DYNAMIC POSITIONING CONFERENCE
September 16-17, 2003

DP Design & Control Systems 1

Introduction to Kalman Filter and its Use in Dynamic Positioning Systems

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Structure

Introduction

The Kalman filter is a widely used algorithm that has been around for more than 40 years. The result of R.E. Kalman's research work was presented in 1960 in a paper entitled *A New Approach to Linear Filtering and Prediction Problems.* R.E. Kalman had the idea of applying the notion of state variables to the Wiener filtering problem. The first application of the Kalman filter was in aerospace when R.H. Battin made the Kalman filter part of the Apollo onboard guidance[1]. In 1976 J.G. Balchen, N.A. Jenssen and S. Saelid wrote a paper on *Dynamic Positioning Using Kalman Filtering and Optimal Control Theory.* This new approach based on the concept of modern control theory was aimed at addressing the disadvantages of PID-controller, such as slow integral action and phase lag in the control loops[2]. Since then Kalman filtering has been widely used in Dynamic Positioning applications.

This paper illustrates the basic concepts behind Kalman filtering in Dynamic Positioning application. We will start the discussion with an overview of Dynamic Positioning and the role of the Kalman filter. The concept of a predictor-corrector estimator will then be introduced and we will present the discrete Kalman filter algorithm and application. In order to illustrate the operation of the Kalman filter an overview of Kalman gains and the evolution of estimate uncertainty are then presented. Finally we discuss some of the considerations to make when implementing the Kalman filter in DP applications. In order to illustrate some of the concepts introduced in the paper a simple example has been created and included in Appendix A.

[1] ***Kalman Filtering – Theory and Practice Using MATLAB®***, 2nd Edition, M. S. Grewal and A. P. Andrews, Wiley-Interscience Publication, 2001

[2] ***A Dynamic Positioning System Based on Kalman Filtering and Optimal Control,*** J.G. Balchen, N.A. Jenssen, S. Saelid, E. Mathisen, MODELING, IDENTIFICATION AND CONTROL, 1980, VOL. 1, No.3

1. Overview of a Dynamic Positioning System and Role of the Kalman Filter

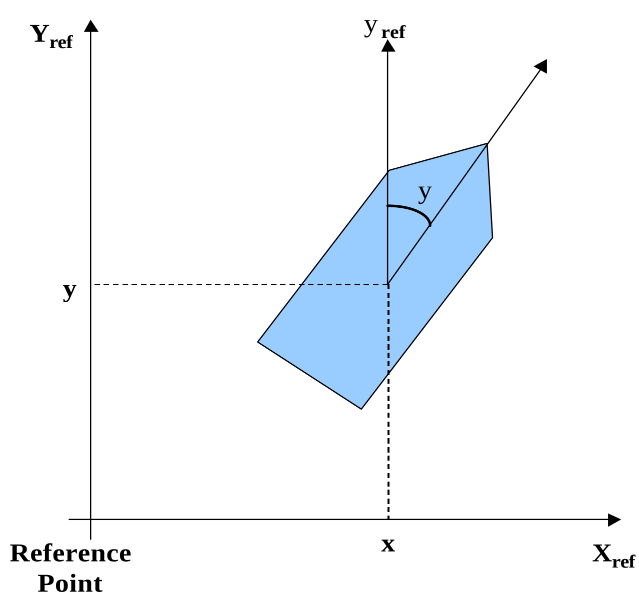
1.1. Objective of a Dynamic Positioning System

According to ABS, by definition, a Dynamic Positioning system is "a hydro-dynamic system which controls or maintains the position and heading of the unit by centralized manual control or by automatic response to the variations of the environmental conditions within the specified limits"[3].

API defines Dynamic Positioning (DP) as "a technique of automatically maintaining the position of a floating vessel within a specified tolerance by controlling onboard thrusters which generate thrust vectors to counter the wind, wave and current forces"[4].

From both definitions we can determine the main **functions** to be performed in order for a dynamic positioning system to control a given vessel position (x,y) and heading (?). These functions are:

- → **Estimate** vessel motion
- → **Measure** vessel response
- → Determine **error** between prediction and measurement
- → Determine **corrective action** to be applied
- → **Calculate and allocate appropriate command** to thrusters to achieve desired corrective action



[3] ***ABS, Guide for thrusters and dynamic positioning systems***, 1994 Section 3, 3.2.4.

[4] ***API Recommended Practice for Design and Analysis of Station keeping Systems for Floating Structures***, 1995

Concentrating on vessel position and heading, the main functions typically performed by a DP System are summarised in figure 1: the vessel position and heading are estimated based on the vessel model, the forces acting on the vessel, and on the position and heading measurements returned by the position reference systems and gyros. Based on the difference between the desired position (and heading) and the estimated position (and heading), the control command to the thruster system is calculated and allocated to the appropriate thrusters. The thrusters then provide the necessary forces to counter the external forces and moment acting on the vessel and maintain the vessel on location (with the desired heading), using the power coming from the power system.

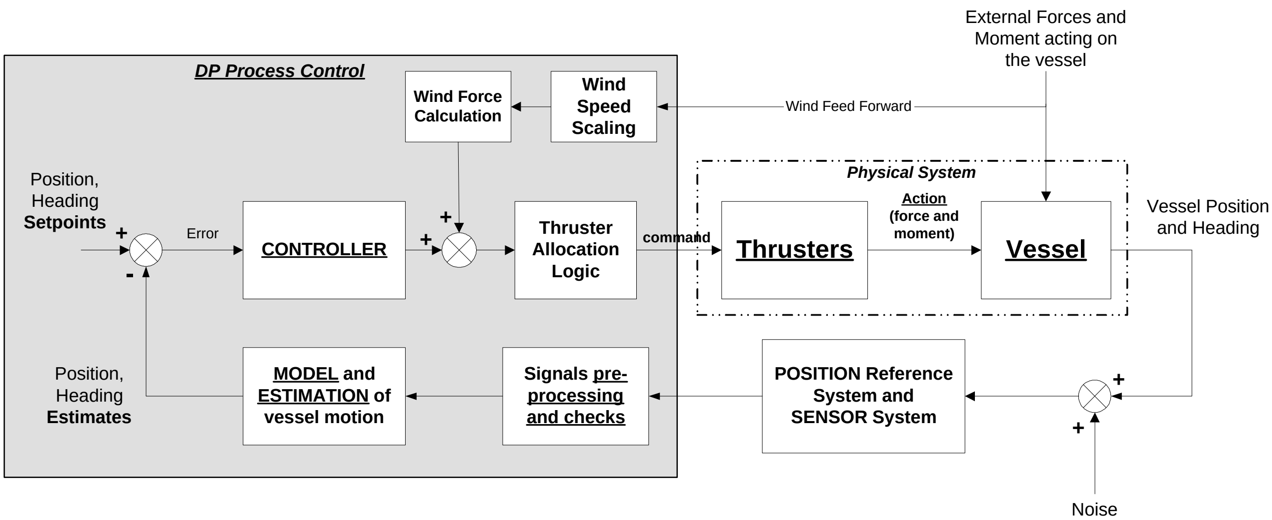


Figure 1 - Functional overview of a Dynamic Positioning control system

Let's now concentrate on one specific function of a Dynamic Positioning system: the **estimation** of the vessel state. The vessel is considered as a ***dynamic system***. Its ***state*** can be defined by a set of variables that explicitly represent all the important characteristics of the vessel at any given time. In a DP application it's important to identify the motion variables of the vessel (**position** and **speed**) as well as the environmental variables influencing the motion. The state of the vessel therefore usually consists of position (x, y), heading (?), velocities (in surge, sway and yaw) as well as steady-state current.

So position (x,y) is only part of the state of the vessel. For the purpose of simplifying our discussion we will confine ourselves to the estimation of vessel position.

1.2. Estimating Vessel Position

In order to estimate vessel position, the DP system uses information taken from:

- ***Position Reference Systems*** (also referred to as Position Measurement Equipment) – for example DGPS, Acoustics.
- Its own internal ***model*** based on physical description of the vessel.

Each ***Position Reference System*** returns a measured position of the vessel. Signals are converted to a common format and then validated before being used by the DP System (since Position Reference Systems are subject to failures which may result in erroneous output, a measurement 'validation' mechanism has to be in place). This is referred to as Signal pre-processing in figure 2 below.

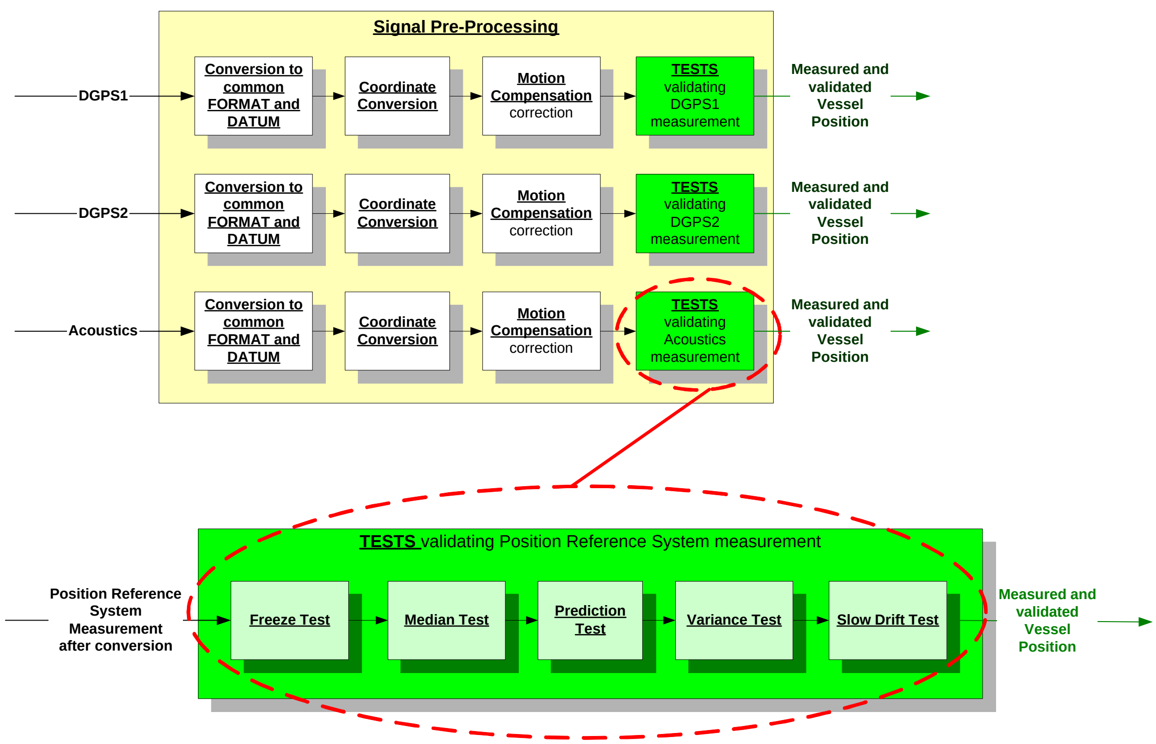


Figure 2 – Typical Position Reference System Pre-Processing Routines

Please note that like any other measurement, the measurements from Position Reference Systems are **noisy**. The source of noise depends on the sensors used and on the method used for measuring position. As a result different types of Position Reference Systems will have different noise characteristics. DGPS and Acoustics are good examples of position reference systems with different update rates and noise characteristics.

The ***model*** also provides some information on the state of the vessel. The model contains a hydrodynamic description of the vessel. In other words the model is used to describe the reaction of the vessel based on external forces acting on it. The vessel model is a set of equations of motion that is used to predict the motion of the vessel when known forces and moment are applied. In order to separate the wave induced oscillatory part of the motion from the remaining part of the motion, the total vessel motion is modeled as the added outputs of a low-frequency model (LF-model) and a high-frequency model (HF-model)[5]. The HF-model represents oscillatory wave components in the vessel motion. The LF-model represents motions induced by wind, thrust and current in surge, sway and yaw. The low frequency portion of the model is controllable by means of thrusters. Figure 3 is a simplified block diagram of a typical vessel model.

In order to achieve good performance of the DP system the model of the vessel has to be as detailed as practically possible. The parameters of the model are verified during sea trials ("tuning" of the model). However the model only represents *some* aspects, and cannot fully capture the entire physics behind vessel motion and dynamics. So the model of the vessel should only be considered as an approximation of the "real thing" and is not perfect.

[5] ***A Dynamic Positioning System Based on Kalman Filtering and Optimal Control,*** J.G. Balchen, N.A. Jenssen, S. Saelid, E. Mathisen, MODELING, IDENTIFICATION AND CONTROL, 1980, VOL. 1, No.3

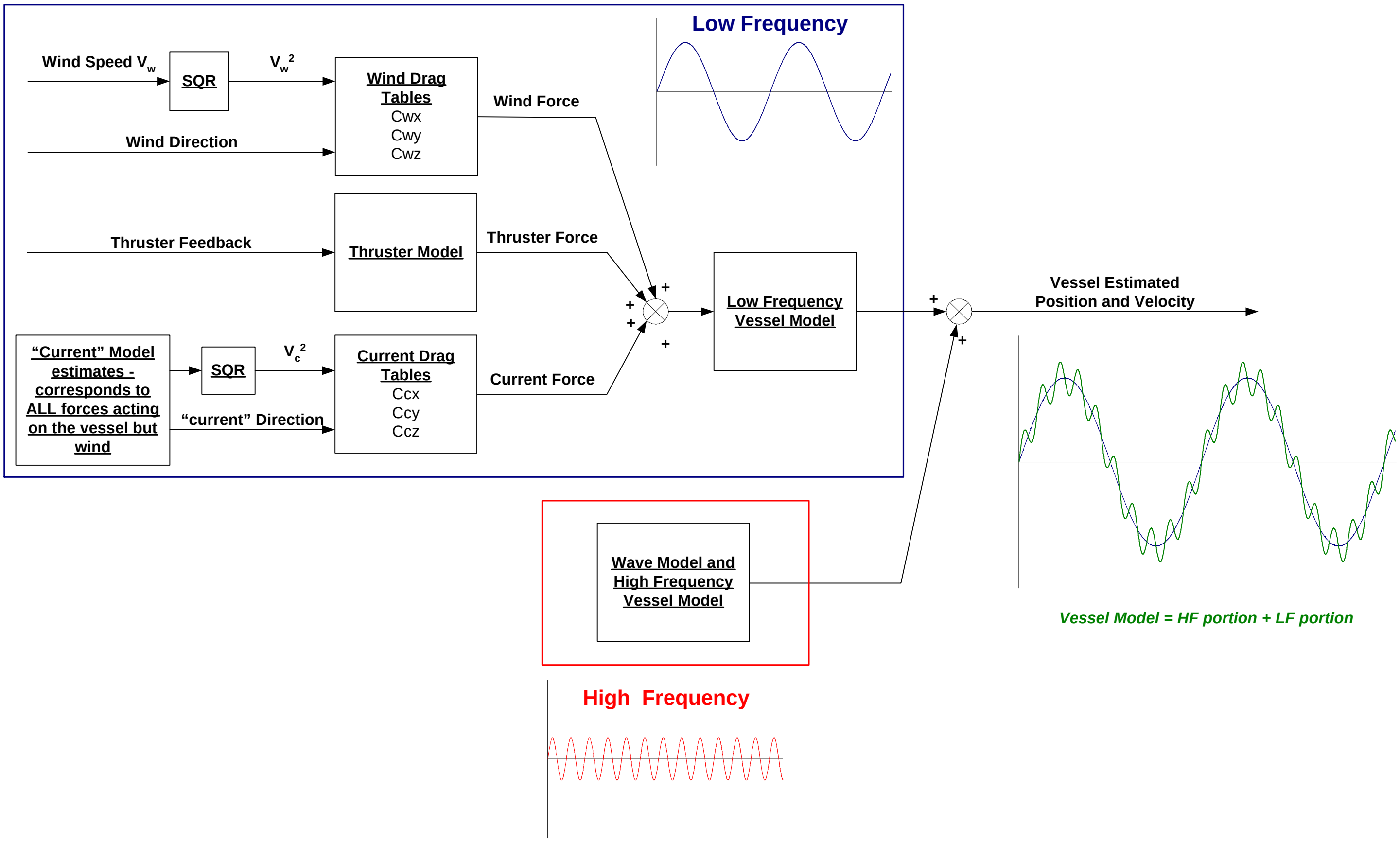


Figure 3 – Block Diagram of Typical Model (simplified)

1.3. State Estimation Problem Formulation

The estimation problem solved by the Kalman filter can be expressed as follows: how do you **optimally** **estimate** the state of a vessel with an approximate knowledge of the vessel dynamics (**imperfect mathematical model)** and with **noisy measurements** from sensors and position reference systems? What is the **best state estimate** you can get out of all that?

1.4. Kalman Filter: First Functional Definition

A Kalman filter is, in fact, the answer to the state estimation problem formulated above. In a Dynamic Positioning application a Kalman filter is used to estimate the state of the vessel (for which a dynamics model has been developed) based on noisy measurements from reference systems and sensors.

This is a first "functional" definition of the Kalman filter. We will define further this type of filter in the paragraphs that follow. In order to fully appreciate the other attributes of this type of filter, we need to review its operation.

2. The Discrete Kalman Filter

2.1. Definition of terms

- Let's first consider the **system dynamic model**. This equation describes the behavior of the vessel). It would be of the following type:

$$x_k = A \cdot x_{k-1} + B \cdot u_k + w_{k-1} \qquad \textbf{\textit{(2.1.1)}}$$

x_k is the state that we're trying to estimate. Typically in DP application the state consists of vessel position and heading (x,y,?), associated velocities (v_x, v_y, $v_?$) and steady-state current forces acting on the vessel (C_x, C_y, $C_?$). So it would be a 9x1 matrix in our application[6].

A is a n x n matrix that relates the state at step k-1 to the state at step k, in the absence of any driving function or process noise. This is a description of how the state changes between measurements. In a DP application that would be a 9x9 matrix if you consider the state described above. This matrix A is given by the mathematical model of the vessel.

B is a [n x l] matrix that relates the control input u_k to the state x_k. This matrix B is given by the mathematical model of the vessel.

$\mathbf{U_k}$ represents the control input (from the thrusters in DP application).

$\mathbf{W_k}$ represents the ***process noise*** or ***model uncertainty***. We will assume that the process noise is white[7] and with a normal distribution of zero mean and Q variance (see figure 4):

$$p(w) \sim N(0, Q)$$

Q is called *Process Noise Covariance*. **It represents the** *uncertainty* **in the process or model.**

[6] Please note that in some cases pitch and roll are added to the *state*.

[7] By definition ***white*** noise is a signal that does not repeat and that has a frequency spectrum that is continuous and uniform over a specified frequency band. White noise has equal power per hertz over that frequency band.

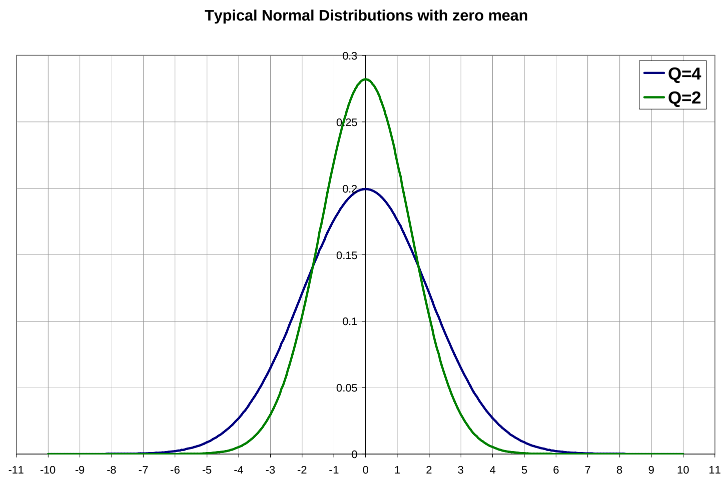


Figure 4 – Typical Normal Distribution with Zero Mean

- Let's also consider the **measurement model** for a measurement $\mathbf{Z_k}$ (in our case a measurement would be given by a position reference system or a sensor).

$$z_k = H \cdot x_k + v_k \qquad \textbf{\textit{(2.1.2)}}$$

H is a [m x n] matrix that relates the ***state*** to the ***measurement*** z_k. It describes how the measurement depends on the state.

v_k represents the ***measurement noise***. We will assume that the measurement noise is independent from the process noise, white and with the following normal probability distribution:

$$p(v) \sim N(0, R)$$

R is called ***Measurement Noise Covariance***. It represents the ***uncertainty*** in the measurement.

Please note that both the ***process noise covariance Q*** and the ***measurement noise covariance R*** can change with time, however they are considered constant in most applications of Kalman filters in Dynamic Positioning system.

At this stage we need to introduce $\hat{x}_k^-$ which is the ***a priori estimate*** at step k given knowledge of the process prior to step k. This is the estimate of our vessel state at step k given our knowledge of the state prior to step k.

The ***a posteriori state estimate,*** $\hat{x}_k$, is the state estimate at step k given measurement z_k . This is the estimate of our vessel state at step k based on the measurement received from a Position Reference System.

2.2. Predictor-Corrector Structure

We are going to introduce in this section the ***Predictor-Corrector*** structure of the Kalman filter. In order to see why the Kalman filter is said to have this type of structure, let's look at its operation.

- **Prediction**

The first step of the Kalman filter operation is called the **Prediction** step, or **Time Update** step, or **State Estimate Extrapolation**. The Kalman filter is going to predict the state of the system based on the current state and the model. Using the state dynamic model presented in equation 2.1.1 the Kalman filter determines the *a priori* estimate during this prediction step. We have the following prediction equation:

$$\hat{x}_k^- = A \cdot \hat{x}_{k-1} + B \cdot u_k \qquad \textbf{(2.2.1)}$$

In addition at this step the Kalman filter projects what is called the ***error covariance*** P_k^- The error covariance can be considered as the uncertainty of this first prediction of the state.

$$P_k^- = A \cdot P_{k-1} \cdot A^T + Q \qquad \textbf{(2.2.2)}$$

- **Correction**

The next step is called the **Correction** step, or **Measurement Update** step, or **State Estimate Observational Update**. The Kalman filter is going to correct or update its first prediction obtained at step 1 based on the measurement received from the Position Reference System. Please note that at this stage the Kalman gain K_k is calculated. We will come back in paragraph 2.3. on how this gain is computed. The result of this second step is a new estimated state of the system, the *a posteriori* state estimate (as defined above). We can see from the formula below that this *a posteriori* state estimate is in fact the *a priori* state estimate plus a correction factor which is proportional to the difference between the measurement and the measurement prediction. This is why we call this second step the **correction** step. Please note that this difference between the measurement and the estimated measurement is called **innovation** or **residual**. This is an important part of the Kalman filter. We will use the terminology **residual** in the rest of the paper.

$$\underbrace{\hat{x}_k = \hat{x}_k^-}_{} + \underbrace{K_k \cdot \left(z_k - H \cdot \hat{x}_k^-\right)}_{} \qquad \textbf{(2.2.3)}$$

***a priori* state estimate obtained at step 1**

Correction = Kalman gain K_k multiplied by residual [difference between measurement and measurement prediction]

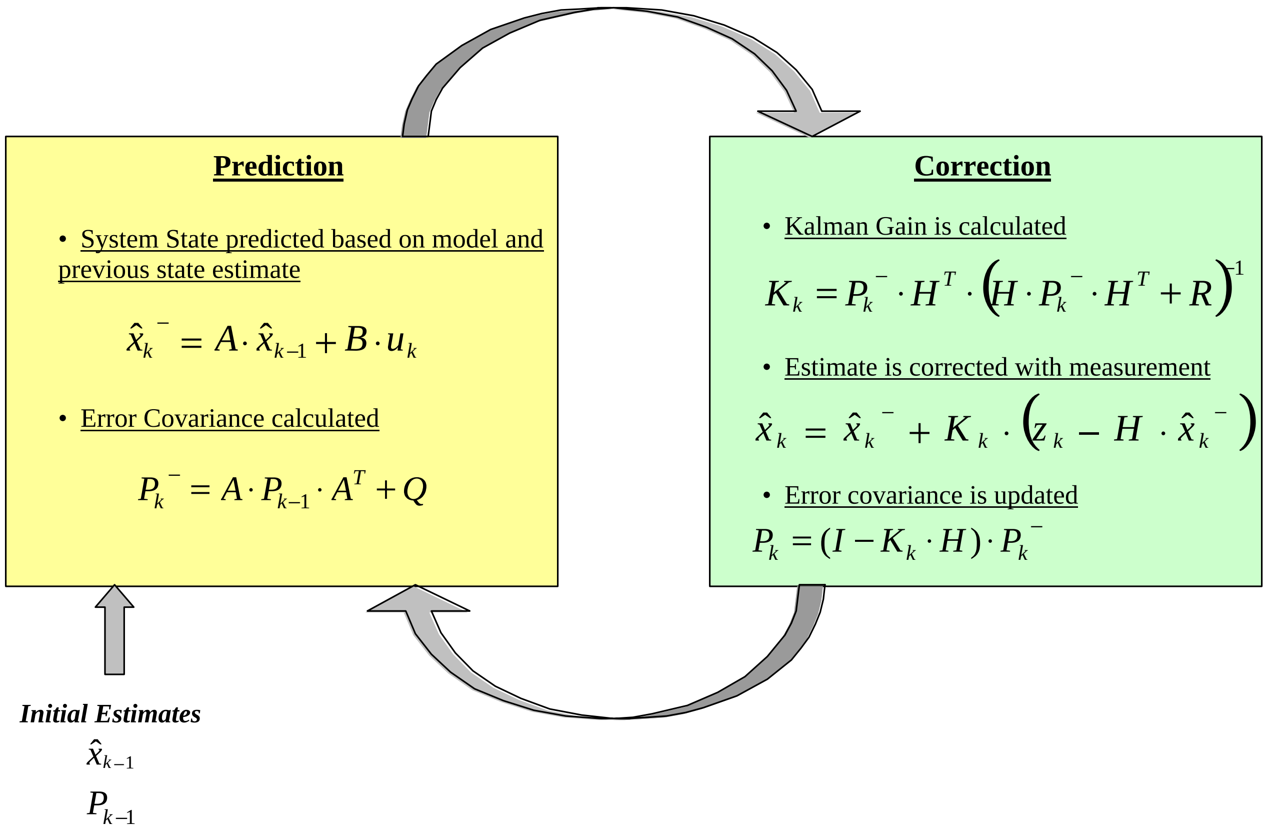


Figure 5 – Predictor-Corrector Structure of Kalman Filter with Equations

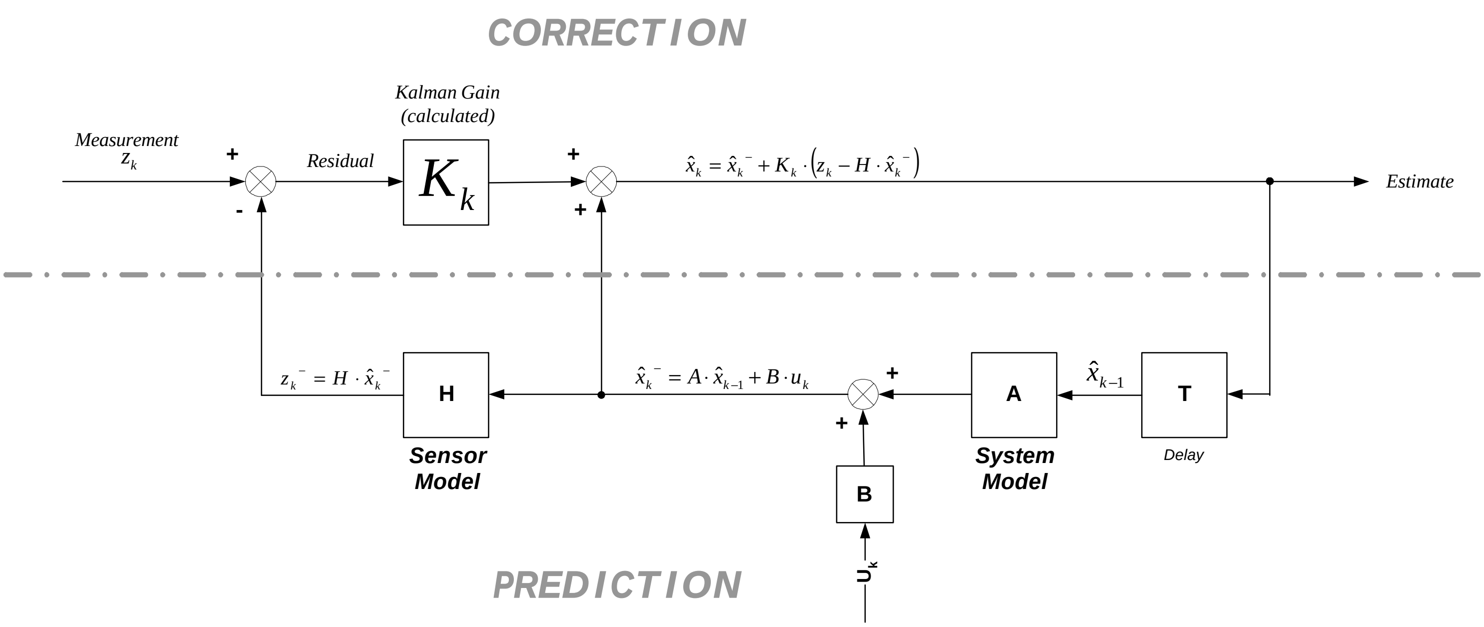


Figure 6 – Simplified Block Diagram of Kalman Filter

But what does that mean for our DP System?

Let's come back to the main objective of our Kalman filter: to estimate the state of the vessel. Let's take position as an example for one of the vessel state.

Based on the vessel model, and using the previous position estimate of the vessel, the prediction step of the Kalman filter gives us a prediction of the vessel position. Based on the forces acting on the vessel, on the vessel model and on the previous position estimate, this is where the DP system thinks the vessel is.

A measurement comes in from one of the position reference systems. That measurement is going to be used to refine the prediction previously calculated. The second step, or correction step, compares the measurement with the measurement prediction calculated by the DP system. There has to be a mechanism in place to give less "weight" to an inaccurate measurement and more "weight" to a very good (compared to the model) and valid measurement: this is what the Kalman gain (computed by the Kalman filter) does. The previously calculated position prediction is then corrected by a factor equal to the Kalman Gain multiplied by the difference between the estimated position and the measured position (this difference is also called innovation or residual as defined above).

And the process goes back to the prediction step and starts over.

A simplified schematic of a Kalman filter for our DP application is given in figure 7. Please note that the implementation of the Kalman filter presented in the figure below is only one of several possible implementations.

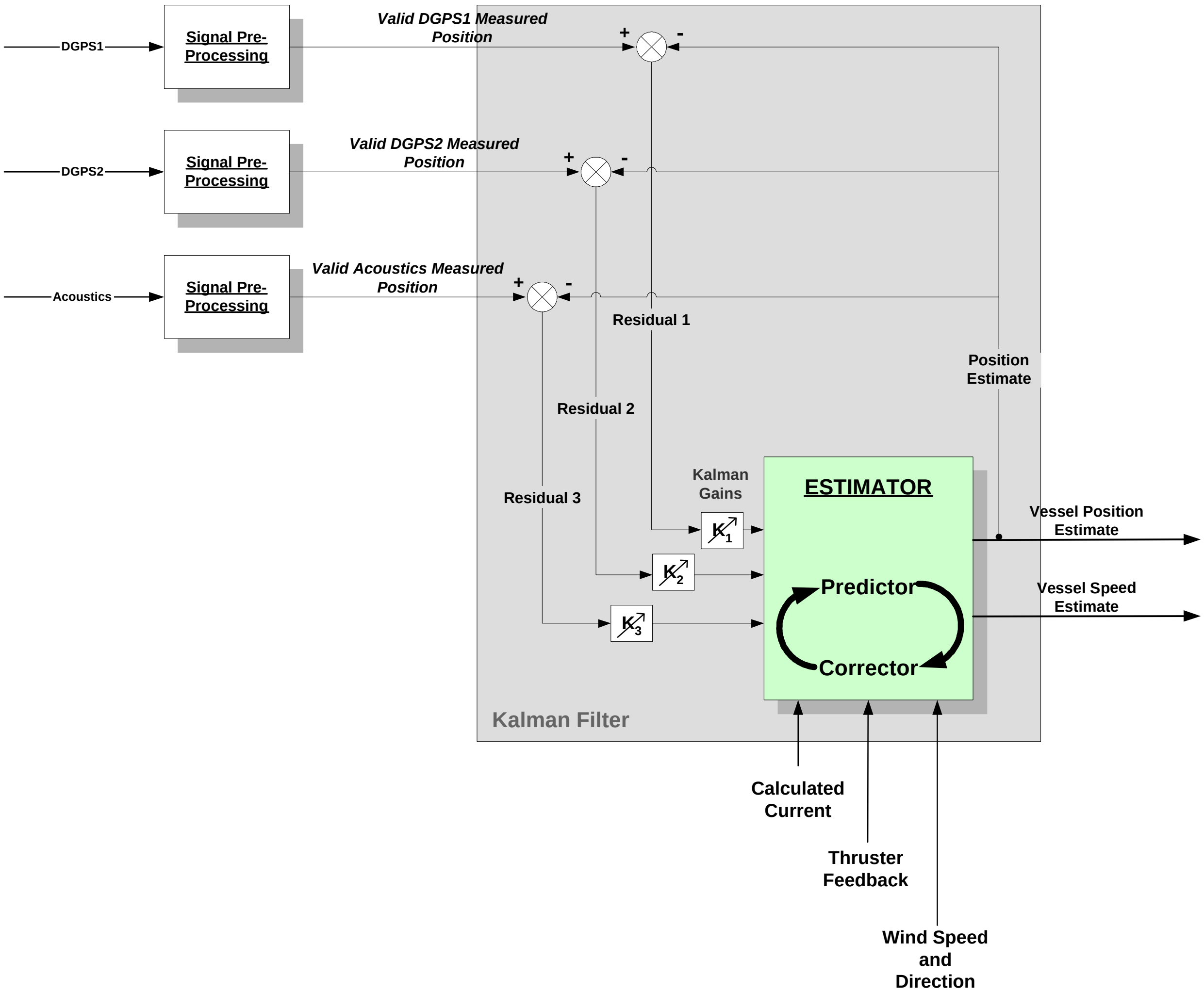


Figure 7 – Simplified schematic of Kalman filter (one possible implementation)

So you can consider the Kalman filter in our DP application as the most efficient way to "blend" measurements coming from Position Reference Systems and Sensors and information coming from the vessel model. The end result or output of the filter is an optimal estimate of the state of your vessel. An example of the estimated position (East) taken from the Deepwater Frontier drillship is given in figure 8. Figure 9 shows the Innovations or Residuals (for each of the Position Reference Systems) and the Kalman gains for the same example. We can see with this example how the "blending" works and how the estimated position is derived.

Please note that applying Kalman gains on each individual Position Reference System is one way of implementing Kalman filter. Another implementation would be to apply the Kalman gain on a "weighted" measurement (all positions returned by the different reference systems are "blended" by a pooling system and that "weighted" average is then sent to the Kalman filter).

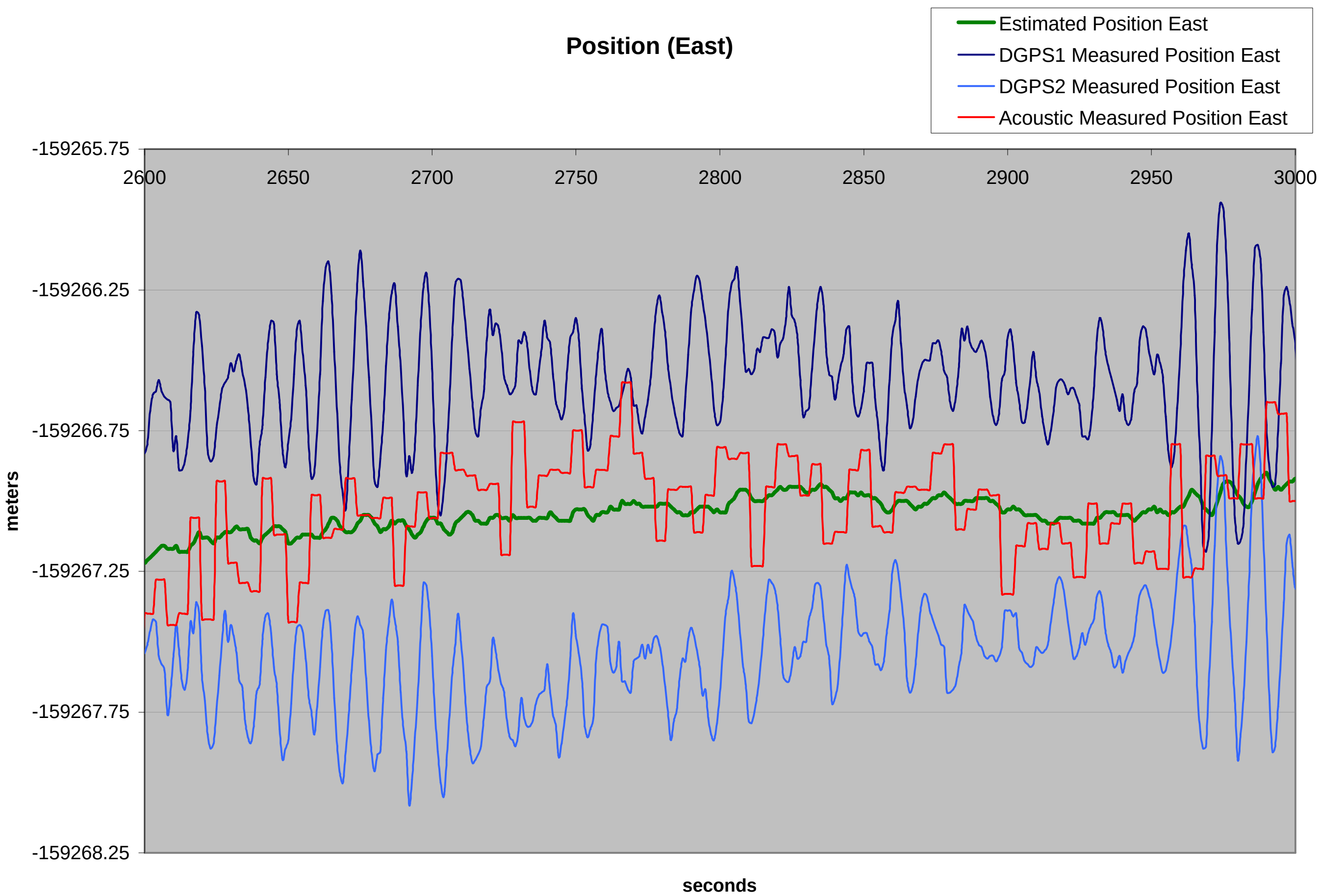
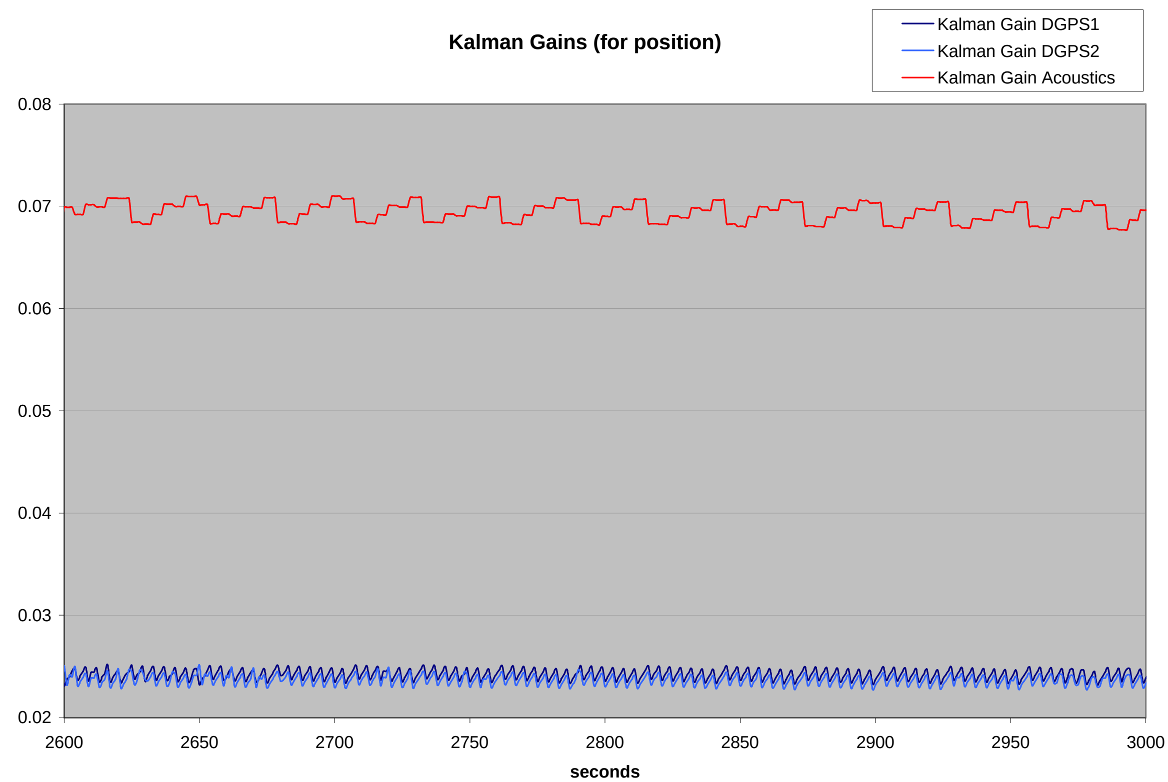


Figure 8 – Example of Estimated Position (East) compared to Position Reference System measurements (after pre-processing) – Deepwater Frontier, Brazil



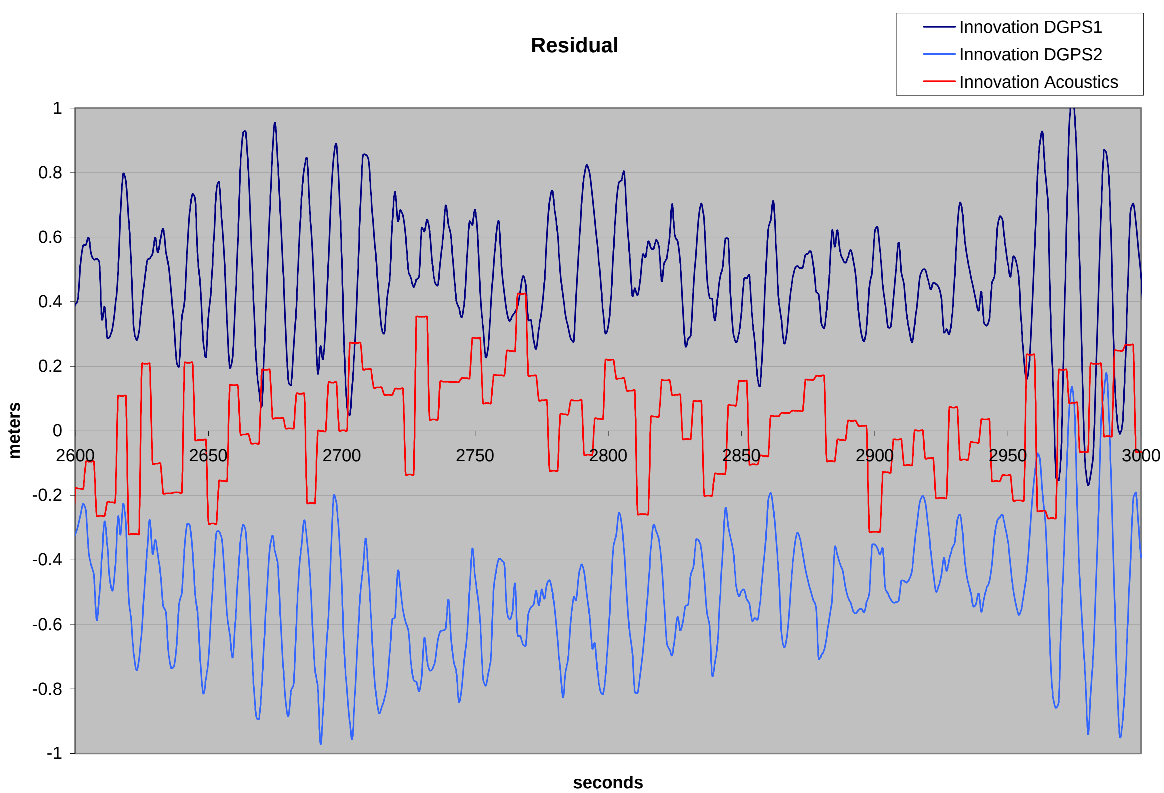


Figure 9 – Kalman Gains and Residual for each Position Reference System

2.3. Kalman Gain

The Kalman gain (K_k) is computed by the Kalman filter so that the *a posteriori* error covariance is minimized. That means that the gain is calculated so that the uncertainty in the state estimate is minimized. One form of the calculated Kalman gain is given by formula 2.3.1.[8]:

$$K_k = P_k^- \cdot H^T \cdot \left(H \cdot P_k^- \cdot H^T + R\right)^{-1} \qquad \textbf{\textit{(2.3.1)}}$$

A good way of defining in simpler terms the Kalman gain is to describe its 'behavior'. The Kalman gain is indeed calculated based on how much we weight the measurement that comes in **and** how much we weight the model.

- If the model is excellent (model uncertainty is small) and the measurement is very noisy (measurement uncertainty is high), then the Kalman gain will be small. By calculating a small gain the Kalman filter voluntarily decreases the effect of a measurement that comes in. So the filtered position will have less tendency to "follow" the measurements (which are noisy). This is apparent when looking at the following *update equation*:

$$\hat{x}_k = \hat{x}_k^- + K_k \cdot \left(z_k - H \cdot \hat{x}_k^-\right) \qquad \textbf{\textit{(2.3.2)}}$$

- If the error covariance is very large and the measurement noise covariance is negligible compared to the error covariance, then going back to formula 2.3.1.:

$$K_k = H^{-1}$$

Please note that in theory H cannot be inverted. We use the notation H^{-1} only for the purpose of describing the behavior of the Kalman gain in an extreme case.

If you then replace K_k by H^{-1} in equation 2.3.2 giving the *a posteriori* state estimate, you get:

$$\hat{x}_k = H^{-1} \cdot z_k$$

This means that in case the error covariance is very large then the *a priori* estimate will not be used in the update and only the measurement will be used.

[8] ***Stochastic Models, Estimation and Control***, Volume 1, Peter S. Maybeck, Dept of EE Air Force Institute of Technology, Academic Press, 1979

Another important point to take into account is the fact that Position Reference Systems are different in nature as we alluded to in the first part of this paper. By different in nature we mean that they have different noise characteristics (this is especially true when you compare DGPS with Acoustics) and different update rates. So how do you "merge" or combine two different Position Reference Systems efficiently (taking into account their differences)? If nothing is done the Kalman filter will give too much weight to DGPS which has low high frequency noise and a high update rate. A solution to this problem is given in ***Improved DP Performance in Deep Water Operations Through Advanced Reference System Processing and Situation Assessment*** by Nils Albert Jenssen (Kongsberg Simrad)[9]. The solution proposed uses quality figures from each position reference system and applies Kalman gain on the individual measurements.

2.4. Typical Evolution of Error Covariance or Estimate Uncertainty with time

This paragraph illustrates the evolution of the Error Covariance (identified as P_k previously) or Estimate Uncertainty with time as the DP system receives valid measurement updates from the Position Reference Systems. This section was added as a means to visualize how a Kalman filter operates.

First let's start with an extreme case: no measurement is received. In this case (also known as dead reckoning) the Kalman filter only relies on the vessel model to estimate the state of the vessel. Let's take position in this case to simplify. What do you think will happen to this position? Will you trust this estimate? Well as time goes by the estimate will be less and less correct simply because the model of your vessel is only an approximation of the real behavior of the vessel. Therefore the estimate uncertainty will increase with time in this particular case.

Now imagine that a position reference system is selected and returns a valid measurement of vessel position. The Kalman filter will want to give this new information a high weight to correct what it thinks is a very approximate estimate (remember that the estimate uncertainty is high as described before). Therefore the calculated Kalman gain will be high and the Kalman filter will give a high weight to the residual. This makes sense and confirms the Kalman gain behavior described on page 14.

All this evolution of error covariance is described on the hypothetical graph in figure 10.

[9] ***Improved DP Performance in Deep Water Operations Through Advanced Reference System Processing and Situation Assessment***, 1997 MTS DP Conference, Nils Albert Jenssen (Kongsberg Simrad)

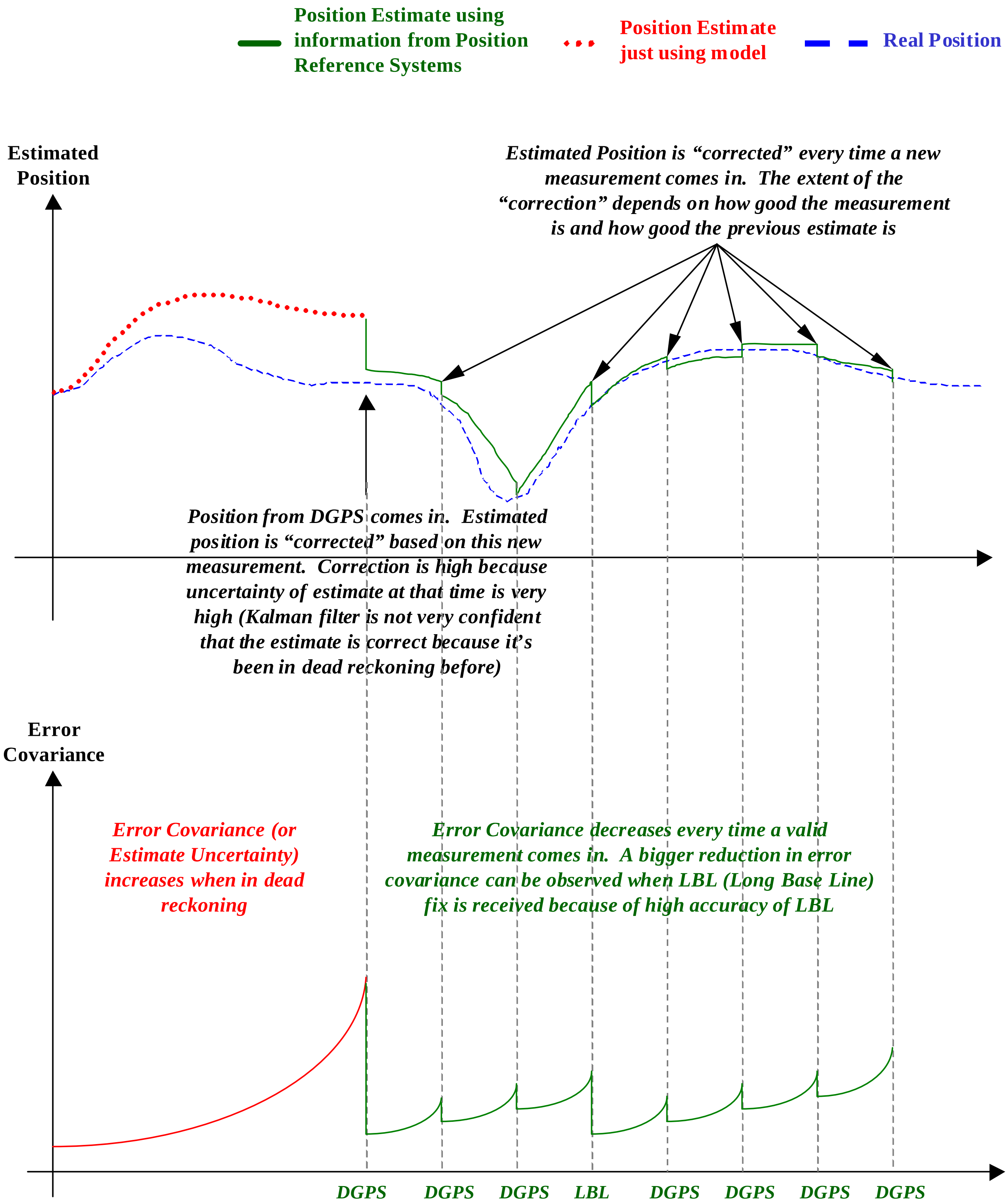


Figure 10 – Evolution of position estimate uncertainty with time

2.5. What makes the Kalman Filter optimal?

First of all Kalman filtering is a very convenient filter for online real time processing. As previously defined a Kalman filter is a recursive filter. That means that all the past "history" is contained in the a priori estimate. So another way to look at that is to consider that each new estimate incorporates all previously calculated estimates. So you don't need to store all previous estimates. Whether the filter works on estimate #1 or on estimate #1000 the computations and work performed by the filter is the same. That's why this type of filter is very convenient for online real time processing. It is also relatively easy to implement.

A Kalman filter is also optimal in the sense that it uses any valid measurement that comes in. So it uses all available (and valid) data and then applies the appropriate weight to it. So any valid measurement that comes in will be used and will contribute to the estimating the state of the system.

In parallel to the recursive nature of the filter it is also interesting to note that Error Covariance calculations and the Kalman Gain calculations do not depend on the measurements coming in. That means that these two values can be computed ahead of time. The overall algorithm is really tailored to real-time applications.

2.6. Summarizing: Complete Definition of a Kalman Filter and of the Extended Kalman Filter

As a way to summarize all the notions introduced above, we can give a better and more complete definition of a Kalman filter.

A Kalman filter is a ***linear estimator***. It is used to estimate the state of a ***linear*** dynamic system by using measurements ***linearly*** related to the state of the system but corrupted with noise.

A Kalman filter is a ***recursive data processing algorithm***. It is a software tool that does not require all previous data to be kept in memory. All previous "history" is in fact captured in the most recent estimate of the state of the system. This is an important characteristic when it comes to implementing this type of algorithm in computers.

Finally this type of filter is ***optimal***: it calculates the best possible estimate (minimum variance) for the state of the system.

Please note that in DP application the standard ***linear*** approach will not work because the vessel model has non-linearities (quadratic drag effect and velocity-position relations). In order to take these non-linearities into account an ***Extended Kalman Filter*** is used. It uses the same principle as a standard Kalman filter but linearizes about the current mean and covariance.

3. Implementation of Kalman Filter and items to consider

A Kalman filter does not function properly when the Kalman gain K becomes too small, but the measurements still contain information for the estimates. In such condition the filter is said to diverge. It is then necessary to determine how well the filter functions and how to tune it. A tuned Kalman filter will have optimal performance[10].

3.1. Starting point for checking Kalman Filter operation

The residual provides the starting point for checking the filter operation. As indicated in *Digital and Kalman Filtering* by S.M. Bozic a necessary and sufficient condition for a Kalman filter to be optimal is that the residual is zero-mean and white. Various statistical tests can be done to check that. An example of innovation is given on figure 11.

For a specific system like a DP system it is necessary to specify the system model in terms of parameters, noise statistics and initial conditions. However, as stated in the first section of this document, the model is not perfect and is only an approximation of the actual physical system. The same is true with the noise statistics and parameters. These errors can cause the Kalman filter to diverge.

[10] ***Digital and Kalman Filtering***, 2nd Edition, S.M. Bozic

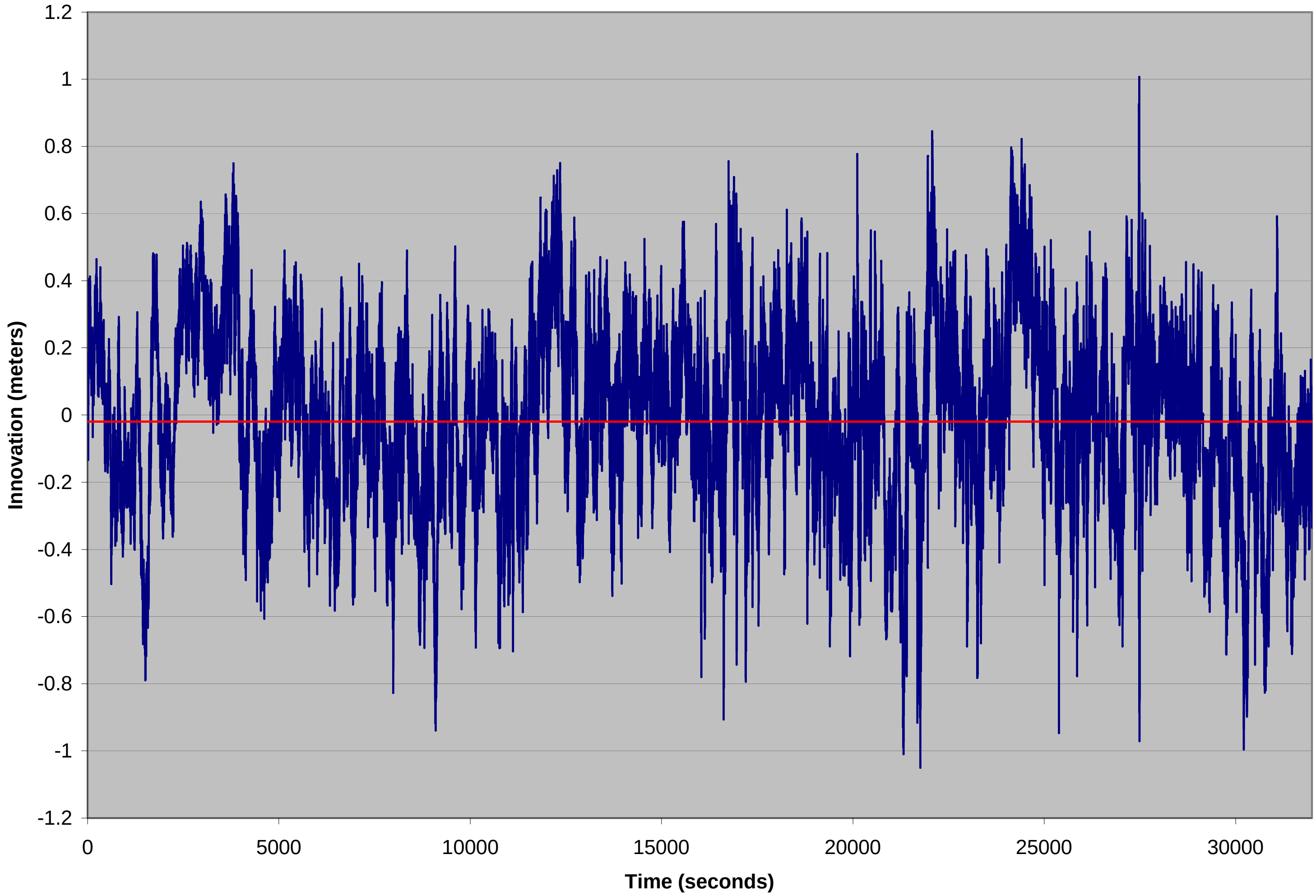


Figure 11 - Example of residual (or innovation) on the Deepwater Pathfinder (Innovation East on DGPS1)

3.2. 'Tuning' the Kalman Filter

Kalman filter performance can be improved by adjusting the ***process noise covariance Q*** and the ***measurement noise covariance R***. As indicated in the two equations below, we can clearly see that adjusting these two values will have consequences on the Kalman gain.

$$P_k^- = A \cdot P_k \cdot A^T + Q$$

$$K_k = P_k^- \cdot H^T \cdot \left(H \cdot P_k^- \cdot H^T + R\right)^{-1}$$

Please note that in conditions where Q and R are constant (this is usually the case in our DP system application) the estimation error covariance and the Kalman gain will stabilize quickly and then remain constant[11].

If Q is too large, then the Kalman gain will be too high and as a result the estimates will have a tendency to follow the measurements "too much" and they will bounce around a lot. If Q is too small, it's exactly the opposite.

If R is too large, then the Kalman gain will be too small and the filter will not take into account the new measurements as much as it should. If R is too small, it's the opposite.

Some DP Systems allow the operator to "adjust" the Kalman filter to be either more relaxed or tighter. These pre-determined filter settings make use of different Q and R values in the filter as explained above. For example a "tight" filter will have a large R and a small Q. In that case the state estimate (output of Kalman filter) will not give too much weight to the measurements from the position reference systems and sensors. A "relaxed" Kalman filter on the other hand will have a small R and a large Q and as a result will follow more position reference systems and sensor measurements.

It is important to note that adjusting the Kalman filter settings will have consequences on the overall behavior of the DP system and of the vessel.

[11] ***An Introduction to the Kalman Filter***, University of North Carolina at Chapel Hill, Greg Welch and Gary Bishop.

Conclusion

Kalman filters are an important part of a DP system. This very mature and standard algorithm is used to estimate the vessel state based on noisy measurements from Position Reference Systems and sensors and an imperfect model of the ship. The recursive nature of that filter, along with the fact that the estimated solution output by the filter is optimal make that filter very interesting from an implementation point of view. The Kalman filter is applied in a lot of different fields apart from the Dynamic Positioning world. You find Kalman filters in GPS, in navigation systems, in cars, in airplanes just to name a few applications. The main disadvantage of the Standard Kalman filter is that it is a linear filter. As a result the Standard Kalman filter does not handle well highly non-linear systems. The Extended Kalman filter was developed in an effort to treat this issue, but the overall solution still does not apply to highly non-linear processes. New types of non-linear filters might in the future replace Kalman filters… Or maybe not.

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Appendix A – Simple Example: Estimating a Constant

Let's take a simplified example to illustrate some of the elements presented above. This example was built using Excel. It does not represent any physical description of Kalman filter applied to dynamic positioning, but it illustrates with numbers how a Kalman filter operates.

Let's take the East position of your vessel as an example. Imagine that the vessel is not moving and not subject to any forces (!). Let's assume that the correct East position of the vessel is 100. A position reference system is sending a measured position to the Kalman filter. This position reference system is noisy.

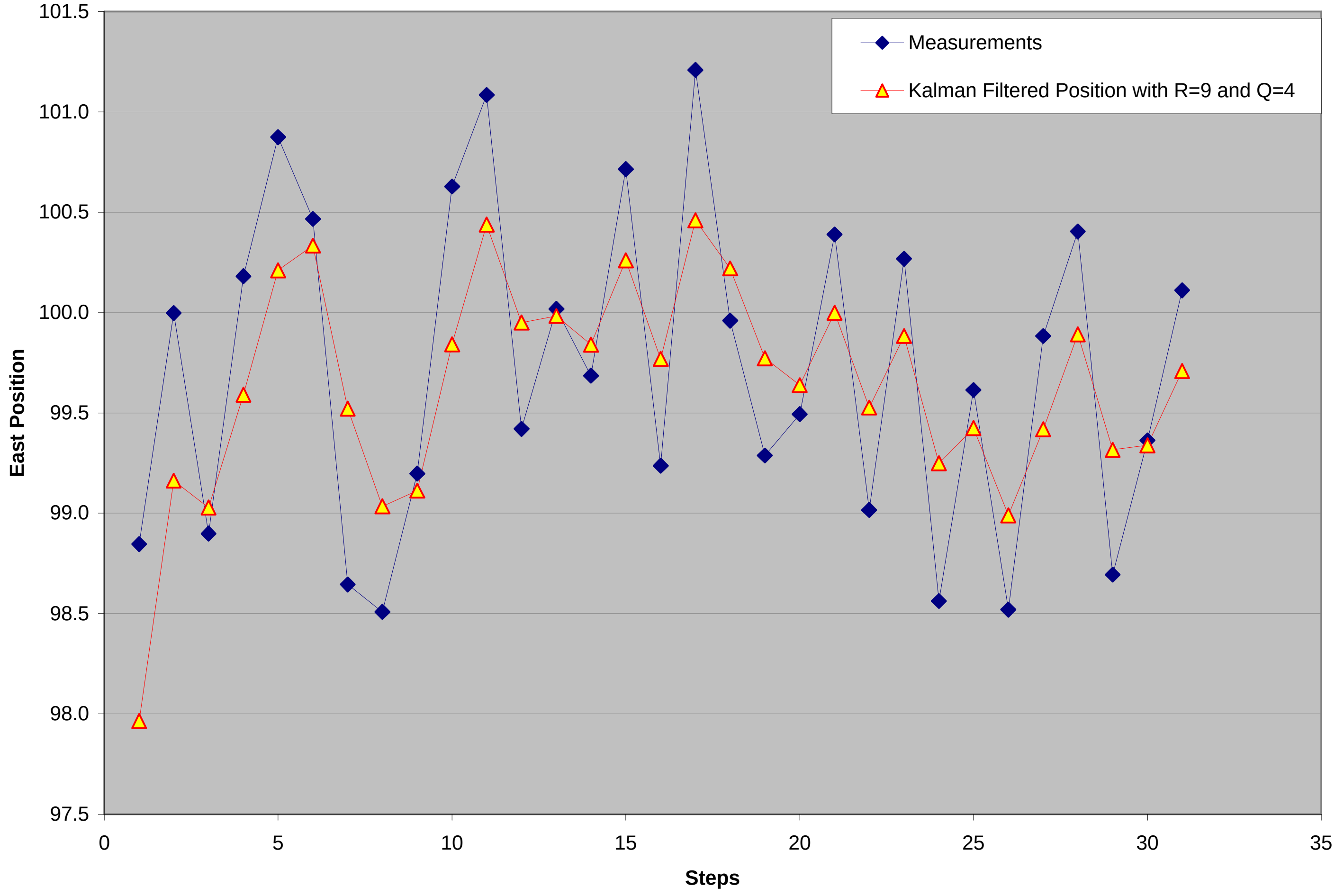
The equation describing the behavior of our vessel would be of the type given in equation 2.1. The "state" of our vessel is the East position. Because the vessel is fixed, the East position does not change with time and therefore A = 1. There's no control input and so $u_k=0$. Because the Position Reference System returns a measurement that is directly the East Position (our state) we have H = 1. As previously identified we will assume that both the process noise and the measurement noise are independent of each other, white and with the following normal probability distributions:

$$p(w) \sim N(0, Q)$$
$$p(v) \sim N(0, R)$$

The following table and graph illustrate the "filtering" aspect of the Kalman filter.

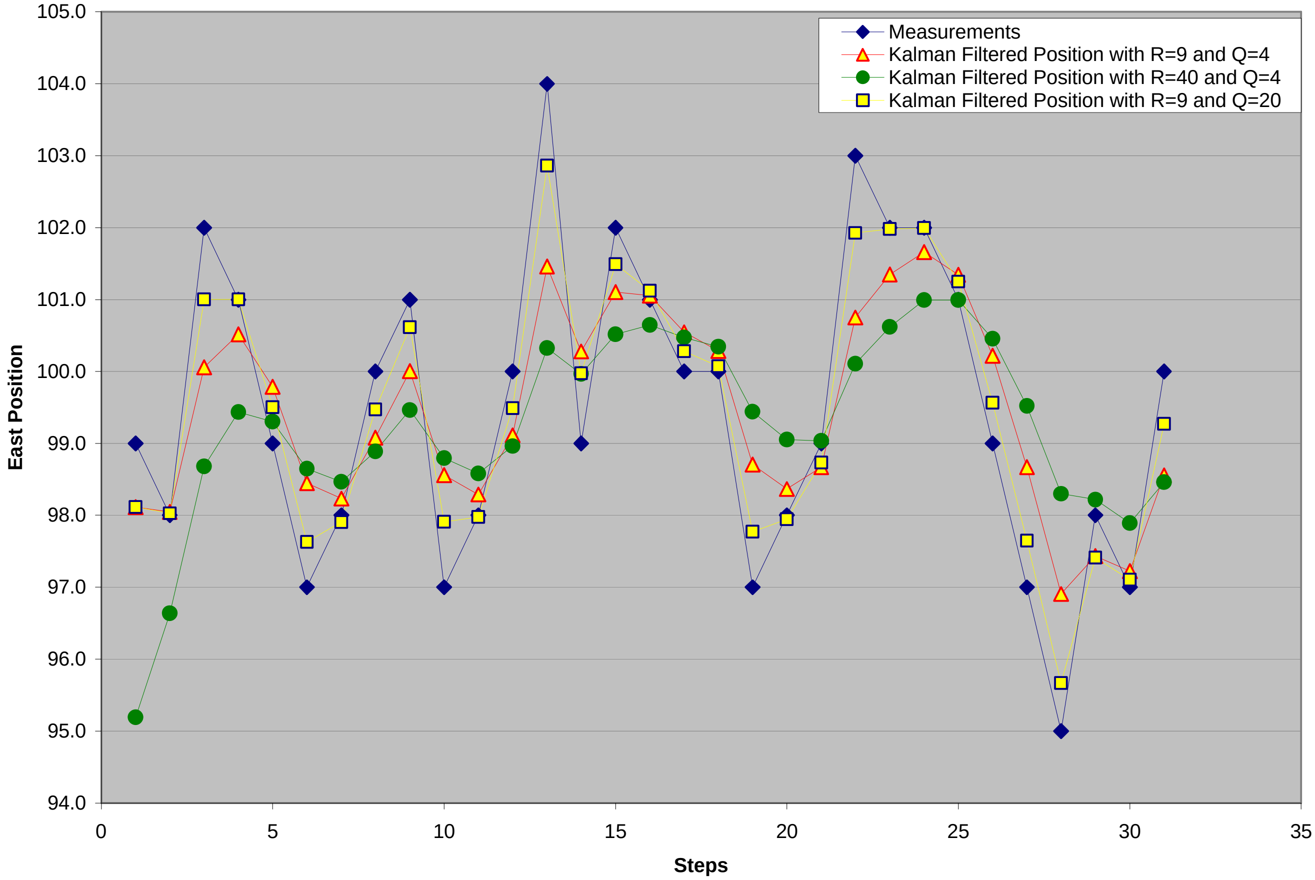
Initial conditions: East Position Estimate = 0 and Error Covariance = 1000
Values used in example: A=1, H=1, R=9, Q=4

Step	East Position Estimate before measurement $\hat{x}_k^- = A \cdot \hat{x}_{k-1}$	Error Covariance before measurement k $P_k^- = A \cdot P_{k-1} \cdot A^T + Q$	East Measurement returned by Position Reference System z_k	Kalman Gain $K_k = \frac{P_k^- \cdot H^T}{(H \cdot P_k^- \cdot H^T + R)}$	East Position Estimate after measurement $\hat{x}_k = \hat{x}_k^- + K_k \cdot (z_k - H \cdot \hat{x}_k^-)$	Error Covariance after measurement k $P_k = (I - K_k \cdot H) \cdot P_k^-$
0	**0.0**	**1000.0**	98.8	1.0	98.0	8.9
1	98.0	12.9	100.0	0.6	99.2	5.3
2	99.2	9.3	98.9	0.5	99.0	4.6
3	99.0	8.6	100.2	0.5	99.6	4.4
4	99.6	8.4	100.9	0.5	100.2	4.3
5	100.2	8.3	100.5	0.5	100.3	4.3
6	100.3	8.3	98.6	0.5	99.5	4.3
7	99.5	8.3	98.5	0.5	99.0	4.3
8	99.0	8.3	99.2	0.5	99.1	4.3
9	99.1	8.3	100.6	0.5	99.8	4.3
10	99.8	8.3	101.1	0.5	100.4	4.3
11	100.4	8.3	99.4	0.5	99.9	4.3
12	99.9	8.3	100.0	0.5	100.0	4.3
13	100.0	8.3	99.7	0.5	99.8	4.3
14	99.8	8.3	100.7	0.5	100.3	4.3
15	100.3	8.3	99.2	0.5	99.8	4.3
16	99.8	8.3	101.2	0.5	100.5	4.3
17	100.5	8.3	100.0	0.5	100.2	4.3
18	100.2	8.3	99.3	0.5	99.8	4.3
19	99.8	8.3	99.5	0.5	99.6	4.3
20	99.6	8.3	100.4	0.5	100.0	4.3
21	100.0	8.3	99.0	0.5	99.5	4.3
22	99.5	8.3	100.3	0.5	99.9	4.3
23	99.9	8.3	98.6	0.5	99.2	4.3
24	99.2	8.3	99.6	0.5	99.4	4.3
25	99.4	8.3	98.5	0.5	99.0	4.3
26	99.0	8.3	99.9	0.5	99.4	4.3
27	99.4	8.3	100.4	0.5	99.9	4.3
28	99.9	8.3	98.7	0.5	99.3	4.3
29	99.3	8.3	99.4	0.5	99.3	4.3
30	99.3	8.3	100.1	0.5	99.7	4.3



Measurements
Kalman Filtered Position with R=9 and Q=4
East Position
Steps
101.5
101.0
100.5
100.0
99.5
99.0
98.5
98.0
97.5
0
5
10
15
20
25
30
35

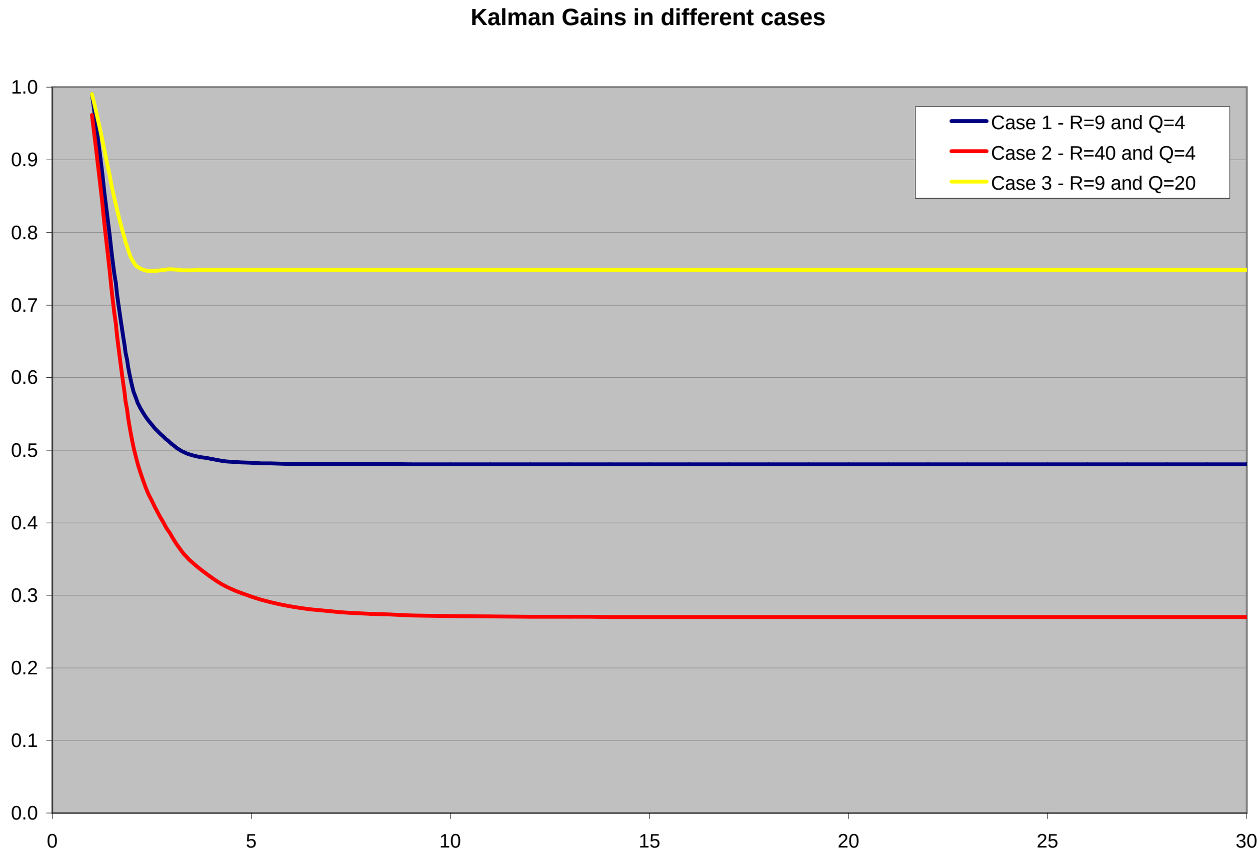
We can also visualize with this example the effect of changes in the Process Noise Covariance Q and the effect of changes in the Measurement Noise Covariance R.



A value of R larger than Q means that the uncertainty of the measurement is way higher than the uncertainty of the process, and therefore the filtered position will follow less the measurements.

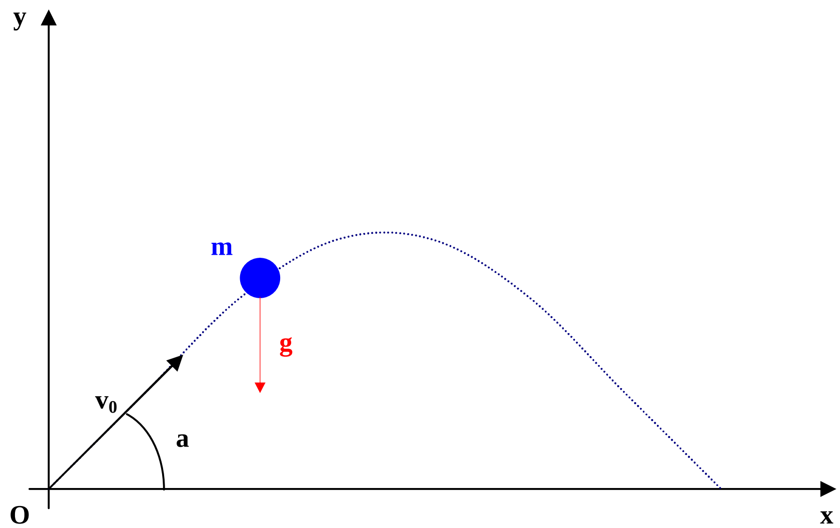
A value of Q larger than R means on the other hand that the uncertainty of the process is way higher than the uncertainty of the measurement. And therefore the filtered position is going to follow more the measurements.

It is also interesting in this simple example to see how the Kalman Gain stabilize (as mentioned in the paper, when Q and R are constant then the Kalman Gain will quickly stabilize).



Appendix B – Simple Example of State Dynamic Model Equation

Let's consider an object in flight moving in a given coordinate system. We will not take into account any other force other than gravity. In the original projection the object is traveling at speed v_0 with an angle of a to the horizontal axis.



We know that: ? Forces = Mass x Acceleration

$m.g = m.a$

$a = g$
$v = v_0 + g.t$
$r = r_0 + v_0.t + ½ g.t2$

$r_0 = (0;0)$
$v_0 = (v_0 . \cos a ; v_0 . \sin a)$
$g = (0 ; -g)$

$$\begin{cases} x = v_0 . \cos a.t \\ y = v_0 . \sin a.t - ½ g.t2 \end{cases}$$

In this example, we're trying to track the position and the velocity of an object in the air. The position and velocity are the state parameters. Let $(x_k ; y_k)$ be the position of the object at step k and $(vx_k ; vy_k)$ its speed at step k. We can determine the following relations:

$$\begin{cases} x_{k+1} = x_k + vx_k \ . \ t \\ y_{k+1} = y_k + vy_k \ . \ t - ½ \ g.t2 \\ vx_{k+1} = vx_k \\ vy_{k+1} = vy_k – g.t \end{cases}$$

In this example, if we chose:

$$X_k = \begin{bmatrix} \mathbf{x_k} \\ \mathbf{y_k} \\ \mathbf{vx_k} \\ \mathbf{vy_k} \\ \mathbf{1} \end{bmatrix}$$

We can see that:

$$X_{k+1} = A.X_k$$

with

$$A = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{t} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{t} & \mathbf{½g.t^2} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{g.t} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$