a_0}{a_c})^{1-r_i}$	nLDE, LLD, nLIA, nSC, few Appl, G	40
Valluri theory	Valluri	1961	Crack growth and dislocation, fracture mechanics (sANA)	$\frac{\mathrm{d}a}{\mathrm{dN}} = \mathrm{Cf}(\sigma)\mathrm{a}$	nLDE, LLD, some Appl, G	41
Scharton–Crandall theory	Scharton and Crandall	1966	Crack growth fracture mechanics (sANA)	$\frac{\mathrm{d}a}{\mathrm{dN}} = \mathrm{a}^{m+1}\mathrm{f}(\sigma_{ij})$	nLDE, LLD, some Appl, G	43

[a]CON, conceptual; PHE, phenomenological; EXP, experimental; EMP, empirical; ANA, analytical; sANA, semi analytical
[b]LDE, linear damage evolution; LLD, load level dependent; LIA, load interaction accountable; SC, small amplitude cycle damage accountable; Appl, application(s); S, simple; G, general; C, complicated; the suffix 'n' stands for 'not' or 'non'

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Table 2 Summary of cumulative fatigue damage theories: DCA, refined DLDR and DDCA

Model	Model developer	Year	Physical basis[a]	Expression	Characteristics[b]	Ref.
Damage curve approach (DCA)[c]	Manson and Halford	1981	Effective microcrack growth (PHE)	$D = \Sigma r_{iq}$ with $q = (N_i/N_r)^{\beta}$ and $\beta = 0.4$	nLDE, LLD, nLIA, nSC, some Appl, G	44,12
Refined DLDR[c]	Manson and Halford	1981	Based on DCA and linearization (EMP)	$D_{I} = \Sigma(n_{I}/N_{I})_i\ N_{I} = N - N_{II}\ B = 0.65$ $D_{II} = \Sigma(n_{II}/N_{II})_i\ N_{II} = BN(N_r/N)^{\alpha}\ \alpha = 0.25$	Two-stage LDE, LLD, nSC, nLIA, many Appl, S	44, 12
Double damage curve approach (DDCA)[c]	Manson and Halford	1986	Based on both DCA and refined DLDR (EMP)	$D = \Sigma[(pr_i)_k + (1 - p_k)r_{kq}]^{1/k}$, $A = 0.35$, $k = 5$, $p = A(N_r/N_i)^{\alpha}/[1 - B(N_r/N_i)^{\alpha}]$, $B = 0.65$, $\alpha = 0.25$, $\beta = 0.4$	nLDE, LLD, nLIA, nSC, some Appl, C	12

[a]CON, conceptual; PHE, phenomenological; EXP, experimental; EMP, empirical; ANA, analytical; sANA, semi analytical
[b]LDE, linear damage evolution; LLD, load level dependent; LIA, load interaction accountable; SC, small amplitude cycle damage accountable; Appl, application(s); S, simple; G, general; C, complicated; the suffix 'n' stands for 'not' or 'non'
[c]The constants were obtained based on experiments with Maraging 300CVM steel, SAE 4130 steel, and Ti–6Al–4V alloy

Table 3 Summary of cumulative fatigue damage theories: hybrid theories

Model	Model developer	Year	Physical basis[a]	Expression	Characteristics[b]	Ref.
Stress version[c]	Bui-Quoc *et al.*	1971	Hybridization, endurance limit change (sANA)	$D = \frac{1 - \gamma_e}{1 - \gamma_{ec}} = \Sigma \frac{r_i}{\left(r_i + (1 - r_i)\frac{\gamma_i - (\gamma_i/\gamma_u)_m}{\gamma_i - 1}\right)}$ $m = 8$	nLDE, LLD, nLIA, SC, some Appl, C	209
Strain version[c]	Bui-Quoc *et al.*	1971	Transplanted from the stress version (sANA)	$D = \frac{1 - \lambda_e}{1 - \lambda_{ec}} = \Sigma \frac{r_i}{r_i + (1 - r_i)\frac{\lambda_i - (\lambda_i/\lambda_f)_m}{\lambda_i - 1}}$ $m = 8$	nLDE, LLD, nLIA, SC, some Appl, C	52
Fictitious load modification[c]	Bui-Quoc	1981	Endurance limit change, to account for load interaction effects (sANA)	$D = \Sigma \frac{r_i}{r_i + (1 - r_i)\frac{\lambda_i' - (\lambda_i'/\lambda_f)_m}{\lambda_i' - 1}}$ $m = 8$	nLDE, LLD, LIA, SC, few Appl, C	51,55
Cycle-ratio modification[c]	Bui-Quoc	1982	Endurance limit change, to account for load interaction effects (sANA)	$D = \Sigma \frac{r_{i'}}{r_{i'} + (1 - r_{i'})\frac{\lambda_i - (\lambda_i/\lambda_f)_m}{\lambda_i - 1}}$ $m = 8$	nLDE, LLD, LIA, SC, some Appl, C	55, 56

[a]CON, conceptual; PHE, phenomenological; EXP, experimental; EMP, empirical; ANA, analytical; sANA, semi analytical
[b]LDE, linear damage evolution; LLD, load level dependent; LIA, load interaction accountable; SC, small amplitude cycle damage accountable; Appl, application(s); S, simple; G, general; C, complicated; the suffix 'n' stands for 'not' or 'non'
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Table 4 Summary of cumulative fatigue damage theories: recent theories based on crack growth

Model	Model developer	Year	Physical basis[a]	Expression	Characteristics[b]	Ref.
Double exponential law (1st version)	Miller and Zachariah	1977	Two-stage crack growth (PHE)	$N_{I,1} = N_{f,1}\left(\frac{r_1 + r_2 - 1}{r_2}\right)$	nLDE, LLD, nLIA, nSC, a few Appl, C	91
Double exponential law (2nd version)	Ibrahim and Miller	1980 1981	Two-stage crack growth (PHE & EMP)	$r_2 = (1 - r_1)\left(\frac{1}{1 - r_{I,1}}\right)\ln\left(\frac{a_{I,1}}{a_f}\right)/\ln\left(\frac{a_0}{a_f}\right)$ $D = \frac{a}{a_f} = \left(\frac{a_f}{a_f}\right)^{(1-r)/(1-r_f)}$	nLDE, LLD, nLIA, nSC, some Appl, C	92 93
Short crack theory	Miller	1982	MSC, PSC, E–P fracture mechanics (sANA)	$\frac{da}{dN} = A(\Delta\gamma)^{\alpha}(d - a)$ for MSCs: $a_0 \le a \le a_t$ $\frac{da}{dN} = B(\Delta\gamma)^{\beta}a - C$ for PSCs: $a_t \le a \le a_t$	Clear physical basis, it is difficult to determine the micro-parameters involved	95–99
Ma–Laird theory	Ma and Laird	1989	Crack population (PHE)	$D = \Sigma(P_i/P_{crit}) = K\Sigma n_i[(\Delta\gamma_p/2)_i\alpha_i - (\Delta\gamma_p/2)_{limit}]$	LIA, few Appl, not universal, G	104
Vasek–Polak model	Vasek and Polak	1991	Microcrack kinetics equivalent crack length (PHE & sANA)	$D = 2D_c r$ for initiation: $D \le r \le 1/2$ $D = D_c + \frac{D_c}{m}[e^{m(2r-1)} - 1]$ for propagation: $1/2 \le r \le 1$	Two-stage nLDE, LLD, nLIA, nSC, few Appl, C	105

[a]CON, conceptual; PHE, phenomenological; EXP, experimental; EMP, empirical; ANA, analytical; sANA, semi analytical
[b]LDE, linear damage evolution; LLD, load level dependent; LIA, load interaction accountable; SC, small amplitude cycle damage accountable; Appl, application(s); S, simple; G, general; C, complicated; the suffix 'n' stands for 'not' or 'non'

Table 5 Summary of cumulative fatigue damage theories: models based on modifying life-curve

Model	Model developer	Year	Physical basis[a]	Expression	Characteristics[b]	Ref.
Subramanyan model	Subramanyan	1976	Convergence to the knee-point (CON)	$r_i = 1 - \{r_{i-1} + [r_{i-2} + \ldots + (r_2 + r_1^{\alpha_1})^{\alpha_2} \ldots]^{\alpha_{i-2}}\}^{\alpha_{i-1}}$	LDE, LIA, nSC, some Appl, G	106
Hashin–Rotem theory	Hashin and Rotem	1978	Two types of convergence (CON)	Formulation based on static strength point, and formulation based on endurance limit point	LDE, LIA, nSC, some Appl, G	107
Bound theory	Ben-Amoz	1990	Upper and lower bounds of convergence lines (sANA)	Bounds formed by Miner rule and Subramanyan model; bounds formed by DLDR and Subramanyan model; and statistical bounds	LDE, LIA, nSC, few Appl, C	109–111
Leipholz's approach	Leipholz	1985	Experimental determination (EXP)	Modified life curve is obtained from repeated multi-level block tests	LDE, LIA, SC, a few Appl, G, E	112,113

[a]CON, conceptual; PHE, phenomenological; EXP, experimental; EMP, empirical; ANA, analytical; sANA, semi analytical
[b]LDE, linear damage evolution; LLD, load level dependent; LIA, load interaction accountable; SC, small amplitude cycle damage accountable; Appl, application(s); S, simple; G, general; C, complicated; the suffix 'n' stands for 'not' or 'non'

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Table 6 Summary of cumulative fatigue damage theories: energy-based damage theories

Model	Model developer	Year	Physical basis[a]	Expression	Characteristics[b]	Ref.
Plastic strain energy (hysteresis energy)	Kujuwski and Ellyin	1984	Convergence, plastic strain energy, (CON)	In the plastic strain energy vs life diagram, isodamage curves converge to the apparent fatigue limit, rather than to the original limit	LDE, LIA, SC, few Appl, G, E	124
Total strain energy	Golos and Ellyin	1987	Convergence, total strain energy, (sANA)	In the total strain energy vs life diagram, isodamage curves converge to the apparent fatigue limit, rather than to the original limit	LDE, LIA, SC, some Appl, G, E	11,126,127
Bui-Quoc model	Bui-Quoc	1973	Constant total plastic energy at failure (CON)	$W_f = \Sigma n_i \Delta W_i = \frac{2K'M_{n'+1}}{n'+1} \Sigma r_i N_{fi}^{1-C(n'+1)}$	Energy version of LDR, few Appl, G	137
Radhakrishnan approach	Radhakrishnan	1978	Crack growth rate is related to plastic energy (CON)	$r_m = 1 - \sum_{i=1}^{m-1} \frac{W_{fi}}{W_{fm}} r_i$	Another energy version of LDR, few Appl, G	138
Kliman Theory	Kliman	1984	Block spectrum, similar basis to the above (CON)	$D_b = \frac{W_b}{W_{fR}} = \frac{1}{W_{fR}} \Sigma \Delta W_i n_{bi}$	Also energy version of LDR, a few Appl, G	140
Niu theory	Niu *et al.*	1987	Strain hardening and plastic strain energy (sANA)	$D = \Phi^{1/[(n'+\alpha)(1+\beta)]} = \Sigma r_i^{1/(n'+\alpha)}$	nLDE, LLD, LIA, SC, some Appl, C	135, 136
Leis model	Leis	1988	Related to two exponents in strain-life equation (CON)	$D = \frac{4\sigma_f'}{E} (2N_f)^{2b_1} + 4\sigma_f'\epsilon_f'(2N_f)^{b_1+c_1}$	nLDE, LLD, LIA, SC, nAppl, G	133

[a]CON, conceptual; PHE, phenomenological; EXP, experimental; EMP, empirical; ANA, analytical; sANA, semi analytical
[b]LDE, linear damage evolution; LLD, load level dependent; LIA, load interaction accountable; SC, small amplitude cycle damage accountable; Appl, application(s); S, simple; G, general; C, complicated; the suffix 'n' stands for 'not' or 'non'

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Table 7 Summary of cumulative fatigue damage theories: continuum damage mechanics approaches

Model	Model developer	Year	Physical basis[a]	Expression	Characteristics[b]	Ref.
Chaboche model	Chaboche	1974	All these models are based on the effective stress concept in CDM. The differences are the number of variables assumed in the damage rate equation and boundary conditions	$D = 1 - [1 - r^{1/(1-\alpha)}]^{1/(1+\beta)}$	All are in similar characteristics such as nLDE, LLD, potential to account for SC and LIA, increasing Appl, Convenience depends on the parameters used	158
Lemaitre–Plumtree model	Lemaitre and Plumtree	1979		$D = 1 - (1 - r)^{1/(1+p)}$		163
Lemaitre–Chaboche model	Lemaitre and Chaboche	1990		$D = \Sigma r_i^{1/(1-\alpha)}$		145,162
		1992		$D = 1 - (1 - r)^{1/[1+\alpha(R)]}$		164
Wang model	Wang	1990		$D = D_c - (D_c - D_0)(1 - r)^{1-\beta}$		165
Wang-Lou model	Wang and Lou					
Li–Qian–Li model	Li *et al.*	1989	Besides above, a dislocation variable is involved (ANA)	$D = 1 - (\frac{\lambda_i}{\lambda_0})_p$	Similar characteristics to above, nAppl, C	166
Three-dimensional CDM model	Chow and Wei	1991	CDM approach in three-dimensional space (ANA)	A damage effective tensor was introduced and a generalized three-dimensional isotropic CDM model was proposed based on effective stress concept	Similar characteristics, nAppl, very C	171

[a]CON, conceptual; PHE, phenomenological; EXP, experimental; EMP, empirical; ANA, analytical; sANA, semi analytical
[b]LDE, linear damage evolution; LLD, load level dependent; LIA, load interaction accountable; SC, small amplitude cycle damage accountable; Appl, application(s); S, simple; G, general; C, complicated; the suffix 'n' stands for 'not' or 'non'

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Table 8 Summary of cumulative fatigue damage theories: other approaches

Model	Model developer	Year	Physical basis[a]	Expression	Characteristics[b]	Ref.
Surface layer stress approach	Kramer	1974	Surface layer stress change (PHE)	$D = \Sigma(\sigma_{si}/\sigma_s^*)$	LDE, LIA, SC, some Appl, G	172,173
ES–Miner rule	Ikai *et al.*	1989	Internal stress and effective stress evolutions (sANA)	Applied stress can be resolved into internal and effective stresses. The internal stress is representative of the fatigue resistance of a material while the effective stress is responsible for the fatigue damage	LDE, LIA, SC, few Appl, C, E	180–183
Overload damage model	Topper *et al.*	1990	Crack opening and closure (PHE)	$D = \Sigma D_{ol} + \Sigma D_{ss} + \Sigma D_{int}$	LLD, LIA, SC, some Appl, G, E	191,192
Plastic strain evolution model	Azari *et al.*	1984	Plastic strain evolution and accumulation (PHE)	$D = \Sigma\left(\frac{\Delta\epsilon_p - \Delta\epsilon_{po}}{\Delta\epsilon_{pf} - \Delta\epsilon_{po}}\right)_{i-1/C}$	nLDE, LLD, nLIA, nSC, few Appl, G	194
Fong theory	Fong	1982	Assuming a linear damage rate (CON)	$D = \Sigma(e_{kr_i} - 1)/(e_k - 1)$ for $k \neq 0$	nLDE, LLD, few Appl, G	195
Landgraf model	Landgraf	1973	Strain version of LDR involving mean stress (CON)	$D/\text{reversal} = \frac{1}{2N_f} = \left[\frac{\sigma_f' - \sigma_m}{\epsilon_f' E}\left(\frac{\Delta\epsilon_p}{\Delta\epsilon_e} + \frac{\sigma_m}{\sigma_f'}\right)\right]_{1/(b-c)}$	LDE, nLIA, nSC, σ_m involved, few Appl, G	196
Plastic work based damage model	Kurath *et al.*	1984	Plastic work, LDR, load interaction (CON)	$D_b = \sum_{i=1}^{k} \frac{2n_i}{(2N_f)_i}\left(\frac{\Delta\sigma_i}{\Delta\sigma_h}\right)_{1/d}$	LDE, LIA, nSC, few Appl, G	198
Unified approach	Pasic	1992	CDM and fracture mechanics (ANA)	This is an approach combining fracture mechanics with damage mechanics	Only conceptual nAppl, C	201
PSB version of LDR	Cordero *et al.*	1988	Persistent slip band density (PHE)	$\Sigma(n_i/N_{fi}) = DS = 1 + D_2/D_1$	LDE, nLLD, few Appl, C	202
Micro-damage mechanics model	Inoue *et al.*	1987	Involving PSB parameter (ANA)	$D(\vec{N}, n) = [\Psi^*(\vec{N})/\Psi^*(\vec{N})_{max}](n/N_f)_{ck/m}$	nLDE, LLD, few Appl, C	203
A model based on resistance-to-flow	Abuelfoutouh and Halford	1989	Change in resistance-to-flow (ANA)	$dX/dN = \pm J_{3+b}\exp(cX + d)$	nLDE, LLD, LIA, SC, few Appl, C	204
A correction approach	Buch *et al.*	1982	Using correction factor C	$N'' = N_{cal}''(N_{exp}'/N_{cal}') = N_{cal}''C$	Relying on experiment, some Appl	205,206

[a]CON, conceptual; PHE, phenomenological; EXP, experimental; EMP, empirical; ANA, analytical; sANA, semi analytical
[b]LDE, linear damage evolution; LLD, load level dependent; LIA, load interaction accountable; SC, small amplitude cycle damage accountable; Appl, application(s); S, simple; G, general; C, complicated; the suffix 'n' stands for 'not' or 'non'

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