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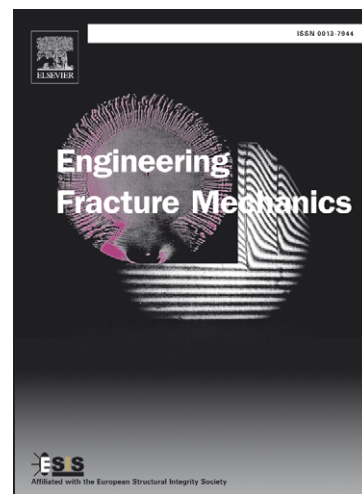
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H. Riahi, Ph. Bressolette, A. Chateaneuf

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Random fatigue crack growth in mixed mode by stochastic collocation method

Riahi H.^{a,b}, Bressolette Ph.^{a,*}, Chateauneuf A.^a

^a Clermont Université, Université Blaise Pascal, EA 3867, LaMI, BP 10448, 63000 Clermont-Ferrand, France

^b Laboratoire de Génie Mécanique, Ecole Nationale d'Ingénieurs de Monastir, 5019 Avenue Ibn El Jazzar Monastir, Tunisie

Abstract

Fatigue crack growth is uncertain, either for cracking rate or direction. The stochastic models proposed in the literature suffer from limited applicability or lack of physical meaning. In this paper, a new stochastic collocation method is proposed to solve mixed mode fatigue crack growth problem with uncertain parameters. This approach has the advantage of non-intrusive nature methods, such as Monte-Carlo simulations, since it allows us to decouple the stochastic and the mechanical computations. The proposed numerical implementation is very simple, as it requires only repetitive runs of deterministic finite element analysis at some specific points in the random space. The method describes a precise approximation of the mechanical response corresponding to the fatigue life, in order to assess the stochastic properties, namely the statistical moments and the probability density function of fatigue life. The performance of the stochastic collocation method for dealing with this kind of problems has been evaluated through two numerical examples, showing the high performance for practical applications. Moreover, the proposed method is extended in the last example to the failure probability assessment, with respect to the target service life.

Keywords: Probabilistic fatigue crack growth; Uncertainties; Stochastic collocation; Reliability.

*Corresponding author: Tel: +33 473 407 526; Fax: +33 473 407 494;
E-mail address: philippe.bressolette@polytech.univ-bpclermont.fr

1. Introduction

Most of engineering components and structures are subject to cyclic loading during their service life, which may lead to fatigue damage. Starting from a microscopic defect, fatigue crack growth takes place and may lead to catastrophic failure of structural components; it is recognized that 50% to 90% of the observed failures are due to fatigue damage and crack propagation. The fatigue process corresponds to a sequence of complex phenomena, such as work hardening and localized damage, which firstly lead to nucleation of microscopic cracks, then to a dominant macroscopic crack which grows until the remaining material cannot support the applied load. For many mechanical structures, such as aircrafts, the crack growth period represents the major part of the total fatigue life. In this context, the present work aims to assess the propagation part of the lifetime.

The process of Fatigue Crack Growth (FCG) is uncertain by nature, due to variability of material properties and composition, fluctuations of load intensity and direction, uncertainties in geometrical properties, and changes in operating conditions, such as temperature and humidity. Even in ideal conditions of laboratory testing, fatigue tests show considerable amount of scatter in crack growth [1], which can be explained by the strong dependence on the microscopic structure of the material. For this reason, the use of

traditional deterministic approaches cannot assess properly the fatigue life of structural and mechanical components. That is why the probabilistic modeling of FCG has been intensively developed in the last two decades [2-10]. An extensive list of technical literature representing the state-of-the-art concerning those models is given by Yao et al [2].

Among others, Tsurui et al [3] proposed a stochastic crack growth model based on closed form solution of Fokker-Plank equation which describes the temporal variation of crack length distribution. Bogdanoff and Kozin [4-7], have developed the stochastic model named the B-model, where the FCG is considered as a cumulative damage problem discretized in time and space. Based on the Markov chain theory, the main idea of this model is to discretize the crack size in several pre-defined states, assuming that the cracked component is subject to repetitive identical damage period (“duty cycle”) during which the accumulated damage and the probability distribution of damage at the end of each “duty cycle” depends, in a probabilistic manner, only on the “duty cycle” itself, in addition to and the amount of damage accumulated at the start of the “duty cycle”. Thus, using the mathematical proprieties of Markov chains, the probability distribution of the crack size at any time during the service life is completely determined by the transition matrix and the probability distribution of the initial crack size. It is noted that, because of its simplicity, the B-model has been widely used in various applications dealing with probabilistic fatigue crack growth [8, 9, 10]; recently it has been applied to model the fatigue damage evolution in composite materials subjected to cyclic mechanical loading [11]. However, it has faced criticism from many researchers for its lack of consistency

with respect to the physics of fatigue crack growth phenomena, since it has a purely mathematical basis [12].

Another category of stochastic models, known as physical approach, is based on randomizing the traditional crack growth equations such as that of Paris-Erdogan one, by introducing random variables or processes leading to random differential equation. We can mention the example of Yang and Manning's model [13-17]. This model has been proposed in order to overcome the lack of physical meaning of the B-model. The stochastic modeling is performed by adding a random factor to the formulation of the adopted crack propagation law. As a result, this latter was transformed into stochastic equation. It has been suggested to model the random factor by stationary lognormal stochastic process [17]. This kind of model has been widely used up to date and has inspired many other models. For example, Wu and Ni [17] and Casciati et al. [18] have employed this model to study the uncertainty of the fatigue crack growth rate with the initial crack length taken as deterministic. Beck et al [19] have combined this model with a random process approach to deal with the problem of overload failure of a structural component under random loading and under random crack growth. Although the Yang and Manning model has been the subject of several studies, the difficulty lies in obtaining a closed-form solution of the stochastic equation. To overcome this difficulty, Paris-Erdogan crack growth law is sometimes simplified in order to obtain the probability distribution of the loading cycles for any crack size reached during the FCG process. The advantage of Yan and Manning's model lies in its generality. In the same category, the polynomial model proposed by Ni [20] can be also mentioned. This model was developed, in order to obtain a better compromise between the adequate representation of the FCG

physics and the numerical simplicity. Consequently, the polynomial modeling could be included in the probabilistic modeling. Madsen et al [21] employed the approach adopted in the Yang and Manning's model but with a random factor defined by a random variable describing the variations between the mean values in different specimens and a positive random process describing variations from the mean value along the crack path within each specimen. The same model has been applied in the work of Ortiz et al [22] in conjunction with time series analysis. Sobczyk [23-25] proposed the cumulative jump model which represents the crack growth process as a discontinuous random process consisting of random number of jumps, with random magnitudes. In further studies, the cumulative jump model has been extended to model curvilinear random fatigue crack growth [25] and to deal with random fatigue crack growth problem with retardation [26]. In the stochastic model proposed by Min and Qing-Xiong [27], the material parameters of Paris-Erdogan crack growth law are considered as random variables and the distribution of the FCG rate for a given crack length is obtained by Monte-Carlo simulations. In the paper of McAllister and Ellingwood [28], a stochastic model based on Bayesian approach has been proposed to assess the fatigue damage in miter gate structures. This study has shown that the probabilistic fatigue analysis can be used to evaluate the fatigue performance of welded miter gate structures and for inspection and maintenance planning. In the same way, Shoji et al. [29] have proposed a study on the evaluation of the reliability of fuselage skin rivet splices with multiple-site fatigue cracks. The uncertain parameters of the model have been identified using a Bayesian approach based on in-service inspection data. Also, Cross et al. [30] have developed a Bayesian technique for simultaneous estimation of the equivalent initial flaw size (EIFS) and crack growth rate

distribution. This technique provides a joint posterior probability distribution of the EIFS; the parameters of the crack growth rate are obtained using samples generated by Markov Chain Monte-Carlo methods and the distribution of the fatigue life for a given crack length is obtained from these samples. Liu and Mahadevan [31] have proposed a new methodology based on equivalent initial flaw size distribution (EIFS). They have combined this methodology with probabilistic fatigue crack growth analysis to predict the fatigue life of smooth specimens. They have shown that their methodology is independent of the applied load level and the predictions are compared to experimental data for several metallic materials. Recently, Castillo et al. [32] have proposed a new probabilistic model for crack propagation under fatigue loading and they have shown the compatibility of this model with Whöler field models. The parameters of this model are obtained by experimental testing.

From the above literature review, it can be concluded that the large majority of stochastic FCG models are limited to opening crack growth mode (ie. mode I). However, in practice the FCG is mainly a mixed mode propagation and the opening mode models represent a very special case that is rarely encountered in engineering.

As an alternative to these methods, the present paper proposes a stochastic collocation method to solve FCG problems in the probabilistic framework. In addition to its numerical efficiency, the proposed method has the advantage of dealing with random crack propagation in mixed mode. The herein developed model allows us to evaluate the first statistical moments, the probability density function of the FCG and the probability of failure. As it will be shown later, this method can be easily implemented in order to

solve efficiently this kind of problems with a good level of accuracy. In addition, it provides results with low computational times compared to Monte-Carlo simulations.

The present paper is organized in four sections: section 2 describes the problem under consideration and presents the stochastic collocation as a tool to solve this problem. In section 3, we present the adopted methodology to construct the finite element model dealing with mixed mode FCG. The convergence and the effectiveness of the stochastic collocation method are investigated through numerical applications in section 4.

2. Stochastic collocation method

In this section, the propagation of uncertainties in the mechanical model, defined by finite element analysis, is considered. In this context, the model output has to be characterized in terms of the uncertainties related to the input parameters. To deal with this kind of problems, two steps are required: the first one consists of developing a probabilistic model to describe the uncertainties related to the input parameters, and the second step aims at computing the statistical characteristics (i.e. first statistical moments) and the probability density function of the output parameters. In our work, the stochastic collocation method [33, 34] is used to estimate the mean μ_z and the covariance C_z of the mechanical response. This method is mainly based on two traditional techniques, namely: the interpolation using Lagrange polynomials and the numerical integration by Gaussian quadratures. In the following, these two techniques are briefly described after rewriting the standard formulation of the problem.

2.1. Standard formulation

In the following, we consider a mechanical system whose behavior is described by a finite element model with M input random parameters y_1, \dots, y_M . Let z_1, \dots, z_N be N observations representing the mechanical responses of the considered system. The relation between the input vector $\mathbf{y} = (y_1, \dots, y_M)^T$ and the output vector $\mathbf{z} = (z_1, \dots, z_N)^T$ is defined by the finite element model. Mathematically speaking, this link can be represented by a measurable function denoted by f , defined from \mathbb{R}^M to \mathbb{R}^N and verifying the following relationship:

$$\mathbf{z} = f(\mathbf{y}) \quad (1)$$

In general, the input parameters, such as those representing the material properties (Young's modulus, toughness, Paris law parameters, ...) and the geometry (crack length, ...), can be appropriately described by lognormal probability distributions. Let \mathbf{y} be a lognormal random variable \mathbf{Y} , defined in \mathbb{R}^M with mean $\boldsymbol{\mu}_Y \in \mathbb{R}^M$, and with covariance matrix $\mathbf{C}_Y \in \mathbb{R}^{M \times M}$:

$$\boldsymbol{\mu}_Y = E[\mathbf{Y}] = \begin{bmatrix} \mu_{Y_1} \\ \mu_{Y_2} \\ \vdots \\ \mu_{Y_M} \end{bmatrix}; \quad \mathbf{C}_Y = E[\mathbf{Y}\mathbf{Y}^T] - \boldsymbol{\mu}_Y\boldsymbol{\mu}_Y^T = \begin{bmatrix} \sigma_{Y_1}^2 & C_{Y_1Y_2} & \dots & C_{Y_1Y_M} \\ C_{Y_2Y_1} & \sigma_{Y_2}^2 & \dots & C_{Y_2Y_M} \\ \vdots & \vdots & \ddots & \vdots \\ C_{Y_MY_1} & C_{Y_MY_2} & \dots & \sigma_{Y_M}^2 \end{bmatrix} \quad (2)$$

where $\mu_{Y_i} = E[Y_i]$ and $C_{Y_iY_j} = E[Y_iY_j] - \mu_{Y_i}\mu_{Y_j}$ denote respectively the mean of Y_i and the covariance of Y_i and Y_j (coordinates of rank i and j) of the vector \mathbf{Y} ($E[\cdot]$ denotes the mathematical expectation).

Before undergoing further developments, it is convenient to standardize the random variables, by probabilistic transformation \mathbf{T} to the standard Gaussian space. In our case,

the physical input parameters are represented by lognormal random variables \mathbf{Y} which may be statistically correlated. By applying the probabilistic transformation, we can get the image of the physical variables as independent normalized variables \mathbf{X} in \mathbb{R}^M , ($i = 1, \dots, M$), with equal cumulative probability function [35]. The transformation \mathbf{T} is a measurable function defined from \mathbb{R}^M into \mathbb{R}^M , such that:

$$\mathbf{Y} = \mathbf{T}(\mathbf{X}) \quad (3)$$

The general formula of the function \mathbf{T} can be found in [35]. For the case of lognormal input variables, let $\boldsymbol{\eta} = (\eta_1, \dots, \eta_M)^T$ be a real vector in \mathbb{R}^M , and $\boldsymbol{\Gamma} = [\Gamma_{ij}]$ a symmetrical and positive definite matrix in $\mathbb{R}^{M \times M}$, such that:

$$\eta_i = \ln \left(\frac{\mu_{Y_i}}{\sqrt{1 + a_{Y_i}^2}} \right); \Gamma_{ij} = \ln \left(1 + \frac{C_{Y_i Y_j}}{\mu_{Y_i} \mu_{Y_j}} \right) \quad (4)$$

with $a_{Y_i} = \frac{\sigma_{Y_i}}{\mu_{Y_i}}$. The probabilistic transformation \mathbf{T} can be expressed as:

$$\mathbf{y} = \mathbf{T}(\mathbf{x}) \Leftrightarrow \begin{cases} y_1 = \exp[\eta_1 \cdot (\mathbf{Lx})_1] \\ \vdots \\ y_M = \exp[\eta_M \cdot (\mathbf{Lx})_M] \end{cases} \quad (5)$$

with $(\mathbf{Lx})_i = \sum_{j=1}^M L_{ij} \cdot x_j$ where $\mathbf{L} = [L_{ij}]$, in $\mathbb{R}^{M \times M}$, is the lower triangular matrix derived from Choleky's factorization, such that $\boldsymbol{\Gamma} = \mathbf{L}\mathbf{L}^T$.

As the input parameters \mathbf{y} are rather probabilistic, equation (1) can be re-written in terms of the input and output random variables \mathbf{Y} and \mathbf{Z} , respectively:

$$\mathbf{Z} = \mathbf{f}(\mathbf{Y}) \quad (6)$$

By substituting Eq. (3) in Eq. (6), we can write:

$$\mathbf{Z} = \mathbf{g}(\mathbf{X}) \quad (7)$$

where \mathbf{g} is a measurable function from \mathbb{R}^M into \mathbb{R}^N such that $\mathbf{g} = \mathbf{f} \circ \mathbf{T}$.

It is clear that, to characterize the probabilistic distribution of \mathbf{Z} , we have to define the set (\mathbf{f}, \mathbf{Y}) . Let us return to the basic problem, that is to find the statistical characteristics of \mathbf{Z} .

On the basis of Eq. (7), the mean $\boldsymbol{\mu}_Z$ and the covariance matrix \mathbf{C}_Z are given by:

$$\boldsymbol{\mu}_Z = \int_{\mathbb{R}^M} \mathbf{g}(\mathbf{x}) \varphi_X(\mathbf{x}) d\mathbf{x} \quad ; \quad \mathbf{C}_Z = \int_{\mathbb{R}^M} \mathbf{g}(\mathbf{x}) \mathbf{g}(\mathbf{x})^T \varphi_X(\mathbf{x}) d\mathbf{x} - \boldsymbol{\mu}_Z \boldsymbol{\mu}_Z^T \quad (8)$$

where $d\mathbf{x} = dx_1 \dots dx_M$ and $\varphi_X = (\varphi_X(\mathbf{x}), \mathbf{x} \in \mathbb{R}^M)$ is the probability density function of the standard Gaussian variable \mathbf{X} , given by:

$$\varphi_X(\mathbf{x}) = (2\pi)^{-\frac{M}{2}} \exp\left(-\frac{\|\mathbf{x}\|_M^2}{2}\right) \quad (9)$$

where $\|\cdot\|_M$ denotes the Euclidean norm in \mathbb{R}^M .

Equations (8) are the standard formulations of the problem which can be well evaluated with a stochastic collocation method based on Lagrange polynomials with quadrature points of Gauss-Hermite type.

2.2. Interpolation by Lagrange polynomial

Let $\mathbf{x} = (x_1, \dots, x_M)$ be a set of points in \mathbb{R}^M , Π_M be the space of all M -variate polynomials with real coefficients, and Π_M^p be the subspace of polynomials of order less

or equal to p . For the set of collocation points $\{\mathbf{x}_i\}_{i=1}^q$, $i = (1, \dots, q)$, $\mathbf{x}_i \in \mathbb{R}^M$, and the set of real constants $\{b_i\}_{i=1}^q$, $b_i \in \mathbb{R}$, the Lagrange interpolating polynomial is defined by $L \in \Pi_M^q$ that satisfies $L(\mathbf{x}_i) = b_i$, $i = 1, \dots, q$.

In the one-dimensional case ($M = 1$), it is always possible to define the polynomial for any set of collocation points $\{x_i\}_{i=1}^q$:

$$L(x) = \sum_{i=1}^q b_i L_i(x) \quad (10)$$

where the set of polynomials $\{L_i\}_{i=1}^q$ forms a $(q - 1)$ basis of the subspace Π_M^q , whose elements are defined by:

$$L_i(x) = \prod_{\substack{k=1 \\ k \neq i}}^q \frac{x - x_k}{x_i - x_k} \quad (11)$$

A specificity of Lagrange polynomial is given by the following property:

$$L_i(x_k) = \delta_{ik}, \quad 1 \leq i, k \leq q \quad (12)$$

where δ_{ik} represents the Kronecker operator, which is equal to 1 when $i = k$ and 0 when $i \neq k$. It should be noted that, in the multidimensional case ($M > 1$) and for any set of collocation points $\{\mathbf{x}_i\}_{i=1}^q$, the existence and the uniqueness of Lagrange polynomials cannot be guaranteed.

In the following and in order to simplify the presentation, we will consider the case of $M = 1$. Consequently, the function g will be defined from \mathbb{R} into \mathbb{R}^N and the parameter X represents a scalar random variable. As mentioned above, the first step is to build the approximation of the function g , which is carried out by projection onto a basis

of Lagrange polynomials with order $p = q - 1$. We can therefore define the response approximation by:

$$\mathbf{g}(x) \cong \sum_{i=1}^q \mathbf{g}_i \cdot L_i(x) \quad (13)$$

This approximation is called *the stochastic response surface*. According to Lagrange polynomials property (Eq. 12), the coefficients \mathbf{g}_i are easily computed by collocation and we get $\mathbf{g}_i = \mathbf{g}(x_i)$, where x_i , ($i = 1, \dots, q$), are the collocation points. By replacing the function \mathbf{g} by its response surface (Eq. 13) in the expressions of the statistical moments $\boldsymbol{\mu}_Z$ and \mathbf{C}_Z (Eq. (8)), we get:

$$\boldsymbol{\mu}_Z = \int \mathbf{g}(x) \boldsymbol{\varphi}_X(x) dx \cong \sum_{i=1}^q \mathbf{g}_i \int L_i(x) \boldsymbol{\varphi}_X(x) dx \quad (14)$$

$$\begin{aligned} \mathbf{C}_Z &= \int \mathbf{g}(x) \mathbf{g}(x)^T \boldsymbol{\varphi}_X(x) dx - \boldsymbol{\mu}_Z \boldsymbol{\mu}_Z^T \\ &\cong \int \left(\sum_{i=1}^q \mathbf{g}_i L_i(x) \right) \left(\sum_{j=1}^q \mathbf{g}_j L_j(x) \right)^T \boldsymbol{\varphi}_X(x) dx - \boldsymbol{\mu}_Z \boldsymbol{\mu}_Z^T \\ &= \sum_{i=1}^q \sum_{j=1}^q \mathbf{g}_i \mathbf{g}_j^T \int L_i(x) L_j(x) \boldsymbol{\varphi}_X(x) dx - \boldsymbol{\mu}_Z \boldsymbol{\mu}_Z^T \end{aligned} \quad (15)$$

For the seek of simplicity, let us denote the integrals in the above equation as \mathbf{I}_i and \mathbf{I}_{ij} , such that:

$$I_i = \int L_i(x) \varphi_X(x) dx \quad ; \quad J_{ij} = \int L_i(x) L_j(x) \varphi_X(x) dx \quad (16)$$

The statistical moments μ_Z and C_Z can thus be re-written as:

$$\mu_Z \cong \sum_{i=1}^q g_i I_i \quad ; \quad C_Z \cong \sum_{i=1}^q \sum_{j=1}^q g_i g_j^T J_{ij} - \mu_Z \mu_Z^T \quad (17)$$

From equations (17), it is obvious that computing the statistical moments μ_Z and C_Z can be done by evaluating the integrals I_i and J_{ij} (Eq. 16), which can be done by choosing a convenient set of collocation points $\{x_i\}_{i=1}^q$. These points should be chosen such that the quadrature formula:

$$\int L_i(x) \varphi_X(x) dx = \sum_{k=1}^q \omega_k L_i(x_k) \quad (18)$$

becomes exact for any polynomial $L_i \in \Pi_M^{2p+1}$, where Π_M^{2p+1} is the sub-space of polynomials with order up to $2p+1$. The quantities x_k and ω_k , ($k = 1, \dots, q$), are respectively the collocation points and quadrature weights, derived from the equality in equation (18). The function φ_X are the probability density function of the variables X . In the standardized normal space, this function φ_X is simply given by:

$$\varphi_X(x) = (2\pi)^{-\frac{1}{2}} \exp\left(-\frac{\|x\|^2}{2}\right) \quad (19)$$

and equation (18) verifies the Gauss-Hermite quadrature rule.

2.3. Computation of the collocation points

The first step to compute the collocation points is to build the Gauss-Hermite orthogonal polynomial basis $\{H_k\}_{k=0}^{q+1}$, which is simply performed by applying the recurrence relationship:

$$\begin{cases} H_{k+1}(x) = x \cdot H_k(x) - k \cdot H_{k-1}(x) \\ H_0(x) = 1, H_1(x) = x \end{cases} \quad (20)$$

As stated by the Gauss quadrature theorem, the set of quadrature points $\{x_i\}_{i=1}^q$ are the roots of the polynomial H_{q+1} . From Eq. (20), we can easily observe that the roots of the polynomial H_{q+1} includes those of H_q ; we can therefore start by computing the root of H_1 to find the roots of H_2 and so on, until obtaining the roots of H_{q+1} .

After defining the collocation points $\{x_i\}_{i=1}^q$, the weights $\{\omega_i\}_{i=1}^q$ are obtained by replacing the polynomial L_i in the quadrature rule (Eq. (18)) by the polynomial H_k for $k = 0, \dots, q$, leading to a $(q+1) \times (q+1)$ linear equation system:

$$[A] \cdot \{w\} = \{b\} \quad (21)$$

where the quantities $[A]$, $\{w\}$ and $\{b\}$ are defined by:

$$[A] = \begin{bmatrix} H_0(x_0) & \dots & H_0(x_q) \\ H_1(x_0) & \dots & H_1(x_q) \\ \vdots & \ddots & \vdots \\ H_q(x_0) & \dots & H_q(x_q) \end{bmatrix}; \quad \{w\} = \begin{Bmatrix} \omega_0 \\ \omega_1 \\ \vdots \\ \omega_q \end{Bmatrix}; \quad \{b\} = \begin{Bmatrix} \int H_0(x) \varphi(x) dx \\ \int H_1(x) \varphi(x) dx \\ \vdots \\ \int H_q(x) \varphi(x) dx \end{Bmatrix} \quad (22)$$

By taking into account that $H_0(x) = 1$ and the set $\{H_k\}_{k=0}^{q+1}$ of Hermite polynomial is a orthogonal basis, the above linear system (Eq. 21) can be considerably simplified. By multiplying each element b_k of the vector $\{b\}$ by the polynomial H_0 , we can obtain, for $k = 0, \dots, q$, the following expression:

$$b_k = \int H_k(x) \varphi(x) dx = \int H_k(x) \cdot H_0(x) \varphi(x) dx = k! \delta_{k0} \quad (23)$$

Consequently, the right hand side of the linear system becomes $(1, 0, \dots, 0)^T$, and we can easily obtain the set of weights $\{\omega_i\}_{i=1}^q$ by using a standard technique of numerical solution of linear systems.

2.4. Statistical characteristics of the mechanical response

The random mechanical response can be characterized by the two first statistical moments, namely μ_Z and C_Z , and by the probability density function. The statistical moments μ_Z and C_Z of the mechanical response can be computed by Eq. (17) in terms of I_i and I_{ij} respectively. Having the sets of collocation points and integration weights, $\{x_i\}_{i=1}^q$ and $\{\omega_i\}_{i=1}^q$ respectively, these integrals can be evaluated by the quadrature rule (18).

$$I_i = \int L_i(x) \varphi_X(x) dx = \sum_{k=1}^q w_k \cdot L_i(x_k) \quad (24)$$

$$I_{ij} = \int L_i(x) L_j(x) \varphi_X(x) dx = \sum_{k=1}^q w_k \cdot L_i(x_k) \cdot L_j(x_k) \quad (25)$$

For the one-dimensional case, the statistical moments $\boldsymbol{\mu}_Z$ and \mathbf{C}_Z are given by:

$$\boldsymbol{\mu}_Z \cong \sum_{i=1}^q \sum_{k=1}^q \mathbf{g}_i \cdot w_k \cdot L_i(x_k) \cong \sum_{k=1}^q \mathbf{g}_k \cdot w_k \quad (26)$$

$$\mathbf{C}_Z \cong \sum_{i=1}^q \sum_{j=1}^q \sum_{k=1}^q \mathbf{g}_i \mathbf{g}_j^T w_k \cdot L_i(x_k) \cdot L_j(x_k) - \boldsymbol{\mu}_Z \boldsymbol{\mu}_Z^T \cong \sum_{k=1}^q \mathbf{g}_k \mathbf{g}_k^T w_k - \boldsymbol{\mu}_Z \boldsymbol{\mu}_Z^T \quad (27)$$

In the case of multidimensional random space ($M > 1$), the function \mathbf{g} is defined from \mathbb{R}^M to \mathbb{R}^N and $\mathbf{x} = (x_1, \dots, x_M)$ is the vector of realizations of the random variables, defined in \mathbb{R}^M ; the tensor product can be applied to deal with these quantities. In order to simplify the formulation, we assume that the solution is spanned by Lagrange polynomials of order $p = q - 1$, in each direction of the multidimensional random space. The stochastic response surface can thus be obtained by:

$$\begin{aligned} \mathbf{g}(\mathbf{x}) &\cong \sum_{i_1=1}^q \dots \sum_{i_M=1}^q \mathbf{g}_{i_1 \dots i_M} (L_{i_1} \otimes \dots \otimes L_{i_M})(\mathbf{x}) \\ &\cong \sum_{i_1=1}^q \dots \sum_{i_M=1}^q \mathbf{g}_{i_1 \dots i_M} L_{i_1}(x_1) \times \dots \times L_{i_M}(x_M) \end{aligned} \quad (28)$$

where \otimes denotes the tensor product of the one-dimensional polynomial spaces with fixed polynomial orders in all dimensions, \times is the usual symbol of multiplication in \mathbb{R} and

$\{L_{i_k}\}_{i_k=1}^q$ are the Lagrange polynomials of order $q - 1$ derived from the collocation points $\{x_{k,i_k}\}_{i_k=1}^q$ corresponding to the density function $\varphi_k = (\varphi_k(x_k), x_k \in \mathbb{R})$ of the

variable X_k ($k = 1, \dots, M$). As stated in the one-dimensional case, the coefficients,

$\mathbf{g}_{i_1 \dots i_M} \in \mathbb{R}^N$ are obtained by collocation, by replacing \mathbf{x} in Eq. (28) by the collocation

point $x_k = (x_{1,i_1}, \dots, x_{M,i_M})$ and using the Lagrange polynomials property

$$L_{i_k}(x_{k,i_l}) = \delta_{i_k i_l}:$$

$$g_{i_2 \dots i_M} = g(x_{1,i_1}, \dots, x_{M,i_M}) \quad (29)$$

Finally, the statistical moments μ_Z and C_Z of the mechanical response can be evaluated as following:

$$\mu_Z \cong \sum_{i_2=1}^q \dots \sum_{i_M=1}^q \omega_{i_2} \dots \omega_{i_M} g_{i_2 \dots i_M} \quad (30)$$

$$C_Z \cong \sum_{i_2=1}^q \dots \sum_{i_M=1}^q \omega_{i_2} \dots \omega_{i_M} g_{i_2 \dots i_M} g_{i_2 \dots i_M}^T - \mu_Z \mu_Z^T \quad (31)$$

Having the stochastic response surface (Eq. (13)), the probability density function can be easily constructed by Monte-Carlo simulations. This procedure is accurate and efficient since it requires very low computing time.

2.5. Numerical implementation

The numerical implementation of the stochastic collocation method can be summarized by the flowchart in figure 1. Firstly, we choose a number q of collocation points and compute the q^M responses (Eq. 29) of the mechanical system. Then, we construct the stochastic response surface (Eq. 28) using the Lagrange polynomials based on the set of collocation points. Finally, we compute the statistical moments by Gauss quadrature rule (Eqs. 30 and 31) and construct the PDF using Monte-Carlo simulations on the obtained stochastic response surface (Eq. 28).

3. Finite element modeling of mixed mode fatigue crack growth

In this section, we present the numerical model used to deal with the mixed mode fatigue crack growth.

3.1. Numerical computation of the stress intensity factor

In finite element analysis, several numerical methods have been proposed to evaluate the fracture parameters in cracked body Ω . In the present work, the stress intensity factors are calculated by the G - θ method, firstly introduced by Destuynder [36]. On the basis of virtual crack extension technique, the G - θ method gives the stress intensity factors through the computation of the strain energy release rate G , representing the decrease of the total potential energy W_p during the growth dA of the crack. Based on G - θ , the quantity G can be computed by:

$$G = -\frac{dW_p}{dA} = \int_{\Omega} \text{Tr}(\sigma \nabla \varepsilon \nabla \vec{\theta}) d\Omega - \int_{\Omega} w \text{div} \vec{\theta} d\Omega \quad (32)$$

where w denote the strain energy density, σ and ε are the stress and the displacement fields respectively, $\vec{\theta}$ is the virtual displacement vector and Tr denotes the symbol of the trace of tensor.

3.2. Bifurcation criterion

In practice, due to the loading and geometrical conditions, fatigue cracks attempt to propagate in mixed mode; in other words, the cracks follow curved paths during their propagation. In order to define the bifurcation angle of the crack during propagation,

there exist mainly three criteria: the maximum circumferential stress $\sigma_{\theta_{max}}$, the minimum strain energy density $S_{\theta_{min}}$ and the maximum energy release rate $T_{\theta_{max}}$. In our study, the maximum circumferential stress criterion $\sigma_{\theta_{max}}$ was implemented in order to predict the crack bifurcation angle. The maximum circumferential stress criterion [37] states that the crack propagate in the plane perpendicular to the direction in which the stress component $\sigma_{\theta\theta}$ is maximum.

$$\sigma_{\theta\theta} = \frac{2}{\sqrt{2\pi r}} \left[K_I (1 + \cos(\theta)) \cos\left(\frac{\theta}{2}\right) - 3 \cdot K_{II} \sin(\theta) \cos\left(\frac{\theta}{2}\right) \right] \quad (33)$$

where K_I and K_{II} are the stress intensity factors for mode *I* and *II* respectively, θ is the crack orientation angle and r is the distance from the crack tip.

Based on the maximization of Eq. (33) with respect to the crack orientation angle θ , the bifurcation angle θ_0 is obtained by:

$$\operatorname{tg}\left(\frac{\theta_0}{2}\right) = \frac{1}{4} \left(\frac{K_I}{K_{II}} \right) \pm \frac{1}{4} \sqrt{\left(\frac{K_I}{K_{II}} \right)^2 + 8} \quad (34)$$

3.3. Fatigue crack growth model

Usually, the FCG rate $\frac{da}{dN}$ is represented by an empirical equation with parameters defined by direct fitting of crack size “ a ” vs. number of loading cycles “ N ” plots obtained from experimental data. This empirical equation defines a functional relationship between the fatigue crack growth rate and different parameters such as loading and material parameters, etc. Generally, this functional relationship can be written as follow:

$$\frac{da}{dN} = f(\Delta K, R, K_{Ic} \dots) \quad (35)$$

The basic model of the fatigue crack growth rate was first proposed by Paris and Erdogan [38], who assumed that the FCG rate depends on the stress intensity factor range ΔK . The Paris-Erdogan equation corresponding to the mixed mode fatigue crack growth is written:

$$\frac{da}{dN} = C \cdot (\Delta K_{Ieq})^m \quad (36)$$

with:

$$\Delta K_{Ieq} = \cos\left(\frac{\theta}{2}\right) \cdot [\Delta K_I \cdot (1 + \cos(\theta)) - 3 \cdot \Delta K_{II} \cdot \sin(\theta)] \quad (37)$$

where a is the crack length, N is the number of loading cycles, C and m are material parameters and ΔK_{Ieq} is the equivalent stress intensity factor (ie. equivalent mode I).

This law has been extended to account for variety of parameters such as loading ratio R effect and the fracture toughness of the material K_{Ic} . In this context, Forman et al. [39] modified Eq. (36) and suggested the form:

$$\frac{da}{dN} = C \cdot \frac{(\Delta K_{Ieq})^m}{(1-R) \cdot K_{Ic} - \Delta K_{Ieq}} \quad (38)$$

To predict the fatigue life of structures experiencing cyclic loading, the number of loading cycles can be obtained by integrating the FCG rate (Eqs. 36 or 38):

$$N_c = \int_{a_0}^{a_c} \frac{1}{f(\Delta K, R, K_{Ic} \dots)} da \quad (39)$$

where a_0 is the initial crack length and a_c the crack length at failure. In this work, the integration of the FCG rate is performed numerically and the failure is observed when the

crack length reaches the critical value a_c . The steps used to perform mixed mode FCG analysis are summarized in figure 2.

3.4. Assessment of structural reliability

The stochastic collocation method allows us to construct efficiently the PDF of the fatigue life and therefore to get the analytical expression of the stochastic response surface. These two advantages allow us to perform reliability analysis either by integration of the PDF or by coupling with FORM method [21]. In this subsection, we use the FORM technique to assess the reliability of the engineering structure under consideration.

To carry out reliability analysis, one important step is to define the failure criterion that represents the frontier between the failure and the safety. According to the model developed in [40] and based on the Forman's law the safety margin is expressed as:

$$H(Y) = \int_{a_0}^{a_c} \frac{(1-R) \cdot K_c - \Delta K_{\text{eq}}}{(\Delta K_{\text{eq}})^m} \cdot da - C \cdot N \quad (40)$$

where Y is the vector of random variables representing the uncertain parameters in the model, a_0 and a_c are the initial and the critical crack sizes respectively and N denotes the number of loading cycles.

One note that, the chosen safety margin is equivalent to the following failure criterion:

$$a_c - a_N \leq 0 \quad (41)$$

where a_N is the crack size after N cycles of loading. By expressing a_N using the Forman law we obtain the expression defined by Eq. (40).

The corresponding failure probability P_f can be written as:

$$P_f = \text{Prob}(H(Y) \leq 0) \quad (42)$$

To compute this probability, different numerical techniques can be used. In this paper, we apply the First-Order Reliability Method (FORM) [21], which gives an estimation of the failure probability in terms of the reliability index β , as proposed by Hasofer and Lind [41]. The FORM failure probability is given by:

$$P_f \approx \Phi(-\beta) \quad (43)$$

where Φ is the cumulative probability function of standard normal distribution.

4. Applications

The stochastic collocation method is applied to fatigue crack growth problems with uncertain input variables. The mechanical response corresponds to fatigue life, which is represented by the number of load cycles at failure. The numerical applications focus on the effect of uncertainties on the mechanical response through the calculation of the statistical moments and the probability density function. The random variables are lognormally distributed and correlations are considered in the second example. The efficiency of the stochastic collocation method is demonstrated by comparison with Monte-Carlo simulations.

4.1. Application 1 : Modified SEN specimen

4.1.1. Structural configuration

Based on the example given [42], we consider a modified four-point bending SEN specimen, in which a hole of radius $R_c = 5.2 \text{ mm}$ is introduced in the neighborhood of the initial crack. The location of the hole with respect to the initial crack is defined by the parameters X_c and Y_c (Figure 3).

Due to this geometrical defect, the stress state in the vicinity of the crack is modified and consequently the trajectory of the crack does not follow a straight line but curves toward the hole.

We consider that the applied load amplitude denoted by P and the two parameters C and m of the fatigue crack growth law are the uncertain input parameters of the problem. They are modeled as independent lognormal random variables as given in Table 1.

The crack growth is performed by using a constant increment length $\Delta a = 1.5 \text{ mm}$ and the FCG life is recorded when the crack length reaches a critical value denoted by $a_c = 16.5 \text{ mm}$.

4.1.2. Finite element model

In order to predict the fatigue crack path, a finite element model was developed using Cast3m software [44]. The analysis was performed under plane stress hypothesis and we use quadratic elements. As shown in figure 4, the mesh is refined near the crack tip and at the boundary of the hole. The initial mesh is composed by 674 elements with 1482 nodes.

In order to highlight the influence of the geometrical defect on the crack trajectory, we simulate the propagation of the crack for four different configurations of the hole location. Each one is defined by a combination of the parameters X_c and Y_c (Table 2).

Figure 5 shows the fatigue crack path obtained for each configuration of the geometrical defect. For the four studied configurations, the fatigue crack is attracted by the hole position.

4.1.3. Convergence of statistical moments

Let us consider the convergence analysis of the first four statistical moments (mean, standard deviation, skewness and kurtosis) of the mechanical response with respect to the number of collocation points. To compute these quantities, the stochastic collocation method based on i th-order Lagrange polynomials was used. The number of collocation points has been varied from 2 to 8.

Figure 6 highlights the statistical moments of the fatigue crack growth life as a function of the number of collocation points. One note that the obtained results concerns the first configuration as mentioned in table 2. It can be seen that the convergence of the mean and the standard deviation is ensured by 4 collocation points for each input random variable. However, it can be observed that the convergence of the skewness and the kurtosis is slower and requires 6 collocation points. We can conclude that, for high order statistical moments, more collocation points are needed to obtain accurate results.

In order to evaluate the accuracy of the obtained results, Monte-Carlo simulations are directly applied to the finite element model using 10^4 samples, was taken as reference

solution. This limited number of samples used in the direct Monte-Carlo simulations is justified, because the FEM is time consuming. However, we have verified the convergence of the statistics. The results show that for the first statistical moments of the stochastic collocation method are close to those given by Monte-Carlo simulations, although the number of samples is low.

Table 3 provides a comparison of the number of FEM calls and the corresponding CPU times required by different computation methods to obtain the statistical moments.

We can observe that the stochastic collocation methods $L-2$ to $L-8$ provides more and more accurate results, for all statistical moments, but far more FEM model calls are required too and CPU time become large. However, Monte-Carlo simulations are too time-consuming to be interesting.

4.1.4. Probability density function comparison

We propose now to study the convergence of the PDF with respect to the number of collocation points. Figure 7 shows the corresponding plots obtained by 10^5 Monte-Carlo simulations on the stochastic response surface (Eq. 28). The same graph shows the PDF obtained by 10^4 Monte-Carlo simulations applied directly on the finite element model.

We can see that starting from a number of collocation points equal to 5, the probability density function is already very close to the one given by the reference solution. Note that the fluctuations observed in the PDF obtained by the Monte-Carlo simulations directly applied to the mechanical model is due to the limited numbers of simulations employed.

4.1.5. Effect of the coefficient of variation

Figure 8 shows the convergence of the four statistical moments of the fatigue life when increasing the coefficients of variation of the uncertain input parameters. We are interested in studying the effect of the variability level of the uncertain input parameters on the convergence of statistical moments of the fatigue life. For this purpose, we choose to increase the COV in the range [5%-20%].

It can be seen that, for a given level of accuracy (which can be defined by the number of collocation points), an increase of the COV of the input parameters requires the use of more collocation points to achieve the convergence. The computation cost in term of number of numerical model calls increase with the COV of the input parameters and the order of the statistical moments.

4.2. Application 2: Engineering structure

4.2.1. Structural configuration

In this example, we consider an engineering structure whose geometry is depicted on figure 9a. The structure contains a corner crack and is subjected to cyclic pressure with amplitude P applied as shown in figure 9a. The material of the structure is elastic with the

following mechanical properties; Young's modulus $E = 30000 \text{ ksi}$, Poisson's ratio $\nu = 0.30$ and fracture toughness $K_{Ic} = 70. \text{ ksi}\sqrt{\text{in}}$.

In this application, we consider that the fracture toughness K_{Ic} , the applied load P and the two parameters C and m of FCG law are random. The parameters C and m correlated. The statistical characteristics of this random variables are listed in table 4.

4.2.2. Finite element model

As in the previous examples, in order to predict the fatigue crack growth life, a finite element model has been implemented using Cast3m software. The FCG analysis is performed using 6-node plane stress elements. The initial mesh is composed of 1010 elements and 2125 nodes as presented in figure 9b.

Before applying the stochastic collocation method, the finite element model should be validated for FCG analysis. For this purpose, the deterministic path was studied. The crack bifurcation angles θ_0 obtained at each step of crack growth were compared to the results of two other software (QUEBRA2D and FRANC2D) [43] (Table 5).

The crack growth is performed by using a constant increment length $\Delta a = 0.075 \text{ inch}$ and the fatigue crack growth life was recorded when the crack length reaches a critical value denoted by $a_c = 0.60 \text{ inch}$. In this exemple, the fatigue crack growth rate has been represented by the Forman's law (Eq. 38). As could be seen, the results obtained by the different software are in good agreement. Consequently, the finite element model is approved.

4.2.3. Convergence study

As for the previous example, we are interested in the convergence of the statistical moments of the fatigue life. Table 6 summarizes the obtained results.

It can be seen, that stable convergence is achieved for the all the computed quantities. However, the convergence speed is influenced by the order of the statistical moment. As an example, the convergence of the standard deviation is achieved by 4 collocation points, while for the kurtosis the convergence was insured by 6 collocation points.

Figure 10 compares the probability density functions obtained by the stochastic collocation method for different values of collocation points. It can be observed that convergence is achieved for only 5 collocation points for each input random variable.

4.2.4. Reliability analysis

In this subsection, we consider the same uncertain parameters as defined in table 4. In order to evaluate the efficiency of the proposed method to carry out reliability analysis, we choose to compute the failure probability P_f with respect to the target operation life $N = 15 \times 10^3$ and using the procedure presented in section 4.

Firstly, we are interested in convergence of the failure probability with respect to the number of collocation points. The obtained results are given by table 7.

It is showed that the convergence of the failure probability is well reached with a reasonable number of collocation points, six in this case which represents 6^4 FEM calls.

At convergence, the failure probability is estimated by $P_f = 9.6349 \times 10^{-2}$, and the coordinates of the design point in the random physical space are $K_{Ic}^* = 60.539 \text{ ksi}\sqrt{\text{in}}$, $C^* = 7.19852 \times 10^{-9}$, $m^* = 3.1048$ and $P^* = 11.275 \text{ ksi}$.

We propose now to evaluate the structural reliability with respect to the number of loading cycles N . For this purpose, we consider a reference period of service life equal to $N = 26 \times 10^3$ cycles and to ensure the convergence we use 6 collocation points.

Figure 11 highlights the evolution of the probability failure P_f as function of the number of loading cycles N .

It can be observed that the failure probability increases as the design life increases. If we consider a target failure probability $P_f = 0.1$, the damage accumulated during the FCG process is acceptable when the design life N is lower than 15×10^3 cycles.

5. Conclusion

This paper presented a stochastic collocation method applied to solve mixed mode fatigue crack growth problems with uncertain input parameters. The study aims to evaluate the effect of these uncertainties on the variability of the fatigue crack growth life.

In this way, the first four statistical moments and the probability density function of the fatigue crack growth life are estimated. The obtained results were compared to those given by Monte-Carlo simulations and have shown that fast convergence can be achieved for the statistical moments and accurate PDF could be obtained. The presented results show that the stochastic collocation method is efficient to study stochastic problems: this

approach is easy to implement (non intrusive as Monte-Carlo simulations that is to say, based on a series of computations using the deterministic model as a black box) and not time consuming (as the number of input random variables is low).

Throughout the applications performed in this study, the dimension of the random space was limited to four. However, for more input random variables, the cost of the method in terms of computation time grows rapidly. One of the significant advantages of the stochastic collocation method lies in the fact that a stochastic response surface is obtained, which allows to perform a reliability analysis by coupling with an optimization algorithm. In this context, a successful attempt was made in the last application to assess the structural reliability by coupling the stochastic collocation method with the First Order Reliability Method (FORM). However, additional efforts are needed to develop more effective structural reliability analysis. Since the proposed method allow us to obtain the PDF of the crack length at any time of the service life, it can be possible to integrate additional events like non-destructive inspection.

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Nomenclature

a	the crack size
a_0	the initial crack size
a_c	the critical crack size
Δa	the increment of the crack size
θ	the crack orientation angle
θ_0	the crack growth angle
N	the number of loading cycle
N_c	the fatigue life time (the number of loading cycle at failure)
ΔN	the increment of the number of loading cycle
E	the Young's modulus of the material
ν	the Poisson's ratio of the material
C, m	the parameters of the Paris law of the fatigue crack growth
P	the applied load
R	the load ratio
G	the strain energy release rate
W_p	the total potential energy
\vec{u}	the displacement vector field
σ	the stress field
ε	the strain field
K_I, K_{II}	the stress intensity factors with the fracture mode I, II
K_{Ieq}	the equivalent mode I stress intensity factors
K_{Ic}	the mode I fracture toughness of the material

X	the vector of the random standard normal variables
Y	the vector of the random input variables
Z	the vector of the random output variables
μ_Y, μ_Z	the mean value vector of the random vectors Y and Z , respectively
C_Y, C_Z	the covariance matrix of the random vectors Y and Z , respectively
ρ	the coefficient of correlation
φ	the probability density function of the standard normal distribution
Φ	the cumulative probability function of the standard normal distribution
Π_M	the space of all M -variate polynomials with real coefficients
Π_M^p	the subspace of polynomials of order less or equal to p
$\{L_i\}_{i=1}^q$	the Lagrange polynomial basis
$\{H_i\}_{i=1}^q$	the Gauss-Hermite polynomial basis
$\{x_i\}_{i=1}^q$	the set of collocation points
$\{\omega_i\}_{i=1}^q$	the set of weight integration
$r = [r_{ij}]$	the symmetric and positive definite matrix
$L = [L_{ij}]$	the lower triangular matrix derived from Cholesky's factorization
g	the stochastic response surface
H	the limit state function
β	the reliability index
P_f	the probability of failure
FCG	the Fatigue Crack Growth
COV	the Coefficient Of Variation
PDF	the Probability Density Function

MCS	the Monte-Carlo Simulations method
$L-i$	the stochastic collocation method using i th-order Lagrange polynomials
FORM	the First Order Reliability Method
FEM	the Finite Element Model
$G-\theta$	the G-Theta method
$E[.]$	the mathematical expectation
$\ \cdot\ _M$	the Euclidean norm in \mathbb{R}^M
δ	the Kronecker operator

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List of figures:

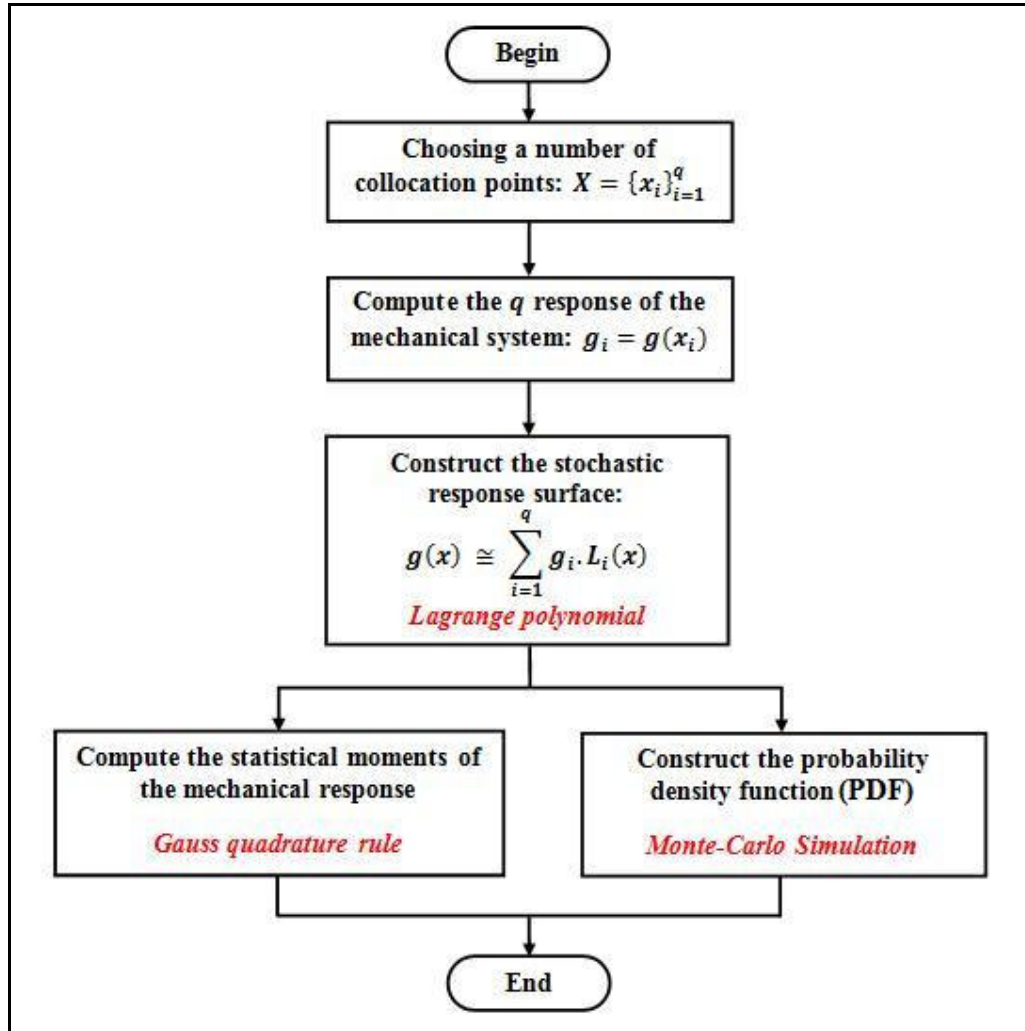


Figure 1: Flow-chart of the stochastic collocation method

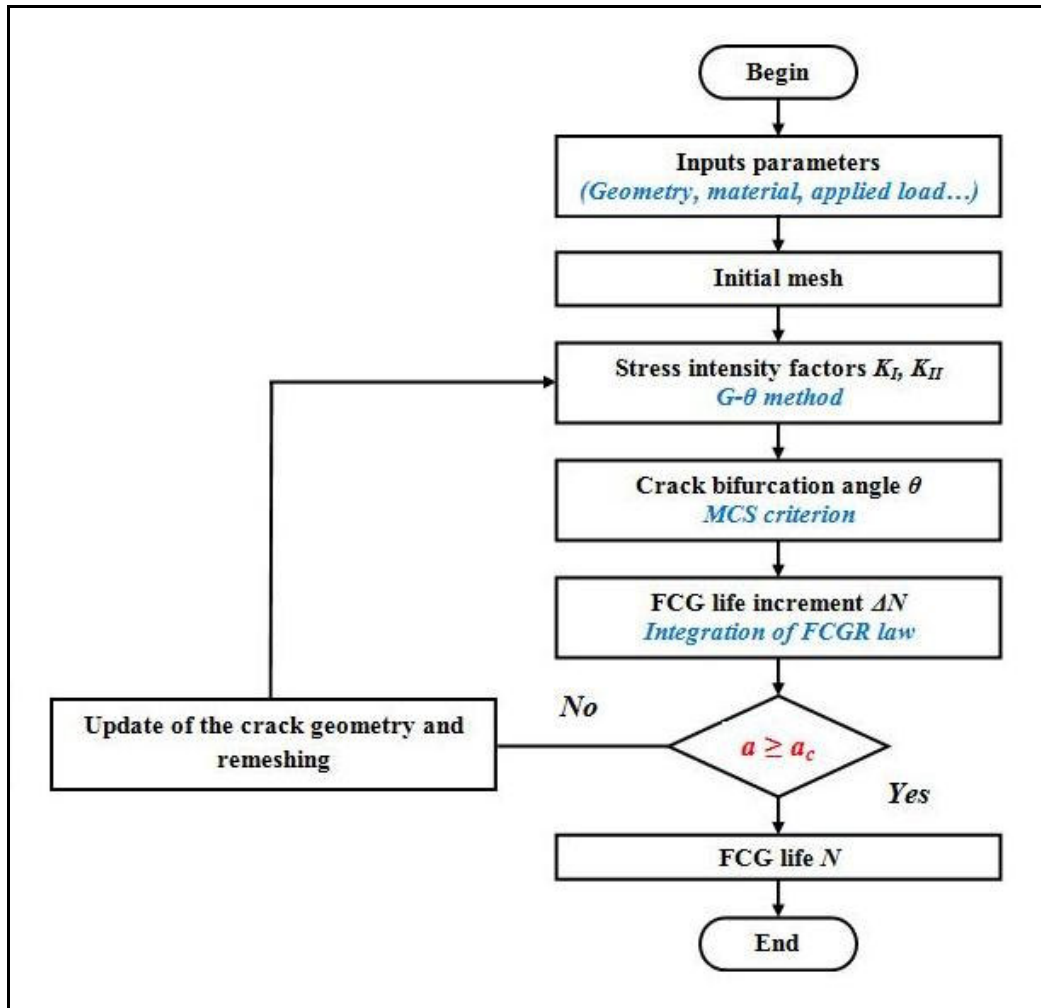


Figure 2: Flow-chart of mixed mode fatigue crack growth analysis

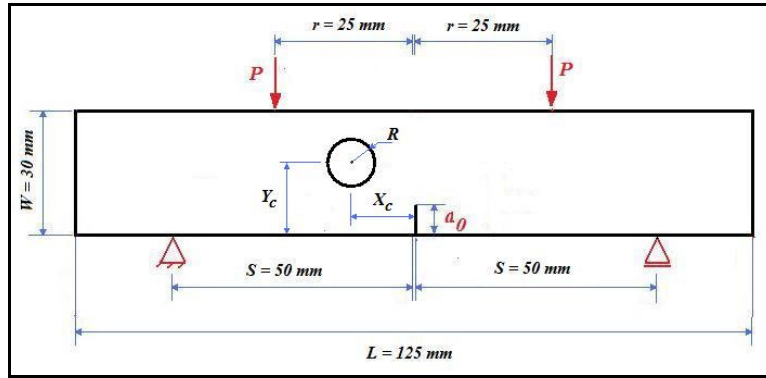


Figure 3: Modified SEN specimen loaded in four-point bending

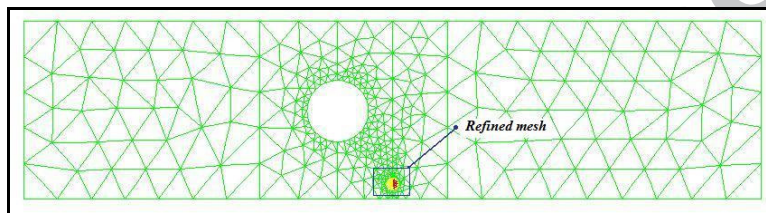


Figure 4: Finite element mesh of the modified SEN specimen

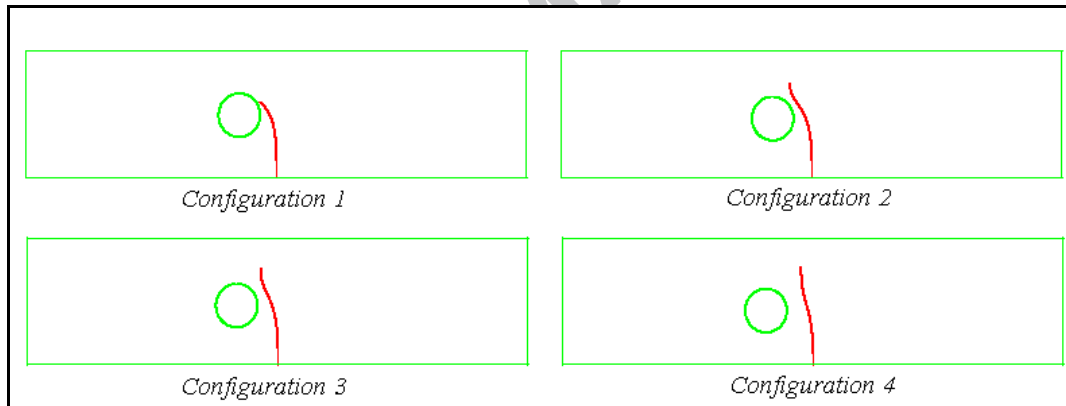


Figure 5: Predicted crack path for each configuration of the hole

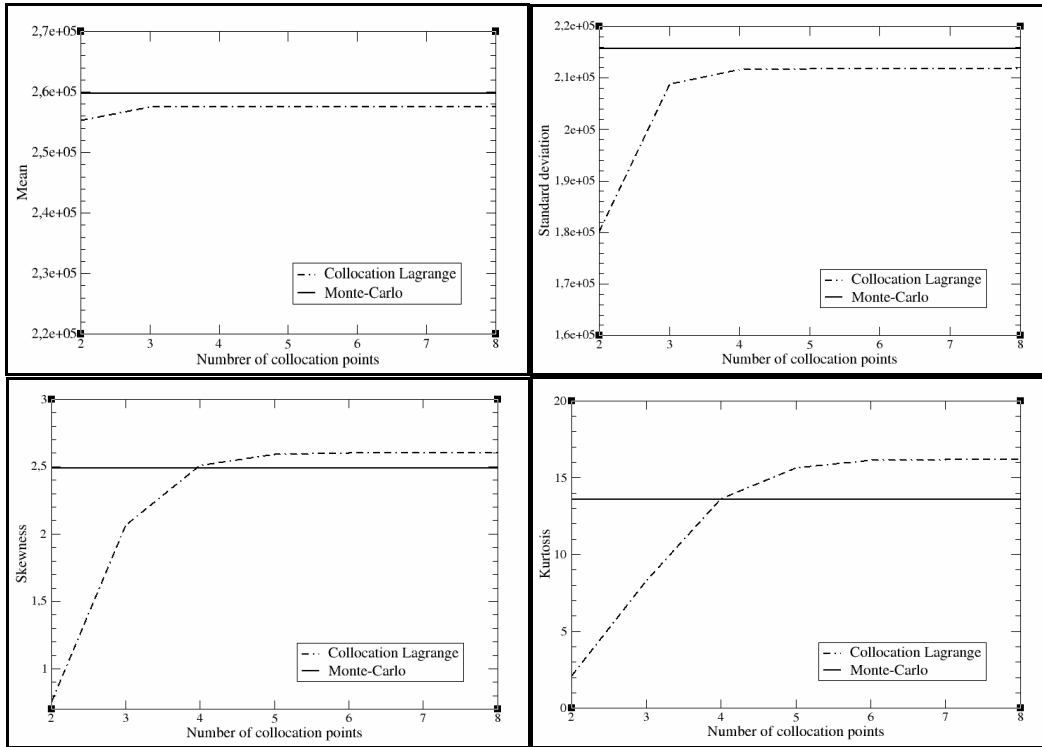


Figure 6: Convergence of the statistical moments (Configuration 1)

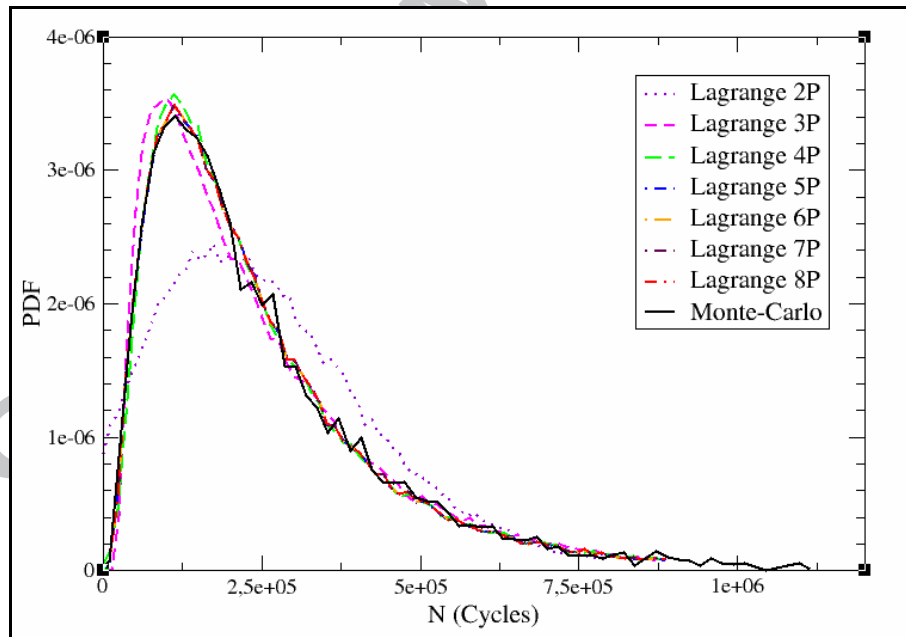


Figure 7: Fatigue life PDF (Configuration 1)

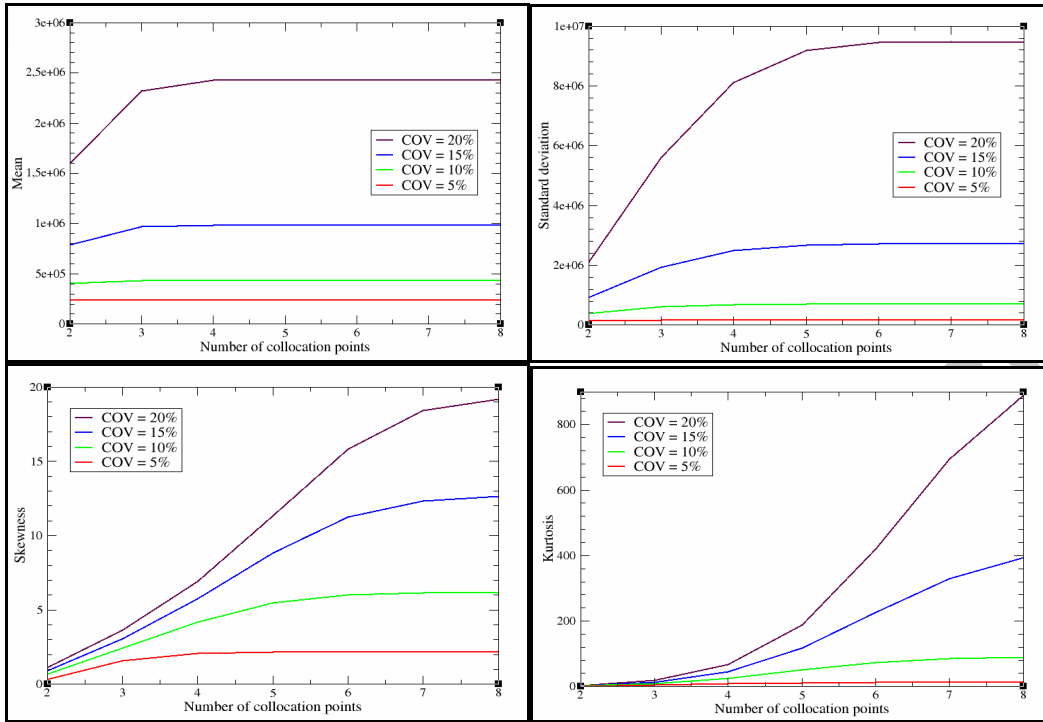


Figure 8: Convergence with respect to the COV of input variables

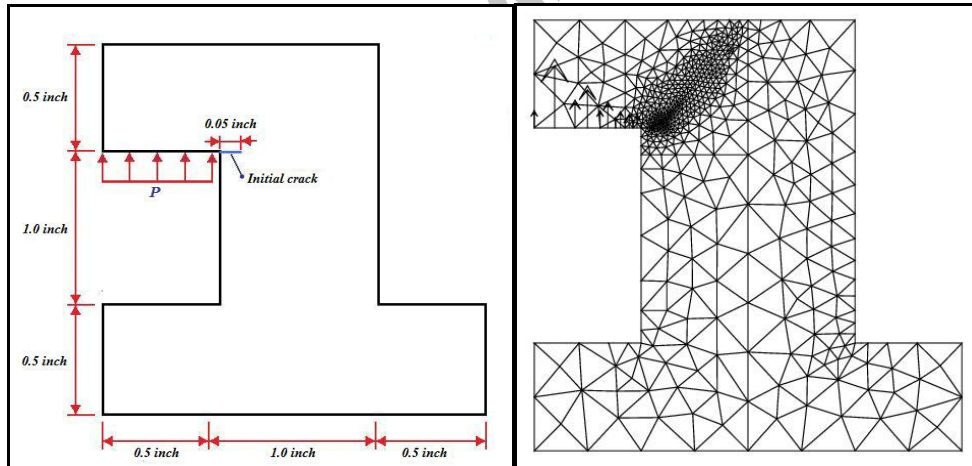


Figure 9: a) Geometry of the structure, b) Finite element mesh

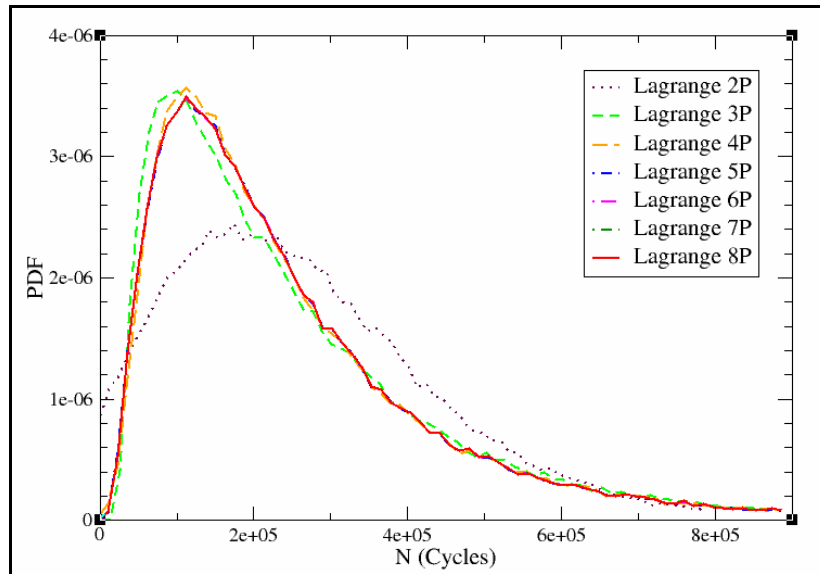


Figure 10: Fatigue life PDF

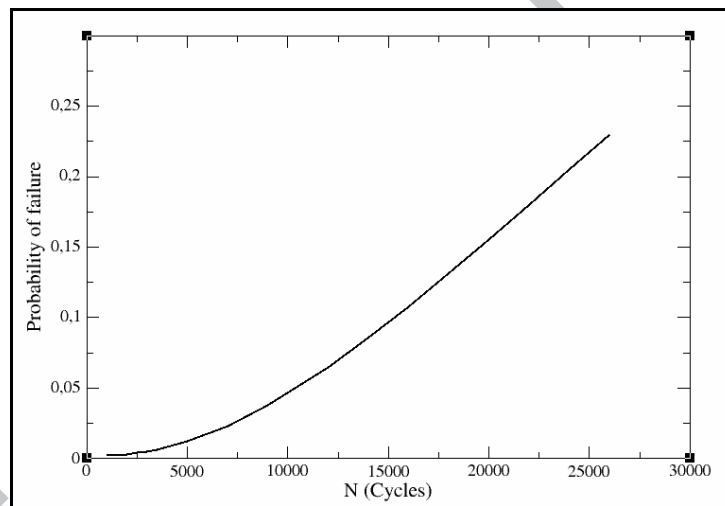


Figure 11: Probability of failure P_f with time

Liste of tables:

Table 1: Statistical characteristics of the uncertain input parameters

Variable	Distribution type	Mean	COV (%)
P (kN)	lognormal	11	10 %
c	lognormal	8.5×10^{-11}	5 %
m	lognormal	4.2	5 %

Table 2: Values of the parameters X_c et Y_c

Configuration	X_c	Y_c
1	9.3 mm	14.8 mm
2	9.8 mm	14 mm
3	10.3 mm	14 mm
4	11.8 mm	12.8 mm

Table 3: Number of FEM calls and corresponding CPU times

Computation method	Number of FEM calls	CPU
L - 2	2^3	34'23
L - 3	3^3	1'89
L - 4	4^3	4'52
L - 5	5^3	8'96
L - 6	6^3	15'82
L - 7	7^3	25'99
L - 8	8^3	40'21
MCS	10000	23h47

Table 4: Statistical characteristics of the uncertain input parameters

Variable	Distribution type	Mean	COV (%)
K_{Ic} (ksi $\sqrt{\text{inch}}$)	lognormal	70	20 %
P (kst)	lognormal	10	15 %
c	lognormal	7×10^{-9}	10 %
m	lognormal	3.00	5 %

Table 5: Crack growth angle obtained by Cast3m, QUEBRA2D and FRANC2D

Step	Crack bifurcation angle $\theta_b(^{\circ})$		
	Cast3m	QUEBRA2D [41]	FRANC2D [41]
1	29.740	29.940	31.400
2	5.1603	5.0700	4.3700
3	2.3712	2.2400	2.4500
4	3.0068	2.9900	3.5800
5	3.7203	3.7500	3.2300
6	4.4512	4.8800	4.0400
7	5.1171	4.9500	6.4300

Table 6: Statistical moments of the fatigue life

$(COV_{K_C} = 20\%, COV_P = 15\%, COV_C = 10\%, COV_m = 5\%, \rho = 0.3)$					
Computation Method	Statistical moments				Number of FE model calls
	Mean	Standard deviation	Skewness	Kurtosis	
L - 2	68506.817	59799.917	1.531	1.943	2 ⁴
L - 3	68970.415	66198.273	2.723	12.031	3 ⁴
L - 4	68977.331	66648.915	3.003	17.088	4 ⁴
L - 5	68977.456	66671.908	3.042	18.353	5 ⁴
L - 6	68977.482	66672.898	3.045	18.574	6 ⁴
L - 7	68977.454	66672.886	3.046	18.604	7 ⁴
L - 8	68977.429	66672.837	3.046	18.608	8 ⁴
MCS	68714.398	67035.901	3.247	23.737	10000

Table 7: Convergence of the failure probability P_f

Number of collocation points	Number of FEM calls	Failure probability P_f
2	2 ⁴	8.36708 10 ⁻²
3	3 ⁴	0.11930
4	4 ⁴	9.55324 10 ⁻²
5	5 ⁴	9.55663 10 ⁻²
6	6 ⁴	9.63491 10 ⁻²
7	7 ⁴	9.63491 10 ⁻²
8	8 ⁴	9.63491 10 ⁻²