THEORY OF DIDACTICAL SITUATIONS IN MATHEMATICS

DIDACTIQUE DES MATHÉMATIQUES, 1970–1990

Mathematics Education Library

VOLUME 19

The titles published in this series are listed at the end of this volume.

THEORY OF DIDACTICAL SITUATIONS IN MATHEMATICS

DIDACTIQUE DES MATHÉMATIQUES, 1970–1990

by

GUY BROUSSEAU

Edited and translated by

NICOLAS BALACHEFF, MARTIN COOPER,
ROSAMUND SUTHERLAND AND VIRGINIA WARFIELD

KLUWER ACADEMIC PUBLISHERS
NEW YORK / BOSTON / DORDRECHT / LONDON / MOSCOW

eBook ISBN: 0-306-47211-2
Print ISBN: 0-792-34526-6

Created in the United States of America

Visit Kluwer Online at: http://www.kluweronline.com
and Kluwer's eBookstore at: http://www.ebooks.kluweronline.com

TABLE OF CONTENTS

EDITORS' PREFACE

On the occasion of the celebration of "Twenty Years of *Didactique* of Mathematics" in France, Jeremy Kilpatrick commented that though the works of Guy Brousseau are known through texts referring to them or mentioning their existence, the original texts are unknown, or known only with difficulty, in the non-French-speaking world. With very few exceptions, what has been available until now have been interpretations of the works of Brousseau rather than the works themselves. It was in response to this need that two of us, in the euphoria of an unforgettable Mexican evening at the time of the 1990 PME conference, decided to undertake the task of translating into English most of the works of Guy Brousseau.

The ceuvre is immense, and once past the initial moments of enthusiasm, with the accompanying ambition to produce the entire of it, we recognized the need to choose both the texts and a method of proceeding. As far as the texts go, we chose to take the period from 1970 to 1990, in the course of which it seemed to us that Brousseau had forged the essentials of the Theory of Didactical Situations. But even there the collection is huge. So, after an initial translation of most of the publications of the period, we carved out a selection, retaining the texts which gave the best presentation of the principles and key concepts of the Theory. At the heart of the book we put two works which demonstrate in detail the articulation between theoretical work and experimental research which is the source of the richness of the Theory of Didactical Situations.

The texts we chose, which came from a variety of sources, occasionally overlapped. In the interests of creating a book rather than a collection of papers, we have permitted ourselves to recompose some of them to avoid redundancies and fuse some previously distinct texts. It was, however, out of the question to rewrite absolutely everything, and the attentive reader will observe that a few redundencies remain. Perhaps, though, they will prove to be an aid to comprehension. We have composed preludes and interludes to situate the chosen texts and clarify the construction of the book. And finally, footnotes here and there fill out references from the original texts or elucidate for the reader certain points which seem to us particularly specific to the French educational context of the research presented.

In the domain which interests us, language plays an essential rôle, and at times words resist being translated. Thus we have had to make some choices to take into account distinctions which exist in French but are difficult to translate—as, for instance, that of "savoir" and "connaissance". The most important of these choices are pointed out in the course of the text. But the English language itself has variations depending on the country in which it is spoken and the culture in which it has developed. To take this into account, the initial duo invited in two "first readers" who swiftly turned into full-fledged editors. The work of translation was thus illuminated by the English of Great Britain, Australia, and the United States.

Finally, this work could not have been accomplished without the collaboration of Nadine Brousseau, who accompanied us in our research and found the answers to many questions, and Guy Brousseau, who produced for this translation a number of amendments which a connoisseur may spot.

BIOGRAPHY OF GUY BROUSSEAU

Guy Brousseau was born at Taza, Morocco, on February 4th, 1933. As the son of a soldier, he had an early education marked by frequent changes from one school to another. In 1948, he received his secondary school diploma and earned admission to *l'École Normale d'lnstituteurs* (Normal School) at Agen (in *Lot et Garonne*). There he completed his studies for the first level of the *Baccalauréat.* This in turn earned him admission to *l'École Normale* in Montpellier where he was awarded a *Baccalauréat* with distinction in elementary mathematics in 1950. He was fascinated by both mathematics and physics. At that point, he also obtained a scholarship to study higher mathematics at Toulouse in order to prepare for entrance to the *École Normale Supérieure* of St. Cloud. At the end of one academic year, however, he decided to abandon his studies and return to the *École Normale* at Agen to undertake a year of professional education. He knew already what interested him and what he wanted to study: the way in which children learn mathematics. He explained this to his mathematics professor, Mr. Duclos, who told him, "Then it is not mathematics that you must do, but psychology!". Brousseau replied, "No, it is not psychology!"

In 1953, Brousseau was appointed teacher in a small village in the *Lot et Garonne* region, in a one-class school in which he taught all subjects to students whose ages ranged from 5 to 14. During the year, he married Nadine Labeyue, whom he had met on his first day at the *École Normale* at Agen. (He reports with relish that he may have been only a fifteen-year-old in short pants, but he had great powers of discernment!)

At the beginning of the next school year, in October 1954, he and Nadine were appointed to a "double position" in a two-class school where he taught students in the "Middle Class" (9–10 year-olds) and those in their final year (14 year-olds). It was during this period that his reflections on the acquisition of mathematics and the teaching of mathematics really started.

In 1953, Brousseau took part in a controversy about the "teaching of calculation". The issue was the comparison of Freinet's modern school of thought with traditional methods. During the next two years (from 1954 to 1956), he taught, observed children, prepared sheets of lessons, analyses and problems and continued to learn mathematics. This activity was interrupted in October 1956, when he was called up for military service. He was stationed briefly in Paris, where he was able to take some courses at the Sorbonne from Mr. Pisot (in which the theory of sets and of structures was revealed to him) and at the *Conservatoire Nationale des Arts et Métiers* (CNAM) from Mr. Hocquemgheim. In Algeria, in 1958, he created worksheets which he was anxious to try out in his class on his return.

In January 1959, having completed his military obligations, Brousseau returned to the same village, and to his work as a teacher. From January, 1959 to October, 1961, he experimented and created new texts, all the while carrying out the tasks of all sorts which were expected of teachers at that period. He wrote in a series of grey notebooks (his *cahiers gris*), sometimes all through the night: lesson plans, problems, reflections. The grey notebooks became a kind of talisman for him. As of April, 1996, he figured to have filled some 5750 pages ("with whatever, with everything, and especially with nothing", as he put it.)

In October, 1960, he was encouraged by an article in *Sciences et Avenir,* in which he learned that in Belgium G. Papy was teaching Bernstein's Theorem to future kindergarten teachers (with students 3 to 6 years old).

A more important, determinant event took place in May 1961: Brousseau's encounter with Lucienne Félix[1] , who had written an article in the pedagogical review, *l'Education Nationale.* The content of this article and of her works dedicated to elementary education so completely agreed with Brousseau's beliefs that he decided to write to the author. She replied, asking him to send her copies of his texts to look at. He had, at that time, about three hundred lesson-sheets. Mme. Félix strongly encouraged him to continue this work, and following the exchange of some letters she suggested that he participate in the conference of CIEAEM, which was being held in Switzerland that year. There he had minor disagreements with G. Choquet, who was leaving the presidency, but met G. Papy and W. Servais. He presented the results of his observations of a class where "a more modern teaching of mathematics" was being attempted and returned from the meeting full of enthusiasm.

He continued his writing, producing a textbook for fourth and fifth grades. The primary-school inspector whom he consulted before conducting a larger experiment in the class had himself to obtain authority from the General Inspector, Mr. Degeorge, who was very reluctant and replied that this teacher (G. Brousseau) ought to start by studying mathematics (which he had, in fact, started and which he continued to do).

During this year, Brousseau also became interested in the mathematical problems posed by the management of agricultural business, which he worked on with the farmers in the village. At issue were problems of optimisation (applications of operational calculation) or of representations and treatment of numerous variables which must be considered in order to modify the practices of polyculture. He became aware at that point of the importance and the difficulty of the diffusion of mathematical knowledge in the population at large.

In January, 1962, teachers holding an elementary-mathematics *Baccalauréat* were invited to apply for one year's training to teach mathematics in a general teachers' college in which there was a shortage of teachers. Brousseau was accepted and granted leave for the following school year (1962–63) to attend this course in Bordeaux. At the same time, he applied to the university to resume officially his mathematical studies (interrupted in 1952). In June 1963, he was admitted to the IPES and the MGP[2] . He continued his studies at the university without ceasing in his efforts to disseminate new mathematical ideas. In this way he was introduced to Professor Colmez, who took an interest in him, then, by Lucienne Félix, to Professor Lichnèrowicz who introduced him to the editor, Georges Dunod. Dunod's publishing company published his manual for teachers of the *Cours Préparatoire*[3] , which appeared in 1965.

In 1964, Professor Lichnèrowicz proposed to Brousseau a thesis topic: "the limit conditions of an experiment in mathematical pedagogy", and introduced him to Pierre Greco, who co-directed the work. In order to treat the subject correctly, Brousseau proposed to the director of the *Centre Régional de Documentation Pédagogique* (CRDP) of Bordeaux, to create within his establishment a *Centre de Recherche sur l'Enseignement Mathématique* (CREM). The director, Mr. la borderie, accepted. With the help of a number of people from the university and teachers from the *École Normale d'Instituteurs,* Brousseau published a large number of "notebooks" intended to suggest innovations to teachers, but also to spell out the conditions for the emergence of real research.

Research on these

- *technical conditions* (theories, methodology, fields of experience),
- *sociological conditions* (administrative organisation, contacts among researchers, subjects, domains of reference, fundamental concepts), and
- *pedagogical conditions* (acceptable forms of teaching, ethical aspects)

led him to deepen his knowledge in various domains, and to seek the guidance of specialists whom he encountered at the University of Bordeaux or thanks to the activities of the CRDP: cinematic linguistics and semiology with Christian Metz, later the Science of Education with J. Wittwer, etc.

In February, 1968, at a colloquium at Amiens, Brousseau presented, with J. Becker and J. Colmez, the results his reflections on Professor Lichnérowicz's topic. What had developed was a project for the creation of an *Institut de Recherche sur l'Enseignement des Mathématiques* (IREM) made up of three components: a colloquium open to all teachers of mathematics; teaching, information and documentation of professors of mathematics (of all levels and in all materials); research on the teaching of mathematics. The component "research on the teaching of mathematics" was itself composed of two activities:

– the first which one could today call "engineering and workshops of *didactique*" (and which has been called, at different points, "innovation", "action research", "production of materials, projects and curriculum"...)

– the second which was dedicated to experimental and theoretical research on the teaching of mathematics and was to obey the academic rules of research. In addition to the "application" of the fundamental domains of teaching, it was necessary to create a theoretical instrument for the integration and co-ordination of these efforts. The project was studied in a systemic perspective. In order to permit it to function, its relations with all the organisations concerned were examined. Academic research on what was to become "experimental epistemology" and then "*didactique* of mathematics" (after nearly having been called "didactology of mathematics") was at the time completely non-existent—even inconceivable. It was necessary to propose one or more initial theoretical models (to be demolished, but consistent and large enough), research methods, an initial group of researchers ... and systems for re-opening the discussion. The most delicate point was imagining a relationship between the researchers and their object of study—teaching—which would neither compromise the academic research nor be detrimental to the teaching. The creation for this purpose of a *Centre pour l'Observation de l'Enseignement des Mathématiques* (COREM), distinct from the experimental schools and pilot schools of the period, became the kernel of the project. The COREM would provide the milieu in contact with which a composite team would be able to carry out the first whorls of the spiraling development of this academic research.

In July, 1968, the International Commission on Mathematics Education (ICME) held its first congress. The congress was at Lyon, where Maurice Glayman and his group had also prepared a form of IREM which was to be created in January, 1970 (with Paris and Strasbourg). The IREM of Bordeaux was created in October, 1970. Guy Brousseau, a licensed teacher of mathematics, was recruited by the University as an Assistant in Mathematics to participate in the realisation of the announced project.

The first elements of Brousseau's Theory of Didactical Situations were communicated in a session of the Congress of the *Association des Professeurs de Mathématiques de l'Enseignement Public* (APMEP) at Clermont-Ferrand in 1970. The first article on methodology (on quasi-implications) was published in the bulletin of psychology of the University of Paris in 1969. The first example of a predictive mathematical model relative to a modification of teaching, experimentally verified (the teaching of the calculation of multiplication and division), was communicated in 1973 at the Sixth International Congress on the Science of Education.

1972 marked the creation of' a school for observing children learning mathematics. This Jules Michelet School at Talence developed progressively into a school for the observation of the teaching of mathematics (the COREM).

Research was carried out there (1970–74) on the teaching of the natural numbers, the operations on the natural numbers and fundamental structures. Other topics were the teaching of probability and statistics, in collaboration with P. L. Hennequin, statistical studies under the direction of H. Rouanet (1973–74), the teaching of rationals and decimals (1973–80),...

In 1975 a doctoral program in *didactique* was created at Paris, Strasbourg and Bordeaux. Brousseau was in charge of theses and several principal courses. He played a leading rôle in the late seventies in the development of *Didactique des Mathématiques* as a scientific discipline. Among the essential events of this period we should mention especially the creation of a National Seminar in Paris held three times a year, a Research Summer School and the establishment in 1980 of the international journal *Recherches en Didactique des Mathématiques.*

Member of the Department of Mathematics of the University of Bordeaux 1 since 1970, Guy Brousseau, after having twice decided against presenting different works as theses in areas which he did not favour, submitted his *Thèse d'Etat* and was granted his doctorate in 1986. He wrote his thesis with constant interaction with Professor Bernard Malgrange who supported him and, says Brousseau, "allowed him to clarify his ideas and to improve his texts".

Brousseau has been a Professor at the IUFM of Bordeaux since 1990. He is currently Director of the *Laboratoire Aquitain de Didactique des Sciences et des Technologies* of the University of Bordeaux.

NOTES

1. Lucienne Félix (1901-1 994) was the author of numerous works of mathematics, including, among many others: "Mathé*matiques modernes et enseignement élémentaire*" (Modern mathematics et elementary teaching), *"L'exposé moderne des mathématiques élémentaires"* (Modern presentation of elementary mathematics), Geometry textbooks for all levels of high school, an Analysis textbook for the most advanced level of high school and a number of articles in journals on the teaching of mathematics. She designed a course on Henri Lebesgue (whose assistant she had been), "*Constructions Géométriques",* and wrote a book *"Message d'un mathématicien*—Henri Lebesgue" on the occasion of the centennial of his birth.
2. IPES was a kind of institute allowing students access, through a competition, to a state position while they prepared themselves to become secondary school teachers. The training in IPES was essentially in the discipline, in this case mathematics. MGP, *"Muthématiques Générales et Physique",* was at this time the diploma obtained after the first two years of studies at the university.
3. First year of schooling.

PRELUDE TO THE INTRODUCTION

Our initial intention had been to start the book directly with Chapter 1, since it is an article which introduces and presents the foundation and the methods of the Theory of Didactical Situations. Knowing, however, the difficulties frequently encountered by researchers when they first discover Brousseau's work, we feared that that entry into the book would discourage some readers. Indeed, we could have provided a warning like that offered by a colleague from Québec who wrote as a note to the publication of a text of Brousseau: "The lecture has been accepted by the Editorial Board, which decided to publish it, encouraging our readers to persevere in their appropriation of a terminology which is at times new for them. The article [here part of Chapter 5] is so rich that it is worth the effort required to get all the way to the end of it". Such a warning seemed, however, likely to prove counterproductive. Actually, we do not think that it should come as a terrible surprize to the reader for a scientific text to require some effort to read. Mathematicians especially must be aware that the reading of mathematics is in itself work. Nonetheless, this particular difficulty seemed worthy of attention. Our decision was to supply as an introduction the text, "The Race to 20", in which Brousseau introduces most of the main features of the Theory of Didactical Situations in such a way as to provide the reader with an intuitive understanding of their meaning.

"The Race to 20" is a situation designed and used by Brousseau in the early seventies. As Perrin-Glorian emphasizes (1994, p. 106), it was the generic example on which he first built and developed several aspects of his theory, and which he used in this period in order to illustrate it. The presentation speaks to the teacher as well as to the researcher or the mathematics educator.

The key role played by the relationships between the functioning of student knowings, made evident by the student's behaviours, and the characteristics of the situation is well described. It is at the core of the whole theory. The roles of the teacher and the related didactical problems are identified and described in a way which prepares the reader for the powerful concepts of *Didactical contract* and of *Institutionalization* which were coined later by Brousseau.

The Editors

INTRODUCTION

SETTING THE SCENE WITH AN EXAMPLE: THE RACE TO 20*

In a general way, thanks to the analysis of systems, teaching situations can be described and classified in terms of exchanges between students, teachers and the *milieu*. As an illustration, let us study the lesson on "The Race to 20". From this study, we shall derive a general classification of didactical situations.

1. INTRODUCTION OF THE RACE TO 20

The aim of this lesson was to revisit division (in circumstances in which the "meaning" of the operation did not conform to the one learned earlier) and to foster the discovery and demonstration, by the children, of a sequence of theorems.

1.1. *The game*

The game is played by pairs of players. Each player of a pair tries to say "20" by adding 1 or 2 to the number given by the other. One of the pair starts by saying "1" or "2" (for example, "1"); the other continues by adding 1 or 2 to this number ("2" for example) and saying the result (which would be "3" in this example); the first person then continues by adding 1 or 2 to this number ("1" for example) and saying the result (which would be "4" in this example); and so on.

1.2. *Description of the phases of the game*

This first section is a nutshell description of the phases of the game, all of which will be discussed in more detail later. We include it in order to make the reading easier.

Phase 1: Explanation of the rules
The teacher explains the rules of the game and starts playing a round at the chalk-board against one of the children, then relinquishes her place to a second child.

* Brousseau, G. (1978a) Etude locale des processus d'acquisition en situations scolaires. *Etudes sur l'enseignement élémentaire* (Cahier 18, pp. 7–21). Bordeaux: IREM et Université de Bordeaux 1. *Editors' note:* This is the text of a lecture given in Barcelona in 1977.

Phase 2: Playing One-against-one
The class then divides into pairs; members of each pair play against each other. They write the numbers chosen on a sheet of paper on opposite sides of a line. This phase should consist of about four rounds and take no more than ten minutes.

Remark: During this phase, the children apply the rules.
Certain children, without being conscious of it, realise that saying numbers at random is not the best strategy; they test the constraints of the game at the level of action and immediate decisions, and provide themselves with a sequence of examples. Some of the chidren discover implicitly the advantage of saying “17”.

Phase 3: Playing Group-against-group (six to eight rounds, 15 to 20 minutes)
The children are divided into two groups. In each one, the teacher nominates one child as team representative for each round, naming her at random. Any child might be called upon to defend her group at the chalkboard, in front of the whole assembly; if she wins, her group will receive one point.

The children very quickly realize the necessity of planning together and discussing within each group so as to share strategies among themsleves. Some recognize right from the beginning that “You have to say ‘17’”.

Phase 4 : Game of discovery (20 to 25 minutes)
The teacher then asks the children to put forward propositions. These are the discoveries that they made which allowed them to win. The teacher writes these discoveries on the chalkboard as they are presented by each group in turn; they are then verified by the other group and either accepted or rejected. If they are accepted, they remain on the chalkboard.

Proposed discoveries	
team A	team B

Score	
A	B

Figure 1

Each child who puts forward a proposition must prove to an opponent that it is either true or false, either by playing or by an intellectual demonstration.

In order to make the game more interesting, the following rule can be adopted:

– each proposition accepted by the class is worth one point;
– each proposition proved false is worth three points to the group that does so.

Remark : If the game of discovery grinds to a halt (the children find no more propositions to put forward), the teams can return to playing “Who will say ‘twenty’?”.

Observation of a particular lesson: Very quickly the following propositions were made:

– if I write “17”, I am sure of winning;
– if I write “14”, I am also sure of winning.

The discoveries then became much more minor.

Example : If I say "16", my opponent can say "17" and will win. If I say "18", I shall lose, etc.

At this point, the discovery game was abandoned in favour of the team "20" game. At the end of two rounds, the children discovered that when they said "11", then "14", then "17", they would win. Very intense discussion arose about "5": one child who had written 5 during the first game played in pairs had lost. Another child showed her that if she knew how to play, she could win by saying "5". If she did not know how to play, clearly she might lose. These proofs were always carried out by further play (starting with 5, for example). After an hour, the children had discovered that, to win, they must say 2, 5, 8, 11, 14 or 17.

1.3. *Remarks*

This situation was reproduced under observation sixty times. In addition, each of its phases was the object of experimentation and clinical study[1]. The observations were carried out over a period of three years by a group from the IREM of Bordeaux consisting of a Teaching Assistant in Mathematics, seven academic psychologists and a middle school teacher specializing in computer science. Despite the non-directive and non-informative nature of the teacher's interventions, the lessons proceeded in a remarkably comparable manner for the first two phases. As a result, we are able to introduce our remarks as properties of the situation.

These remarks are clinical observations (CO), statistical observations (SO) or axioms (AS) or theorems (TS) of the Theory of Didactical Situations.

1. Strategies and discoveries are used implicitly before being formulated so as to respond to the needs of an ongoing action. Example: The sequence 5, 8, 11, 14, 17 occurs disproportionately often well before the students have formulated the necessity of "playing 14". (SO)

2. Formulation takes place after conviction and before proof[2] in order to respond to the needs of communicating an action. Several formulations precede the proof and are supported at the same time by effectiveness and rationality. (SO)

3. Established theorems do not immediately serve to support each other; their articulation is discovered only at the end. The same proof is discovered several times, even by the same child. (SO)

4. It is the children who lose a round who most want to explain their failure or the conditions for success. (CO)

5. Proof gets its mathematical value when it has been tested as a means of convincing and is obligatory for being convinced; this can be negotiated only among "equals", between children. The teacher must send the questions back to the groups. (AS)

6. Explanation must be necessary, technically and sociologically; if the result is obvious or—as it was here in the beginning—generally accepted, only a recipe is obtained. (TS)

2. FIRST PHASE OF THE LESSON: INSTRUCTION

Thus, in the situation that we just went through, the communication started with an *instruction.*

The teacher talked about a game; she communicated a message which contained the rules of the game so that the students could internalize and apply them. This message did not contain any new words; it is assumed that the children understand the terms and their organization (that is to say, the phases).

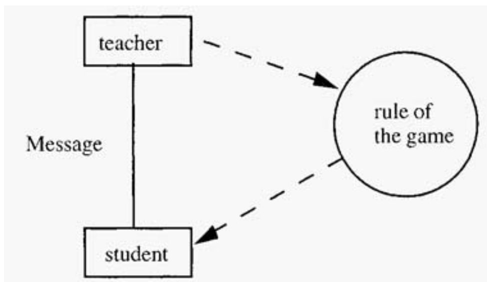


Figure 2

The statement of the whole set of rules might have been too long for some children. So the communication of the message is accompanied by an action of the child (by making her play).

The teacher simulates the situation which the child will meet during the course of a normal game. In this case, the situation for the child is the sequence of numbers in the game.

Example: The child plays 1; the teacher, 3; the child, 5; the teacher, 6. For the child, the situation at this moment consists of the sequence of moves illustrated below:

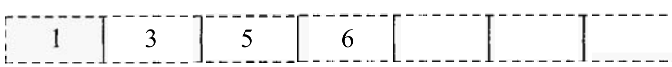


Figure 3

This is a source of information for her and she acts on the situation by deciding to add a number.

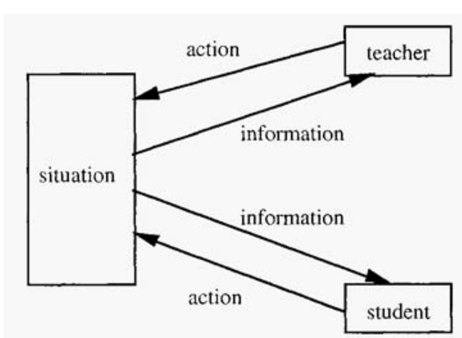


Figure 4

At the same time and while she is playing, the teacher comments on the decisions and illustrates the rules of the argument. She talks about these rules by matching them with the circumstances of the situation.

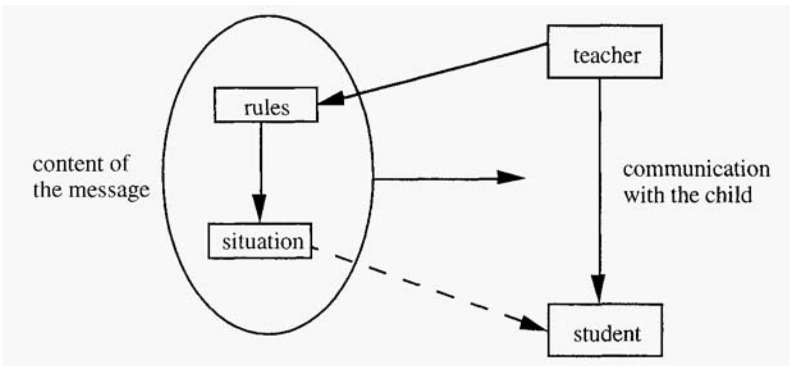


Figure 5

Earlier, when the instructions were given, the teacher stated the rules and the child had to imagine the corresponding game-situations. Now, the rules are communicated to her with the corresponding and alternating circumstances; 2 is played and 3 is communicated. The aim of this sequence is still the communication of an instruction but it has slipped into an action phase.

Remark: The teacher wants to communicate to the child a rule of action that she has in her repertoire. She transforms this rule into a message appropriate to the medium by means of which she can communicate with the child. Here, she communicates by sound. She therefore expresses the rules in the spoken code of the children's language.

For the child, this message is a source of information (which she interprets following her own codes) starting with which she reconstructs a message having meaning. The meaning given by the child does not necessarily coincide with the meaning the teacher intended to convey.

In the case of an instruction, it is hoped as well that the pupil interprets the content of the message communicated by the teacher as a rule of action. She must therefore internalize and remember it.

The aim of practising a game at the same time as giving the intruction is to ensure that the rules internalized by the child are the same as those given by the teacher; action reduces the ambiguity of the message by introducing feedback.

We call an influence of the situation on the pupil "feedback". The child receives this influence as a positive or negative sanction relative to her action, which allows her to adjust this action, to accept or reject a hypothesis, to choose the best solution from among several (the one which improves the satisfaction obtained during the action).

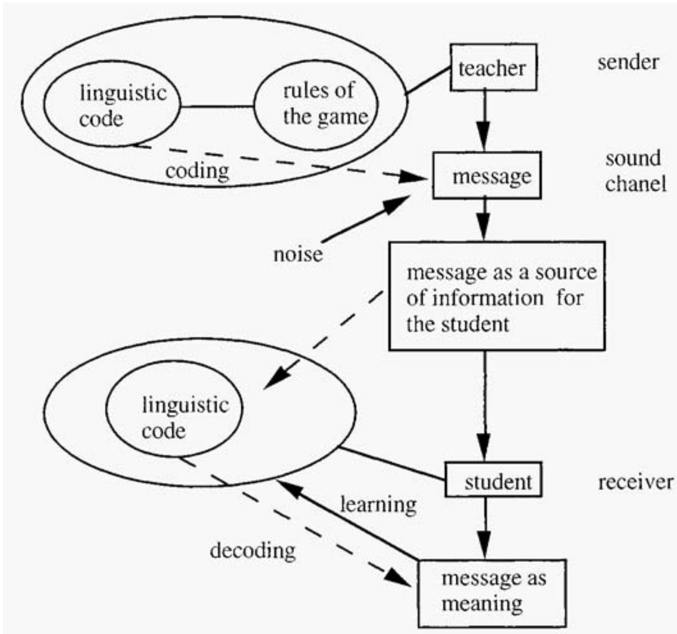


Figure 6

Remark:
This feedback must be closely associated with the learning which the teacher is trying to make happen.

Here, the communication of the instruction is controlled by the teacher with the help of feedback; if the student doesn't apply an instruction correctly, then she hasn't learned it correctly (it must then be repeated to her).

The student, for her part, has feedback which is the evaluation given by the teacher about the validity of the rules and the propositions that she has made.

3. ACTION—SITUATION, PATTERN, DIALECTIC

3.1. *First part of the game (one against the other)*

The first part of the game realizes a situation typical of what we call a "situation of action".

The children play two by two, one against the other; each student is faced with a situation: the sequence of numbers already played. When her partner has played, she must make a decision and act on the situation by proposing a number herself (after having, in this case, analyzed the situation and drawn information from it).

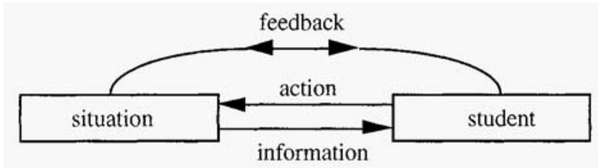


Figure 7

At each move, the situation is modified (it is modified by the partner). At the end of a few moves, the penalty occurs: the round is won or lost.

Within a situation of action, everything that acts on the student or that she acts on is called the "*milieu* ". It can be that it includes neither the teacher nor another student. This is a very general pattern. Nearly all teaching situations are particular cases of it.

3.2. *Dialectic of action*

While the student is playing new rounds, she develops strategies, that is to say reasons for playing one number rather than another. For instance, she will play 10 rather than 9 because she thinks, incorrectly, that the game has something to with decimal counting. Perhaps she plays 13 because she considers it to be a lucky, magic number, or 17 because she has intuitively noted that she has already won after playing 17 or, on the other hand, that she found herself in bad shape after her opponent had played 17.

She may also adopt the strategy that consists of randomly choosing one of the two possible numbers.

Generally, a strategy is adopted by intuitive or rational rejection of an earlier strategy. A new strategy is therefore adopted as the result of experimentation. It is accepted or rejected following the student's evaluation of its efficacy; this evaluation can be implicit.

The sequence of "situations of action" constitutes the process by which the student forms strategies, that is to say, "teaches herself" a method of solving her problem.

This succession of interactions between the student and the *milieu* constitutes what we call a "dialectic of action". We use the word "dialectic" rather than the word "interaction" because, on the one hand, the student is capable of anticipating the results of her choices and, on the other hand, her strategies are, in a way, propositions confirmed or invalidated by experimentation in a sort of dialogue with the situation.

In the course of this "dialectic of action", the child organizes her strategies, and constructs a representation of the situation which serves as a "model" and guide for her when making her decisions. This "model" is an example of relationships between certain objects that she has perceived as relevant in the situation.

Example: At the start of a game, all numbers appear to the child to be of equal importance. At the end of this phase, when she begins to know that when she plays 17 she will win, the choice of other numbers (18 or 19) will not seem relevant to her (they have no rôle in her decision).

The set of relationships, "If I play 14 or 17, I can win", can remain more or less implicit; the child plays according to the model before being able to formulate it. (In general, six or seven rounds of the game are needed between the moment when the number 17 is played in a preferential way (third round) and the moment when certain children are likely to state, "To win, I must play 17!".

We use the term "implicit model" to describe the set of relationships or rules according to which the student takes her decisions without being able to be conscious of them and a fortiori to formulate them (which does not mean that a rule of action always appears without one's being able to formulate it).

The majority of didactical situations arise from a particular scheme of action. But we reserve the term "dialectic of action" in its strict meaning for didactical situations which do not make it necessary for the child to formulate the model used.

Some implicit models correspond, at the end of the learning, to pieces of "know-how" which have been taught.

Observation: This implicit model does not coincide with the whole of the know-how. It can happen that during the learning of algorithms the child develops, unknown to the teacher, incorrect models which justify (in a good or not-so-good way) the know-how acquired. (TS and SO). It was observed, for example, that as early as the third round children play "17" in preference to "16" or "18". In the sixth round, they preferentially play 14 long before stating why. (SO)

4. FORMULATION—SITUATION, PATTERN, DIALECTIC

4.1. *Second part of the game (group against group)*

In this second part of the game, the children are formed into two equal groups, and two different phases can be alternately observed:

a) when the group representative is at the chalkboard and playing;
b) when there is discussion within the group.

In the former phase, a child who is not at the chalkboard records all the information present by observing what the two representatives write down, but she can neither act (play, herself) nor intervene (transmit information to her teammate).

Whoever is playing at the chalkboard is in the didactical situation of action.

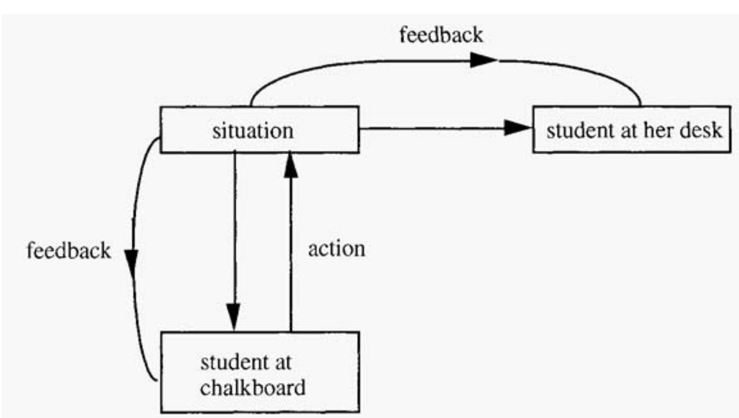


Figure 8

During the phases of discussion within the groups, the *milieu* for each of the students consists of all the rounds played and, in particular, the last round written on the chalkboard.

In order to win, it is not enough for a student to know how to play (that is to say, have an implicit model)—she must indicate to her teammates which strategy she proposes; this is the only way she has of acting on the situation to come.

Each child is therefore led to anticipate, that is to say, to be conscious of the strategies which she would use (first phase, 2a).

Her only means of action is to formulate these strategies. She is subjected to two types of feedback:

– an immediate feedback (at the time of formulation) from the people with whom she has the discussion, who show that they do or do not understand her suggestion (phase 2b);
– a feedback from the *milieu* at the time of the next round played, if the formulated, applied strategy is a winning one or not.

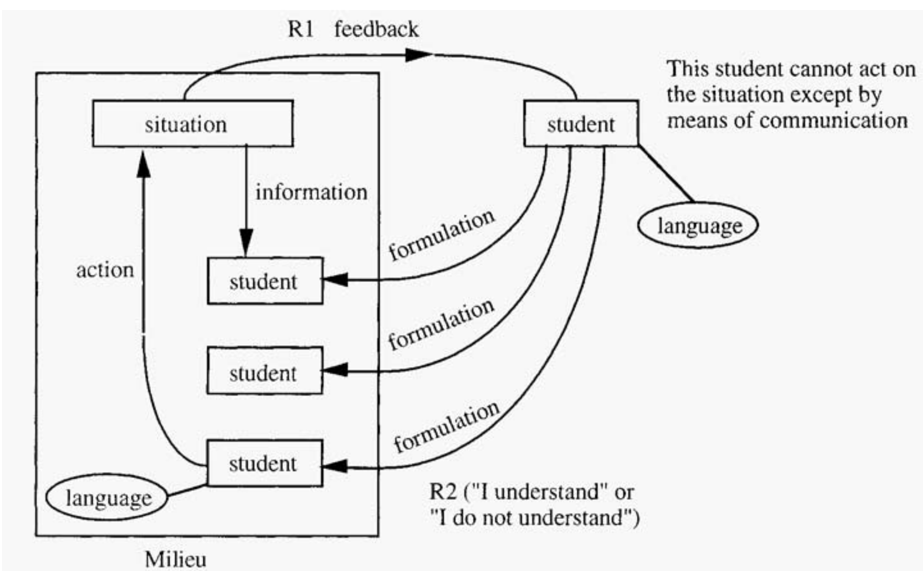


Figure 9

The formulation of a strategy is the only means that a student has of getting it to be applied by the student at the chalkboard. This second phase (b: discussion within the group) achieves what we call the didactical situation of formulation. Figure 9 is a special case of the scheme of action, a case in which this action is possible only if it is given a formulation. In the present case, the formulations are very easy and do not give rise to the construction of any special language. (Every child is capable almost immediately of stating, “You have to play such-and-such a number”.)

4.2. *Dialectic of formulation*

A dialectic of formulation would consist of progressively establishing a language that everyone could understand, which would take into account the objects and the relevant relationships of the situation in an adequate way (in other words, by permitting useful reasoning and actions). At each moment, this constructed language would be tested from the point of view of intelligibility, ease of construction and the length of the messages that it allows to be exchanged. The construction of such a language or code (repertoire, vocabulary, sometimes syntax) into an ordinary language or a formalized language makes possible the explanation of actions and models of action.

The scheme of formalization is governed by the laws of communication:

- qualititative conditions of intelligibility (on repertoire and syntax),
- quantitative conditions of intelligibility (on utterance, noise, ambiguity, redundancy, capacity for collating and control conditions for beginning redundancy, etc.)

This requires the following terms and notions to be used with ease:

- "You played such-and-such a number after she played such-and-such a number"
- "You should have..."
- "We need to play,... you played..."

It requires, as well, the distinction between "to want to" and "to be able to", between what will certainly happen and what could happen. If some of the children whom we are addressing haven't sufficient mastery of this language (perhaps because they are too young), the dialectic of formulation will function for them; they will have to construct an efficient vocabulary (this will therefore be a learning exercise only for those who need it).

During this phase, it can happen that one student's propositions are discussed by another student, not from the point of view of the language (the message is or is not understood) but from the point of view of the validity of the content (that is to say, its truth or its efficacy).

Example:

"We must play 15", says one child.
"No, because in one round, I played 15 and I lost",
responds another child. Or,
"Why must we play 17?"

We call these spontaneous discussions about the validity of strategies "validation phases". They appear here as means of action. The students use them as means of encouraging their partners to carry out the proposed action. The means of transmitting conviction can vary widely (authority, rhetoric, pragmaticism, validity, logic).

In the dialectic of formulation, these means are out of the student's control and remain implicit, unlike other validations which appear as the goal or object of study. In order to obtain the latter, one must organize a new type of didactical situation.

5. VALIDATION—SITUATION, PATTERN, DIALECTIC

5.1. *Third part of the game (establishment of theorems)*

In this third part of the game, the pupils are still divided into two groups—A and B, the same groups as for the previous part.

Instructions:

> "We are going to have a theorem competition. We are all mathematicians, and we are going to cooperate in the advancement of science by adding 'true' statements which we are all sure about, and which will be useful for winning.
> To add a theorem, first you have to make some statement, which we will label a 'conjecture'. When this conjecture has been accepted by everybody, then it will become our theorem.
> I will write the conjectures proposed alternately by Group A and group B on the board this way:

Statement under study		**Statement accepted**	**Score A**	**Stake**	**Score B**
Team A	Team B				
Prop: 17 wins	*Opp: OK*	*17 wins*	1		

Figure 10

> The group that proposes a conjecture puts one point at stake which I will put in the "at stake" column. Depending what happens after that, the point will be added to the score of Group A or Group B.
> You may confer with each other before proposing a statement. I'll explain the rest of the rules as the game goes along.

Remark: Unlike phases 1 and 2, this one does not proceed in a standardized way. Only the rules and results stay the same. But they can only be explained to the students progressively, as the game goes along, and depending on how it goes. Listing off the rules is fussy and hard to understand. Here we will communicate them to the reader by describing the development of a fictional class.

Teacher: Group A, can you make a declaration which is true and will help you win? You may consult with each other before you make your proposal. You are the Proposers.

Pupil in A: You can be sure of winning if you say 17.

Teacher: I will write on the board "17 is winning number", *proposer A..*

Pupil in B: Yeah, that's what we wanted to say!

Another pupil in B: But I don't agree! People have played 17 and lost. I can play 17 and lose if I want to.

Teacher: Wait! When a conjecture has been written up on the board, the other team—B in this case—has to decide

– whether to declare it true, in which case the other group wins the stake of one point;

– whether to declare it false, in which case that group becomes in turn a proposer, but of the opposite conjecture. For example, if group B decides to challenge A's conjecture, I will write "17 is not a winning number" *proposer B.* Then there will be two points at stake.

– you can also just say you doubt it.

Discussion among the children...

Pupil in B: And if we say we doubt it?

Teacher: You become the opposer. You put two more points in play, to be added to the stakes.

The opposer can

– make the proposer play 5 rounds of the Race to 20 in which the proposer is required to apply her conjecture. Points won by the proposer or opposer are marked on the score. The opposer can make the proposer go on playing until one or the other retracts her conjecture. The other one then adds to her score the points at stake.

– Ask the proposer for a suitably convincing mathematical proof. In this case, the one who convinces the other gets 5 points, the one who gets convinced gets 2 points.

Then everybody is "supposed" to apply this declaration in the rounds we'll play later. The conjecture becomes a *theorem*.

Pupil b in B: We four do not agree with the rest of group B. We want to say we doubt the theorem.

Teacher: You want to make the A's play starting with 17?

Pupil b in B: Ummm... no! We want to request a mathematical proof!

Other pupils in B: No! No! It's a sure thing—you gotta play 17! We're going to lose points!

Teacher: OK—as an example, you are going to carry out a discussion between members of Group B with everyone listening following the same rules. Obviously there are no points at stake, since you're all in the same group.

Teacher to the others in B: Figure out the proof that A could give b so you can save your points. In any case, people can always ask for proofs of theorems, even when everybody agrees to them.

5.2. The attitude of proof, proof and mathematical proof

The reasons that one child can give in order to convince another, those that can be accepted without "loss of dignity", must be drawn out progressively, constructed, tested, formulated, discussed and agreed upon.

Doing mathematics does not consist only of receiving, learning and sending correct, relevant (appropriate) mathematical messages.

To state a theorem is not to communicate information, it is always to confirm that what one says is true in a certain system; it is to declare oneself ready to support an opinion, to be ready to prove it.

It is therefore not a question only of the child's "knowing" mathematics but of using it as a reason for accepting or rejecting a proposition (a theorem), a strategy, a model, that which requires an attitude of proof. This attitude is not innate. It is developed and sustained by particular didactical situations which we shall now discuss.

In mathematics, the "why" cannot be learned only by reference to the authority of the adult. Truth cannot be conformity to the rule, to social convention like the "beautiful" and the "good". It requires an adherance, a personal conviction, an internalization which by definition cannot be received from others without losing its very value. We think that knowledge starts being constructed in a genesis of which Piaget has pointed out the essential features, but which also involves specific relationships with the *milieu,* particularly after the start of schooling. We therefore consider that for the child, making mathematics is primarily a social activity and not just an individual one.

The passage from natural thought to the use of logical thought like that which regulates mathematical reasoning is accompanied by construction, rejection, the use of different methods of proof rhetorical, pragmatic, semantic or syntactic.

The consideration of a proof is a reflexive attitude. The proof must be formulated and present while being considered, and therefore most often written, and must be able to be compared with other written proofs also dealing with the same situation.

In general, proof will be formulable only after having been used and tested as an implicit rule either in action or in discussion.

5.3. Didactical situation of validation

The child is therefore dealing with relationships between a "real" situation, concrete or not ["a" in the diagram], and one or several statements ["b"] formulated about the subject of this situation.

These statements can be messages previously exchanged at the time of a dialectic of formulation, bearing on strategies and descriptions as well as on judgements.

Example: "You say 'We must play 14' [b]. But in this round, I played 14 and I lost [a]".

The child must make statements about these relationships. The favourable situation will therefore obey a scheme of formulation. But these statements must be subject to a judgement on the part of the interlocutor, who cannot be, as previously, a simple receiver. This means that the interlocutor must be able to provide feedback to the child who is judging; she must be able to protest, reject a reason which she judges false, prove in her turn. They must therefore both be in a *priori* symmetrical positions as much from the point of view of the information which they retain as that of the means of feedback. In addition, discussion between teacher and student is highly disadvantageous, even when the teacher practises a refined *maieutic* in order to reduce her authority.

If one wishes to avoid having sophistries, rhetoric and authority take the place of consistency, logic and the efficacy of proof, one must not let the discussion lose touch with the situation which reflects the students' discourse and gives it meaning. Motivation must make this double confrontation (R_1 and R_2) necessary.

Example in "The Race to 20":
A statement can be confronted with reality in several ways:

- one can take examples from rounds of the game (that does not always provide a proof);
- one makes the other person play the strategy which she proposes;
- one makes the other person play against the strategy that she rejects;
- one profits from intellectually checking out an error.

Figure 11 results from these considerations.

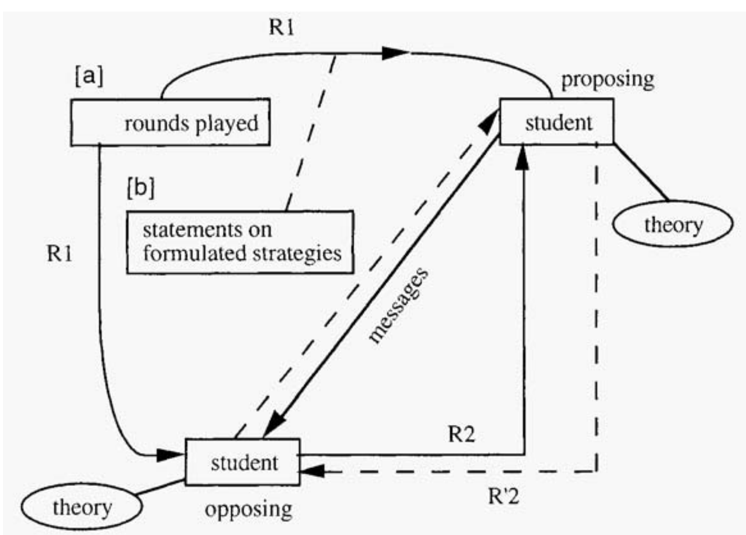


Figure I1

Remark: This pattern is characteristic of social situations involving the construction or reorganization of a repertoire, and in particular of a repertoire of theorems, i.e., a theory.

5.4. Dialectic of validation

The didactical scheme of validation motivates the students to discuss a situation and favours the formulation of their implicit validations, but their reasoning is often insufficient, incorrect, clumsy.

They adopt false theories, accept insufficient or false proofs.

The didactical situation must lead them to evolve, to revise their opinions, to replace their false theory with a true one. This evolution has a dialectic character as well; a hypothesis must be sufficiently accepted—at least provisionally—even to show that it is false.

The system of proof functions alternatively

- as an implicit means; for example children tacitly accept an unformulated fact or a method of proof (logical or implicit model);
- as a means of explicitly communicating a reason being advanced;
- as an object of study consciously subjected to logical, semantic or pragmatic proof.

In the reality of the classroom, it is not possible to establish an order of appearance. In this fashion, "a mathematical theory" is progressively constructed. "Progressively" does not mean either following an axiomatic pattern or even by a regular increase in the number of theorems.

Examples of good and bad reasons:

> *Intellectual reasons:* If I play 17, I win because my opponent can play 18 or 19 and I play 20 in both cases.
>
> *Semantic reasons:* If I say 15, I lose because every time I have played 15, I have lost.
>
> *Pragmatic reasons:* By playing 14, I win; proof, let us do it, I win (the child tests what she says by really playing a game).

Children must be given the chance to discover their errors. We shall see later that this is a necessity in the construction of knowledge.

We have distinguished within the didactical situation several kinds of feedback. They correspond to the different types of proof (intellectual, semantic, pragmatic). A dialectic of validation will consist of various particular dialectics of action or of formulation (in order to establish a terminology, for example).

it is clear, at any rate, that a dialectic of validation is itself a dialectic of formulation and therefore a dialectic of action.

Remarks: What are the "yields" of the various phases in the production of theorems?

The action phase, carried out by itself, would produce a significant number of winning numbers (implicit theorems) after a certain number of rounds, as shown in the following table:

Number of rounds necessary for the emergence of a theorem

Theorems	20	17	14	11	8	5	2
Situation of action	1	3	9	15*	26*	46	—
Situation of formulation after action	1	4	16	—	—	—	—
Situation of proof	1	6	10	10	—	—	—
Institutionalization	1	4	9	10	10	10	10

* "11" appeared as a theorem in the fifteenth round, then disappeared after the thirtieth. The theorem "8" only held up for about fifteen rounds, then disappeared. The theorem "5" appeared in round 46.

Figure 12

General results and conclusions: Discoveries are made starting at the end, "20", and working backwards. They occurred in sequence, but slow down as they get further away. The scheme of proof is not self-reinforcing, and does not get re-applied very well, though reasoning by recurrence appears in the situation of proof. Formulations do not appear until later than the corresponding implicit theorem. Informal debates among students in phase two do not produce any new theorems and do not reinforce convictions—on the contrary, they cause the students who do have a proof to become dubious. The situation of validation permits the organization of proofs and of a mathematical proof which itself follows the progression of the game. Institutionalization is in any case necessary to shore up the practices and their use elsewhere.

For the game of "The Race to 7" there is a stochastic model of simple learning which gives an account of the learning curve observed. For "The Race to 20" there is none. The pupil implements cognitive strategies. For example, she establishes the terms starting at the end, then plays systematically up to a number six or seven less than the last theorem and proceeds from there by trial and error.

NOTES

1. For example, 140 children played and re-played (some of them quite a number of turns in the conditions of the first phase).
2. *Editors ' note:* "mathematical proof" is used to translate the French "*démonstration*". The word "proof", in French "*preuve*", stands for a more general meaning, not restricted to mathematics, which refers to a discourse intending to establish the validity of a statement or of a result.

CHAPTER 1 PRELUDE

The following chapter is the text of a synthesis written by Brousseau in the year 1984–1985 for the dissertation he submitted in fulfillment of the *Thèse d'Etat* which he defended in December 1986. It was the first comprehensive presentation of the Theory of Didactical Situations.

The purpose of this text was to gather the concepts Brousseau had coined in the course of more than 20 years of research, to formulate them and to organise them in a coherent theoretical framework. Brousseau, in an abstract for the corresponding article, which is actually more a foreword than an abstract, presents the method he has chosen for elaborating this presentation:

"The method of exposition chosen is quite slow because it makes the introduction of each new concept dependent upon three distinct *problématiques*[1].

The first is relevance. What is at issue first is to describe certain kinds of human relationships in such a way that the concepts of *didactique* are made to appear in order to serve as useful means of description. The new examples that the community of didacticians have been accumulating over the past ten years have allowed the "demonstration" of some didactical phenomena (aging, the effect of the contract,...), but these "observations" appear either excessively trivial or completely strange or singular, if they are not linked up with each other to the point where they provide a real method of analysis of any teaching phenomenon.

This view calls for a second *problématique,* that of exhaustiveness. Here the issue is for all the relevant phenomena to be taken into consideration.

The third *problématique* is that of consistency. It may be the newest because if teachers in their professional practice use relevant concepts which tend to allow the treatment of all the cases, they do not—and they are not supposed to—take responsibility for the consistency of these concepts."

The Editors

CHAPTER I

FOUNDATIONS AND METHODS OF *DIDACTIQUE**

1. OBJECTS OF STUDY OF *DIDACTIQUE*

1.1. *Mathematical knowledge and didactical transposition*

Established knowledge appears in many forms. It can, for example, take the form of questions and answers. For mathematics, one of the classical forms is axiomatic presentation.

In addition to its more apparent scientific virtues, this presentation appears to be marvellously well adapted to the needs of teaching. Axiomatics make it possible for us to define the objects of study in terms of previously introduced notions, thus allowing the organization of the acquisition of new items of knowledge in relation to that already acquired. It thus promises the student and her teacher a way of ordering their activities and accumulating in the shortest time the maximum number of items of knowledge which are reasonably close to the experts' knowledge. Clearly, to complete the process, one must fill in with examples and problems whose solution requires the use of the knowledge in question.

But such a presentation removes all trace of the history of this knowledge, that is, of the succession of difficulties and questions which provoked the appearance of the fundamental concepts, their use in posing new problems, the intrusion of techniques and questions resulting from progress in other sectors, the rejection of points of view found to be false or clumsy, and the very many quarrels about them. It hides the "true" functioning of science, which is impossible to communicate and describe faithfully from the outside, and replaces it with an imaginary genesis. To make teaching easier, it isolates certain notions and properties, taking them away from the network of activities which provide their origin, meaning, motivation and use. It transposes them into a classroom context. Epistemologists call this *didactical transposition.* It has its uses, its inconveniences and its rôle, even for the construction of science. It is at the same time inevitable, necessary and, in a sense, regrettable. It must be kept under surveillance.

1.2. *The work of the mathematician*

Before communicating what she thinks she has discovered, a mathematician must first identify it. It is not easy, within the maze of thoughts, to distinguish what has

---

* Brousseau G. (1986) Fondations et méthodes de la didactique des mathématiques. *Recherches en didactique des mathématiques* 7 (2) 33–115.

the potential of becoming new knowledge of interest to others—proofs obtained are rarely those of the original conjectures. A whole readjustment of similar and related knowledge, old or new, must be started.

In addition, all irrelevant reflections must be suppressed—the traces of one's errors and haphazard progress. One must conceal the reasons which led her in these directions and the personal influences which guided success. One must skillfully contextualize even ordinary remarks, while avoiding trivialities. One must look, too, for the most general theory within which the results remain valid. Thus, the producer of knowledge depersonalizes, decontextualizes and detemporalizes her results as much as possible.

This work is essential if the reader is to be able to gain knowledge of these results and convince herself of their validity without having to go through the same procedures in order to discover them, and at the same time to benefit from the possibilities offered by their use.

Other readers also transform these results, reformulate them, apply them, and generalize them according to their needs. Occasionally, they destroy them by identifying them with previous knowledge, by including them within stronger results, or by simply forgetting them; or even by showing them to be false. Thus, from the time of its discovery, the organization of knowledge depends on the requirements of communication imposed on the author. The knowledge never ceases being modified for the same reasons, so much so that its meaning changes quite profoundly. Didactical transposition develops to a large extent within the scientific community and is carried forward in the general culture (in the *noosphere*[2], more exactly). The functioning of this community is based on the relationships that exist between the personal and contextual investment and renewal of mathematical questions, and the rejection of this investment for the production of a text of knowledge that is as objective as possible.

1.3. *The student's work*

The intellectual work of the student must at times be similar to this scientific activity. Knowing mathematics is not simply learning definitions and theorems in order to recognize when to use and apply them. We know very well that doing mathematics properly implies that one is dealing with problems. We do mathematics only when we are dealing with problems—but we forget at times that solving a problem is only a part of the work; finding good questions is just as important as finding their solutions. A faithful reproduction of a scientific activity by the student would require that she produce, formulate, prove, and construct models, languages, concepts and theories; that she exchange them with other people; that she recognize those which conform to the culture; that she borrow those which are useful to her; and so on.

To make such an activity possible the teacher must imagine and present to the students situations within which they can live and within which the knowledge will appear as the optimal and discoverable solution to the problems posed.

1.4. *The teacher's work*

The teacher's work is to some extent the opposite of the researcher's; she must produce a *recontextualization* and a *repersonalization* of the knowledge. It must become the student's knowledge, that is to say, a fairly natural response to relatively particular conditions, conditions that are essential if she is to make sense of this knowledge. Each item of knowledge must originate from adaptation to a specific situation; we do not create probability theory, for example, in the same kind of context and relationship with the *milieu* as that in which we invent or use arithmetic or algebra.

The teacher must therefore simulate in her class a scientific microsociety if she wants the use of knowledge to be an economical way of asking good questions and settling disputes, and if she wants language to be a tool for mastering situations of formulation and mathematical proofs to be a means of convincing classmates[3].

But within the story which the students are reliving the teacher must also provide the means of discovering the cultural and communicable *knowledge* that she wanted to teach them. The students, in their turn, must *redecontextualize* and *re-depersonalize* their knowledge in such a way as to identify what they produce with the knowledge which is in current use in the scientific and cultural community of their time.

It is of course a simulation; it is not "true" scientific activity, in the same way that knowledge presented in an axiomatic fashion is not "true" knowledge.

1.5. *A few preliminary naive and fundamental questions*

This evocation of communication of knowledge appears fairly classical. Yet it calls for a few remarks and raises some interesting questions.

First of all, there is a strong emphasis on the social and cultural activities which condition the creation, the practice and the communication of knowledge and *knowings*[4].

The classical approach considers the subject's cognitive activity as central; it must first of all be described and understood in a relatively independent way. It furthermore assumes, at least implicitly, that knowledge about the knowledge necessary for teaching must first be established in an independent manner, by means of mathematics and epistemology, for example. The same goes for knowledge about social aspects specific to education, etc. So the classical approach consists of deducing consequences for teaching from this preliminary knowledge; this is done directly, that is to say, with the sole support of "naïve" reflections.

Here, there is more than one nuance: does knowledge imported from fundamental disciplines itself allow, independently and without modification, the explanation of teaching phenomena and the production in a controlled manner of the desired modifications? Must we, on the contrary, create new concepts, a field of knowledge and related methods in order to study didactical situations?

One of the fundamental hypotheses of *didactique* consists of claiming that only the global study of situations presiding over the manifestation of knowledge allows us to choose and connect knowledge from different origins; knowledge necessary to understand the subject's cognitive activities, as well as the item of knowledge she uses and the way in which she modifies it.

A second, stronger, hypothesis is that the above study of (didactical) situations must in the end allow the derivation or modification of the necessary concepts currently imported from other scientific fields.

Does there exist a "didactical variety" of the concepts of meaning, memory, structure, decimal, etc., unknown in linguistics, psychology or mathematics?

Teaching, also, is conceived of as a social project: that of causing a student to appropriate knowledge which is established or in the process of becoming established. This point of view brings back to the core of the concern of teaching the cultural and political discussion of knowledge, treating it, however, more as an object of study which is part of the situations than as a preliminary philosophical consideration.

Isn't learning essentially an individual act? Is it necessary to place it in such a large context in order to understand it? Isn't individual tutoring a sort of optimum condition that only economical conditions prevent from being realized?

Even accepting that knowledge about situations for the implementation, adaptation and teaching of knowledge can play a certain technical rôle in terms of means for teaching, an important question remains: once elevated to the rank of a cultural object, won't this knowledge profoundly disturb communication and perhaps even the construction of knowledge? The latter is built, as we have seen, on the rejection or forgetting of the circumstances which gave rise to it.

Why isn't the possession of knowledge itself, together with general knowledge from the social sciences, some common sense and, of course, some pedagogical skills that no training can truly produce, sufficient for all teachers with all students, as it is for a few?

We can question, then, to what degree this reference to the functioning of research is necessary for, and relevant to, the study of learning and, especially, the study of teaching. To what extent is there a similarity and under what conditions?

It appears that a good epistemological theory accompanied by good didactical engineering is essential for answering these questions.

Didactique *studies the Communication of knowledge and theorizes its object of study, but it can take up this challenge only if the following two conditions are satisfied:*

- *that it make evident the specific phenomena which appear to be explained by the original concepts it proposes;*
- *that it indicate the specific methods of proof which it uses for that purpose.*

These two conditions are essential if didactique *of mathematics is to be able to take charge of its object of study in a scientific manner and thus to allow controlled actions on teaching.*

2. PHENOMENA OF *DIDACTIQUE*

Some of the phenomena connected with the control of didactical transposition could have been discussed with respect to different settings; the same phenomenon can govern the intimacy of a private lesson or concern a whole community for generations.

Identification of these phenomena comes down to the construction of a "model" of protagonists face to face, and of the relationships and constraints which bind them, showing that the interplay of these constraints does indeed produce the effects and developments observed.

It is more convenient in a relatively short text to take examples already known to the reader than to explain in all their complexity cases actually observed.

2.1. *The Topaze effect and the control of uncertainty*

The first scene of Marcel Pagnol's famous play *Topaze* illustrates one of the fundamental processes. Topaze is giving a dictation to a weak student:

> "Two lambs were safe in a park". Being allowed neither to accept too many major mistakes nor to give the expected spelling directly, he "suggests" the answer by concealing it in more and more transparent didactical codes: "Te-woo... lambs... were safe in a park". For the student, it was originally a spelling and grammar problem: "Te-woo lambs... were..."—the problem has completely changed![5] Confronted with repeated failure, Topaze begs for a sign of understanding and negotiates a drop in the conditions under which the student will end up writing the correct spelling. We can guess that he might follow this by requiring the recitation of the rule, then by having it copied out several times. The total collapse of the teaching act is represented by a simple command: put a "w" in "two". The teacher ends up taking responsibility for the essential part of the work.

The answer that the student must give is determined in advance; the teacher chooses questions to which this answer can be given. Of course, the knowledge necessary to produce these answers changes, as does its meaning. By choosing easier and easier questions, the teacher tries to achieve the optimum meaning for the maximum number of students. If the target knowledge disappears completely, we have the *Topaze effect.* The preservation of the meaning throughout the changing of the questions is controlled by the teacher's knowledge of the discipline being taught, but the choice of the learning situations and their management, usually left to teachers' "common sense", is currently the object of active research which deals as much with theory as with didactical engineering[6].

2.2. *The Jourdain effect or fundamental misunderstanding*

The Jourdain effect—so called by reference to the scene in *le Bourgeois Gentilhomme*[7] in which the philosophy teacher reveals to Jourdain what prose and vowels really are—is a form of Topaze effect.

In order to avoid debating knowledge with the student and possibly acknowledging failure, the teacher agrees to recognize the indication of an item of scientific knowledge in the student's behaviour or answers, even though these are in fact motivated by ordinary causes and meanings.

The whole humour of the scene is based on the absurdity of this repeated use of the power of scientific speech to make familiar activities sacred.

Example: The student asked to perform rather strange manipulations with jars of yoghurt or coloured pictures is told, "You have just discovered a *Klein group*".

In a less crude way, the wish to insert knowledge into familiar activities can lead the teacher to substitute another *problématique* for the true, specific one—for example a metaphoric or metonymic one which doesn't give a correct meaning to the situation. Often the two *problématiques* are present and juxtaposed, and the teacher tries to obtain "the best" compromise.

Certain pedagogical methods centred on the preoccupations of the child often bring about this effect, but the reforms of the Sixties and the use of mathematical structures which they proposed clearly provided a powerful incitement to play this game.

At the same time, structuralist ideology offered it an epistemological justification. This, then, consists of a double Jourdain effect. The first is at the level of the student's relationships with the teacher: the student deals with an example and the teacher sees a structure in it; the second is at the level of the relationships between didacticians[8] or mathematicians and the teacher. A philosophical, scientific justification of the former is superimposed on the practice of the latter and makes it "sacred": recognition of the structure has become a scientific activity.

2.3. *Metacognitive shift*

When a teaching activity has failed, the teacher can feel compelled to justify herself and, in order to continue her activity, take her own formulations and heuristic means as objects of study in place of genuine mathematical knowledge.

This effect can be iterated several times; it can concern a whole community and constitute a veritable process escaping from the control of its actors. The most striking example is probably that of the use of graphics in the Sixties to teach structures, a method attributed to G. Papy.

At the end of the Thirties, set theory departed from its initial scientific function to become a way of teaching which satisfied teachers' need for a metamathematics and a fundamental formalism. Teachers then had to invite students to a semantic control of the theory (then called "naïve"). In order to avoid errors, it wasn't enough to apply axioms; one needed to know what one was talking about and to know paradoxes attached to certain usages in order to avoid them. This control differed from the usual, more "syntactic" mathematical control. This use of set theory, already didactical, was to make possible for other theories an axiomatic presentation whose negotiation would more classical.

This means of teaching became the teaching object for younger and younger children. Semantic control was based on a "model"[9] which goes back to Euler[10], which makes reference to various graphical representations. This “model” was actually not a correct model; it did not allow the expected control and caused teaching difficulties. Because of these difficulties, this “method” of teaching in its turn became a teaching object and was overloaded with conventions and specific language which were themselves taught and explained at every stage of presentation. With this process, the more the teaching activity produced comments and conventions, the less the students could control the situations which were being put to them.

This is the effect of *“metacognitive shift”*. It would be naïve to believe that common sense would have made it possible to escape the quite extravagant consequences to which the process led. The force of didactical effects is uncontrollable as long as the teacher is unable to withdraw herself from the obligation of teaching at all costs. The greater the number of members of the public engaged in the negotiation, the more the process eludes “naïve” control.

Thus common sense, like any other corrective factor, cannot play a rôle in social processes without the mediation of an adequate social structure. It has been shown that this type of “error” is neither silliness nor, in most cases, ignorance of mathematics as a discipline. It is more or less to the extent that disease may be attributed to errors of behaviour—if I may be allowed to use a bold metaphor.

2.4. *The improper use of analogy*

Analogy is an excellent heuristic means when the person using it takes responsibility for it. But its use in the didactical relationship makes it a formidable way of reproducing Topaze effects. It is, however, a natural practice. If a few students have not learned, they must be given a second attempt with the same subject matter. They know that. Even if the teacher conceals the fact that the new problem is similar to the previous one, students will look for similarities—this is a legitimate activity—so that they can carry forward, in its current state, the solution that they have already been given. This response doesn’t indicate that they find that the solution fits the given question, but only that they have recognized indications, perhaps quite exogenous and not controlled, that the teacher wanted them to produce it.

They find the solution by reading the didactical indications and not by involving themselves in the problem. And it is in their interest to do so because after several failures with similar but unidentified, unrecognized problems, the teacher will rely on these suddenly renewed analogies to reproach the student for her obstinate resistance (this effect is used by Devos in his sketch about the two ends of a piece of wood[11]): “I’ve spent no end of time explaining it!”.

2.5. *The aging of teaching situations*

The teacher finds it difficult to reproduce the same lesson, even with new students. The exact reproduction of what she did or said previously doesn’t have the same effect and usually the results are not as good. But, perhaps as a consequence, she also feels a certain reticence towards this reproduction. She feels quite a strong need to change at least the formulation of her explanations or her instructions, the examples, the exercises, and if possible even the structure of the lesson. These effects increase with the number of reproductions and are all the stronger as the number of interactions between the teacher and the student increases. Lessons which include an explanation followed by exercises or a simple instruction followed by a learning situation which does not require the teacher's interventions age less quickly. This effect was directly observed at the *Ecole Jules Michelet* in Talence[12] on several occasions in which teachers were encouraged to reproduce a given lesson. But the efforts at renewal attempted by teachers who are free in their work are an equally reliable indicator and can be easily observed.

This phenomenon, like the earlier ones, can be observed at the level of a class but also of an educational system in general, or among other partners:

The programmes and the ministerial directives (or the curricula in other countries[13]) are more or less *the* unique means of presenting to teachers the didactical requirements of society and *the* means of agreeing on the division of tasks among them. As regards the complexity of the processes of control, these generally quite short texts, which must leave open the essentials of relevant questions, appear to be entirely inadequate. Their periodic modifications appear completely ludicrous when we compare them with each other, or when we measure them against the importance that teachers and administrations seem to give them. Primary school texts since the 1980s offer only slight differences on essential matters and differ only in nuances.

Programme modifications are the projections of teachers' desires for the renewal of their didactical situations in response to the aging of their lessons.

The enormous disproportion between introducing novelty in this way and the surprising stability of teaching practices is also a sign of the constraints which intervene in the regulation of ageing. The educational system takes a long time to respond to any modification and feedback is weak and uncertain. The best guarantee against drift is therefore a hefty inertia. But the activity of teaching itself demands an intense personal investment on the part of the teacher and this investment can be maintained only if it is renewed. Reproduction therefore demands a certain renewal that risks compromising future reproductions. The means of obtaining equilibrium not being known, the system tends to bring renewal to bear on factors which do not have much influence on the main object of teaching; modification of the curriculum follows a procedure similar to those of obsolescent fashions in clothing.

This question of aging and the effect of *didactical time*[14] raises a fundamental question for *diductique:* What really is reproduced during the course of a lesson?

A teacher who reproduces *the same history,* the same sequence of the same activities and the same statements on her part and on the students' part—has she reproduced the same *didactical event* producing the same effects from the point of view of meaning?[15] No naïve way exists of differentiating a good reproduction of a lesson—which under the same conditions gives an identical development and also the same meaning to items of knowledge acquired by the student—from a bad reproduction of this lesson—which under the same conditions gives an identical "development", but a different meaning to acquired items of knowledge. In the second case, the similarity of the development of the lesson is obtained by the teacher's discrete but repeated intervention, which transforms the whole situation without apparently modifying its "history".

Knowing what is being produced in a teaching situation is precisely the object of *didactique.* It is not a result of observation, but one of analysis based on the knowledge of phenomena which define what they leave unchanged.

3. ELEMENTS FOR A MODELLING

These different phenomena can be observed just as well in the particular relationships between two people as they can in more complex relationships involving organisms and hundreds of people[16].

Is it possible to "model" a whole educational system by means of a "teaching" system defined by some of the relationships which it maintains with a "taught" system, which itself represents hundreds of students whose diversity seems to be precisely the primary source of teachers' difficulties? This is an unavoidable gamble for the theorization process.

The problems raised by the systemic approaches to which this method is similar will be discussed later.

The form in which we briefly described these phenomena has laid a foundation for their modelling. It is now a question of identifying the fundamental relationships which we must consider.

It is advisable, however, to refrain from an excessive, premature formalization. A more rigorous formulation will occur at a subsequent stage.

3.1. *Didactical and adidactical situations*

In the most general conception of teaching, knowledge is a correspondence between good questions and good answers.

The teacher sets a problem which the student must solve. If the student answers, she thereby demonstrates that she knows; otherwise the need for knowledge becomes apparent, and that calls for information, for teaching. *A priori,* any method that allows the memorization of favourable associations is acceptable.

The *Socratic maieutic* limits these associations to ones which the student can make herself. The purpose of this restriction is to guarantee the student's understanding of the knowledge, since she produces it. But one then has to assume that the student already possessed this knowledge—either that she always had it (*reminiscence*) or that she is constructing it herself by means of her own isolated activity. All behaviours in which the teacher herself does not give the answer are acceptable as a means of bringing this knowledge to the student.

The Socratic framework can be improved if we assume that the student is able to draw her knowledge from her own experiences, by her own interactions with her *milieu,* even if this *milieu* is not organized with learning in mind. The student learns by looking at the world (empiricist-sensualist hypothesis) or by making hypotheses or the kind her experience lets her choose (a-priorist hypothesis) or in a more complex interaction consisting of assimilation and accommodation such as described by Piaget.

The student learns by adapting herself to a *milieu* which generates contradictions, difficulties and disequilibria, rather as human society does. This knowledge, the result of the student's adaptation, manifests itself by new responses which provide evidence of learning.

This psychogenetic Piagetian process is the opposite of scholastic dogmatism. The one seems to owe nothing to didactical intention, whereas the other owes everything to it. By attributing to "natural" learning what is attributed to the art of teaching according to dogmatism, Piagetian theory takes the risk of relieving the teacher of all didactical responsibility; this constitutes a paradoxical return to a sort of empiricism! But a *milieu* without didactical intentions is manifestly insufficient to induce in the student all the cultural knowledge that we wish her to acquire.

The modern conception of teaching therefore requires the teacher to provoke the expected adaptation in her students by a judicious choice of "problems" that she puts before them. These problems, chosen in such a way that students can accept them, must make the students act, speak, think, and evolve by their own motivation. Between the moment the student accepts the problem as if it were her own and the moment when she produces her answer, the teacher refrains from interfering and suggesting the knowledge that she wants to see appear. The student knows very well that the problem was chosen to help her acquire a new piece of knowledge, but she must also know that this knowledge is entirely justified by the internal logic of the situation and that she can construct it without appealing to didactical reasoning. Not only *can* she do it, but she *must* do it because she will have truly acquired this knowledge only when she is able to put it to use by herself in situations which she will come across outside any teaching context and in the absence of any intentional direction. Such a situation is called an *adidactical situation*[17]. Each item of knowledge can be characterized by a (or some) adidactical situation(s) which preserve(s) meaning; we shall call this a *fundamental situation.* But the student cannot solve any adidactical situation immediately; the teacher contrives one which the student can handle. These adidactical situations arranged with didactical purpose determine the knowledge taught at a given moment and the particular meaning that this knowledge is going to have because of the restrictions and deformations thus brought to the fundamental situation.

This situation or problem chosen by the teacher is an essential part of the broader situation in which the teacher seeks to devolve to the student an adidactical situation which provides her with the most independent and most fruitful interaction possible. For this purpose, according to the case, the teacher either communicates or refrains from communicating information, questions, teaching methods, heuristics, etc. She is thus involved in a game with the system of interaction of the student with the problems she gives her. This game, or broader situation, is the *didactical situation.*

Within the situation which she is experiencing, the student does not distinguish at once between what is essentially adidactical and what is of didactical origin. The final adidactical situation of reference, the one that characterizes the knowledge, can be studied in a theoretical way, but in the didactical situation, for the teacher as well as for the student, it is a sort of ideal towards which they are trying to converge. The teacher must always help the student strip the situation of all its didactical artifices as quickly as possible so as to leave her with personal and objective knowledge.

The didactical contract is the rule of the game and the strategy of the didactical situation. It is the justification that the teacher has for presenting the situation. But the evolution of the situation modifies the contract, which then allows new situations to occur. In the same way, knowledge is what is expressed by the rules of the adidactical situation and by the strategies. The evolution of these strategies requires productions of knowledge which in their turn allow the design of new adidactical situations. The didactical contract is not a general pedagogical contract. It depends closely on the specific knowledge in play.

In modern *didactique,* teaching is the devolution to the student of an adidactical, appropriate situation; learning is the student's adaptation to this situation. We shall see further on that one can conceive of these situations as formal games and that this conception favours the understanding and mastery of teaching situations.

3.2. *The didactical contract*

Thus, in all didactical situations, the teacher attempts to tell the students what she wants them to do. Theoretically the transition from the information and the teacher's instructions to the expected answer must require students to bring the target knowledge into play, whether it is currently being learned or whether it is already known. We know that the only way to "do" mathematics is to investigate and solve certain specific problems and, on this occasion, to raise new questions. The teacher must therefore arrange not the communication of knowledge, but the *devolution* of a good problem. If this devolution takes place, the students enter into the game and if they win learning occurs.

But what if a student refuses or avoids the problem, or doesn't solve it? The teacher then has the social obligation to help her and sometimes has to justify herself for having given a question that is too difficult.

Then a relationship is formed which detennines—explicitly to some extent, but mainly implicitly—what each partner, the teacher and the student, will have the responsibility for managing and, in some way or other, be responsible to the other person for. This system of reciprocal obligation resembles a contract. What interests us here is the *didactical contract,* that is to say, the part of this contract which is specific to the "content", the target mathematical knowledge.

That is why we cannot give details of these reciprocal obligations here; besides, it is in fact the *breaking of the contract* that is important. Let us examine some immediate consequences.

– The teacher is supposed to create sufficient conditions for the appropriation of knowledge and must "recognize" this appropriation when it occurs.
– The student is supposed to be able to satisfy these conditions.
– The didactical relationship must "continue" at all costs.
– The teacher therefore assumes that earlier learning and the new conditions provide the student with the possibility of new learning.

If this learning does not occur, the student is put on trial for not having fulfilled what was expected of her, but so is the teacher for not having fulfilled what was expected (implicitly) of her.

Let us recognize that this interplay of obligations is not exactly a contract. First, once it claims to concern the result of the teaching action, it cannot be made completely explicit. There are no known, recognized, sufficient ways of allowing the construction of new knowledge or of ensuring, against all resistance, the student's appropriation of the target knowledge. And if the contract rests only on the rules of the teacher's or the student's behaviour, scrupulously respecting it will condemn the didactical relationship to failure.

The teacher must, however, accept responsibility for the results and ensure that the student has the effective means for acquiring knowledge. This "making sure" is fallacious but essential if she is to be allowed to engage the student's responsibility. Similarly, the student must accept responsibility for solving some problems whose solutions she has not been taught, although she does not see *a priori* the choices that are offered her and their consequences, and she is therefore involved in an obvious instance of juridical irresponsibility.

We shall see that a totally explicit contract of this kind is doomed to failure. In particular, clauses concerning the breaking and the stake of the contract cannot be written in advance. Knowledge will be exactly the thing that will solve the crises caused by such breakdowns; it cannot be defined in advance. However, at the moment of such a breakdown, everything happens as if an implicit contract were linking the teacher and the student; surprise for the student, who doesn't know how to solve the problem and who rebels against what the teacher cannot give her the ability to do—surprise for the teacher, who reasonably thought that she performed sufficiently well—revolt, negotiation, search for a new contract which depends on the new "state" of knowledge, acquired and desired.

The theoretical concept in *didactique* is therefore not the contract (the good, the bad, the true, or the false contract), but the hypothetical *process of finding a contract.* It is this process which represents the observations and must model and explain them.

3.3. *An example of the devolution of an adidactical situation*

In a microcomputer game, young (five-year-old) children have to use the mouse to lead rabbits into a meadow and ducks into a pond, one at a time. The rules of manipulation do not present insurmountable difficulties for this age group. The children can interpret the disappearence of an animal from one place and its re-appearance in another place as corresponding to a displacement. But soon something other than a manipulation according to the rules comes into question; in order to develop the student's ability to count a set, the teacher wants her to identify all the rabbits one after the other, *before* directing them towards the meadow. The sequence of the operations to be carried out is not given in the instructions, it is up to the student. The devolution of this task is done step by step.

First stage: *pure play*
The students have not yet understood that from among the possible outcomes of the game, some are desirable (e.g., all the rabbits go into the meadow to dance *en ronde),* and others are not (e.g., the forgotten rabbits become red and growl). The children play, they can prod the rabbits, and they are happy to cause an effect, whatever it is.

Second stage: *devolution of a preference*
The students have understood very well what the desired effect is (for example, all the effects of false manipulations have been eliminated), but they attribute the results, good or bad, to a sort of fate or chance.

This kind of interpretation is adequate for several games—such as "la bataille" or "petits chevaux"[18]—where pleasure comes from the expectation of what fate holds, the player not making any decision.

Third stage: *devolution of a responsibility and a causality*
To accept responsibility for what happens to her, the student must consider what she is doing to be a choice selected from among various possibilities and must then envisage a causal relationship between the decisions she has taken and their results.

At this stage, students can, with hindsight, see that the development of the game could have been different. This assumes that they can remember some of their actions and, more precisely, which were relevant and which were not.

This devolution is delicate; the majority of children are ready to accept from the teacher the idea that they are responsible for the result of the game even though they are incapable of establishing that they could have obtained a better result by a suitable choice on their part. Now, only the knowledge of this linkage would justify the transfer of responsibility.

If the student solves the problem quickly enough, the fact that she has accepted her responsibility *a priori* appears to be only a necessary prologue to learning. The latter has just justified this giving of responsibility after the fact, by giving the student the means of taking it over and, finally, of escaping culpability.

But for the student who cannot overcome the difficulty and use the knowledge to relate her action to the obtained results, the giving of responsibility must be re-negotiated for fear of provoking feelings of culpability and unfairness that very quickly become detrimental to subsequent learning and to the notion of causality itself.

Fourth stage: *devolution of anticipation*
The relationship between the decision and the result must be considered before the decision is made; the student herself then takes on anticipations which exclude any hidden intervention. Even if it is not yet entirely mastered, this anticipation is seen as being the cognitive responsibility of the player and not merely her social responsibility.

Fifth stage: *devolution of the adidactical situation*
To win the "rabbits game", the student must correctly count a set. But it is not enough for her to do it once "by chance". She must know how to reproduce it at will in various circumstances. She must be conscious of this capability to reproduce and must have at least an intuitive knowledge of the conditions which give her a good chance of succeeding. The student must recognize the games that she has just learned to play. But what she knows has not been named, identified and, especially has not been described for her as a "fixed" procedure. Thus, devolution does not bear on the object of teaching but on situations that characterize it. This example was chosen in order to distinguish carefully the different components of devolution. Enumeration is not a mathematical concept that is culturally very strong. It intervenes in teaching only later, with different languages and *problématiques.* Neither vocabulary nor formal knowledge therefore interfere with the object of teaching.

Before this learning, the child could have "enumerated" sets by moving the objects or by marking them in some way that would ensure that she always has a convenient representation of the set remaining to be counted. But here, she must carry out the same task mentally, her representations must be extended to cover a far more complex intellectual control: to look for a first rabbit which is easy to identify, and then another one in such a way as to keep in mind that these two have already been taken; to look for another one quite near the first two and forming some configuration with them (small group, line, etc.), letting her not "lose sight" of them while looking for a fourth, which in its turn enters into the structure so that she does not take a rabbit that has already been caught and letting her know if there are any left, etc.

This "task" cannot be described as a procedure, nor even "demonstrated" because to enumerate the elements of a set *in front of* the child gives her no idea of the means of control that she must acquire.

In this example, the devolution of the adidactical situation can be observed independently of the devolution of the teaching objective (which cannot take place at this time). Neither the teacher nor the student can identify, except by the success of a complex task, what is taught, what is to be understood or known.

A little later, enumeration, as a production, can become an object of study for the student. She can recognize the animals which are the same or different, those which are correct or those which fail, she can conceive and compare methods and know—afterwards—the teaching objective linked with the rabbits game. She will be able to tackle problems of enumeration and combinatorics that are closer to scientific problems and then define what she must learn, what she must solve, and what she is asked to know. This devolution of objects of study, objects of knowledge and objects of teaching could be interpreted as devolution of adidactical situations of another kind.

3.4. *The epistemology of teachers*

The teacher, then, has to formulate a method for making the answer explicit for the student: how to *answer* with the help of previous knowledge, how to *understand* and *build* new knowledge, how to "*apply*" previous lessons, how to *recognize* questions, how to learn, guess, solve, etc. Thus she refers to an implicit functioning of mathematics or to a model (like elementary geometry) constructed for the use she is making of it, that is, to solve the conflicts of the didactical contract.

This "epistemology of the teacher" (for professional use) must also be the epistemology of the student and her parents. It must be present in the culture to allow justifications to function and be accepted. The teacher is not free to change it as she pleases. It is understood that it has little chance of being consistent, so as to serve as a basis for a didactical theory.

To teach it, then, a teacher must reorganize knowledge so that it fits this description, this "epistemology". This is the beginning of the process of modification of knowledge that changes its organization, its relative importance, its presentation and its genesis, following the needs of the didactical contract. We called this transformation *didactical transposition.*

Let us notice that *a priori* the empirical practice of mathematics teaching, no matter what the scientific quality of the teachers is, does not lead them spontaneously to the construction of a correct simulation of the genesis of notions. On the contrary, there is a great temptation to cut out the double work (of recontextualization and redecontextualization) and to make students learn a *text of knowledge* directly. To respect the other obligations of the contract, problems are certainly given to the students but their solutions can be found by procedures that make the economy of knowledge specific to the notion (as in the example of analogy). The solution is hidden under a didactical fiction known by the student, which serves at the time of the negotiation. Since the teacher must "prove" to the student that it is possible for her to answer and to learn the target knowledge, she must at least be able to tell her how, "*a priori*". Indeed, if the solution is articulated as it would be in a mathematical text it includes the correct scientific justification of the result, but many students get the answer not by means of the desired mathematical reasoning, but by decoding the didactical convention.

3.5. *Illustration: the Diénès effect*

The study of the conceptions of Diénès[19] and the echoes which they have awakened in teachers in the framework of the reforms of the 1970s gives an excellent demonstration of this subject (Maudet, 1982).

By means of his “psychodynamical process”, Diénès proposed a teaching model founded on the recognition of similarities among “structured games” and on the schematization and the formalization of these guided “generalizations”.

This was in fact nothing but a description and a systematization of certain teaching practices already in use, such as the repetition of problems or similar examples so as to induce a standard answer. But it went hand-in-hand with a translation into mathematical terms: similar problems became “isomorphs” and a generalization became a “passing to the quotient”. Set theory and fundamental structures became the means of describing all elements of the teaching situation which, in return, illustrated them perfectly.

This translation implied a systematic confusion between the structure of the situation (the game), the structure of the task, the intellectual process, and the knowledge itself (as a mathematical structure). It thus led implicitly to the establishment of the foundations of mathematics, as they were conceived at the time, as a universal model, as well as a means of description and organization of mathematics (logic), a means of its construction and functioning (epistemology), a means of explaining the student’s psychological functioning in the subject (cognitive psychology), a means of describing the learning process and the steps in the development of knowledge (genetic epistemology) and finally the didactical means of bringing this learning about.

The spontaneous epistemology of teachers was thus suddenly justified, “made sacred” by its reformulation in “scientific” terms and, miraculously, it was in agreement with all the domains likely to contest it. This was one reason for the initial success of the Diénès proposals.

Such a *didactique* is independent of the content. It even induces the teacher to emphasize the irrelevant variables of the mathematical situation (those which do not modify it) to the detriment of the specific conditions (“principle of variability”). And, finally, it is only a method of presenting knowledge that favours its memorization.

The most evident fact in the use of this method is that only converts to the method are likely to make it work successfully. All “servile” use of the Diénès materials leads to failure and deception.

Analysis in terms of the didactical contract can offer an explanation of this fact.

The didactical method of Diénès, being based on the “psychodynamic process”, does not explicitly leave initiatives other than the choice of materials, the presentation of activity sheets, the encouragement of their use, and so on, to the teacher. The method must operate by virtue of an internal process of the subject, *unavoidable* as soon as its initial conditions are satisfied: offering structured games repeatedly, asking for schematization, etc. It thus relieves the teacher of the technical responsibility for bringing about the desired learning. She can present her exercises, wait, and eventually provide answers accompanied by a meagre explanation, move on to the next activity sheet, organize the corresponding game; but the teaching contract no longer ties her to the evolution of the cognitive behaviour of which the “game” is supposed to take charge. On the contrary, she must leave the student to think for herself. However, the games of Diénès are often unsatisfying, because they postulate that the rules given to the student (to play) are the same as those that she must learn; the structure of the game and what knowledge "is" are identical! Thus, understanding the rule, which is a condition for action, first requires the student to possess the knowledge that we are claiming to teach her. If the teacher first taught the rule, the game would then be transformed into an exercise. To avoid this, she tries to make the students guess the rule—an activity which is not theorized in the psychomathematical process.

But the theoretical and practical insufficiency of Diénès's games does not explain why failures are observed less frequently among converts to the method than among conscientious but uncommitted users. A teacher who has confidence in the psychodynamical process is content to give students activity sheets and games, and waits until the predicted effect, generalization or good formalization, is produced. It is produced badly because of the breaking off of the negotiation that goes hand-in-hand with the decline of pressure from the teacher.

The teaching contract can subsist if the teacher worries about the student's quantitative results but the articulation of knowledge and its genesis remain ignored. On the other hand, the "militant" action of a teacher who has decided to show that the method is effective leads her to re-open this discussion. The insufficiency of the proposed adidactical situations as far as the justification and meaning of the target knowledge are concerned does not prevent the teacher's discourse from giving them sufficient meaning and scope for learning, but in certain cases it causes failure at the level of the contract.

It is true, however, that if the situations were mathematically incorrect, no devolution would allow Diénès's games to produce the intended knowledge. The problem remains open for "good" situations. In any case, Diénès's teaching methods do allow results to be obtained, but for reasons different from those suggested by the theory that accompanies them.

This analysis shows the use that can be made of the notion of contract in attempting an explanation of a phenomenon related to the teachers' epistemology. Important problem: wouldn't every method or situation predicted to be efficient by any "psychological law" or *"didactique"* whatsoever, which would free the teacher from didactical negotiation, bring about the same effect?

The more the teacher is assured of success by means of effects that are independent of her personal investment, the more she is likely to fail! We call this phenomenon, which shows the necessity of integrating the teacher-student connection in any didactical theory, the *Diénès effect.* And this conclusion raises a more difficult question: Is the epistemology of teachers impossible to bypass?

3.6. *Heuristics and* didactique

It is clear that we do not know the necessary, minimum conditions that will give the maximum meaning to the student's activity, which are nevertheless sufficient to allow her to satisfy her contract. We do not know an effective genetic epistemology which would allow the good management of these negotiations, so the teacher and the student are often reduced (unconsciously, of course) to short-term measures such as:

– problem substitution which can result in the Topaze effect or, more drastically, the Jourdain effect;
– improper use of analogy, metacognitive shift, etc.

Now, the teacher must provide ways of solving problems (theoretical knowledge, for example) and must take into account the fact that the methods already taught could well allow the construction of the solution. The teacher must therefore act as if she knows how solutions to new problems are constructed, starting from (taught) knowledge. And one day, she must also talk about these methods, how to recover them, how to recognize them, and so on.

Does her action presuppose an epistemology? She will be obliged to produce it, to reveal it! Why did the student make a mistake? How can she avoid subsequent ones? How can she find the solution?

"The algorithm" constitutes a tool for clearing a blockage and solving didactical conflicts in the sense that it momentarily allows a clear division of responsibilities. The teacher shows the algorithm, the student learns it and "applies" it correctly; otherwise she must practise, but her uncertainty is almost zero. We assure her that a whole class of *different* situations exists in which the algorithm gives a solution (the conflict will appear again when it becomes necessary to choose an algorithm for a given problem).

The algorithm is practically the only "official" means of clearing a blockage in that teaching methods related to the algorithm are made explicit. It serves as a unique, or almost unique, model for any cultural approach to teaching.

We must therefore expect the student to receive all the teacher's indications in the same mode, as "effective" methods for solving problems (such as algorithms), even if the teacher chooses the indications in such a way that they rekindle the student's search, encourage it and help it *without* meddling with the essentials of what must remain under her control. Thus, information of the heuristic type will be requested, given and received in the midst of a misunderstanding: as vague suggestions for the one, as knowledge comparable with algorithms or mathematical theorems for the other.

With this Art of problem solving, essentially based on introspection, the teacher would like her student to learn how to find solutions, while the student expects algorithms.

Now, what the teacher would like to present to the student as opportunities for typical investigation is only a collection of cultural objects, a collection of problems whose solutions are known and itemized by heuristics. The student is therefore justified in receiving this set of problems as if it were knowledge. In this sense, as Glaeser (1984–1985) strongly emphasizes, "heuristics cannot be taught since its content is the unpredictable and creative part of every problem-solving process. We can only allow students to be trained to the heuristic in such a way as to familiarize them with investigative situations" (*ibid.* p. 151). But then the process remains blocked! The teacher should not, for example, invite the student to make use of the thought-processes listed by Polya (1957) which she recognizes as having used herself at the time of her success as a mathematician.

On occasion, however, there is no great harm in *giving* information or advice. "Draw a figure.... Introduce suitable notation.... Look at the unknown!... Here is a problem related to yours and solved before. Could you use it?... Could you restate the problem?... Go back to definitions..." (*ibid.* pp. xvi–xvii). On the contrary, one must acquire such habits.

But Polya's first recommendation in "How To Solve It", is: "You have to understand the problem." (*ibid.*) The contract shifts; now the search for information or lateral suggestions becomes an *acknowledged* didactical method, possibly something to be required of a student who claims to be investigating, but of whom one doubts the real activity.

In turn, the teacher is led to clarify these methods, to classify them, to identify them, to define them, to account for their efficiency. She might thus choose problems which are the best examples for allowing her to illustrate these methods, to apply them, to make them work. But she cannot restrict mathematical problems to those for which an almost automatic application of a procedure given in advance provides the solution. The student then searches out which "procedological"[20] suggestion is the correct one. The circle is closed. "Heuristics" have replaced or taken a place alongside the theorems and theories from among which the means of solving a problem must be chosen, but the problem remains and so does the didactical contract. Why not, then, look for second-order heuristics? (!)

This path encourages a type of recurrent (heuristic) shift comparable to the *metacognitive shift.* It is possible also to identify a *metamathematical shift* which consists of substituting for a mathematical problem a discussion of the logic of its solution and attributing all sources of error to it.

The process which we have just described is therefore a tendency resulting naturally from the needs of the didactical contract. It is easy to find repeated examples of it in the history of teaching. It is clear, also, that there is nothing inescapable; reticencc, then resistance, becomes gradually stronger as the shift becomes large. It seems that, as for the effect of the metacognitive shift, the only antagonistic force is epistemological vigilance.

As in the case of analogies, the use—naïve or systematic—of heuristics is an excellent means of looking for solutions to problems (heuristics being the means by definition and by excellence), provided that it is put into practice under the responsibility of the user. Any credit given *a priori* to a particular method is a source of often bitter disappointment, which makes it unsuitable for the didactical contract. Following Glaeser (1984–1985), we use the term *procedology* to describe "the whole repertoire of tested recipes [tested over the stock of classical problems] that teaching inculcates" (*ibid.* p. 151) which are neither theorems nor metatheorems. Teaching doesn't seem to have the explicit mission of inculcating these recipes, and we prefer to assume that it does so under the pressure of the didactical contract.

On the other hand, I propose to put forward the term *algorithmic procedures* "which appear [...] as sub-programmes of a heuristic search" (*ibid.*) for all the things that, in the didactical contract, have a tendency to play the same rôle, including heuristics or original ideas, in that they are presented or used as recipes. It is the didactical function and didactical presentation which retain or remove the value of a procedure. More exactly, it is the nature of the contract which takes shape on their behalf. Like the Diénès effect (for the teacher), telling the student that an automatic (or almost automatic) method exists for establishing a family of results, *even if it is true,* tends to relieve her of the fundamental responsibility for the control of her intellectual work, thus blocking the devolution of the problem, and most often making the activity fail (and moreover allowing the student to contradict and contest the method if she wants to do so).

I think I should emphasize what we have just shown:

– there is no difference in nature between a cautious, legitimate use of Polya's "normative heuristic" for the purpose of mathematics education, and a refined second-order procedology—only a difference of degree in the acceptance of the shift under the pressure of the contract (or to move towards the student);
– there is no reason to declare it a *priori* illegitimate for the teacher to give indications of this nature (such as what we have called "teachers' epistemology"); in the absence of an authentic science of *didactique*, we can consider them to be an inevitable professional necessity.

It is more important to understand the antagonistic conditions which influence the equilibrium between the opposing tendencies (no information—too much information).

This analysis gives rise to the following statement: heuristics might be no more than a rationalization based on the teacher's epistemology, a didactical invention for the needs of the contract, taken over and developed by mathematicians by way of a spontaneous epistemology.

4. COHERENCE AND INCOHERENCE OF THE MODELLING ENVISAGED: THE PARADOXES OF THE DIDACTICAL CONTRACT

Envisaging teaching as the devolution of a learning situation from the teacher to the student has allowed us to discover certain phenomena. The attempt to model this devolution as the negotiation of a contract allows us to explain a great many of these phenomena and to predict others.

The result of these processes causes us to consider the teacher as a player faced with a system, itself built up from a pair of systems: the student and, let us say for the moment, a *"milieu"* that lacks any didactical intentions with regard to the student.

In the student's "game" with the *milieu,* knowledge is the means of understanding the ground rules and strategies and, later, the means of elaborating winning strategies and obtaining the result being sought.

In the teacher's game with the student-*milieu* system, the didactical contract is the means of establishing the basic rules and strategies and, later, of adapting them to changes in the student's game.

To each piece of knowledge, and perhaps to each function of a piece of knowledge, there must corrcspond specific situations (problems) and probably didactical contracts. The evolution of the players and of the game—unlike games having fixed rules—leads to some questioning of both knowledge and the didactical contract.

This *didactique* is precisely the basis of the constitution of knowledge in that the knowledge articulates the specific and the general. Before systematizing and going more deeply into this modelling, it will be useful for us to examine its *coherence.* This study will allow also the *clarification* of the functions or relationships that it would be proper to represent (by rules) and of the difficulties of the enterprise.

This paragraph will allow us to present the methodology of *didactique* more clearly.

Considering teaching as devolution of responsibility for the use and construction of knowledge to the student leads to some paradoxes that it is helpful to point out.

4.1. *The paradox of the devolution of situations*

The teacher must make sure that the student solves the problems set in order to evaluate, and make it possible to make the student evaluate, whether she has accomplished her own task.

But if the student produces her answer without having had herself to make the choices which characterize suitable knowledge and which differentiate this knowledge from insufficient knowledge, the evidence becomes misleading. This occurs particularly when the teacher is induced to tell the student *how* to solve the given problem or what answer to give. The student, having had neither to make a choice nor to try out any methods nor to modify her own knowledge or beliefs, has not given the expected evidence of the desired acquisition. She has given only an illusion of it. The teacher has the social obligation *to teach* everything that is necessary about the knowledge. The student—especially when she has failed—asks her for it.

And therefore, the more the teacher gives in to her demands and reveals whatever the student wants, and the more she tells her precisely what she must do, the more she risks losing her chance of obtaining the learning which she is in fact aiming for.

This is the first paradox; it is not exactly a contradiction, but the knowledge and the teaching plan will have to proceed under a façade.

So the didactical contract faces the teacher with a true paradoxical injunction: everything that she undertakes in order to make the student produce the behaviours that she expects tends to deprive this student of the necessary conditions for the understanding and the learning of the target notion; if the teacher says what it is that she wants, she can no longer obtain it.

But the student, also, faces a paradoxical injunction: if she accepts that, according to the contract, the teacher teaches her the result, she does not establish it herself and therefore does not learn mathematics; she does not make it her own. If, on the other hand, she refuses all information from the teacher, then the didactical relationship is broken. Learning implies, for her, that she accepts the didactical relationship but that she considers it as temporary and does her best to reject it. We shall see, further on, in what way.

4.2. *Paradoxes of the adaptation of situations*

Let us accept that the meaning of a piece of knowledge originates to a large extent from the fact that the student acquires it by adapting to the didactical situations which are put (devolved) to her. We shall assume also that for every piece of knowledge there exists a family of situations to give it an appropriate meaning.

In certain cases, there are fundamental situations that are accessible to the student at the required time. These fundamental situations will allow her quite quickly to create a correct conception of the knowledge which can be inserted, when the time comes and without radical modification, into the construction of new knowledge.

But let us suppose that some piece of knowledge exists for which the above conditions are not fulfilled; there exist no situations sufficiently accessible, sufficiently efficient and in sufficiently small numbers as to allow students of any age to have access straight away, by adaptation, to a form of the knowledge that could be considered correct and definitive. It is then necessary to accept stages in the learning process. Knowledge taught by adaptation in the first stage will temporarily be not only approximate but also partly false or inadequate.

The teacher will then find herself confronted with new paradoxes such as the following.

4.2.1. *Maladjustment to correctness*

Even if the knowledge taught in the first stage is necessary in order for a later stage to be undertaken, the teacher must expect to find herself blamed for allowing or creating these mistakes. Reproach will come as much from her students as from teachers of higher grades, unless a tradition or cultural negotiation exonerates her.

Under the hypothesis envisaged, an alternative exists: the teacher gives up teaching by adaptation; she teaches knowledge directly in accordance with scientific requirements. But this hypothesis implies that she must give up providing a meaning to this knowledge and obtaining it as an answer to situations of adaptation because then students will colour it with false meanings.

The teacher has the choice of either teaching formal, meaningless knowledge or teaching more or less false knowledge that will have to be corrected. Intermediate choices could blend the two disadvantages and even complicate them.

The student who is being taught on the one hand a “scientific” piece of knowledge and who is being presented on the other hand with inadequate situations of reference is in a position to observe all sorts of contradictions and maladjustments between these two teaching objects. The knowledge that she obtains by understanding is even false or different from what we intend to teach her.

The distinctions that are established between theoretical knowledge and practical knowledge are perhaps often only a simple consequence and a recuperation of this purely didactical difficulty. Here, again, the student is faced with a paradoxical injunction; she must understand AND learn; but in order to learn she must to some extent give up understanding and, in order to understand, she must take the risk of not learning. Taking knowledge *and* its genesis (true or fictional) as a teaching object, and thus teaching knowledge *and* its meaning is not a perfect solution either.

4.2.2. *Maladjustment to a later adaptation*

Memorization of formal knowledge, largely meaningless, can be very costly in terms of learning exercises. These exercises must not reintroduce too much meaning, a fact that reinforces their difficulty. The representation which the student makes of the mathematical knowledge and its functioning is profoundly perturbed as a consequence. The more the student has been drilled in formal exercises, the more it is difficult for her, later, to restore a fruitful functioning of concepts so acquired. "Application" of learned, ready-made knowledge goes badly because the logic of the articulation of the acquisitions which compose it is exclusively that of the knowledge itself and because the rôle of situations has been excluded *a priori.*

Let us examine the alternative choice, that of a temporarily erroneous understanding of knowledge obtained by adaptation to "introductory" problems. It will be necessary to go over this knowledge again and modify it.

A new paradox appears: if the student has adapted *well* to the situations put to her, she has better understood the reasons for her answers and the relationship between her knowledge and the problems. It will therefore be more difficult, later on, to change this knowledge so as to make it correct and complete.

We have just shown that for some knowledge it is quite predictable that "going over" it again and modifying it will be much more difficult, because it was better learned, better understood and better consolidated during the first stage.

No doubt this fact is due to matters of a psychological nature: it is all the more difficult to change habits or opinions, since they are more intimately connected with more personal, more numerous, older activities.

But it could also be for a more directly epistemological reason. The over-adaptation of "knowledge" to the solution of a particular situation is not necessarily a factor favourable to the solution of a new situation. Too strong a differentiation, too large a dependence on direct "knowledge", and the evolution of knowledge becomes impossible. The initial knowledge creates an obstacle. Some of these obstacles are inevitable and constitutive of knowledge—others are the result of a didactical over-investment.

Thus, under the hypothesis that some piece of knowledge is not accessible to all students by a reasonably quick adaptation to a fundamental, sufficiently *correct* situation, the teacher finds herself confronted by a new paradox. Whether she chooses formal teaching or teaching by adaptation, the more she insists on the learning of intermediate knowledge, the more she risks hindering further teaching. Conversely, if she abandons the establishment, the institutionalization of what is acquired, even partially, the student will find no support for the steps that follow. In some cases, the better the student adapts to an intermediate didactical situation, the greater is her maladjustment to the ensuing stage.

It is probably this phenomenon which causes teachers of higher grades to use more elementary content only in the form of procedures or algorithms and, if meaning is to be considered, to do it in situations with vocabulary and methods as different as possible from those of earlier levels.

4.3. *Paradoxes of learning by adaptation*

4.3.1. *Negation of knowledge*

Is the hypothesis that the student could construct her knowledge by a personal adaptation to an adidactical situation consistent?

Let us imagine that the teacher devolves to the student a source of auto-controllable questions[21] or a problem. If the student solves this problem, she can think she has done so by the normal application of her earlier knowledge. The fact that she has solved the problem will seem to her to be proof that there was nothing new to learn in order to do so. Even if she is aware of having replaced an old, culturally-identified strategy by another of her own "invention", it will be difficult for her to claim that this "innovation" is new knowledge; what point is there in identifying it as a method since it seems it can be produced easily whenever necessary? How could a subject alone distinguish, from among all the decisions that she has made, between those which are detachable from the situation and could serve as they are in other situations, and those which are purely incidental and local?

The social conditions of learning by adaptation, by rejecting the principle of the intervention of knowledge from a third party to produce the answer, tend to make it impossible to identify this answer as new, and thus as corresponding to an acquisition of knowledge.

The subject deems trivial the question whose answer she knows insofar as she has no way of knowing whether others have considered it before, or whether no one knew how to answer it, or even whether other questions resemble it or are linked to it by the fact that they could be answered because of this one, etc. Some external person must therefore look at her activities and identify those which are interesting and have a cultural status. This *institutionalization* is in fact a complete transformation of the situation. To choose certain questions from among those we know how to solve, to place them at the heart of a *problématique* which confers the status of more or less important knowledge on the answers that these questions require and to relate them to other questions and other knowledge, ultimately constitutes the best part of scientific activity. This cultural and historical work differs totally from what it seemed necessary to leave to the student, and it comes back to the teacher. It is not, thus, the result of adaptation by the student.

In some ways, adaptation contradicts the idea of the creation of new knowledge. Conversely, knowledge is almost the cultural recognition that direct knowing is impotent to solve some situations naturally (by adaptation).

4.3.2. *Destruction of its cause*

Situations permitting the student's adaptation are most often repetitive by nature; the student must be able to make several attempts to investigate the situation with the help of her representations and to draw consequences from her failures or her more or less accidental successes. The uncertainty into which she is plunged is a source of both anguish and pleasure. The reduction of this uncertainty is the aim of intellectual activity and is its driving force. But knowing the *solution* in advance—that is to say, having transformed sufficient and particular answers into methods that give the answer every time—destroys the uncertain nature of the situation, which then loses its interest. Knowing therefore deprives the student of the pleasure of seeking and finding a "local" solution. Adaptation—by means of the knowledge—thus coincides with the renunciation of an uncertainty which in the end is pleasant. The student's adaptation tends to destroy the motivation that produces it as it tends to remove all meaning from the situation that instigates it. It should therefore end quickly and, ultimately, not occur from the moment that a process becomes necessary.

The simple image of adaptation to external disturbances is not sufficient for representing the phenomenon of learning. It leaves no room for two elements essential for maintaining the process:

- on the one hand, the creation of an intrinsic motivation which stimulates the student to search for another "occasion" for adapting herself without attempting to adapt the *milieu* to herself;
- on the other hand, the subject's internal adaptation without external disturbances and without real "activity" (as, for example, the resolution of the subject's internal contradictions generated by the assimilation of new schemes, as discussed by Piaget).

4.4. *The paradox of the actor*

Can the teacher escape devolution, the direct intention of teaching some particular knowledge? Can she escape the didactical situation? After all, it would perhaps be enough for her to be a mathematician and to behave as such in front of and with the student. The progressive participation of the student in this activity could allow her to learn mathematics as a direct cultural activity, without shift of language or method, and without transposition either. The student would learn mathematics as she learned her mother tongue. Can the cultural *milieu* take the place of the teacher without being locally didactical? Can the didactical system be envisaged without teachers?

Indeed, numerous works have shown the important rôle of the familial, social, cultural *milieu* in individual differences in behaviour and success in school.

The child can probably learn many things if the mathematical activity of family members produces discussions and questions which are accessible to her; in particular, she will pick up from them methods, requirements, habits and the detection of difficulties; that is to say, information of an epistemological nature. But when a project involving the personal learning of precise knowledge takes shape, the child will once again become a student and the fundamental systems will reappear, one will be the teacher and the other the taught. Whether it is a spontaneous or an institutional project, the teacher cannot escape the devolution of knowledge.

This knowledge whose text already exists is no longer a direct production of the teacher, it is a cultural object, quoted or requoted. And its reproduction at the desired moment is much more comparable to a theatrical act replayed for the benefit of the student, and then by the student herself, than to an experience lived through with her. Even if the student can *live* her learning, the teacher, for her part, is necessarily an actor since she knows in advance what she wants to teach. That is not only a metaphor; the teacher really is an actor—with or without text—occupied with making her re-production of knowledge live for her student.

This approach partly answers the initial question, but transforms it and breaks it up:

i) Must the teacher "do" the mathematics that she wants to teach, in the same way that the actor might feel the sentiments that he or she wants to share with the audience?
ii) Must the teacher re-phrase her script around an outline each time, as in the *Comedia del Arte,* or must she stick to the well proven "script"?

On this last point, Diderot (1773), in a well-known study, stated the inherent paradox about the actor's activity: the more the actor feels emotions he wants to display, the less he is able to allow the audience to share the feeling because, being a "continuous observer of the effects that he produces, the actor becomes a sort of spectator of spectators as well as being what he is himself and can thus perfect his game" (*ibid.* pp. 186–187).

This paradox can be extended to the teacher. If she herself produces her mathematical questions and answers, she deprives the student of the possibility of acting. She must therefore ignore time, leave questions without answers, use those which the student gives her and integrate them into her own process by giving them a larger and larger place. This idyllic scheme can work so long as the teacher produces new knowledge, but if the knowledge is determined in advance, this "liberty" becomes nothing but an actor's performance, and the student is invited to be another actor, restricted to a script, or at the very least an outline which she is not supposed to know about. Some pedagogical schemas postulate the necessity for the teacher herself not to know the knowledge to be constructed (transmitted) in such a way as to be better capable of managing the passage from ignorance to knowledge in a convincing way. The existence of these schemas is proof of the relevance of our analysis. It is easy to show their illusory character (which doesn't mean that all enterprises of this type are failures, but that they are successful only under other conditions).

As our study shows (Brousseau and Otte, 1991), Diderot's paradox applies to the teacher in an extended way, and it is perhaps more fundamental and more acute than for the actor. Among other things, the explanation of actors' resistance to this analysis can be extended to those observed in the world of teachers.

The "paradox" in the sense of Diderot is apparently absurd because it is contrary to generally accepted opinions but is nevertheless basically true. We have given it a narrower meaning: our paradoxes are sorts of functional contradictions between an apparently exhaustive game of decisions and their result.

The resolution of these paradoxes just as much as the explanation of observed phenomena is one of the aims of a theory of situations as well as being a means of testing its consistency.

5. WAYS AND MEANS OF MODELLING DIDACTICAL SITUATIONS

Here, the purpose is to disclose the instrument of modelling—the game—and then to discuss what the relationships will be between these "models" and the reality that they describe.

These relationships are not those of an original, which would be the fundamental game as a model, with its copy, which would be the didactical reality, and where difficulties could be ascribed to differences introduced by a "wrong" response of the players. On the contrary, these relationships leave room to be challenged by observations and are susceptible to falsification. This systemic approach that we are proposing will be illustrated by means of a discussion of the initial fundamental subsystems that have to be taken into consideration. We shall show that the need to introduce a *"milieu"* system into the student's didactical game is not a *reification* of the model (the instruments of the game) nor the product of an observation, but that of an internal necessity.

5.1. *Fundamental situation corresponding to an item of knowledge*

Modelling a teaching situation consists of producing a game specific to the target knowledge, among different subsystems: the educational system, the student system, the milieu, etc. There is no question of precisely describing these subsystems except in terms of the relationships they have within the game. Before we specify the type of game to be used, the two major goals of the modelling must be identified.

5.1.1. *With respect to the target knowledge*

The game must be such that the knowledge appears in the chosen form as the solution, or as the means of establishing the optimal strategy:

- Is knowing this property the only way of shifting from a given strategy to another one?
- Why would the student look for a way of replacing this strategy with that one?

– What cognitive motivation leads to the production of such-and-such a formulation of a property or to such-and-such a mathematical proof?
– Is the given reason for producing this knowledge better, more correct, more accessible or more effective than any other reason?

This type of question can be asked *a priori.* Initially, the answers can be drawn from the logic of the game, from the history of science, or from mathematical or didactical analysis: the game specific to an item of knowledge must justify its use or appearance in a way which conforms to theoretical *didactique.*

5.1.2. *With respect to teaching activity*

The "game" must allow a representation of all situations observed in the classroom (if not specific progressions), even the less "satisfying" ones, so long as they manage to make the students learn one form of the target knowledge. It must be able to generate all the alternatives, even the most degenerate ones. These may be obtained by the selection of values of certain variables that are characteristic of this game[22].

The general concepts of *didactique* must allow the establishment of the relative meanings of these different variants and the explanation and prediction of their effects on the type of knowledge that they will allow a person to acquire, on the development of the teaching activities that they discriminate and on the quality of their result.

Conversely, they should allow a piece of knowledge to be matched with the conditions that justify it, that make it necessary, in its different forms.

Adjustment of these conditions in terms of what we know of the child's epistemology and psychology, of linguistics, or of sociology is a reasonable objective of *didactique.*

To provide an experimental counterpoint to the thoughts of epistemologists or theoreticians of knowledge is a legitimate ambition. But to pretend that every knowledge-producing activity can be considered to be an "economical" behaviour in a game which can be made explicit is out of the question. Moreover, knowledge is always amply overdetermined. It is only a question of models, accepted as such.

5.2. The notion of "game"

To model the vague notion of "situation" by that of "game" requires that this word to be given a precise meaning. Its five main definitions are all related to the elements to be represented.

i) The first characterizes the set of relationships, the "hyposystem", to model: "entirely free physical or mental activity, generally based on convention or fiction, which, in the mind of the one who performs this activity, has no other purpose than itself, no other goal than the pleasure that it provides". [Definition 1]

This definition presents essentially *a player* capable of taking pleasure, of imagining a fiction and of establishing conventions and *relationships* with an undefined *milieu.* She produces an activity upon which her pleasure depends. But, above all, the definition emphasizes the quasi-isolated character of the system to which it refers. We admit that there can exist a *"Deus ex machina"* of which the player must not be aware. For her, therefore, the activity is gratuitous. But how can we reconcile this idea of an action motivated by pleasure which is nevertheless gratuitous? Aren't all actions really motivated by pleasure? We shall interpret the sentence in the following way: decisions and actions *during* the game are determined only by the pleasure that the player derives from accomplishing them or by the pleasure that she derives from their effects, but the decision to take part in the actual game is not motivated by any goal. We shall return to this notion of lack of motivation further on.

Alongside this first meaning, we find four others:

ii) The game is "the organization of this activity within a system of rules defining a success and a failure, a gain and a loss" (Lalande, 1972, p.546). [Definition 2] This is the "game"[23].

iii) It is also, and we shall often be using the word in this sense, "whatever is used for playing, the instruments of the game", and occasionally one of the states of the game determined by a particular setting of the instruments of the game. [Definition 3]

iv) It is sometimes "the way in which one plays", the "play"[24]. In cases where procedures are concerned, we like the terms "tactics" or strategy better. [Definition 4]

v) Finally, it is the set of positions from among which the player can choose in a given state of the game (following meaning 2), and by extension, in mechanics for example, the set of possible positions and thus movements of a system, of an organ, of a mechanism that has furthermore been subjected to certain constraints. [Definition 5]

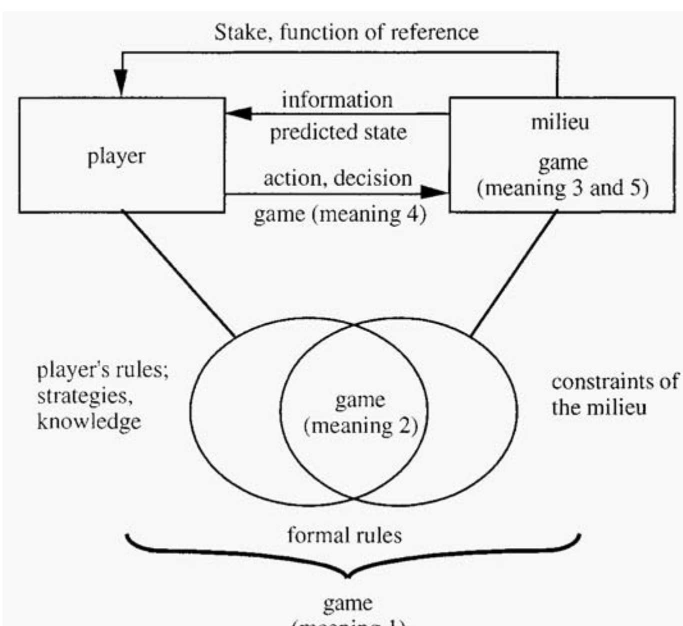


Figure 1

The relationships among the different meanings are shown in Figure 1. Let us recall that formally, a "game" played by *k* persons (for example) is the structure defined by the following:

a) a set X of distinct "*positions*" containing objects and relevant relationships;
b) a mapping Γ of $X \to \Gamma(X)$ which, to every state $x \in X$ associates the set $\Gamma(x)$ of permissible positions from among which the player who has the move can choose, starting from the state x. Γ thus represents the rules of the game;
c) an initial state I and one or many final states F (such that $\Gamma^{-1}(I) = \emptyset$ and $\Gamma(F) = \emptyset$);
d) a set **J** of *k* players and a mapping θ of JxX into **J**, which, for each state x of the game, designates the successor to the move $\theta(j,x)$ of player j;
e) a function called gain, or stake or preference, which is a mapping of A, the subset of X *containing* F, into **R**.

This definition is not general and examples of games which require a different, appreciably more complex modelling, may be encountered. For example, it is suitable for games of chess or chance but not for rôle-playing games.

It is nevertheless sufficient to allow the definition of some didactical terms:

– *A round* is a finite sequence of states $(x_i)_{1 \le i \le n}$ in X such that $x_1 = \mathbf{I}$ $x_n \in F$ and $\forall i\ x_{i+1} \in \Gamma(x_i)$ The "*allowed states*" are the positions in X which can be part of a round (in chess, disallowed states are sometimes called "a fairyland").
– *A strategy* S is a mapping X→X that determines a player's choices from all the permissible states S(x) E $\Gamma(X)$; since there are *k* players, *k* strategies are sufficient to determine a round.
– *A tactic* T_A, will be a mapping of a subset A of X into X such that $x \in A$ and $T_A(x) \in \Gamma(X)$ A strategy is then a tactic that is defined over all elements of X.
– A player's state of "*knowing*", C, will be characterized by a mapping of X into $\Gamma(X)$ such that $(\forall(x) C(x) \in \Gamma(X))$. A (non-zero) "knowing" strictly narrows the players' choices. (This definition can be compared to that of information.)
– A "determining" knowledge reduces the player's choice from a certain number of states (respectively, all states) to a single state, and thus characterizes a tactic (respectively, a strategy).

Acquisition of knowledge, as the consequence of received information (or of learning) for example, is a modification of the state of knowledge: an ordered pair (C, C′)—in fact, C at the instant t, C′ at the instant $t+\partial t$.

More often than not, to be consistent with the theory of information, it is assumed that $C'(X) \neq C(x)$, that is to say, the knowledge *reduces* the subject's uncertainty by removing the possibility of choice.

In order to model the modifications of the student's knowing, however, it is necessary to imagine that she does not envisage *all* the permitted positions at each moment (even though they are allowed by the rules of the game) and that a modification of her state of knowing may consist not of reducing her uncertainty but, instead, of increasing it at the instant $t+\partial t$ by the consideration of new possibilities that are available for her to choose from. This consideration prevents the strict use of the theory of information.

– *Model for action*[25]: We shall call every strategy or calculation procedure giving rise to a strategy (or a tactic) a "model for action".

We could likewise use the word "representation" to denote "what" in a particular game gives rise to states of knowing, and what will allow the knowing to be predicted.

The first advantage of a model of this kind is to allow in precise cases the *a priori* consideration of "all" sequences of responses and the comparison of their effectiveness.

A *winning strategy* provides, against any defence by the opposition, a round with positive payoff, but various characteristics can be evaluated, such as

– its cost, for instance the number of moves leading to the end of the round;
– the gain that it provides.

It will nevertheless be possible for a *non-systematically-winning strategy* to be better than another one in terms of the risk of loss that it entails, the gains that it allows one to hope for, etc. Game theory then allows the study of the dilemmas that arise. The majority of fundamental learning exercises are envisaged with "Nature" as a partner.

The construction of models for action allows one to go much more closely into the analysis of the subject's possible behaviour, as we show in many examples (*cf.* the thesis of Ratsimba-Rajohn, 1981). Thus, the study of the adequacy of a situation for a particular piece of knowledge has the aim of showing that the optimum strategy can be brought about by this piece of knowledge and not by another one. Reciprocally, it then becomes possible to state hypotheses about the variables of the situation and their influence on strategies and changes of strategies (*cf.* the thesis of Bessot and Richard, 1979).

The meaning of a decision or a choice made by the student can itself be modelled with the help of a number of components such as:

a) the set of choices the student considers and rejects as a consequence of the choice made;
b) the set of possible strategies considered and excluded, and in particular the sequence of choices or replacement strategies the student considers;
c) the very conditions of the game that appear to be determining the choice made, and in particular the space of situations brought about by the values of the pertinent variables which give the decision a character of optimality, validity, or relevance.

5.3. *Game and reality*

5.3.1. *Similarity*

In her "real" life, the subject organizes her actions according to her interests, within the framcwork of unknown, changing rules; in contrast to these serious, professional or private activities, game situations occur in which she can choose her rules, give herself up to pleasure, free herself from other constraints.

However, there are several examples in which the precise description of the functioning of certain social, financial, economic, military, etc., relationships is clarified and facilitated by simulation by means of a game. The game situation is often an excellent model of a real situation. This is why the game can be a powerful entertainment and a symbol of life; it resembles it! At the same time, one can master constraints which, in the real life overwhelm the player, and this liberty plays a fundamental rôle in balancing the frustrations that they cause.

Let us, for example, examine the "doll game" reported by Freud. The child makes the doll disappear under a piece of furniture: "*fort*" and then, at will, makes it reappear: "*da*"[26]. We can see the relationship that this game can have with the mother's uncontrolled and certainly unpredicted appearances and disappearances. But it would be wrong to think that the doll represents the mother and that the child reproduces or imitates with no further relationship to everyday life.

For the child, the interest lies in the fact that, in the model (the game), she *controls* the doll's movements whereas, in real life, she has no control over her mother's appearences. This game with the doll allows her to re-live the distressing situation of separation from her mother: "*fort*", controlling the emotional effect by creating, at will, the joy of her return: "*da*". Of course, this reproduction of pleasure is linked to the process and the doll must disappear in order to be able to reappear. However, it is controlled by the child: that is the fundamental requirement. An "automatic" doll that appears at regular intervals would not play the same rôle. As soon as the child could predict the appearences, the game would cease to interest her.

A doll which makes random appearances (that can be neither ordered nor predicted) would be distressing, too "realistic", that is to say, too close to the symbolized situation of the mother. When the child discovers that the doll will certainly reappear soon, when she watches out for it, and when it actually does appear during this waiting period at an unexpected moment — under these conditions, the baby bursts into laughter, particularly if she discovers that someone is manipulating the doll in a mischievous way. But this laughter is a defence reaction, more like the mocking derisive laughter of the powerless than the pleasant laughter of someone who feels in control of the situation. Laughter, yes; but the laughter provoked by distress on the way to being overcome.

The game is a symbol in the sense that it "sufficiently" resembles life. It solicits the same kind of possibility of action from the player, the same kind of emotion and motivation, but it differs from life because in the game most conditions which, indeed, oppress and elude the player are overcome. Similarity is one way of giving a meaning to dissimilarity.

5.3.2. *Dissimilarity*

It could thus be thought that the fundamental separation that places playing and living, or more exactly wish and reality, opposite to each other has been justified and explained, allowing one to be seen in relationship to the other:

– conventional and symbolic games playing their rôle within the game of life,
– the game a symbol of life.

But according to Lacan[27], the symbol created to balance frustrations and tensions arising from relationships with the desired object actually inherits their frustrating characteristics. As soon as the doll appears when and where one wishes, for example, the business of taking control is finished and the game as such disappears. Thus, the game which remains a game by definition doesn't satisfy the player and creates the need for a new round of a new game, or for a new symbol. Following Lacan the relationships with the symbol must therefore themselves be balanced by the creation of a new symbol, and therefore the chain of meaning is open.

The game must be either totally controlled and thus rejected as a desired object, or otherwise reproduced endlessly. These two characteristics are very important:

– a "game" in which the player controls all the issues and all the results, and achieves a sure win, would offer no uncertainty and would leave no place for the simulation of the uncertainties of its "model".

If "a complicated game is not usable as it is in the classroom ... an analyzed game is a dead game"[28]—a game cannot be entirely gratuitous. Facing the player, there must be an opponent, a *milieu,* a law of nature, that in some way stops her from obtaining the desired result at every move.

5.4. *Systemic approach of teaching situations*

The systemic approach of teaching situations appears to be suitable to the extent that the sub-systems considered—the person taught and the educational system—are immediately identifiable as actors.

It offers some benefits to the extent that the consideration of sub-systems allows either an appreciable simplification of the study of the problems presented or the isolation of a few of the problems which can be solved within these sub-systems. It proves to be essential if the entire set of didactical phemonena can be taken on in this way and can therefore claim to provide a theoretical foundation.

But it presents a certain ambiguity, and there is a danger of its just being the instrument of a projection onto reality of the model thought up by the researcher.

The (systemic) classical approach of teaching situations places the emphasis on these concrete systems, face-to-face (the teacher, the student), and their functions, their characteristics. It leads to the examination, using the model of social functioning, of the manner in which these functions are undertaken and internalized. The observed difficulties will then be attributed to bad responses to the needs of the system. This reasoning constitutes a *reification* (Berger and Luckmann, 1966), that is to say that the abstract schema and reality "must" coincide and that there is no longer any place for experiments and falsification. On the other hand, the decomposition into sub-systems, envisaged here, has as its objective the definition of *games* which allow the communicating partners' opposed strategies to be co-ordinated. We shall therefore find the student's games with her didactical environment as regards knowledge, the games of the teacher who plays with the student's games, etc. It is a question of postulating the object of didactical study and of proving its existence. The method appeared as early as 1970 in the works presented below[29], and subsequently improved in an empirical manner. The method is very close to that advocated by Crozier and Friedberg (1977, p. 216) for the study of social and political systems:

"If one can... discover sufficiently stable strategies inside a set [of people] and if one can, furthermore, discover games, the rules of the games and the regulations of these games under which these strategies can effectively be considered to be rational, one has, at the same time, the proof that this group could be considered as a system and precise answers about its mode of government" (*ibid.* p. 215).

It would still be naïve to believe that the construction of these "concrete systems of action" which articulate the play of the actors facing each other, according to Crozier and Friedberg, can proceed without unceasing confrontation with reality. Anticipation of the relevance of elements retained to explain a phenomenon obviously includes the risk of locking oneself into the prior categories that one has accepted as the starting point because they conformed to accepted ideas. And it is only by ceaseless analytical work on the meaning of numerous natural or provoked observations and on local and global methodological requirements that one can link these observations to hypotheses about the games relative to which they are rational. and about the didactical system that contains these games.

6. ADIDACTICAL SITUATIONS

6. 1. *Fundamental sub-systems*

6.1.1. *Classical patterns*

To start with, the didactical game puts a first player, the educational system—the teacher, bearer of the intention to teach a piece of knowledge—into relationship with a second player, the person being taught, the student. Above, we have shown the need for the teacher, real or internalized.

Is it possible to define the didactical game while limiting oneself to these two sub-systems? Several patterns have been proposed for this purpose including Osgood's communication pattern[30] and Skinner's school conditioning pattern[31].

In the communication pattern, the educational system is a transmitter of information and the student a receiver who, with the help of her repertoire, decodes the messages that she receives. The teaching consists of triggering the creation of new elements to be added to the repertoire by means of messages formed exclusively from the repertoire of the receiver so that they are intelligible (see Figure 2). It is clear that the rule for a teacher who adopts this model would be never to introduce a new piece of knowledge except by a known method of construction based on known concepts. What is communicated is only knowledge in its cultural form and, in the case of mathematics, in its axiomatic form. This model is insufficient for many reasons. For example, it does not allow the meaning of these memorized messages to be defined in any other way than as the reformulations of previous messages. Why would these messages be memorized anyway?

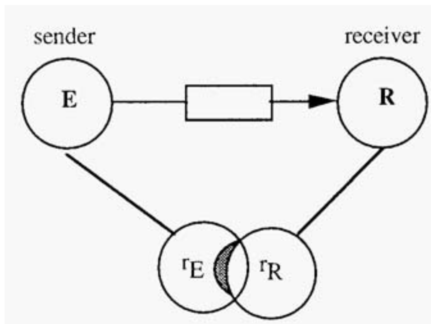


Figure 2

Learning "... by statistical apprehension in which *semanteme* that are the most frequently put forward by the transmitter gradually filter into the receiver's repertoire, at the same time modifying it...", imagined by Moles (1 967, p.111), resurrects the sensualist, empiricist hypotheses refuted a long time ago. This interpretation assumes also that in every message addressed to her, the student will recognize what is a new piece of knowledge to learn. What transparency!

Behaviourism (see Figure 3) replies to this objection by putting forward a learning pattern made up of two sub-systems: the student who influences (we shall say "who acts upon") the *milieu,* and the *milieu* which "informs" or sanctions the student.

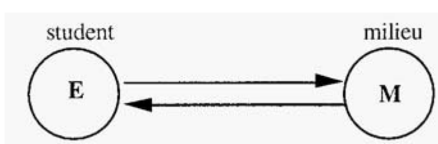


Figure 3

The student, disturbed by the influence of the *milieu,* attempts to annul these sanctions by modifying the *milieu* and/or by engaging in learning by which she herself is changed. Of course, the teacher is a part of the *milieu* and can even substitute for it. This pattern supposes that knowledge can be expressed in the form of a list of stimulus-response ordered pairs. This thesis was first refuted by Chomsky and Miller (1968) in connection with the teaching of the mother-tongue, which has been recognized as capable of being produced at best only by a finite automaton and by a stimulus-response model. Despite Suppes's objection (for every finite automaton, there exists a stimulus-response model which is asymptotically equivalent to it), Nelson and Arbib's consideration about the speed at which these models converge has condemned them[32].

6.1.2. *First decomposition proposed*

Without for the moment rejecting the above reductions, we see that it is necessary to consider two distinct types of games:

a) the student's games with the adidactical *milieu,* which allow the specification of what the function of the knowledge is after and during the learning—these games are obviously specific to each piece of knowledge,
b) the games of the teacher as an organizer of these student's games (insofar as they are also specific to the target knowledge). These games concern at least three partners, and generally four: the teacher, the student, the student's immediate environment and the cultural *milieu.* The teacher's game (see Figure 4) in each concrete system of action defines and gives a meaning to the student's game and to the knowledge.

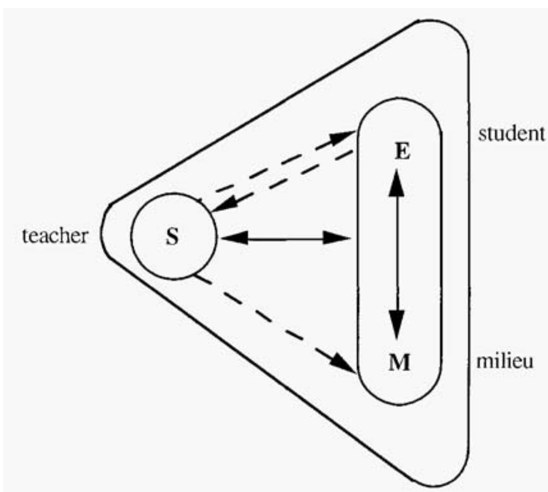


Figure 4

It follows from this definition that the teacher's two major types of game are *devolution,* which we have presented already, and *institutionalization.* In devolution, the teacher puts the student in an adidactical or pseudo-adidactical situation. In institutionalization, she defines the relationships that can be allowed between the student's "free" behaviour or production and the cultural or scientific knowledge and the didactical project; she provides a way of "reading" these activities and gives them a status. These two types of negotiation are quite distinct. Concerning the former, the above discussion certainly provided hints about the elements of choice, its rules and its stakes, which allow it to be modelled in terms of games. The latter, more strongly related to the didactical contract, is the object of current work[33]. It would be better to return to this topic after we have studied the adidactical situation.

6.1.3. *Necessity of the "adidactical milieu" sub-system*

Thus in the general case, the didactical situation cannot be modelled as a simple communication or a simple social interaction. It is necessary to make another system intervene.

This need arises from one of the clauses of the didactical contract itself, which involves the project in its own destruction: it is understood from the beginning of the didactical relationship that a moment must arrive when it will be broken. At that moment, at the end of the teaching, the taught system will, with the help of the learned knowledge, be assumed to be capable of facing systems without didactical intentions. The knowledge taught to the student is then assumed to make it possible for her to read her relationships with these systems as new *adidactical situations* and by this means bring them an appropriate answer. The *milieu* is the system opposing the taught system or, rather, the previously-taught system.

This reading can in reality happen in different ways, but coherence requires that we model it in the form of "games" recognized as similar to those which the student knows. The taught system can then make decisions supported by its knowledge and, furthermore, can establish relationships between the two meanings, the old and the new. On the other hand, she can read these adidactical situations in the form of new "games" calling for new answers without reference to those she knows. In any case, she can see in these situations the occasion to ask herself new questions and, possibly, questions without any answers for her.

Conversely, the didactical situation must include, either in reality or simply evoked, a reprcscntation of these future relationships. It must include and make explicit another system, distinct from the educational system, which will represent "the *milieu*". As the student's progress gradually continues, this cultural and didactical rcprcsentation of the *milieu* will be assumed to approach "reality", and the subject's relationships with this *milieu* will have to become free of didactical intentions.

A number of consequences result:

- The didactical relationship always relies on epistemological hypotheses, conscious or not, explicit or not, and coherent or not.
- The analysis of the didactical relationships implies the definition or the recognition of these "fundamental" and adidactical games, bringing together a *milieu* and a player, these games being such that knowledge—a given precise knowledge—will appear as the means of producing winning strategies. For that to happen, one must have at one's disposal a particular and very concrete form of epistemological knowledge.
- At a certain point in the teaching, the student finds herself engaged by her didactical contract in a more or less real relationship with a *milieu* organized (at least in part) by the educational system. This relationship has been organized so as to justify the student's "relevant" production of behaviours which are indicators of the appropriation of knowledge. That is to say that the student's answer must not be motivated by obligations related to the didactical contract but by adidactical necessities of her relationships with the *milieu.*

– The student's relationships with the *milieu* can be conceived of (in particular by the educational system) as playing any of several very different rôles:
  - For example, the adidactical situation can be incapable of provoking learning at all, all didactical virtue being contained in the didactical contract.
  - On the other hand, the effects of the *milieu* on the student, and vice versa, can be expected to be sufficient, alone, to provoke the expected adaptations and learning (we are talking of the learning situation in the strict sense). The educational system then contents itself with choosing, organizing and maintaining relationships which assure the genesis of knowledge of the subject. With the student's permanent return to inquiry about the *milieu,* it does not allow her to remain unaware for long of the fact that the pedagogical contract is devoid of all didactical content.
  - The general case is obviously intermediate and combines a didactical contract and an adidactical situation that can also be a situation of learning by adaptation.

6.1.4. *Status of mathematical concepts*

We have seen that the production and teaching of mathematical knowledge requires an effort to transform this knowledge into institutionalized knowledge, a depersonalization and a decontextualization that tend to blot out the historical situations (the games) which had presided over their appearance. However, these transformations do not lead to a complete loss of their fundamental characteristic, which is to answer questions. Questions, which are the motivation, change; most of them disappear from the body of the theory but remain in problem form. It is clear enough to most mathematics teachers that only problem solving can demonstrate that the student has at least partly acquired the target mathematical knowledge. The range of problems connected with a piece of knowledge continues to change as the theory evolves.

It establishes a sort of dialectic between the capacity of the mathematical theory to solve the stock of existing problems very easily, and the capacity of the stock of problems to make the transmitted knowledge work in a non-trivial way. This dialectic rests on a necessary equilibrium between scientific activity, which tends to pose new questions to be solved and thus increases the field of problems and knowledge, and the communication of this knowledge which leads on to a better theoretical organization which reduces the complexity of the field. This reorganization then makes the old problems trivial and allows the reduction of the field of problems necessary for the understanding of the theoretical knowledge which can then raise new questions.

This system of actions and retroactions does not guarantee a "regular" development of mathematics because, in this domain, an equilibrium can only break down and cause different types of activities. In any case, it shows that the correspondence between problems and knowledge evolves, and is not intrinsic. It is only under the control of a theory about these relationships that adidactical situations can be proposed for teaching.

If, on the other hand, one considers the evolution of mathematical knowledge and concepts, it is quite commonly noticed that it often conforms to a pattern that tends to be justified by the functioning which we have just presented. The final step, the one that puts the concept under the control of a mathematical theory, allows its exact definition in terms of structures in which it intervenes and of the properties that it satisfies. Only this step gives it its status *as a mathematical concept* and protects it from ambiguities and "errors"—but not from re-working or being pushed aside.

This step is generally preceded by a period in which the concept is a familiar, recognized, named object whose characteristics and properties are studied but which has, for various reasons, not yet been organized and theorized—such as the notion of function in the nineteenth century, or that of equation in the sixteenth century, or that of variable in the twentieth century. The functioning and the rôle of these *paramathematical concepts* is rather different from those of *mathematical* concepts. The former are more like tools and the latter objects in the sense coined by Douady in her doctoral thesis[34].

But it can also be said, by taking "a more formal", more "systemic" point of view, that in the absence of recognized mathematical status, the terms used are tools which respond to the needs of identification, formulation and communication, and that their use is based on a semantic control. Mathematicians use them well, not because they possess a definition that would give a "syntactical" control over them, but because they "know them well" and no contradiction which would necessitate their mathematization has appeared. This paramathematical use corresponds to a certain economy of theoretical organization, and therefore to an economy for communication, teaching and problem solving. As long as difficulties do not appear—contradictions (foundations of mathematics at the end of the nineteenth century) or too wide a semantic field (like, for example, the concept of probability just before Kolmogorov)—it is quite acceptable. But the paramathematical stage of a concept is in all likelihood preceded by another one that Chevallard suggested should be called "protomathematical"[35].

It is a question, then, of a certain *de facto* coherence of the preoccupations of mathematicians of a period, of points of view, of methods, of choice of questions, which are articulated very clearly as a concept identified today, but which were not at that time.

Before the concept can be referred to, sufficient indications must certainly demonstrate that these convergent preoccupations were not simply random or the result of mathematical need, but also the result of a choice of mathematicians of the time, and that they perceived these proximities of questions as related, even if they possessed no term with which to identify them. It is therefore not a question of confusing historical conceptions with the names of their mathematical identities. For example, in simplifying them a lot, al-Khowârizmî was concerned with rational numbers but not really with real numbers[36]; on the other hand, Stevin presented all the *problématique* of real numbers and, because of him, the concept reached the protomathematical level.

This hypothesis of an object of knowledge, still implicit but already regulating decisions within a field of questions, rests in fact on *a priori* recognition of the possibility of interpreting mathematical texts with the help of a sort of representation of the work of the mathematician. It therefore implies taking charge not only of mathematics, its history and its classical epistemology, but also of a certain part of *didactique,* to the extent that it claims to model this work.

6.2. *Necessity of distinguishing various types of adidactical situations*

Although it is neither the original nor a copy of it, this classification of different statuses of a mathematical concept corresponds quite well to the differentiation of sub-systems of the adidactical situation.

It is now a question of establishing:

– a classification of the interactions of the subject with the adidactical *milieu;*
– a classification of the types of organization of this *milieu;*
– a classification of the types of function of a piece of knowledge; and
– a classification of the modes of spontaneous evolution of knowledge.

Each classification will have to justify itself sufficiently clearly in its own domain:

– by the considerable and obvious differences between the objects classified;
– by the simplification that it can provide in their description, their analysis and their understanding;
– by the relevance of this classification (and its importance in relation to other possible classifications) for each domain concerned;
– by its completely exhaustive character.

The classes obtained must correspond to, and be able to be organized into, a hyposystem. For example, a certain type of interaction is specific to one type of social and material organization; it favours a certain form of knowing and can also cause it to evolve.

These hyposystems, identified in this way, are aimed at predicting and explaining certain relationships between the interactions which are observed (or which one wants to obtain), knowledge which one verifies or of which one expects acquisition, constraints created by the *milieu,* etc. They are therefore a support for the production of falsifiable hypotheses. They need not be exclusive; even a "real" situation will generally be able to correspond to several which correspond to various components and to varied knowledge at work. They are assumed to cause the student's knowings and questions to evolve separately, but they can depend on each other and be articulated in organized or spontaneous processes or dialectics.

So nothing will require us to take them as a norm, an iron collar in which to constrain the functioning of knowledge and didactical engineering.

These conditions will not be trivially satisfied; in particular, the correspondence mentioned above is not guaranteed in advance, even if it is possible to suspect it or to believe it inevitable. Afterwards, it will appear obvious, but subjecting it to experimental testing has shown at what point these bets become risky.

The novelty of the object of study makes difficult the perception of the relative interest of the questions and the necessity or the contingency of the answers produced.

6.2.1. *Interactions*

The relationships between a student and the *milieu* can be classified into at least three major categories:

– exchange of judgment [3][37];
– exchange of information coded into a language [2];
– exchange of information that is not coded or which is without a language: actions and decisions that act directly on the other performer [1].

These categories arc mutually embedded, for an exchange of judgments is an exchange of particular information, which itself is a particular category of actions and decisions. They are strictly embedded.

i) There exist interactions where the player expresses her choices and decisions, without any linguistic encoding, by actions on the *milieu.* We shall put into this class of interactions those in which messages appear in a code so easy with respect to the action that it will play no rôle in the game, likewise those in which there are exchanges of messages unrelated to the solution of the problem; for example, the player explains herself to or carries on a trivial conversation with a third person without expecting any feedback.

It can also happen that the "player" is a pair of students co-operating in a common enterprise after having exchanged information and opinions. But this composite relationship includes an easily identifiable component of action upon which other interactions having local and temporary purposes are superimposed. If the exchange of information is not *necessary* for obtaining a decision, if the students share the same information about the *milieu,* the "action" component is dominant. [1]

ii) In the same way, there are interactions in which the player acts by sending a message aimed at the antagonist *milieu* without this message signifying her intention to put forward an opinion. It isn't just a question of classifying orders, questions, etc., in this category, but every communication of information as well. Indeed, most information is implicitly accompanied by an affirmation of validity. But to the extent to which the sender does not explicitly indicate this validity, if she does not expect to be contradicted or called upon to verify her information, if the context does not give a certain importance to the question of knowing whether the information is true, how and why or if this validity is likely to be established without difficulty, the message will be classified as simply giving information. Information thus given is assumed to change at least the uncertainty of the *milieu* and in general its "state". [2]

iii) Finally, there are interactions in which the messages exchanged with the *milieu* are assertions, theorems, mathematical proofs, sent and received as such.

The difference between a piece of information and a statement of validity is sufficiently clear and important in mathematics, so there is no point in stressing it here. We shall see later that these statements can themselves be of different types, according to whether they bear on the *syntactic validity* or on the *semantic validity* of the statement contained in the assertion. Pragmatic validity could also be invoked, an assessment of the efficacy of the statement. [3]

It is not necessary to prove here the importance for teaching of distinguishing these three types of production expected from the students. They are designated in the experimental works which this theoretical essay accompanied[38]:

– the first as *"actions"*, inferred without including the formulations or declarations of validity that can accompany them;
– the second as implied "*formulation*", inferred without discussion of proof;
– the third—the term is not very suitable, but has been used for fourteen years[39]—as *"validation"*.

6.2.2. *The forms of knowledge*

The forms of knowledge which control the subject's interactions have been approached in many ways. All tend to contrast the most explicit and best assumed formal knowledge, for example knowledge which is expressed in the "declarative" mode (Skemp[40]), with more implicit forms—representations, patterns, *know-how*—which are expressed in a more "procedural" mode. To this we have added a more strictly linguistic component: the codes and languages that control the formulations.

i) Simplifying a little, the forms of knowledge which allow the *explicit* "control" of the subject's interactions in relation to the validity of her statements are mainly the items of her knowledge that can be expressed and are recognized as such by the *milieu.* They are organized in well determined theories, mathematical proofs and definitions in their most perfect cultural form. [3]

The distinction between knowledge and a knowing depends primarily on their cultural status; a piece of knowledge is an institutionalized knowing. Passing from one status to the other nevertheless involves transformations which differentiate them, which are partly explained by didactical relationships that are associated with them.

But we shall assume as a first approximation that expressable knowings and knowledge intervene in similar ways in the control of the "student's''judgments. In some ways, they form the "code" which helps her to construct, justify, verify and establish her statements of validity.

This justification relates both to the student's deep conviction and to accepted social convention.

Explicit proofs and validations are assumed to support each other as evidence, but their articulation is certainly not automatic. Knowledge and knowing actualize themselves through activities of research or proof in ways that Heuristics is seeking to discover and Arti ficial Intelligence is trying to reproduce. For the time being, they remain rather inaccessible to scientific analysis and *a fortiori* to the subject herself.

One can suppose them to be themselves managed by representations, epistemological or cognitive schemes, implicit models, etc. Differentiation of the types of knowledge that we are attempting must go no further than is necessary for the organization of didactical debate with the student. In all probability, mental activity destroys these fragile distinctions and unifies these modes of control in a complex thought.

It is, however, useful to keep the distinction made in logic between a statement considered as a well formed expression or a set of realizations and the assertion that includes this statement in a metatheoretical declaration about its validity over a given domain or its deducibility within an axiomatic system. In generalizing this distinction, a judgment is composed

- of a description or model expressed in a certain "language" or (in a certain theory) possibly referring to "a reality" (that is to say, to the device of the current game); and
- of a judgment about the adequacy of this description, whether it is a contingency or a necessity and whether it is consistent with the subject's knowledge or the *milieu.*

It is very important not *a priori* to confuse knowledge and knowing, objects of a student's construction activity, with knowledge that describes the relationships which we are seeking to establish as having a unified or identical nature; these distinctions are quite clear in the works on "natural thought" that we have used in several research projects, notably those of Wermuz[41].

ii) The formulation of the descriptions and models in question is regulated by an entirely different type of code. Even if the theory of languages allows the unification of the construction of a statement and the mathematical proof of a theorem, the constant appeal, in mathematical activity, to natural language and to all sorts of other types of representations such as drawings or graphs, requires the distinction of codes and modes of control peculiar to them. [2]

iii) The different types of representations or theorems-in-action[42] which govern the subject's decisions are not very easy to identify, even when it seems that they can be formulated or made explicit by the subject. But many research projects are beginning to show how patterns of behaviour can provide an access to this type of "implicit model". The importance that they play in acquisition remains a largely open problem, very often tackled in too narrow a way. It is certain that these forms of knowledge work in neither a completely independent nor a completely integrated way in the control of the subject's interactions. The study of the relationships that become established between these types of control within the subject's activity and the rôle that they play in acquisition is a branch of psychology, essential for *didactique*—a study to which *didactique* claims, moreover, to contribute. [1]

6.2.3. *The evolution of these forms of knowledge: learning*

Knowledge evolves according to complex processes. To hope to explain this evolution solely by effective interactions with the *milieu* would certainly be an error, for children are able very early to internalize situations that interest them and "perform" very important mental experiences with their "internal" representations. In this way, they settle equally well problems of assimilation (expansion of schemes already acquired by the addition of new facts) or acconiodation (reorganization of schemes in order to understand new questions or resolve contradictions). But the internalization of these interactions doesn't change their nature very much; dialogue with an "inner" opponent is certainly less invigorating than a true dialogue, but it is nevertheless a dialogue. Do distinct forms of learning and evolution of knowledge exist in accordance with the types which we have just discussed? Does theoretical knowledge augment and restructure itself as languages and implicit models do? What rôles do the different forms of knowledge play in the various types of acquisition?

In mathematics, there exists a "conventional" mode of increasing knowledge by means of the game of defining new objects, and listing those of their properties which serve to ask new questions, which introduce definitions, and so on. Even if this "axiomatic and formal" mode cannot be retained as a global model, even by restricting oneself to knowledge, it cannot be rejected entirely, at least as a local model. It cannot, however, be extended to the learning of representation.

On the other hand, modes of learning of the stochastic kind, based on repetition, hardly appear adapted to complex knowledge (of high taxonomic level). Descriptions of the acquisition of language by young children (like those put forward by Alarcos Llorach, 1968, especially pp. 331–332) show that productions peculiar to the subject (like babbling) appear spontaneously, but almost like exercises. that they enter into relationships with the *milieu* and adjust there naturally or are corrected by quite diverse interventions. The steps of this acquisition can be understood only by the global study of relationships between the subject and the *milieu.* For example, passing from one-word phrases to sentences made up of several words is not mere concatenation.

A more precise discussion about distinguishing forms of acquisition specific to the different forms of knowledge mentioned above is outside the framework of this text. Essentially, it appears to contradict neither the epistemological models which Piaget linked to his theory of equilibration nor Bachelard's conceptions. On the contrary, we have shown in many precise studies that the classification of types of situation allows the clarification and extension of these theories[43].

We must emphasize the "dialectical" character of these processes; students' previous conceptions and the problems that are put to them by the *milieu* lead to new conceptions and new questions whose meaning is basically local.

One of the main arguments in favour of different modes of evolution for different modes of knowledge is the one we have advanced above, which relies on the history of mathematics and on epistemology: the evolution of protomathematical concepts, of paraniathematical concepts and of concepts already mathematized are different. Conversely, the study of the evolution of students' knowledge, provided that it is suitably supported by the analysis of the conditions supporting the situations, can cast an interesting light on historical processes and constitute a sort of experimental epistemology. This is what we have attempted to show in many experimental works[44].

6.2.4 *The sub-systems of the* milieu

The fact that different types of interaction with the *milieu* and different forms of knowledge are justified *a priori* and independently allows us to discuss the particularities of the *milieu* which are necessary for them.

By questions like "Why would the student do or say this rather than that?", "What must happen if she does it or doesn't do it?", "What meaning would the answer have if she had been given it?", it is possible to expose the more important conditions which these typologies impose on the *milieu.*

Here again, however, the categories are fairly obvious:

[3] Does the *milieu* include an opponent (or a proponent) with whom the subject must be confronted in order to attain the fixed goal in an exchange of opinions?

[2] Does the *milieu* include a receiver of messages that the student must send in order to attain the target goal?

The answer to these two questions determines the layout of the *milieu* and the rules of the games, which are totally different.

Let us examine it by emphasizing that, for the time being, the *milieu* in question is a real *milieu* and not an imagined or simulated one, and that the teacher is supposed not to intervene. It is clear in this case that the above conditions correspond to very important differences in organization of the class or the *milieu.* To find whom to speak to and what to act on are perhaps the student's main problems.

6.3. *First study of three types of adidactical situations*

6.3.1. *Action pattern*

Figure 1 (page 49) represents the general model of action without an interlocutor. It already provides a grid for reading a real teaching situation.

- Is the "partner" (the *milieu*) perceived as devoid of didactical intentions?
- At every move, must the student effectively chose a state from among several possible ones? Does she know which states she can select from?
- Can the student lose? Does she know that she can? Does she know the final state (the class of the final states) in advance; in particular, does she know the final winning state?
- Does she know the precise rules of the game without knowing a winning strategy ? Can she be taught the rules without being given a solution? (Has she to search for an optimum strategy?)
- Is the target knowledge necessary in order to pass from the basic strategy to a better (or optimum) strategy? Is it the principal means of this transfer?

– Can the student start again? Does the game "gratify" anticipation?
– Has the student any chance of finding out the sought strategy for herself if she borrows it (from another student)?
– Are the system's (adidactical) "answers" to the student's unfavourable choices nevertheless relevant to the construction of the knowledge (do they give indications specific to the error)?
– Is the control of decisions possible (is the student led to notice them, to observe them)?
– Is a reflective attitude useful, necessary for progess in the solution?

With the above points, we are approaching the conditions at the limit of a situation of action; consideration of the validity of a solution is classified as a situation of validation. Indeed, a student who reflects naturally on her game is in an effective adidactical situation of action, but she internalizes and in some way simulates a situation of validation. If the teacher goes on to want the student to have this reflective position, admonition ("Think! Look at what you have done...") will not suffice and the teacher will have to communicate her didactical wishes by means of a situation of validation.

The openness, which we initially represent by an uncertainty of the meaning, in the sense of information theory, is one of the more important conditions thus brought to light. The distinction between the didactical situation and the adidactical situation lets us design open teaching situations of mathematical thinking for the student because the thinking about the knowledge behind the situations has already been "done"[45]. The manipulation of this openness at the level of the whole class is a delicate technical problem, but within reach. For example, managing to keep each child's search Prom being overwhelmed by the work of another child is a didactical problem (and not a pedagogical one).

This reading grid can also assist in the design of new didactical situations. Each proposed game can be examined and compared with those which are already known. It is possible to fix engineering problems, to classify known settings, to regroup similar productions *from the point of view of this modelling,* and to predict new ones. The essential problem which remains in the experimental domain is that of the importance of the realization, or not, of the conditions thus proposed as coming logically or systematically from the possibilities of the model.

The variables that appear in this way have theoretical reasons for being relevant, and economical calculations of complexity or efficiency can specify those reasons[46]. Confrontation with actual occurrences or experimentation such as is practised in most research cannot be bypassed.

The model, from this point of view, has played its rôle for ten years. Its effectiveness is attested by the many original teaching situations that it continues to produce.

When the properties of a situation capable of justifying (or provoking) the implementation of a specific piece of knowledge are better known, it is possible to study the possibilities of the former for causing the latter to evolve. Didactical variables are those which influence learning and whose value the teacher can choose (Engineering—research on teaching numerical calculations). Numerous learning problems have been studied by using this model[47].

6.3.2. *Communication pattern*

The "milieu" includes a system of receiver and/or transmitter with which the player exchanges messages.

We shall assume here that the purpose of these messages is not to act on the receiver (to change it, take power from it, constrain it, etc.) but to act by means of its intervention on the "*milieu*" device.

The student is still engaged in a game with a *milieu* devoid of didactical intentions. If she simultaneously had the information and means of action sufficient to choose by herself the state of the "*milieu*", her messages, having no purpose in the game, could he anything at all. In addition, we assume for the moment that there is only one player, A. The receiver has nothing at stake but to be helpful. The player B makes no choices of her own (See Figure 5).

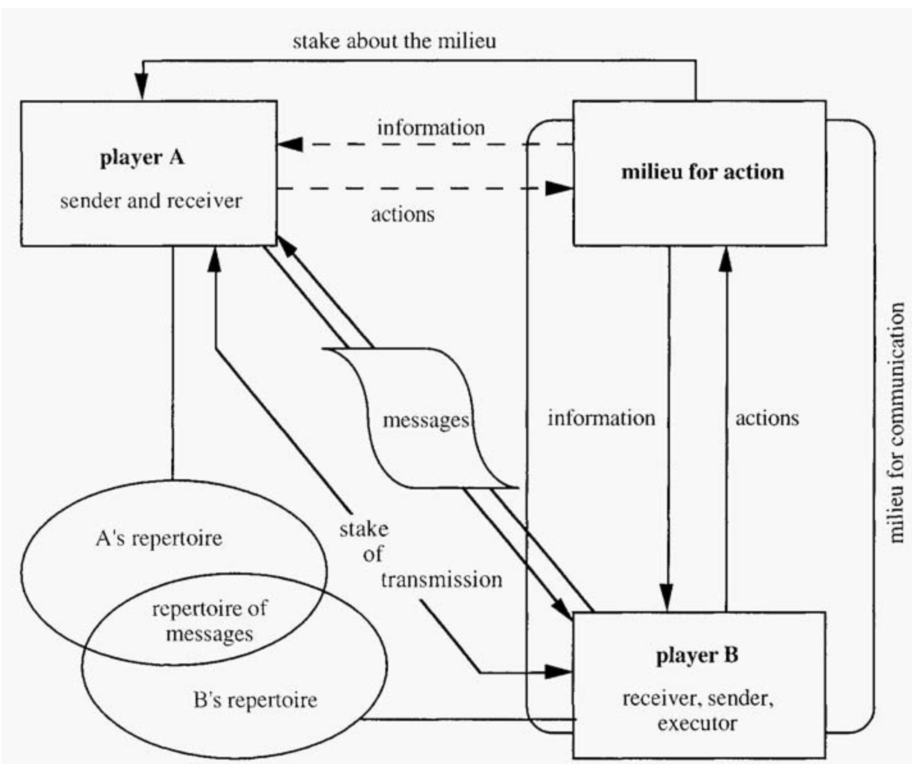


Figure 5

Different cases are obvious:

– insufficient means of action: A must describe to B the action which she had to carry out and often a part of the *milieu* as well so that the message is intelligible,
– insufficient information for A but sufficient means of action: it is B who must describe the *milieu* and A who must decode the description and direct the observation,

– means of action and information insufficient for A.

In the first case, the meaning of the message (the information content necessary for the game) that A sends to B must be represented by the passage (the ordered pair) from the choices of action that the game offers B to the one chosen by A.

In the second case, it is the passage from the choices envisaged by A before the contribution of information by B to those she envisages afterwards.

The fact that B is also a player can have some importance, because it reduces A's freedom and thus the meaning of her action. Of course, she must co-operate with A. The pattern does not vary whether the rôles are well identified or undifferentiated (but then one player may not leave any choice for the other to make, or any action).

The messages exchanged are under the control of linguistic, formal or graphical codes and therefore make them function[48].

The stake of the communication itself is expressed by retroactions which one or the other of the two speakers carries out in order to ascertain whether she has been understood. Their requirements will focus on the conformity to the code (minimal for the intelligibility of the message), ambiguity, redundancy, lack of relevance (superfluous information) and efficiency (the optimum character) of the message.

By judicious combination of a *milieu* (a game in Meaning 4) and suitable conditions for the exchange of the message (having to do with the medium, for example), it is possible to influence the type and meaning of messages obtained from the player. It is also possible to make the code itself evolve; to pass from a formulation in natural language to a formal statement, or from metaphors to systematic descriptions.

These results have been obtained in many research projects, but one must also be prepared for disappointments because of children's enormous capacity for semiological invention[49].

This pattern of adidactical situations offers some benefits in giving meaning to (or for analyzing the meaning of) a message, a formulation. Let us take an entirely theoretical and slightly provocative example: let us look for a (theoretical) social organization such as to provoke the transmission (say by a child of 5 or 6 years, to set the scene) of the formula "13 = 9+4" taken as information and not as an assertion. The sender, S, of the message addresses a receiver, R. If R knows the meaning of both "13" and "9+4", the message can perhaps offer her only information about "=", paradigm of a very restricted set: $\{=, \equiv\}$. The statement will be more informative if one of its terms is unknown to the receiver — for example, if the student is conventionally describing a calculation. (9 is known, as is 4; the result of adding them is 13.)

But if the first term is itself considered to be a message, the message must be sent by a sender, S1, to a receiver, R1, in order to inform her, for example, about the state of a stock. If "9+4" is a message sent by S2 to R2 in *another code* in order to inform her about the state of another stock (or the same one), then the complete message informs R1 that the two messages "13" and "9+4" name the same object. All that remains is to find a game that makes this functioning of six people plausible. Why would R need to know whether or not the two messages in "language 1" and "language 2" designate the same object?

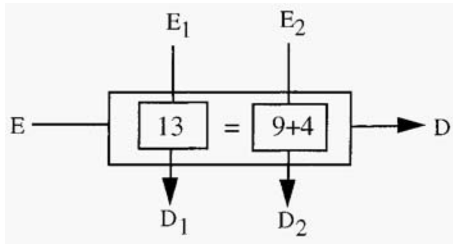


Figure 6

This type of reasoning doesn't always lead to a game which can be used in class (that is the case here), although it has been tried[50], but it is a very efficient way of analyzing the meaning of students' productions and of proposing means of control.

Variants can easily be envisaged as autocommunication that leads more easily, by memorization, to personal codes. Passing from the oral to the written, or from graph to verbalization, has often been observed and has given rise to the study of the influence of several variables[51].

The use of mathematical language in a precise way in deliberate communications between students is certainly one of the best pedagogical outcomes of this sort of situation. It is necessary to emphasize the importance

- of the quality of the game with the *milieu* in order to ensure and to maintain the relevance and the richness of students' discourse;
- of the frequency of use that it creates in communications;
- of the possibility of analyzing the messages produced.

6.3.3. *Explicit validation pattern*

Suitable situations of communication favour the appearance of messages which can have a form very close to that of a mathematical discourse and which are concretely significant for a certain "*milieu*". But these messages do not have the meaning of a mathematical text. Situations of validation involve two players who confront each other over an object of study composed, on the one hand, of messages and descriptions which the student has produced and, on the other hand, of the adidactical milieu which serves as a reference for these messages (see Figure 7). The two players are alternatively a "proponent" and an "opponent"; they exchange assertions and proofs about this ordered pair (*milieu,* messages). This ordered pair is the new device, the "*milieu*"—the game in Meaning 4—of situations of validation. It can appear as a problem accompanied by attempts at solution, as a situation and its model, or as a "reality" and its description.

Even though the informer and the informed have asymmetrical relationships with the game (one possesses information of which the other is ignorant), the proponent and the opponent must be in symmetrical positions, not only with regard to the available means of acting on the game and the messages, but also with regard to their reciprocal relationships, the means of mutually penalizing each other and the stakes as regards the *milieu*-message pair (see Figure 7).

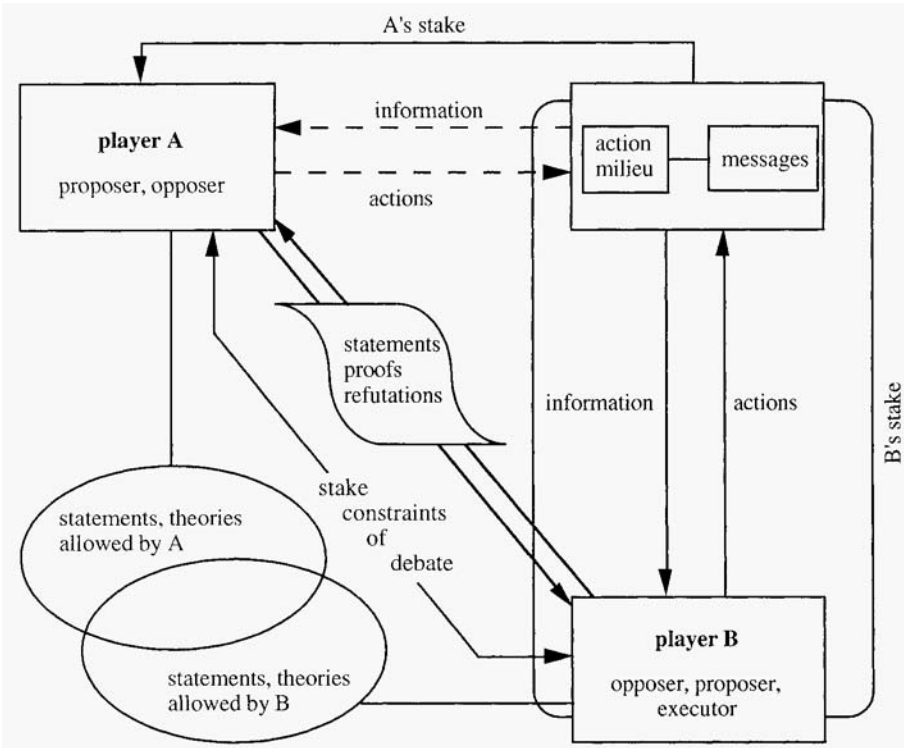


Figure 7

In particular, it should not be possible for one of the players to obtain the agreement of the other by "illegitimate" means such as authority, seduction, force, etc.

The challenge of *didactique* is to produce situations which allow students to use mathematical knowledge as an effective means of convincing someone else (and thus of convincing themselves) while leading them to reject rhetorical means that are not good proofs or refutations.

The exact meaning of mathematical productions is determined by this range of choice. What a theorem says is also what it contradicts; what a mathematical proof says is not only what is assumed to be acknowledged by the proposer and the opposer, but also what might have been contested. Mathematical discourse is constructed partly *against* other processes of acquiring beliefs and knowledge, and not only *with* them.

Let us specify the game of proof.

"A" makes a statement: this is the declaration of a property, knowledge of which is useful for mastering relationships (by A or B) with the *milieu;* the fundamental stake is always to "win" a certain "game". If "B" wants to take possession and avail herself of this property, she must either ''pay'' A each time (this is a recognition of pragmatic validity) or "pay" A once and for all by accepting the proposition to be true.

But she can also engage in the process of refutation if she thinks that A's statement is false.

Pragmatic refutation: she can "make" A play an unfavourable move required by her statement. If the statement is false and if the situation is correct, the move must be a losing one (this is not so easy to obtain). This is the device that makes the counter-example work. B can force A to make her losing move as often as it is her turn to move until A retracts her assertion.

Intellectual refutation: B can also offer A a bargain: she can offer an explicit refutation of her initial statement, which can save her the cost of an expensive obstinacy. If A accepts (she pays a little *a priori*), for example, by giving up her turn to move, B then becomes the proposer. A accepts the refutation, or rejects it. The arguments are always ones that the opposer can receive—just as in every communication the repertoire must be within that of the receiver so that the message may be understood. Thus, a theory is created *"de facto*", as a set of observations accepted by both players. At any moment, it can happen that one of them discovers that a piece of knowledge is not shared although she believed that it was available to her partner. Thus, all assertions of the "theory" are susceptible to being viewed explicitly and to being brought back into question. The theory itself is an object of study and construction.

The technical means of these mathematical proofs and the arguments can be of a semantic order—appropriateness to the *milieu,* to the problem—or of a syntactical order at many levels: mathematical articulation and validity, logical and even formal constitution of the argument; but also of an epistemological nature, as shown by Lakatos (1976).

The use of situations of proof restores a socio-cultural environment which gives body to mathematical discourse. Situations of validation can help the teacher make a genuine small mathematical society take life in the classroom. But by restoring meaning one takes the risk of enhancing the difficulties of those who will not enter into the game, and for whom theorems and even mathematical proofs are only knowledge like any other, to be exhibited occasionally. A situation of validation is not *a priori* the best situation for learning institutionalized knowledge. It can even trigger didactical obstacles and resurrect disturbing epistemological obstacles. It is nevertheless essential as a paradigm for other mathematical situations. It is unfortunate, for example, that the geometry taught in middle schools doesn't use these processes, although it could be justified as the first example of axiomatic thought in relation to a field naturally mastered by other means. Here, ready-made mathematicization is contradictory to the stated didactical objective.

Besides their epistemological or didactical properties, these social situations can present worthwhile advantages in the domain of student motivation—motivation that is quite often transferable. This motivation can manifest itself not only in the case where the situation is genuinely organized and really lived, but also when it is simulated, retold or internalized.

Students co-operate to whatever extent they come to share the same desire for acquiring a truth. They must receive with respect, *a priori,* their opponent's point of view and defend their own without false modesty as long as they are not convinced of the contrary; but if it seems to them that they are wrong, they must learn to change position immediately, without displaced self-respect and at whatever social price.

These situations show the profound anchoring of mathematical activity in rational thought and the educational importance at stake which goes beyond the simple domain of the learning of knowledge. In concrete situations which come within this model, when they are useable, the rather formal character of this "competition" disappears. The constraints are actually present, but not too sharply apparent.

NOTES

1. A *problématique* is a set of research questions related to a specific theoretical framework. It refers to the criteria we use to assert that these research questions are to be considered and to the way we formulate them.
2. *Editors' note:* The term "noosphere " was coined in the twenties by Teilhard de Chardin. Simply stated, it refers to the world of ideas, of productions of the mind. In the late seventies, Chevallard introduced it "as a parody" in *didactique* in the context of the theory of didactical transposition, with the purpose of designating the sphere of people who think about didactical functioning (Chevallard 1985 p. 25; Arsac 1992). For a presentation of the recent development of the theory of didactical transposition, see (Chevallard 1992).
3. *Editors' note:* see Introduction, footnote 1.
4. *Editors' note:* In order to account for the French distinction, frequently made by Brousseau, between "*connaissance*" et *"savoir*", we translate these respectively as "a knowing" and "knowledge". The former refers to individual intellectual cognitive constructs, more often than not unconscious; the latter refers to socially shared and recognised cognitive constructs, which must be made explicit.
5. *Editors' note:* In the original story, Mr Topaze is giving a dictation to a pupil, and from time to time he bends over the shoulder of the little boy to see what he is writing. Seeing a mistake, he tries to obtain the correct spelling: *"Des moutons… moutonss. Voyons, mon enfant, fait un effort. Je dis moutonsse*" (Pagnol, *Topaze,* Scene 1, Act 1). The relevance of this metaphor is due to the fact that in French the "s" of "moutons", which is the plural of "mouton" (lamb), is a silent letter. In order to render this effect for the English reader, we have decided to offer a free translation by Virginia Warfield which substitutes the issue of spelling the word "two" which children often confuse with the spelling of "to" or "too".
6. *Editors' note:* For more references about didactical engineering see (Artigue 1992).
7. *Editors' note:* Molière, *The Bourgeois Gentleman.* Translation by Albert Bermel (1987) NY: Applause. pp. 51–63.
8. *Editors' note:* By this neologism we translate the French expression "chercheurs *en Didactique".*
9. Euler Diagram, Venn Diagram, Papy's *"patates". Editors' note:* Georges Papy used orally the word *patates* (potatoes), but he used to write the word *figure* (figure, in: Papy G. 1964). The word *patates* was adopted by children and teachers. Later Frederic Papy introduced the more precise and elegant term *papygramme* (in: Papy F. 1970).

10. Euler (1768) *Lettres à une princesse d'Allemagne sur divers sujets de physique et de philosophie.* Especially, see Letter XXXIV of February the 14th 1761, and figures 39 to 89.
11. *Editors' note:* A play on words is always difficult to render, especially those of Devos. Nevertheless the following story may give the reader an idea of the metaphor: "Go to a Discount Fabric Store and get a mill end. Now cut the mill end in two. Then you still have an end at each end of the end. Do you follow that? I have spent no end of time explaining it!". The interested reader may consult Devos (1976) pp. 149–151.
12. *Editors' note:* The Ecole Jules Michelet in Talence, near Bordeaux, is a school where Guy Brousseau has organised facilities for observing classroom situations for the purpose of research. (See also the Appendix.)
13. *Editors' note:* In France a curriculum is described by an official programme which presents its content, and programme commentaries which give more details about what is meant and about the ways to fulfil the official expectations. Text books, published by several private publishers, provide many examples of the way in which the curriculum can be understood.
14. The research object of Chevallard and Mercier. *Editors' note:* see for example Chevallard and Mercier (1983) or Mercier (1992), and Brousseau and Centeno (1991).
15. This question is studied in my article, Didactique of decimals: the obsolescence of situations, and is taken up by Michèle Artigue in her thesis on reproducibility. *Editors' note:* See Artigue (1986).
16. *Editors' note:* Brousseau's theoretization is not confined solely to school systems, as it is in this article. Readers could refer to Ressot *et al.* (1993) or Rogalski and Samurcay (1994) for examples about the use of the Theory of Didactical Situations in the case of vocational teaching or professional training.
17. In the sense that the intention to teach leaves it (it is always specific to the knowledge). A non-specific pedagogical situation of an item of knowledge would be called non-didactical. instead of adidactical.
18. *Editors' note:* "The Battle" and "Little Horses" are gambling games whose outcome is determined by chance.
19. *Editors' note:* Diénès (1970).
20. *Editors' note:* Glaeser (1984–1985) p. 151.
21. That is to say, such that the child doesn't know *a priori* how to answer it, but will be able to do so when she has a solution and will know whether it is correct without recourse to the teacher.
22. cf. "Engineering and *didactique"* in Brousseau (1982a). *Editors' note:* On this topic see also Artigue (1992).
23. *Editors' note:* in English in the text.
24. *Editors' note:* in English in the text.
25. I have used these definitions often, especially in the study of measurement strategies with H. Ratsimba-Rajohn (1981). A much more detailed version together with several interesting examples may be found in his thesis, as well as in the thesis of A. Bessot and F. Richard (1979).
26. *Editors' note:* in German in the original text, these words mean respectively "gone" and "there".
27. *Editors' note:* According to Brousseau (personal communication) the original contribution of Lacan on this classical problem is that of showing that the question of meaning is always open, that any symbol is unable to answer the questions which are at the source

of its existence. Concerning the “Fort-Da” game , see especially (Brousseau and Otte, 1991, pp. 32–35).

28. *Editors' note:* Brousseau is recalling here a comment of André Deledicq at an international conference in Saigon in 1975.
29. *Editors' note:* Here Brousseau refers to section 6, and the reader will find an original fomulation of it in Brousseau (1970). The empirical work Brousseau refers to is in particular illustrated by Chapter 4.
30. *Editor' note:* Osgood *et al.* (1957).
31. *Editor' note:* Bramaud du Boucheron *et al.* (1970).
32. Brousseau and Gabinski (1976) gives a fairly complete presentation of this question and an experiment (Maysonnave 1972) brings an experimental verification that consists of comparing the speed of appearence of certain theorems (in action) among children ten years old with the predictions of a stochastic model of teaching. It fits reasonably well in the case of a simple situation (race to 7). The children are even faster than the model predicts, whatever the values of its control parameters. *Editors' note:* see: “The race to 20”, Introduction to this book. For the references given by Brousseau, see Suppes (1969, 1976), Nelson (1969) and Arbib (1976, 1976a).
33. *Editors' note:* See, for example, Rouchier (1991), especially the section *L'institutionnalisution des savoirs dans l 'enseignement des mathématiques (ibid.* pp. 26–65); see also Margolinas (1 993).
34. *Editors' note:* Douady (1984, 1985, 1986).
35. *Editors' note:* Chevallard (1985) pp, 49–56.
36. *Editors' note:* this book, Chapter 2, §1.3; Chapter 4, §1.2.
37. These numerals all refer to the same type of categories: [1] Action, [2] Formulation , [3] Validation.
38. *Editors' note:* see this book, Chapter 4.
39. *Editors' note:* This was written by Brousseau in 1984. It refers to his first presentation of the theory of didactical situations (Brousseau 1970).
40. *Editors' note:* Skemp (1979).
41. *Editors' note:* Wermus (1978).
42. *Editors' note:* the concept of *theorem-in-action* was introduced by Vergnaud; a detailed presentation of the theory of Conceptual Fields can be found in (Vergnaud 1990).
43. *Editors' note:* see in this book Chapter 4; Chapter 5, sect.3.
44. G. Brousseau, *Problèmes de didactique des décimaux. Editors' note:* see in this book Chapter 4.
45. *Editor's note:* Brousseau adds the following clarification: “The point at issue here is best described by the following metaphor: If you release a marble at the edge of a bowl you know that it will end up at the bottom of the bowl, but you cannot predict, much less pre-determine, its trajectory—it might spiral downwards or rock back and forth, or... Thus a situation can be simultaneously closed for the teacher who knows where the marble will end up, and open for the student whose trajectory is all her own.” (Personal communication, September 1995.)
46. *Editors' note:* see in particular Ratsimba-Rajohn (1982).
47. *Editors' note:* Here Brousseau refers to works in a special section of the bibliography of his article. The reader will find them in the general bibliography of this book: Balacheff (1982), Berthelot and Berthelot (1983), Bessot and Richard (1979), Brousseau and Brousseau (1987), El Bouazzaoui (1982), Galvez (1985), Katambera (1986), Maudet (1982), Mopondi (1986), Perez (1984), Quevedo de Villegas (1986), Ratsimba-Rajohn (1981), Rouchier (1980).

48. *Editors' note:* The situations described in this section are the *situations of communication,* the second fundamental type of Brousseau's situations (Brousseau 1970). Brousseau did not introduce this expression in this section, but he mentions it incidentally at the beginning of the following section.
49. *Editors' note:* see in particular the work of Laborde (1982).
50. Editors' note: in particular, such situations were implemented during the year 1971 in classes of the School associated with the *Ecole Normale de Caudérun* (Normal School of Gironde).
51. cf *La création d'un code à l'Ecole Maternelle* and *L'enseignement de la géométrie. Editors' note:* the first indication Brousseau gives refers to Perez (1984), the second indication refers to a theme of research well exemplified by Brousseau (1983a, 1987).

CHAPTER 2 PRELUDE

The idea of the existence of obstacles to the learning of mathematical concepts is touched upon in the previous chapter, in the section 4.2. Brousseau discusses there, among the "Paradoxes of the adaptation of situations", the possibility of student knowings becoming obstacles to further learning. He suggests that some of these obstacles are "inevitable and constitutive of knowledge". This is one of the key constituants of the *problématique* of the Theory of Didactical Situations.

In 1976, in a communication then published in the proceedings of the 28th meeting of CIEAEM held in Louvain la Neuve, entitled "*La problématique et l'enseignement des mathématiques*", Brousseau introduced the concept of epistemological obstacle which he borrowed from the French epistemologist Gaston Bachelard. As Perrin-Glorian (1994 p.112) analyses it: "The notion of obstacle appears very quickly as a necessity of the theory because, as results of learning by adaptation, knowings constructed by students are more often than not local and linked to other knowings in a 'contingent and unjustified' way. They are also temporary and incorrect."[1]

This recognition of the existence of obstacles intrinsically related to the nature of knowledge raises the question of the characteristics of situations allowing them to be overcome. This is why Brousseau claimed in the abstract accompanying the re-print of this fundamental paper in *Recherches en didactique des mathématiques* in 1983: "The identification and characterization of an obstacle are essential to the analysis and construction of didactical situations". The problématique of epistemological obstacles puts mathematics at the core of the Theory of Didactical Situations and explains the privileged links between this field of research and mathematics.

We have put together in the following chapter two texts in which Brousseau presents the concept of epistemological obstacles and analyses its relation to the Theory of Didactical Situations:

- The first one is the fundamental presentation of epistemological obstacles as it appeared in 1976. It is followed by a response of Brousseau to Georges Glaeser[2]. We reproduce this reponse here because of the clarification it provides about the meaning of this concept and its relation to the common sense idea of "difficulty" with which it is sometimes confused.
- The second paper provides some more examples and formulates the more accurate definition by Duroux which is only evoked in the preceding one. This paper introduces some questions concerning the didactical consequences of the recognition of the existence of epistemological obstacles.

The Editors.

CHAPTER 2

EPISTEMOLOGICAL OBSTACLES, PROBLEMS, AND DIDACTICAL ENGINEERING

1. EPISTEMOLOGICAL OBSTACLES AND PROBLEMS IN MATHEMATICS*

1.1. *The notion of problem*

1.1.1. *Classical conception of the notion of problem*

A student isn't really doing mathematics unless she is asking herself questions and solving problems. Everybody agrees about that. The difficulties start when questions arise about knowing what problems must be asked, who asks them, and how they are asked.

In order to simplify these difficulties, it appears that didacticians of mathematics have attempted, for some time, to project the set of imaginable problems on to a sub-space defined by the five following components.

Teacher's methodological intentions
This is the component described at the beginning of the *Livre du problème* by Glaeser and his colleagues: problems of research, problems of training, problems of introduction, etc. (IREM de Strasbourg 1973)

Didactical intentions and objectives
Examples include Bloom's objectives: acquisition of knowledge, better understanding, analysis, and so on. (Bloom 1975).

Mathematical content
Almost always, the question consists of asking the student to establish a true formula in a theory currently being studied. The content of a problem is thus *a priori* definable as an ordered pair (T, f), T being a theory supposedly made explicit during the course, and f the formula to be found, established or located in a mathematical proof within T.

This conception first of all allows certain problems to be placed in relationship to others, in a partially ordered structure (a lattice), provided that we have an axiomatic system appropriate to the theory being taught; discussions about the choice of the best axiomatic system underlie most of the research that has been done on curricula for years[3]. The "best axiomatic system" would be the one which,

---

* Rrousseau, G. (1 983) Les obstacles épistémologiques et les problemes en mathematiques. *Recherches en didactique des mathématiques,* 4/2, 165–198.

with the least effort of learning or teaching, would allow the generation of the set of theorem-problems, of assessment or evaluation, fixed by a social consensus.

Should we provide for several specific theories, to be connected later ("classical" tendency), or for a single unifying general theory from which others are to be derived ("modern" tendency)?

Does one need many weak axioms, well organized, (Dieudonné 1964) or a few powerful axioms (Choquet 1964)? "Obvious" axioms or "highly elaborated" axioms?

In the absence of a suitable theory of knowledge accompanying a relevant learning theory, these discussions have never resulted in experimental scientific studies.

This conception furthermore allows one to distinguish, on the one hand, the ordered pair (T, f) that characterizes the problem and, on the other hand, the mathematical proof of $T \vdash f$, which can be the object of a mathematical or metamathematical study. And this distinction will serve as the basis for a new decomposition of mathematical content, following two different but closely related criteria:

- the application domain (the theory T) as opposed to the mathematical or logical "structure" operating on T;
- the mathematical model (in the sense of mathematical logic), as opposed to the language.

These pairs of opposed characteristics correspond to distinctive features which teachers use spontaneously: abstract–concrete, formal–semantic, theoretical–practical, etc., but their use has never resulted in either useful typologies or objective indices.

Mathematical component

Actually, all attempts at rational, formal description of mathematics are used to try to build intermediate variables, which, without constituting the content itself, will allow it to be generated at low cost.

The conception of problems in the form $T \vdash f$ often leads to the likening of the hypotheses to what is known, the conclusions to what is sought (or vice-versa), and the problem-solving process to a progression that will coincide easily with the mathematical proof being sought.

Some mathematical proofs can be obtained with very little effort by the application of a finite set of specifications known in advance: it is then a matter of an algorithm, an automaton which produces the particular mathematical proof being sought.

In this case, we can construct a description, classical and beautifully simple and gratifying for the teacher, of the student's cognitive activity, of the learning and of the rôle of the teacher: the teacher teaches the algorithm which allows the student to establish the theorems and the student memorizes it.

Heuristic component

But for other mathematical proofs, such algorithms don't exist. In order not to lose the above acquisition model, we can imagine that mathematical proof is driven by "intuitions" which to some extent play the rôle of algorithms. When the implementation of an already constituted theory will provide the sought-after mathematical proof or part of it (a theorem can be applied), it is possible to rationalize these intuitions locally, the choice of theories or structures being itself guided by the heuristics which we can afterwards invoke to justify the method used. Despite their rather ad hoc character, these concepts are not without interest, as shown by other papers (Glaeser 1976, Paquette 1976, Ciosek 1976, Wilson 1976, and Janvier, 1976)[4].

1.1.2. *Critique of these conceptions*

The validity of such a classificatory decomposition is questionable. Regardless of the facilities that it provides, it leads to the acceptance of doubtful presuppositions by separating elements which function together.

The subject

The subject—the student—is absent from this conception, where she appears only as a receiver, an extremely simplified recorder in whom the acquired knowledge produces no appreciable modifications, especially not structured ones.

Sense and meaning

By the same token (and in consequence) the *meaning* of the mathematics disappears; that which produces not only truth, but also interest in a theorem (what Gonseth (1936) called the *idoine* character of a piece of mathematical knowledge); that which ensures that this knowledge exists as an optimal solution in the field defined by a certain setup of constraints relative to the subject and/or the knowledge itself (an object in Thom's sense: a solution to a problem (Thom 1972)); that which reveals the interest of the problem itself; etc.

The meaning of a piece of mathematical knowledge is defined, not only by the set of situations in which this knowledge is realized as a mathematical theory (semantic in Carnap's sense), not only by the set of situations in which the subject has come across it as a means of solving a problem, but also by the set of conceptions, of previous choices which it rejects, of errors which it avoids, the economies it procures, the formulations that it re-uses, etc.

Learning

Axiomatic construction suggests an enchanted learning in which the volume of knowledge—immediately obtained, structured, useable and transferable—swells in an empty space. However ...

- A learned notion is usable only to the extent to which it is connected to others, these linkages constituting its meaning, its label, its method of activation.
- But it is *learned* only to the extent to which it is usable and effectively used, that is to say: only if it is the solution to a problem. These problems, a set of constraints to which the notion responds, constitute its meaning. It is learned only if it is "successful" and it must therefore have a territory in which it can bc put into practice. This territory is only rarely general and definitive.

- Because of this localized use, the notion receives certain particularizations, limitations, deformations of language and meaning; if it succeeds well enough and long enough, it takes on a value, a consistency, a meaning, a development that make its modification, re-use, generalization and rejection more and more difficult. For later acquisitions, it becomes both an obstacle and a support.

All this demonstrates:

- why learning cannot be achieved by means of the classical scheme of continuous and progressive acquisition (such that for any acquisition there exists a finite set of acquisitions which are equivalent to it, each of them providing a quantity of information as small as desired);

and, consequently:

- why the confusion between the algorithm for the establishment of a formula and the algorithm for the acquisition of a piece of knowledge is devoid of a basis.

Algorithm and reasoning
Several examples demonstrate all the disastrous consequences of this confusion for the learning of operations on the natural numbers.

By teaching, by the same procedures and to the same age-group, both a sophisticated theory—that of probability and statistics—and the so-called operation "mechanisms", it has been possible to show that this separation of mechanisms and reasoning is neither necessary nor useful; learning takes place by the trying out of successive, temporary and relatively good conceptions, which must be successively rejected or given an entirely new genesis each time.

If the conditions require it, the student could herself sum up complex activities as "automatisms", drawing from them meaning and possibilities of choice for her activities. But for these automatisms to be used, they must be put into place by the subject herself.

Obstacles
Those works that refer to Bachelard (1938) and Piaget (1975) also show that errors and failures do not have the simplified rôle that we would like them to play. Errors are not only the effect of ignorance, of uncertainty, of chance, as espoused by empirist or behaviourist learning theories, but the effect of a previous piece of knowledge which was interesting and successful, but which now is revealed as false or simply unadapted. Errors of this type are not erratic and unexpected, they constitute obstacles. As much in the teacher's functioning as in that of the student, the error is a component of the meaning of the acquired piece of knowledge.

1.1.3. *Importance of the notion of obstacle in teaching by means of problems*

Interactions
We assume, then, that the construction of meaning, as we understand it, implies a constant interaction between the student and problem-situations, a dialectical inter-

action (because the subject anticipates and directs her actions) in which she engages her previous knowings, submits them to revision, modifies them, completes them or rejects them to form new conceptions. The main object of *didactique* is precisely to study the conditions that the situations or the problems put to the student must fulfil in order to foster the appearance, the working and the rejection of these successive conceptions.

We can deduce from this discontinuous means of acquisition that the informational character of these situations must itself also change in jumps.

Conditions

Under these conditions, the didactical interest of a problem will depend in an essential way on what the student will engage in, what she will put to the test, what she will invest. It will depend on the importance for her of the rejections that she will be led to make, and of the forseeable consequences of those rejections, and on how often she would risk committing these rejected errors and on their importance.

Thus, the most interesting problems will be those which will permit the overcoming of a real obstacle. This is why, in connection with problems, I wanted to examine the question of obstacles in *didactique.*

1.2. *The notion of obstacle*

1.2.1. *Epistemological obstacles*

The mechanism of acquisition of knowledge, as we described it above, could be applied just as well to epistemology or to the history of science as to learning or teaching. In both cases, the notion of obstacle appears fundamental to the consideration of the problem of scientific knowledge. According to Bachelard (1938), who was the initiator of this idea:

> "It is not a question of considering external obstacles like the complexity or the transient nature of phenomena, nor of implicating weakness of the human senses and the human mind; it is in the very act of intimately knowing that there appear by a sort of functional necessity sluggishness and troubles... we know against a previous knowing" *(ibid.* p. 13).

Bachelard studies obstacles in physics and identifies the following: the obstacle of first experience; the obstacle of general knowledge; verbal obstacles; the obstacle of improper use of familiar images; the obstacle of unitary and pragmatic knowledge; the substantialist, realist, animist obstacle; the obstacle of quantitative knowledge.

These obstacles have survived for a long time. They probably have their equivalent in the child's thinking, even though the current material and cultural environment has without doubt somewhat modified the conditions in which they are met[5].

In mathematics, very important epistemological work has been undertaken by Althusser (1967), Badiou (1972), Houzel *et al.* (1976), etc., in conditions similar to those of Bachelard.

This has not yet led to a list similar to Bachelard's, but important features are becoming clearer, as well as classes of obstacles. The notion of obstacle itself is being formed and diversified. It is not easy to put forward pertinent generality on this subject, it is better to do these studies case by case. Alongside the work of recording and describing significant obstacles to the *constitution* of concepts, studies are developing that bear on the characteristics of the functioning of knowledge, *simultaneously as support and as obstacle* (alternatively or dialectically).

In addition, the notion of obstacle tends to extend outside the strict field of epistemology: in *didactique,* in psychology, in psychophysiology, etc.

1.2.2. *Manifestation of obstacles in didactique of mathematics*

Errors

An obstacle is thus made apparent by errors, but these errors are not due to chance. Fleeting, erratic, they are reproduceable, persistent. Also, errors made by the same subject are interconnected by a common source: a way of knowing, a characteristic conception, coherent if not correct, an ancient "knowing" that has been successful throughout an action-domain. These errors are not necessarily explainable.

What happens is that they do not completely disappear all at once; they resist, they persist, then they reappear, and manifest themselves long after the subject has rejected the defective model from her conscious cognitive system.

Example: A student uses the following "theorem": "If the general term of a series tends towards zero, the series converges". Is she distracted'? Is she reciting incorrectly—interchanging hypothesis and conclusion—a theorem from the course? Has she misunderstood the notion of limit? Or that of series? Is it an error about necessary and sufficient conditions?

In looking at this error along with some others, we understand that in an unconscious manner this student carried out a certain reasoning, distorted by an incorrect representation of real numbers that goes back to teaching at the primary and middle school levels. The reasoning is something like the following: "If x_i tends towards zero, there exists a term n beyond which the x_i are negligible. Beyond this n, practically nothing more is added, and therefore the series converges". Maybe this student would not write this reasoning down without realizing that it is wrong, but for all that, it does seem obvious to her, because she depends on certain practices which were constant in the primary and secondary teaching; only "reasonably long" numbers are written explicitly, that is to say, decimal numbers:

$$d = \sum_{i=n}^{m} a_i \times 10^i, \text{ such that } m \text{ and } n < 10 \text{ (more often than not } n \in \{2,3\}).$$

Other numbers are designated by letters or represented—for practical reasons—by a nearby decimal which is presented as the approximate decimal or as even *the* number.

Example: $\pi = 3.14$.

If questions of incommensurability are raised, they are explained as being provocative or paradoxical and, in the end, gratuitous: example, "Does $1 = 0.9999...$?" And among the proofs put forward (generally reasoning by recurrence), the students only accept those in which the number of decimal places stays reasonably small[6].

Everything reinforces the idea that we use only a *discrete* set of numbers, and the false idea that there exists an $n \in \mathbf{N}$, such that $\forall x \in \mathbf{R}$, there exists $d \in \mathbf{D}$ such that $[|x - d| < 1/10^n \Rightarrow x = d]$ (that is: x is "practically replaceable" by d, x – d is zero...).

Does this idea result from a "wrong" definition of decimals carried since the elementary school? We shall come back to this question later on.

Overcoming an obstacle

The obstacle is of the same nature as knowledge, with objects, relationships, methods of understanding, predictions, with evidence, forgotten consequences, unexpected ramifications, etc. It will resist being rejected and, as it must, it will try to adapt itself locally, to modify itself at the least cost, to optimize itself in a reduced field, following a well known process of accomodation.

This is why there must be a sufficient flow of new situations which it cannot assimilate, which will destabilize it, make it ineffective, useless, wrong; which necessitate reconsidering it or rejecting it, forgetting it, cutting it up—up until its final manifestation.

Furthermore, the overcoming of an obstacle demands work of the same kind as applying knowledge, that is to say: repeated interaction, dialectics between the student and the object of her knowledge.

This remark is fundamental to the determination of what is a true problem: it is a situation that allows and motivates this dialectic.

lnformational characteristics of an obstacle

A piece of knowledge, like an obstacle, is always the fruit of an interaction between the student and her surroundings and more precisely between the student and a situation which makes this knowing "of interest". In particular, it stays "optimal" in a certain domain defined by the numerical "informational" characteristics of the situation.

For example, the solution of linear systems by substitution, which is useful when there are only two equations, becomes materially impractical for a sufficiently large number of them.

Knowledge, people and their *milieu* being what they are, it is inevitable that these interactions lead to conceptions which are "erroneous" (or correct locally but not generally). However, these conceptions are controlled by conditions of the interaction that can be more or less modified. It is the object of *didactique* to understand these conditions and to use them.

This observation has important consequences, especially for teaching: if we wish to destabilize an embedded notion. it is advantageous for the student to invest her conceptions sufficiently in situations

- that are numerous and important for her,
- and above all, with sufficiently different informational conditions for a qualitative jump to be necessary.

Example: A six-year-old child knows how to distinguish numbers up to 4 or 5 with the help of procedures based on perception. These procedures very quickly become costly and less reliable as soon as the number of objects reaches 6 or 7. They fail beyond this number. If we try to teach the numbers 6, 7 and 8 in this order, we come up against numerous and increasing difficulties and a period of disarray appears.

On the other hand, if we propose the comparison of sets of from 10 to 15 objects, the perceptive model is so obviously disadvantageous that the child rejects it straight away and comes up with new strategies (term by term correspondence). What we could call intuition is often unconscious understanding of the informational limitations of knowledge modes.

1 .2.3. *Origin of various didactical obstacles*

Origin of an obstacle

We shall now consider obstacles which come up in the didactical system. These obstacles to the student's appropriation of certain notions could be due to several causes. It is difficult to put the blame on only one of the systems of interaction. That is another consequence of the conception of learning described above. Thus, the notion of epistemological obstacle tends, in certain cases, to take the place of that of error of teaching, or of insufficiency of the subject, or of intrinsic difficulty of the knowledge.

However, we can try to distinguish various origins by looking into the subsystem (of the teacher–student–knowledge system), where its modification could overcome the obstacle, even when no modification of other systems would allow it to be avoided.

We shall thus find didactical obstacles:

– of ontogenic origin,
– of didactical origin and
– of epistemological origin.

For the above example (concerning the acquisition of the notion of number) we shall talk rather of a neurophysiological limitation than of an obstacle.

Obstacles of ontogenic origin

Obstacles with an ontogenic origin are those which arise because of the student's limitations (neurophysiological ones among others) at the time of her development. She develops knowings appropriate to her abilities and goals at a particular age. Genetic epistemology provides evidence of stages and means of development (accomodation and assimilation), which at the same time resemble the stages of development of concepts in the rules ofregulations which cause them to appear, and differ from them in the exact nature of the limitations which determine these regulations.

Obstacles of didactical origin

Obstacles of didactical origin are those which seem to depend only on a choice or a project within an educational system.

For example, the current presentation of decimals at the elementary level[7] is the result of a long evolution within the frame of a didactical choice first made by the encyclopaedists and then by the *Convention*[8] (following a conception which goes back to Stevin himself). Because of their utility, decimal numbers were to be taught to everyone as soon as possible, associated with a system of measurement, and related to technical operations with whole numbers. As a result, for students today decimal numbers are "whole numbers with a change of units" and therefore "natural" numbers (with a decimal point) and measures. And this conception, supported by a mechanization by the student, will, right up to university level, be an obstacle to the proper understanding of real numbers, as we mentioned previously[9].

It is characteristic that the principal factor in discriminating among students in a recent questionnaire developed by *IREM de Rouen* is calculation involving decimal numbers and multiplication by a power of ten at the same time. Thus, it is the "understanding" of the very definition of decimal numbers which explains students' behaviour. Nowadays, such obstacles have become both didactical and socio-cultural.

Obstacles of epistemological origin

Obstacles of really epistemological origin are those from which one neither can nor should escape, because of their formative rôle in the knowledge being sought. They can he found in the history of the concepts themselves. This doesn't mean that we must amplify their effect or reproduce in the school context the historical conditions under which they were vanquished.

1.2.4. *Consequences for the organization of problem-situations*

The conception of learning which relies on the study of the development of knowledge in terms of obstacles differs appreciably from the classical conception, especially concerning the rôle and organization of problem-situations. And this, even more than the problem, will play a fundamental rôle in the process.

Motivations—conditions

The posing of a problem consists of finding a situation with which the student will undertake a sequence of exchanges concerning a question which creates an "obstacle" for her, and from which she will derive support for her acquisition or construction of a new piece of knowledge.

The conditions under which this sequence of exchanges is displayed are initially chosen by the teacher, but the process must quickly move into the partial control of the subject, who will, in her turn, "question" the situation. Motivation is generated by this investment and maintains itself by it. Instead of being a simple external motor, in balancing frustration, it builds up both the subject (her word) and her knowledge.

Thus the resolution of a problem will be for the student a kind of experimental path, the opportunity given to "Nature" (here to mathematical concepts) to manifest itself during the student's activities.

Dialectical character of the process of overcoming an obstacle
The process of overcoming an obstacle necessarily includes a series of interactions between the student and her *milieu.* This series of interactions makes sense only to the extent to which they relate to *the same project* (for the student) *with respect to a concept* in the genesis of which they form a stage and on which their meaning is based.

These interactions bring systems of representation into play for the student, and they can often be interpreted as exchanges of messages, even with something as apparently "amorphous" as a problem, because the student is capable of anticipation and can give direction to her actions. Consequently these interactions take on a dialogic character (*a fortiori* when the teacher is involved in it). Moreover, this "exchanged" information is received as facts confirming or denying hypotheses, or even as assertions. If one assumes that a piece of knowledge establishes itself by opposing another one while relying on and replacing it, one will understand that it could be said that the process of overcoming an obstacle has a dialectical character: dialectics of *a priori* and *a posteriori,* of knowledge and action, of self and others, etc.

Organizing the overcoming of an obstacle will consist of offering a situation which is likely to evolve and to make the student evolve according to a suitable dialectic. It will be a question not of communicating a piece information that we wish to teach but of finding a situation in which it is the only satisfactory or optimal one—among those with which it is competing—for obtaining a result in which the student is invested.

That is not sufficient: the situation must immediately allow the construction of an initial solution or of an attempt in which the student invests her current knowledge. If the atttempt fails or is inappropriate, the situation must nonetheless produce a new situation, modified by this failure in an intelligible but intrinsic way; that is to say, not depending arbitrarily on the aims of the teacher. The situation must allow the voluntary repetition of the testing of all the student's resources. It must be self-motivated by a subtle game of intrinsic sanctions (and not by extrinsic sanctions that the teacher links to the student's progress). The unwinding of the learning process cannot therefore be programmed—only the situation and its choice can.

For the didactician, the question in hand is the one of simultaneously identifying a stage of a concept and a situation which poses a question for the student (one of the student's own) to which this stage is a "constructible" answer in the student's system.

In the student's functioning, we have been led to distinguish three types of questions which call for three types of didactical situation.

Different types of problem: validation, formulation, action ...

a) *Questions of validation.*
The student must establish the validity of an assertion. She must address herself as one subject would address another one capable of adapting or rejecting her asser-

tions, asking her to advance proofs of what she is suggesting, or challenging her by advancing other assertions. These exchanges help to make mathematical theories explicit, but also help to establish mathematics as a means of testing them as they are formed.

A proving process is constructed in a validation dialectic that leads the student, successively, to use rhetorical figures spontaneously and then to reject them. The relationships that the student must bc able to establish for this are specific to this dialectic (Brousseau 1970).

A validation problem is much more a problem of comparison of evaluations, of rejection of proofs, than it is of searching for a mathematical proof.

b) Questions of formulation.

For its validation process, thought must be based on preliminary formulations, even if they will have to be modified later. Languages, also, develop in less specific dialectics than those of validation. Communication (and its constraints) play a large rôle in this, one which is partly independent or problems of validation, at least of explicit validation, because, to bring about the *relevance* of language, this communication must be subjugated to the fulfilment of a rôle which submits it to pragmatic validations. It is within this framework that the economical constraints that determine judicious mathematical choices are most visible.

c) Questions of action.

Questions of action or of mathematical decisions are those where the sole criterion is the appropriateness of the decision—the elaboration system of this decision as well as its justification can remain totally implicit. There is no constraint here either of formulation or of validation. It is the most general dialectic; others are only particular cases. It ends up with the subject constructing regularities, schemas, models of action—more often than not unconsciously or implicitly.

Dialectics and obstacles

Certainly, none of these dialectics is independent of the others—on the contrary.

Formulation is often facilitated if an implicit model of action exists: the subject knows better how to formulate a problem if she has been able to solve it.

Action is facilitated by a suitable formulation, as Vygotsky (1983) has shown. Language cuts the situation up into relevant objects and relationships. Action provides one fundamental type of implicit validation, and formulation another.

But, conversely, each domain can be an obstacle to progress within the other domains. Some things are better done than said. Implicit models are better able to take a larger number of facts at the same time, and are more versatile and easier to restructure. Conditions that are too favourable to action make explanation useless. For example, when the Babylonians' hexagesimal systems for astronomical calculations was in use, the need for the decimal point did not arise, nor for the name of the reference unit, for an error of 1 to 60 was unthinkable to people who knew what they were talking about. In the same way, a language that is "too easy" to handle can hinder a necessary reformulation for a long time. (This is Bachelard's verbal obstacle.)

The overcoming of an obstacle very often involves a complete restructuring of models of action, language and proof-system. But the didactician can precipitate these breakdowns by favouring the multiplication and alternation of specific dialectics.

We have lingered too long on generalities. It is not possible to understand reciprocal relationships between obstacles and problems without a specific study.

1.3. *Problems in the construction of the concept of decimals*

1.3.1. *History of decimals*

It is not possible within the framework of this article to present an epistemology of decimals. Such an epistemology remains to be worked out. It is difficult because the facts which must be taken into account are spread over 15 or 20 centuries. At each "stage", we think that there is only one step to take, but this is not so, and it is rarely for want of trying. Research then leads to an understanding of what wonderful features this step had, and often of what it lost relative the preceding state.

The Chinese had a system of decimal measurement in the thirteenth century BC, the Babylonians had position co-ordinates, the Pythagoreans conceived the notion of fractions and Archimedes contributed to the conception of fractions as proportions. Nevertheless, we had to wait for the Arabs (Abû'l-Wefâ, second half of the tenth century AD) to see the notion of proportion applied to fractions and this proportion attempting to identify itself with numbers, and we had to wait for al-Kashî (1427) and, independently, Stevin (1585), for decimals to appear[10].

Stevin used the same notation for the study of geometry—in fact polynomials with integers as coefficients—and this is not by chance. Decimals were used before his time (Bonfils de Tarascon, 1350; Regiomontanus, 1563), but he was the first to suggest the substitution of decimal fractions for rational fractions and to write them down in a way that allowed the development of their calculation by rules known in the domain of natural numbers. "*Chose si simple qu'elle ne mérite pas le nom d'invention*", says this citizen of Bruges so modestly, *"elle enseigne facilement expédier par nombres entiers sans rompuz tous comptes se recontrant aux affaires des hommes*[11]*"*. But he sees all its benefits and asks that "*l'on ordonnat encore légitimement par les supérieurs, la susdite dixième partition a fin que chacun qui voudrait la pourrait user*[12]".

1.3.2. *History of the teaching of decimals*

The "popularization" of decimals thus became a didactical problem and it took two centuries to manage the first step. For example, Gobain in 1711 doesn't mention it in a work written for merchants, and d'Alembert in 1779 presents the question in its mathematical form in the encyclopædia (in the *Decimale* article). In the 1784 edition, Abbé Bossut presents decimals in a naturalist's manner: whole numbers with a decimal point serve to represent measurements. The decimal fraction aspect is relegated to an "appendix". A break developed between decimal fractions and "popular decimals" with their marvellously simple algorithms, which allowed them totally to popularize shopkeepers' book-keeping. The question wasn't settled by the decision of the *Convention:* the stake was too great right through the 19th century; the political aspect of the didactical problem prevailed. Thus Charles X re-introduced a "nouvelle *toise*" of six new *pieds* and retained only the arbitrary norms of the metric system[13].

The efforts at popularization were facilitated by the choice of the metric system. The generosity of revolutionary intentions led to the teaching of "mechanisms" independently of mathematical justifications, (everything that was essential for the citizen had to be provided in three years). These conquests of the 19th century were to create obstacles in the 20th century, where it is no longer a question of delivering instruction, but of educating, of providing understanding.

Active methods applied to the metric system progressively caused decimals as ratios or fractions to disappear; there was something of it left in terms of changing units, but the efficiency for some and the lack of direction for others contributed to making the last justifying discourse disappear.

Today, at least in France, the break is officially accomplished. The curricular programmes of 1970 introduced a (unachieved) construction of rational numbers that consists of constructing these excellent applications from bad operators which are integers. This construction has no use for introduction, for comprehension, nor for the study of decimals which arc constructed independently. The two continents have become separated. And they are especially so in teachers' and parents' conceptions.

1.3.3. *Obstacles to* didactique *of a construction of decimals*

Thus, a renovation of the teaching of decimals will be faced today with numerous technical and socio-economical difficulties. What will be the cost ? We wanted to study only questions of experimental epistemology in children's normal school conditions[14].

Also, the solutions that we are studying are not applicable, in the present state of things, by the teachers. We cannot here give in detail the analysis of all the obstaclcs. I will content myself, therefore, with mentioning the more important ones".

The act of attaching decimals to measurements leads the child to consider them as a triplet (n,p,u): on one side a whole number n and on the other a division by 10^p (that is to say a change of units) and a unit u; for example, 3.25 metres is 325 cm expressed in metres.

The practice of "changing units" means that p and u have privileged interrelationships—to realize this, it is enough to give exercises in which units are changed and a multiplication by a power of 10 is performed at the same time. Decimals act as whole numbers and are no longer detachable from a unit; the object is not the decimal number, but the physical quantity. Then the student can interpret the product of two decimals only in the case, for example, of the product of two lengths, which brings her back to the well known obstacles of concrete numbers: she will find $a^2–a$ hard to conceive and will implicitly drag the dimensions into equations.

Decimals will be implicitly limited to the range of the smallest units currently *in* use (or, better still, have two digits after the decimal point like *French francs*). The child reasons as if there exist elements simply smaller than the tolerable measurement error, and as if all numbers were whole numbers.

"3.25 is 325 with one hundred as the unit" say the official commentaries. All topologic relationships will be disturbed, and for a long time; the child will not find a decimal between 3.25 and 3.26, but, on the other hand she will find in **D** a predecessor to 3.15 which will be 3.14, etc. Even if she corrects her answer on this or that point, intuitive reasoning will be guided by this erroneous model (we find errors on this point, like the one mentioned earlier, right up to university level).

This integration as natural numbers will obviously be reinforced by the study of operations in the form of mechanisms, that is to say, actions carried out from memory, without understanding, done the same way as for natural numbers, with only a small extension for the decimal point.

When done mentally the calculation will follow another route. One calculates the product of the "whole part" and then that of the "decimal part" and then puts the pieces together: $(0.4)^2 = 0.16$, but $(0.3)^2 = 0.9$, and sometimes $(3.4)^2 = 9.16$.

Once again, it is the measurement effect—the whole part matters most; the decimal part does whatever it can.

Obviously, integration into natural numbers does not happen without some difficulties in the case of certain long division problems, which mess up the structure, but the model is not thrown out automatically; it is the numbers "which don't work correctly" which are buried, indicators of errors that must have been made somewhere. They will be rounded off; at best, they will be "bounded" (without even being defined), but the student will be afraid of them.

The implicit definition of decimals that likens them "to natural numbers with a decimal point" will mean to the student that natural numbers are not decimals, but that $0.3\overline{3}$ is a decimal.

One of the worst consequences of this obstacle is often to lead children to take the too-timid and too-late attempts to surmount it for twaddle and empty reasoning (in eighth grade, for example).

1.3.4. *Epistemological obstacles—didactical plan*

The obstacles given above are all of didactical origin. True epistemological and historical obstacles are different.

First, there is the issue of making **N** symmetrical with respect to multiplication. We can conceive a few fractions, but very quickly we want to be able to obtain "all" of them and at least be able to add them and to multiply them by a whole number. *It is essential not to teach the construction, but to pose the problem.* The child must see that she cannot solve it with natural numbers and deduce all the consequences, particularly with regard to order.

We have shown that the ten-year-old child can invent $\mathbf{Q}^+$ to solve this problem (see the "sheets-of-paper'' problem given below[16]). I don't believe that **D** could satisfy her at this moment and I don't see how and why she would invent it.

However, once she has constructed ($\mathbf{Q}^+$, +, <) and is faced with the need to order, for example, or to add a number of fractions, the child can come to prefer the use of decimal fractions and to see that they can be "close" to other fractions (that **D**is a subset of **Q**). We have shown, also, that this is possible with the help of the "explorer problem" and in the didactical process that follows[17].

This problem is the converse of the preceding one. It is no longer about inventing and combining elements of a new, unknown set but, on the contrary, about approaching a known set by means of a well chosen subset.

A text that is currently in preparation will give details of the twenty-five "problems" which constitute the dialectic[18] (which lasts sixty hours), but I hesitated before taking the following examples out of their context and giving them below with insufficient commentary.

Everything is a question of balance. For example, if children "mechanize" calculation within **Q,** the invention of Dis delayed and its use goes badly. If **Q** is not known well enough, **D** is neither constructed nor understood.

It is advisable, for example, to refrain from recognizing known practices too quickly; children know how to locate a rational number between two adjacent decimals well before discovering that this practice is "Division" and instituting it as an algorithm.

Moreover, it is not necessary to leave the main problems at the implicit level for too long. The dialectics of formulation and frequent organization of discussion bring what needs to be known to a conscious level[19].

The second main obstacle is the conception of rational numbers and decimals as ratios and then as linear mappings operating in **Q**. In a favourable situation (see the puzzle problem[20]) children construct this set of mappings, a few at first and then others which it will be necessary to refer to; fractions, or decimals, or natural numbers will lend themselves to this designation. On the whole, the composition of these mappings, and then decomposition on **D,** will provide a model unifying **Q, N** and **D.**

1.4. *Comments after a debate*[21]

Research on epistemological obstacles in mathematics certainly requires an effort of invention because Bachelard's concept is poorly adapted to this domain. But it can prove itself to be fruitful for teaching insofar as:

– the obstacles in question are truly identified in the history of mathematics;
– they have been traced in students' spontaneous models;
– the pedagogical conditions of their "defeat" or their rejection are studied with precision in such a way that a precise didactical project can be proposed to teachers;
– the assessment of such a project can be considered positive.

A few pieces of work, some of which are quite promising, have followed in the same vein since the initial article (Brousseau 1976). We can make a sort of provisional assessment.

Duroux (1982) refines the conditions that knowledge must satisfy before it can be declared an "obstacle" in Bachelard's sense and explains the interest of this concept, which must be distinguished from that of "difficulty".

Very often, it is among the "difficulties" that one must search for indications of an obstacle, but in order to satisfy the first condition, which says that *an obstacle is a piece of knowledge*, the researcher must make an effort to reformulate the "difficulty" that he or she is studying in terms, not of lack of knowledge, but of knowledge (false, or even incomplete, etc.).

Glaeser (1981) takes a contrary point of view[22]. In his article entitled "Epistémologie des nombres relatifs" (the epistemology of relative numbers), he lists as obstacles[23], "inability to manipulate isolated negative quantities" and "the difficulty of giving meaning to isolated negative quantities".

This formulation shows what is missing in Diophantes or in Stevin, *as seen from our own time*, in our present system. Thus, we spot a piece of knowledge or a possibility which was missing in the fifteenth century, which prevents the giving of a "good" solution or an adequate formulation. But this formulation masks the necessity of understanding how problems came up that would have necessitated the manipulation of isolated negative quantities. Were these problems posed? How were they solved? Or was it thought that they could be solved? Were what we call difficulties today seen in the same way then? Why did this "state of knowledge" seem enough, for what set of questions was it reasonably effective'? What advantages resulted from the "refusal" to manipulate negative isolated quantities or what inconveniences was it possible to avoid? Was this state stable? Why were attempts to modify it or, rather, to renovate it doomed to failure at that time'? Maybe until new conditions appeared and "lateral" work was accomplished, but which ones?

These questions are necessary in order to enter into the intimacy of the construction of knowledge, but Glaeser did not pose them and we don't know at all well what the manipulation of isolated negative quantities is founded on; this would be very useful for teaching.

It can be conjectured that the use of "positives" and "negatives" was transferred by the intermediary of an *interpretation* which arbitrarily attributed the status of positive numbers or negative numbers to "magnitudes" that were essentially positive.

Example: In a small business, the entry and exit of products can be recorded as positive or negative by the accountant who considers them. D' Alembert expressed this point of view at length. The use of an isolated negative quantity denies this symmetry and risks the forgetting of the initial convention, or worse, affirms that intrinsically negative objects might "exist"! But, the "relative" character of negative numbers was an important factor in their creation and in their acceptance.

In Glaeser's article, one can find what allowed the formulation of this hypothesis, but it isn't taken into account as such, because the nai've use of the term *obstacle* does not lead him to a method of working. On the contrary, the definition of obstacle proposed in the above article, and refined by Duroux, necessitates research on this knowledge obstacle and the Confirmation of it.

This method automatically and naturally leads to the examination of the second condition. The knowledge obstacle possesses *its domain of validity and effectiveness,* and thus also a domain in which it is *a priori* relevant but where it is shown to be false, ineffective, a source of errors, etc.

Research on corresponding historic indicators is no longer that of difficulties or errors "similar" to our point of view today, but that of failures characterized by a certain knowledge. By plunging it into our present knowledge, we can predict the kind of problems that will be badly posed or badly solved and search for them in history; epistomology tends to become systematic or experimental. The breaking points are no longer *dates of discovery* but *problématiques* and types of knowedge called into use, which can coincide at different moments in more or less neighbouring domains.

And let us observe that it is not enough to identify only the difficulties and failures of the knowledge obstacle, but also and above all its *success,* and thus to go back into history to earlier obstacles.

We must therefore, for example, link up "difficulty" in unifying "number line" (a following obstacle, according to Glaeser) with "inaptitude" (curious formulation, really, if we think about whom it is supposed to qualify) to give meaning to isolated negative quantities.

In fact, it could very well appear that, if great "efforts" were needed to gain acceptance for the dropping of the constant reference to the symmetrical pair (positive natural numbers, natural numbers considered negative) in order to be able to give meaning to an isolated negative number, it would be this same effort which is an obstacle to the "homogenization" of $(\mathbf{N}, +) \cup (\mathbf{N}, -)$, which can appear as a contrary movement.

Duroux notes that the conception of negative numbers as being by nature different from positive numbers can claim the status of an obstacle, but he lets us understand that he cannot come to the same conclusion for Glaeser's "obstacles" because they are not expressed in terms of knowledge. It may not be impossible, and we would then have one of those pairs of obstacles recognized by Bachelard.

It would also have been necessary, still according to our "definition" (third condition)[24], to demonstrate the *resistance* of knowledge obstacles and to explain it, for example by showing how the manipulation of isolated negative numbers had necessitated the conception of "negative numbers" having a nature different from that of positive numbers, and how, as a result, this conception in its turn became an obstacle to homogenization in a new entity of "numbers" (signed). It is probable that these two conceptions functioned as opposite poles with different domains of efficiency, each providing opposition to the development of the other.

It is the meaning of the third condition that stipulates that evidence of resistance to the establishment and to the rejection of a piece of knowledge is essential for establishing it as an obstacle. It seems to me that Glaeser's article opened up interesting paths to explore from this point of view.

The study of domains of validity and resistance to the use of isolated negative quantities could lead to its replacement by an older, more important obstacle: "numbers must be measures of something". Then one cannot use negatives in isolation without the risk of forgetting what they measure. And we can no longer consider **Z** as a homogeneous set since it cannot be a set of measurements. On the other hand, I believe that in a case like this, we must consider all the didactical components linked with the evolution of knowledge in society. Stendhal's[25] objections are not to be confused with d'Alembert's hesitations even if they can be connected with them.

This is expressed by Cornu (1983), who analyzes the notion of limit from this point of view. After having noticed, in a historical study, difficulties which seemed to him to be good candidates for providing obstacles to the genesis of the notion, he looks for corresponding traces in students' behaviour. It becomes clear that these obstacle-conceptions of the students must be studied in the same way, that is to say, using the same conditions, but this time from the point of view of the student, her environment and her culture.

Cornu is close to showing that, in agreement with my general hypothesis, certain of the students' difficulties can be grouped around obstacles attested to by history.

It is in the analysis of resistance and in the debate which has surrounded it that one must look for elements which will allow the identification of obstacles for the students. In any case, it will never be enough to tack—to apply without modification—historical study onto didactical study. It is from this origin, too, that we must draw arguments in order to choose a genesis of a concept suitable for use in schools and to *construct* or "invent" teaching situations that will provide this genesis.

It is important, furthermore, to make sure that the historical and epistemological, psychological and didactical arguments which make up the interest in the study of epistemological obstacles are well and truly independent. The identification of students' difficulties, and then the organization of these difficulties into spontaneous models or into knowledge obstacles can and must happen independently (and even contrary to the hypothesis) of a historical identification. The demonstration of resistances and the overcoming of them in school situations are followed by their explanation, itself tested by experimental study and the realization of the genesis of the concept (and of the situations that produce it). Once again, the source of information can and must be independent of the preceding information at the price of certain methodological precautions.

Convergence, if realized, of independent arguments of these various types constitutes a scientific fact in the full sense of the term. The meaning of this fact then appears clearly:

– On the one hand, it supports the conceptions that I recalled at the beginning of my 1976 article on learning or rather, the psychogenetical, didactical, historical genesis of knowledge (positive aspect of certain errors made by students and therefore the need for problems presenting difficulties, the existence of

accommodation and local or important restructuring, the necessity of a jump in complexity, etc.).
– On the other hand, it allows us to choose from among the numerous "difficulties" that students meet, those, not numerous, that must be identified and repeated explicitly and institutionally, and thus avoid submerging teaching in an inextricable mass of considerations about knowledge: heuristical, mnemotechnical, etc.

These two consequences are very important for teaching.

Although promising, the works referred to above do not establish such "facts" in an irrefutable manner. Indeed, the task is arduous. Bachelard found supporting points in physics that are missing in mathematics (experimental projects, hypotheses, reports) and his analysis ends when the "facts" pass under the control of a true scientific theory. Bachelard was therefore not constructing an epistemology of theories of physics, but one about the establishment of physics.

It was also interesting, using the methods of situation analysis, to carry out research on the changes of status of mathematical knowledge. Chevallard has thus distinguished protomathematical concepts, paramathematic concepts and actual mathematics[26]. These distinctions seem to me essential for advancing in research into obstacles to the extent that they allow the re-attachment of knowledge to its function and to its mode of functioning in scientific activities, and in the sense that obstacles in mathematics are perhaps more often obstacles to change of status than obstacles to knowledge itself.

The explanation of resistance is the heart of research into epistemological obstacles, and there is hardly any method because each explanation is specific to the obstacle in question. It is necessary to follow the step by step attempts at rejection, and remodelling and recovery, and to look for the conditions that existed in order to find among them the mechanisms. This research is nevertheless facilitated in the case of a real obstacle by the fact that *an epistemological obstacle constitutes a part of complete knowledge* in the sense that its rejection must finally be made incontestibly explicit, and in consequence that it leaves marks, deep at times, in the system of knowledge (this is the fourth condition). This shows that it is never the unique fruit of a transitory error, which it might be sufficient to repair, of an ignorance that one could patch up, of a passing fashion and *a fortiori* of an inaptitude! It can result from social, cultural, or economic circumstances; but these "causes" are actualized in conceptions which remain once the causes have disappeared, and which it is not sufficient to forget, because it is at this level that the "debate" must cut off.

It is also for this reason that epistemological obstacles interest the didactician who, *a priori,* has no use for a museum of concepts and out-of-date knowledge. To the extent that the knowledge obstacle is a component of knowledge, where it is present in students' spontaneous models, and to the extent to which an inadequate treatment in teaching leads to repeated and damaging errors, it becomes essential to recognize it and to reject it with the students.

This point of view leads us, without rupture this time, to the heart of the study of the evolution of mathematical theories where the importance of teaching stops being metaphorical and starts asserting itself as an essential component of *didactique,* as emphasized by Balacheff and Laborde (1985) in the introduction to their French translation of Lakatos (1976).

We are only beginning to study and exploit this notion of obstacle and the theories that accompany it. I am convinced that the methods and the *problématiques* that it will allow to be developed, and that I have tried to present here, will bear fruit (particularly for teaching) in the near future.

2. EPISTEMOLOGICAL OBSTACLES AND *DIDACTIQUE* OF MATHEMATICS[27]

2.1. *Why is* didactique *of mathematics interested in epistemological obstacles?*

The transposition into mathematics of the notion of epistemological obstacle, which Bachelard (1938) thought of confining to the experimental sciences, was made possible and even necessary by the development of the Theory of Didactical Situations in the 1970s. It came directly from the concept of the "informational leap" (Brousseau 1974a) and from the "theorems" of *didactique* which flow from it.

A piece of knowledge is the result of the student's adaptation to a situation, S, which "justifies" this piece of knowledge by making it more or less effective, of different pieces of knowledge leading to learning and of the performance of tasks of different complexity. Depending on the values of variables relevant to S, one can envisage the association of each useful piece of knowledge in S with a region of effectiveness (or cost). The upper envelope of this region can include maxima, separated by saddle points (or any other singularity). Thus, in order to make the student create a particular piece of knowledge, the teacher "must" choose values which make this piece of knowledge optimal with respect to competing pieces of knowledge; progression is by leaps and not smooth. For example, if one wishes to encourage the solution of a linear system by means of linear combinations, it is better to choose systems of rank 4 rather than 2 or even 3 for students who know the method of substitution.

This reasoning can be applied to the analysis of the historical genesis of a piece of knowledge as well as to the teaching of it or to a student's spontaneous evolution.

Learning by adaptation to the *milieu* must therefore bring about cognitive ruptures: adaptation and changes in implicit models, in languages, and in cognitive systems.

If her history binds a student—or a cultural group—to a step-by-step progression towards a pass, the principle of adaptation can itself counter the rejection, however necessary, of an inadequate piece of knowledge. This fact suggests the idea that "transitory" conceptions endure and persist.

In a procedure opened up by Gonseth (1936), these ruptures can be predicted by the direct study of situations (the effects of didactical variables) and pieces of knowledge and not only by the (indirect) study of students' behaviour (Brousseau, 1974a, 1976).

According to Salin (1976), however, taking this route requires the re-examination of the interpretation of students' errors and the ways in which they are produced. Until this suggestion was put forward, such errors were all attributed either to erratic disfunctioning or to the absence of knowledge, and therefore viewed very negatively. Recurrent errors should now be envisaged as the result of (produced by and constructed around) conceptions, which, even when they are false, are not accidental but are often positive acquisitions.

From the outset, therefore, researchers should

a) find recurrent errors, and show that they are grouped around conceptions;
b) find obstacles in the history of mathematics;
c) compare historical obstacles with obstacles to learning and establish their epistemological character.

2.2. *Do epistemological obstacles exist in mathematics?*

On the first point (finding recurrent errors), observations of striking errors have been developed: [$(a+b)^2 = a^2+b^2$; $0 \cdot a = a$; $\sqrt{a^2} = a$; $(0.2)^2 = 0.4$; and so on], but their linkage with conceptions depends on statistical methods which necessitate adjustment to standard methods (Cronbach, 1967; Pluvinage, 1977; Gras, 1979). Progress has been made possible by a better definition of the notion of conception based on the Theory of Didactical Situations.

The possibility of provoking the acquisition of different conceptions is demonstrated for rational numbers (Brousseau 1980, 1981; Brousseau and Brousseau, 1987): either *measurement* or *subdivision* is obtained by simple manipulation of didactical variables. Ratsimba-Rajohn (1981) observes how these two conceptions can become mutual obstacles and even coexist for the same student and how an initial conception can be reinforced, rather than rejected, in spite of an *a priori* sufficient informational leap.

On the second point (finding obstacles in the history of mathematics), Glaeser's study (1981) on the history of signed numbers unquestionably shows the interest and the importance of these phenomena of rupture—observable in the history of mathematics—for the understanding of students' difficulties. But it then becomes apparent that Bachelard's model (1938) has to be interpreted before it can be extended to mathematics. Duroux (1982) proposes not a definition, but a list of necessary conditions:

a) An obstacle is a piece of knowledge or a conception, not a difficulty or a lack of knowledge.
b) This piece of knowledge produces responses which are appropriate within a particular, frequently experienced, context.
c) But it generates false responses outside this context. A correct, universal response requires a notably different point of view.
d) Finally, this piece of knowledge withstands both occasional contradictions and the establishment of a better piece of knowledge. Possession of a better

piece of knowledge is not sufficient for the preceding one to disappear (this distinguishes between the overcoming of obstacles and Piaget's adaptation). It is therefore essential to identify it and to incorporate its rejection into the new piece of knowledge.

e) After its inaccuracy has been recognized, it continues to crop up in an untimely, persistent way.

On the third point (comparison of historical obstacles with learning obstacles), the results are starting to appear substantial. On the notion of limit, let us cite the very fine remarks of Berthelot and Berthelot (1983) and Sierpinska's (1985, 1987) important observations; and on the simple continuity of functions, let us mention El Bouazzaoui's (1988) second and recent thesis on the conceptions of teachers, students, and manuals and those which appear in the history of mathematics. This work leaves little doubt: obstacles certainly exist, even if distinguishing them, recognizing them, listing them, and examining their relationships and their causes requires much more discussion and research.

Fundamentally cognitive obstacles seem able to be ontogenetic, epistemological, didactical and even cultural, according to their origin and the way in which they evolve. Perhaps it would be interesting also to differentiate among them according to the form of control of the knowledge (protomathematical, paramathematical, or mathematical) where the rupture is produced.

2.3. *Search for an epistemological obstacle: historical approach*

2.3.1. *The case of numbers*

The history of numbers is rich in examples of epistemological obstacles. For example, heterogeneous measurement, more adaptable to particular social and material conditions, was an obstacle to the installation of a generalized decimal system for a long time, and this has hindered the adoption of a universal metrological system right up to our own times. Much later, systems "assuming" that all fractions could be generated by a small number of them, as was done with natural numbers, controlled the organization and naming of fractions until the end of the Dark Ages and were still obstacles to al-Uqlidisi's first attempts (952) to switch to decimals (Abûl-Wefâ, around 961–976). The constant use of ratios in all the ancient calculations, which is related to the use of fractional measurement, was an obstacle to the formalization of ratio-fractions and to the conception of mappings as numbers. The study of their history shows more clearly the advances such as Euclid's attempt and the inertia and retreats such as those displayed by the neo-Pythagoreans. Did Archimedes know about Archimedian fractions?

The hexagesimal system, another means of simultaneously solving algebraic, topological and meteorological problems in a unique way made room, not without difficulty, for Indian methods, and then Arabic ones, all the while giving support to them.

2.3.2. *Methods and questions*

The way in which *didactique* understands these historical questions must be made precise. It is a question of producing models of situations that take into account all the relevant conditions for the creation of knowledge (known in history) and of organizing them according to logic—logic that can be compared with other requirements (mathematical, psychological, sociological, ergonomical, etc.) and, among other things, with the experience of reproducing this knowledge. In fact, this method no more changes historical method than experiments on techniques of carving stone do for prehistory. Conjectures about what would happen if such-and-such a human group had been able to use such-and-such a piece of knowledge are not venturesome assertions about effective history (and about Cleopatra's nose), but simple working hypotheses about a model that is temporarily under the control of another system of knowledge. This method allows historical questioning and comparisons and ensures the connection of hypotheses, because when an explanation is contradicted, new modelling can give better results than the earlier one within a larger domain.

It is therefore a question:

i) of describing this knowledge and of understanding its use;
ii) of explaining what advantages this use has over earlier uses and which social practices, which techniques and, if possible, which mathematical conceptions it is related to;
iii) of seeing these conceptions in relation to other possible conceptions, particularly those that have succeeded them, so as to understand the limitations, the difficulties and eventually the causes of the failure of this conception, but at the same time the reasons for an equilibrium that seems to have lasted for a considerable period;
iv) of identifying the moment of, and the reasons for, the breakdown of this equilibrium, and then of examining the traces of resistance to its rejection, explaining it, if possible, by the survival of practices, languages or conceptions;
v) of searching for possible resurgences, for unexpected returns, if not in the original forms at least in similar forms, and of understanding the reasons for them.

Alongside historical arguments based on the study of texts, technical and epistemological arguments supported by experiments linked to learning (subject to deontological precautions) can intevene. On the other hand, historical arguments can intervene in choices of teaching under the surveillance of a Theory of Didactical Situations.

2.3.3. *Fractions in ancient Egypt*

As an illustrative example, let us take a fossilized piece of knowledge and examine the characteristics it has for candidature as an epistemological obstacle: the exclusive use of unit fractions for expressing fractions in ancient Egypt. Certainly, this example lacks didactical interest because the “ecological” conditions which permitted it to exist at that time have probably completely disappeared, but here it is used only as an exercise.

2.3.3.1. Identification of pieces of knowledge

In order to express measurements, the Egyptian scribe used natural numbers and the sums of unit fractions; the scribe operated on these numbers only by multiplication or division by 2 in such a way that their (implicit) ratios appeared as sums of powers of 2 (see Figures 1 and 2).

It appears that though certain techniques and conceptions were contributions internal to the scribes’ circle, there were others which must have depended on practices which the calculations were responsible for accompanying. In addition, the scribe’s result was probably subject to a certain amount of verification by administrators, verification that was carried out within the popular system. It is therefore important to know what material manipulations were necessitated by these social activities and how they varied, for example, depending on the quantities in question. The scribe used these calculations as much for inventories of harvests as for sharing of resources, in particular proportional sharing, and especially for exchanges in a civilization without money.

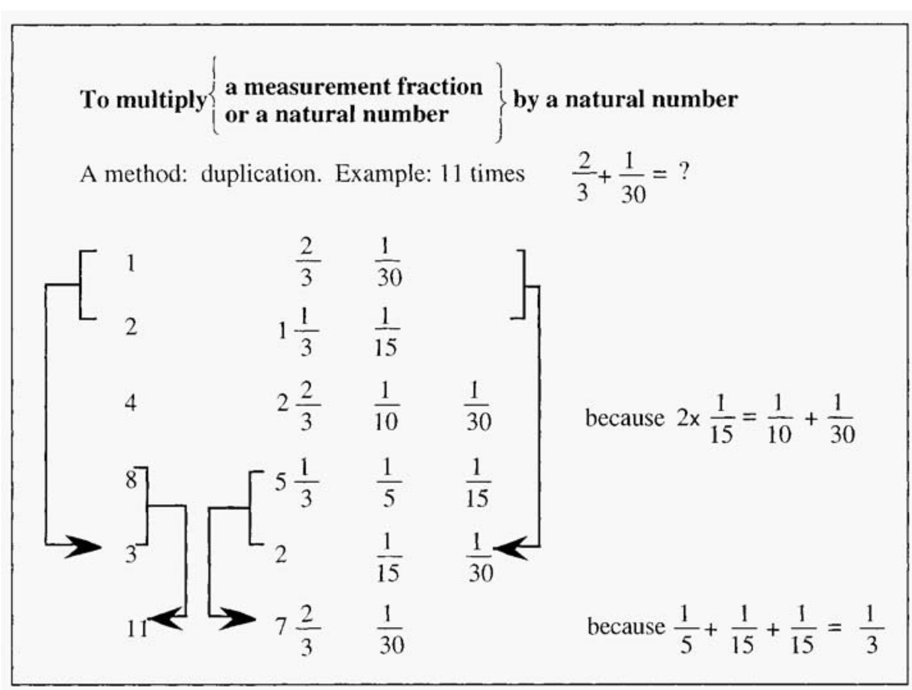


Figure 1

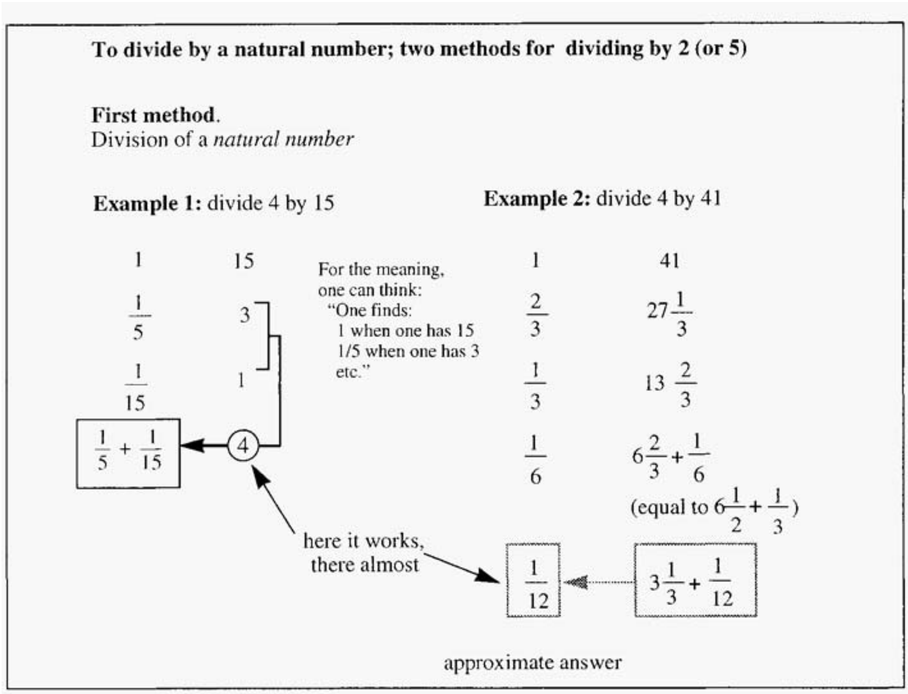


Figure 2

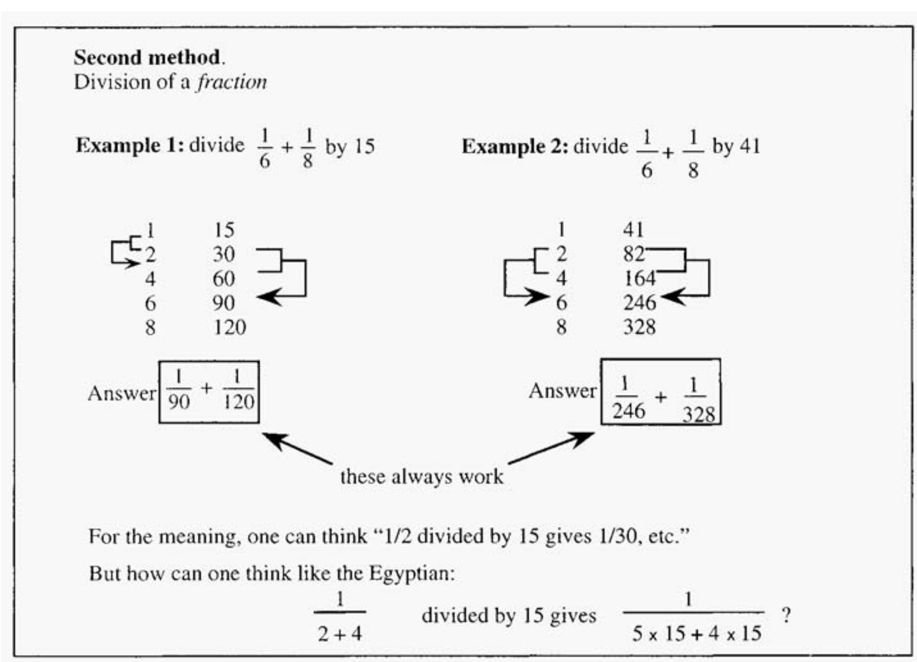


Figure 3

Observation: Historians have too often contented themselves with asking mathematicians to interpret the observed calculations and at best the reasons which could underlie them. Techniques of calculation are interesting, but they might not explain possible survivals too well because they were reserved for a caste which has disappeared along with the economical (absence of money), social and political (centralization) conditions which justified it. The scribes' freedom to modify their system of calculation and estimation was moreover confined within narrow limits that were fixed in terms of compatibility with the social functioning of their caste. Every mathematical invention that they produced remained localized like, for example, the beginning of the Archimedean notation for large numbers. The functioning of knowledge must therefore be put back into its social, economical and technical context, in relationship to the frames of reference which support it.

2.3.3.2. What are the advantages of using unit fractions?

Why, therefore, did the scribes choose the Hieratical system, which was different from the popular old hieroglyphic system? Why did they all accept the unit fractions? And why did they use only the denominators and exclude the numerators? Different choices are known to have been made in other places in neighbouring epoques.

Ancient Egypt knew at least two systems that are potentially equivalent to the "practical" decimal system: the natural numbers and the binary "fractions". Ever since the expression of quantities began to be divided into the name of a number on the one hand accompanied by the name of a unit on the other hand, it has been possible to choose within each "size" the smallest unit that can be manipulated and to express all "possible" quantities by means of natural numbers. Ergonomical conditions that are easy to envisage dictated that such a decision would have had a high cost for a rather small gain. In addition, partition and counting would then appear as different operations, necessitating a theory and specific symbols for each one.

But already calculation was unified; in both cases, products were carried out by duplication, calculations being simply a little more difficult with natural numbers.

The system of Horus's[28] eye is openly based on the fact that division into two equal parts is easy for most sizes when the quantities are manipulable. With the calculation procedures of the epoch, this system theoretically allowed the solution of any problem which we solve today by using decimals. If it was superceded by another system, it is for reasons of conception and adaptation to situations of reference.

The calculation of situations allows the forecast of (and epistemological experiments allow the observation of) various didactical variables in situations of partition and measurement. At least fifteen situations are thus obtained, which lead to different conceptions (Brousseau and Brousseau, 1987). When the quantities are small and easy to compare and to manipulate globally, approximate partition followed by an equalization is the clear choice. The operation can be repeated for larger quantities, but very quickly regular attribution becomes more economical: the number of parts being known, each part is given an equal quantity until all the stock to be divided has been used up, without the need to measure it first (which would require the same manipulation).

When neither the value of a part, nor the number of parts, nor the total quantity is known in advance, the division can nonetheless be prepared by the establishment of small aligned, arbitrary but equal bits. If one wishes to take one-ninth, one counts, "One, two, three, four, five, six, seven, eight, nine" and takes the ninth bit; then one continues by counting all over again, "One, two,..., nine", and takes the ninth bit; and so on for all the multiples of nine. This kind of practice might have been followed quite often in the situations that the scribe had to solve.

The number of languages in which rank-order is expressed by the same word and the same suffix as the name of the unit fraction shows the age, the importance and the widespread nature of this method. It is hardly remembered today except as the ancient way of decimating a beaten army.

Another advantage of the Hieratic system in comparison with the old system is that it assures exact calculations in the scribe's accounts (it allows the writing of any fraction at least in the form of a sum). When food was being divided among the inhabitants of a village, the fact that the diadic system doesn't produce an exact inverse for most natural numbers was translated by various phenomena such as the following: the sum of the numbers representing the parts, calculated with Horus's fractions (right up to a 64th) is not equal to the total quantity distributed. The error increases as the number of inhabitants goes up. The scribe's division itself is always correct and the length of the writing of the quotient can be controlled quite well (see Figure 4).

From the conceptual point of view, the restriction of fractions to unit fractions allows reasoning similar to that used with natural numbers: in calculations, the reasoning about the number of parts is symmetrical to that about the value of one of the parts; thus, the bar written over a numeral to indicate the unit might not be there to indicate the absence of some numerator. In addition, unit fractions are most suitable for proportional division, which the scribe constantly did. In any case, the conception of ancient fractions did not pass directly to the subdivision of the unit, as today's culture would lead us to assume, but rather by way of commensuration, which is much better adapted to the handling of multiples. Actually, unit fractions could be the sign of the passage from decomposition of the multiples into the sum of powers of two to a more global conception.

It is clear that *this construction doesn't permit the conception* that there can be "two"-ninths, and more generally, it does not allow the addition of two unit fractions (as such). In particular, the repetition that typifies the addition of natural numbers does not correspond to any simple operation: if one successively takes one-ninth, then one-ninth (of whatever remains), one obviously does not get 2/9, but 18/81. On the other hand, true repetition models multiplication very well (for example, take one-fifth of one-quarter) and gives an internal result. Fractions larger than unity have no possible meaning.

In order to divide one natural number by another, one successively uses two methods. The first is used until a remainder less than unity is obtained; that is to say, a number on the right that is less than the dividend.

For example: 2 ÷ 41

First step:

1	41
$\frac{2}{3}$	$27\frac{1}{3}$
$\frac{1}{3}$	$13\frac{2}{3}$
$\frac{1}{6}$	$6\frac{2}{3}\frac{1}{6}$
$\frac{1}{12}$	$3\frac{1}{3}\frac{1}{12}$
$\frac{1}{24}$	$1\frac{2}{3}\frac{1}{24}$

less than 2

but $2 - (1\frac{2}{3} + \frac{1}{24}) = \frac{1}{6} + \frac{1}{8}$

Second step:

1	41
2	82
4	164
6	246
8	328

answer: $1\frac{2}{3}\frac{1}{24}\frac{1}{246}\frac{1}{328}$

The exact answer can always be obtained

Figure 4

In principle, this system allows the uniqueness of writing to be conserved and allows the remainders to decrease quite rapidly, thus enabling approximations and comparisons to be made. In fact, the scribes could not optimize their method and obtain this uniqueness except by tradition, by relying on great skill in calculation and unequalled familiarity with this type of fraction. The acceptance of any type of numerator *would have caused this "uniqueness" of writing to have been lost.*

The Babylonian polynomial system proposed a universal solution in which natural numbers and fractions were written with the same symbols. This invention was possible only at the cost of an enormous base (60) which dispensed with indicating the unit of measurement, since an error of 1 to 60 was unthinkable for anyone who knew the reference context, which made it possible to believe in the use of scalar fractions (an error of 1 to 10 is often possible). The writing of simple fractions and their addition was simplified, but calculation with natural numbers proved to be more complicated and it was necessary to use tables. The writing of inverses remained unresolved, but the conception of the fraction with a numerator that is greater than unity was on the way—not that one can say, for all that, that one system "replaced " the other.

2.3.3.3. Does the system of unit fractions constitute an obstacle?

The Egyptian procedures were not abandoned by the Greek astronomers until the second century BC and we discover traces of them right up to the 12th century in the Arab civilization, with the functionaries, the surveyors, the merchants, etc. How can we construct, in terms of modern operations, tables such as those of the scribes' second method of division (Figure 3).

This example shows that an obstacle is made neither by clumsiness nor by really "false" explanations. It is a legitimate adaptation to precise conditions, and it leaves traces within the culture. We do not yet know how to characterize obstacles within a specific meta-language as Bachelard has done.

2.4. *Search for an obstacle from school situations: A current unexpected obstacle, the natural numbers*

Are the natural numbers an obstacle to the conception of rational numbers and decimals?

An error such as "0 x 3 = 3", which occurs very frequently, can be explained from the outset by the fact that this error, if it occurs, will never be corrected during the execution of an operation, unlike "3 x 0 = 0". In fact, it cannot give rise to errors since, instead of having to envisage it, the student simply shifts a partial product[29]. This does not explain why it is produced. It is possible to lay the blame on the reference conception of multiplication:

- 3 ×0: take 0 3 times, is well understood as 0+0+0 = 0;
- but for 0×3, it is a question of taking the quantity 3 0 times; this quantity "must" certainly "exist" and therefore it remains present even though one doesn't want to take it.

This reasoning is the reverse of what Rogalski (1982) describes: a student counts the number of rows and columns of a rectangle; when she has counted the number of rows by using the squares in the first column, she counts the columns, omitting the first one, "because she has already counted the square in the corner". Is the defective conception the same? If "yes", it will then concern the implicit definition of what "0 times" is. But is it a question of the natural scalar? Of the natural ratio? Of the natural mapping? Or of a conception necessitating a richer structure such as the decimals? The answer to these questions depends on the knowledge of the student capable of correcting this error. Various proofs can be put forward to convince a student. For example (nothing can be based on the consideration of ratios between zero and 1):

- 0 times 3 is less than 1 times 3; the product is therefore less than 3; or perhaps $1\times3=3$, $1/2\times3=1.5$, $1/10\times3=0.3$, $1/1000\times3=0.003$,...;
- 0 is 1 minus 1; "0 times" is 1 times less 1 times; 0 times 3 is therefore 1 times 3 less 1 times 3; or perhaps even better formally: $0\times3=(4–4)\times3=(4\times3)–(4\times3)=12–12=0$;
- $0\times3=3\times0=0$;
- and so on.

None of them directly transforms the erroneous conception; none of them offers protection from the unexpected return of this error.

We therefore have an obstacle-candidate and the difficulty consists of identifying the conception which corresponds to it. We can either confine it to this simple local representation or attach it to the whole mathematical structure that underlies it: the natural numbers. This problem has often been identified as fundamental to *didactique;* we possess more experimental methods for distinguishing and separating these concepts than we do for grouping them. It appears reasonable to avoid abiding by an arbitrary decision and to choose the most restrictive structure that explains the error. But if other errors are linked to similar conditions, isn't it also reasonable to enlarge the conception assumed to form an obstacle, so as to obtain a common model? Thus, one can be tempted to group this difficulty with the one which tells the student that multiplying increases, or that which asks her to envisage the product of 0.35 and 0.84.

A method is necessary. In pursuing it in the indicated direction (Ratsimba-Rajohn, 1981; El Bouazzaoui, 1988; Brousseau and Brousseau, 1987), it is possible:

- to establish a "fundamental" situation corresponding to the knowledge in question;
- to seek the didactical variables and the different conceptions that they engender—in particular the one which is assumed to be sufficient to explain the error;
- and then to use factor analysis or more classical statistical methods as a means of identifying groups of students who "separate" these conceptions. If there is no clear discrimination, there is no reason to consider these conceptions as distinct.

Under these conditions, the first of the above difficulties does not belong to the same obstacle as the other two. But let us examine the problem from the theoretical point of view. For a child to "understand" is to take on her own responsibility for establishing and linking phenomena or facts left "independent" by the teacher, the situation, her language and learned knowledge.

For instance, a child can understand the first measurements with the help of counting, understand the properties of order with the help of measurement, verify operations with the help of order ("it" grows, therefore it isn't necessary to divide) or some other operation (multiplication is repeated addition), understand counting thanks to operations or a search for successors... and every possible relationship, true within the set of natural numbers, is good for providing meaning.

This knowledge, constructed by the student personally or thanks to the history of the class, is not entirely institutionalized by the teacher's activity, but some of it certainly is, and deservedly in the context. It is, in every case, essential for the correct functioning of institutionalized knowledge, taught by the teacher.

For the student, these properties are those of numbers in general, of all numbers. It is possible to understand that what mathematicians call the extension of the set of natural numbers into a larger set results in some properties no longer being true for all numbers, or even for any number.

The student is not informed of this rupture because neither the culture—and in particular tradition—nor didactical engineering has yet produced the necessary instruments (exercises, information, concepts, observations, paradoxes, etc.). She therefore make errors and, as they are attached to a certain way of understanding the properties of numbers, these false conceptions persist and the effects of the rupture can be observed for a number of years.

Even more important is the mechanism of this obstacle. It is not taught knowledge that is missing—generally teachers foresee this problem—it is the student's personal way of understanding. She no longer understands because what must be changed is really the means of what she previously called "understanding".

With the set of natural numbers, we certainly have all the characteristics that we have set out for the recognition of an obstacle. An obstacle is obviously unavoidable. Must the term *epistemological obstacle* be reserved for such a type of knowledge?

Must we believe that *every* conception is an obstacle to later learning? Certainly, as we have seen, it is within the realm of their nature. But very few present difficulties which are sufficiently important and common to be treated as such.

It is, moreover, easy to understand how a premature over-teaching can increase the chance of transforming a necessary item of knowledge into an insurmountable obstacle.

As has already been stated, the analysis of situations whose solution appeals to division has led to the identification of fifteen or so which arise from different conceptions. Teachers identify only a small number of them, often only a single one—*partition—of* which they study only two aspects: "the search for one part" and "the search for the number of parts". They ask students to recognize every division problem on the basis of this single conception, whereas the study of different kinds of division (search for the remainder, approximation to a ratio, etc.) with the students doesn't lead to any specific difficulty, even when it comes to generalization.

On the other hand, the use of the natural-numbers model will cause difficulties when decimals are studied. Thus, students try to understand the problem by cutting up numbers in an attempt to bring the problem back to the domain of natural numbers.

For example, if one pays 135.40 francs for 35.75 litres of gasoline, the student envisages the operation as similar to finding the price of one litre if one were to pay 135 francs for 35 litres. But, this procedure wouldn't apply if one were to buy 0.75 litres for 0.40 francs. Division by zero (or even by 1), or of zero by something, poses "resistant" problems which depend, this time, on the conception used (successive subtraction or the inversion of a product, for example) and on the nature of the sizes represented by the numbers (measurements or scalars). From the point of view of *didactique,* these must be treated as an obstacle. For all that, should this difficulty be considered to be an epistemological obstacle? It seems to me that at present we lack much too much information (in particular about the generality of the phenomenon and about its historical incidence) to do so.

In conclusion, students' observed errors, as in historical practice, can be grouped around very particular or, on the other hand, very general conceptions. The identification of the conceptions is a major difficulty for all sectors of *didactique.*

Obstacles must also be considered together from the point of view of their interrelationships. Many of them can coexist, mutually contradicting and successively replacing each other; for example: conceptions about fractions as against those about decimals, or even the "measurement" aspect versus the "relationship" or "mapping" aspects. Rejection of one leads to the other right up to the solution.

We haven't the space here to examine the precise functioning of an obstacle. But such a study would reveal the social and cultural characteristics of obstacles, as much as and even more than their simply psychological and cognitive aspects. In the examples given by Bachelard, the rôle played by a change of practice, context, or reference system must be observed. These conditions are also specific characteristics of the didactical relationship; the greater the gradient of didactical transposition, the greater the difference between the knowledge environments of the two didactical partners, the greater will be the risks of functioning as an obstacle.

2.5. *Obstacles and didactical engineering*

Whatever their origins and their importance, the existence of obstacles poses a certain number of problems of engineering to *didactique.* How can these obstacles be avoided? Must they be avoided? Can they all be avoided? How can those that can't be avoided be overcome?

2.5.1. *Local problems: lessons. How can an identified obstacle be dealt with?*

The students meet an epistemological obstacle; how can their recognition and "overcoming" of it be organized? We can no longer talk about its disappearence. In my opinion, there isn't a standard solution, but what has already been discussed certainly shows the need to establish, simultaneously, didactical situations and all sorts of adidactical situations.

a) The need for validation situations arises from the definition of obstacles; they are the only ones that allow personal integration in the theory being managed. Sociocognitive-conflict situations are of this type. Since the elicitation of the obstacle is essential, situations of formulation can be useful. But, since obstacles frequently appear at the level of implicit models and in spite of a suitable item of knowledge at the level of consciousness, adidactical situations of action are useful as well.

b) Didactical situations are no less necessary. The intervention of the (mathematical) culture through the medium of the teacher is inescapable at different times during the procedure; here too, there are differences from the Piagetian stages.

c) Another essential characteristic is the dialectic character of the negotiation of epistemological obstacles. Changes of framework and relationship with knowledge (tool-object dialectic), in particular, seem to me to be required to play an important rôle (Douady, 1984, 1985).

2.5.2. "Strategic" problems: the curriculum. Which obstacles can be avoided and which accepted?

Ignoring obstacles leads:

- either to choosing when defining a piece of knowledge to use definitions which seem easily accessible to the students, and which can simply be added onto previous knowledge. The functioning of this incomplete knowledge in too limited a context produces a "temporary culture" and obstacles which can be more or less overcome by the student and the teacher but which provoke many difficulties;
- or to choosing to define the knowledge in its definitive form and "organization" as a language with the risk of a usage that is strictly formal and stripped of meaning since it is possible that this language is not adapted to the students' development.

Taking obstacles into account implies the choice of a genesis that the student can produce herself, which doesn't leave in the dark problems that the knowledge being taught has solved. This genesis handles some obstacles and ignores others. It is therefore a question of making a choice among them because it would be absurd to reinstate the sources of the difficulties without purpose and thus increase the number of false trails. In spite of these precautions, such a genesis can only be very complex. For example, it has been shown (Brousseau and Brousseau, 1987) that it is possible for children of ten or eleven years old to acquire directly a correct mathematical knowledge of rational numbers and decimals with all the aspects of their algebraic, topological, ordinal structure, and to handle the main obstacles in a "suitable" way.

In the experimental curriculum, every obstacle is approached in a specific way, but they are all manifested for the students:

- first, in an implicit way within the means of solving situations,
- secondly as curiosities,
- before becoming obects of study, and
- finally as commonplace facts.

This sequence of sixty-five lessons takes up no more time than that generally devoted to the classical teaching of the same type of knowledge. It is, however, rather heavy with respect to what the teachers in subsequent years will use of it, and is totally incommunicable to a teacher working in ordinary conditions.

Should one pay the cost of such an investment? Up to what point?

2.5.3. *Didactical handling of obstacles*

The natural numbers form an obstacle to the conception of decimals. For obvious reasons of the proximity of the writing and structure, this obstacle is more difficult to overcome than the one they pose to the conception of rational numbers. Izorche (1977) shows how students fifteen and sixteen years old identify **D** with **D**2 (or with **D**3), the set of decimals such that $d\cdot10^2$ (or $d\cdot10^3$) belong to **N**, and then **D**2 (or **D**3) with **N.**

This obstacle appears in the form of several difficulties that can be dealt with separately. Most of them are well known:

- the difficulty of accepting that an increase can be obtained by a division and a decrease by a multiplication,
- the difficulty of finding one decimal number between two others, so as to forego finding a successor to a decimal,
- the difficulty of accepting the "double writing" of decimals (for example: 1.5 and 1.4999...),
- the difficulty of conceiving of the product of two scalar decimals,
- the difficulty of conceiving of new types of division.

Let us take the example of errors referred to above where, in order to solve a problem, students intuitively guess required operations that are suitable, by replacing given decimals by approximate whole numbers. They fail when the dividend (or even the quotient) is such that the rounding off doesn't give a usable natural number; if the decimal is smaller than I, or even if it is only less than 2, confusion with subtraction, inversion, errors, or inability to understand the problem reappear.

The fact of having previously distinguished natural numbers from decimals and their applications such as the expression of measurements, etc., and even of "suitably" studying multiplication by decimals less than 1 (cf. Brousseau and Brousseau, *op. cit.*), doesn't completely prevent the phenomenon from occurring[30].

The technique used consists of organizing a competition of problem statements so as to ensure that, within this framework, the children themselves search for the variables of the problems they propose. It is a question of changes that make the problem unfeasible or difficult; for example: "Three girls divide 5 metres of cloth" will be transformed into "3.5 girls divide 5 metres of cloth". This situation effectively changes the student's situation with regard to the obstacle and makes her go through the stages described above.

2.6. Obstacles and fundamental didactics

But obstacles also pose more fundamental didactical problems. In fact, if the student's acquisition of mathematical knowledge *necessarily* occurs according to the scheme of a succession of different conceptions, each more or less forming an obstacle to the following one, then many didactical practices justified by the simply additive classical model must be reviewed and perhaps rejected. But this model affects both internal (within and between classes) and external (between teachers and the society) negotiations with the educational system, in terms of the teaching curriculum.

2.6.1. *Problems internal to the class*

The function of a lesson is no longer only to introduce a new piece of knowledge that it is necessary to learn, which is placed harmoniously next to preceding knowledge, but it is also to encourage the forgetting—or even explicitly the destruction—of old conceptions which had their use but which have become incompatible with this new knowledge.

It is not only a technical problem; the didactical problem is completely different. Not only the diagnosis of errors, their explanation, and the description which follows are modified, but also the reassignment of duties and responsibilities between the teacher and the students.

In order to indicate the size and difficulty of the problem, let us examine, only as examples, two very important aspects of the didactical contract that are open to complete transformation.

a) *How can the teacher accept that the result of her teaching is knowledge* on the student's part, which she knows is not only incomplete but also *false* and will be refuted by what follows. For that acceptance, serious support will be needed, not only from the institution, but also from the culture and from the society. How can the student herself have confidence in a contract which sooner or later will contain such clauses?

b) *The importance of memory of the circumstances of the teaching* cannot be ignored and can no longer be excluded from the teachers' responsibility as it is today.

This point merits an explanation. Epistemological obstacles do not reside in the formulation of institutionalized knowledge (teaching tends to communicate a "clean" knowledge in this respect) but in the representations which the subject—and eventually the teacher—construct in order to ensure the functioning and the understanding of the knowledge. This understanding is linked to the circumstances of the learning and is necessary for the implementation of institutionalized knowledge. The student must therefore remember not only the knowledge which is taught to her but also the circumstances of the learning, this memory being organized in her own way. This memory is at present solely the student's responsibility. The responsibility of the educational system on this subject is limited to the organization of institutionalized knowledge and an *ad hoc* progression. It allows the regulation of questions of temporal dependence in such a way that it is possible for some teachers to produce sequences of worksheets, lessons or exercises without saying anything about their inter-relationships. One could even envisage a succession of different teachers, each doing her hour of teaching knowing nothing at all about the student's past except her institutionalized knowledge. From that point, recognition of the existence of epistemological obstacles and the wish to handle them "officially" in the didactical relationship brings the teacher to recognize her student's history, the historicity of her knowledge and her standing. This recognition causes her, as a participant in and witness of this history, to summon up a different didactical memory. She must remember the context, the examples and the behaviours, and especially the meaning that she causes to develop. Part of this knowledge is institutionalized, at least at the level of the class, and there exists a small inventory of common images held and recognized by the culture. Recollection of some of the students' personal behaviours is essential. This recognition of an item of knowledge in evolution, and of all that this implies, is clearly visible in the experiment on rational numbers cited previously. It is clear that this still personalized and contextualized knowledge cannot be mobilized by students without the support of a witness who has a recollection of the conditions of previous learning.

These *difficulties* are even more visible and acute when they concern *relationships between levels.* Only reference to culturally recognized examples can replace a memory of the context of the previous class; when this reference is impossible, a whole section of the student's acquisition is lost. Ignoring earlier learning is also a way of avoiding the discussion which the revival of old knowledge provokes and therefore of ignoring epistemological obstacles.

2.6.2. *Problems external to the class*

Integration of this new model of didactical communication requires modification of the teachers' epistemology. But this serves as a basis for negotiation between teachers and students, and also with the noosphere and the whole public. One can imagine the magnitude of the social and cultural modifications which would be produced by changes in this domain.

The fact that these items of knowledge, even false ones, can be necessary to support the establishment of definitive knowledge is difficult to accept and negotiate. The idea of over-learning, of lateness of initiation, or of loss accrued by change of class, throws custom into disorder. How can one obtain such a contract without straining even further the difficult, legitimate and necessary control that society has on the communication of knowledge?

NOTES

1. Perrin-Glorian quotes Brousseau.
2. Glaeser reacted to Brousseau's introduction of the concept of "epistemological obstacle" in *didactique* in a discussion about the epistemology of negative numbers (Glaeser, 1981). He objected that such a theorization was premature considering the state of our knowledge at the time. The issue raised by Glaeser and made explicit in his reaction to (Brousseau 1976) is whether it is legitimate or not to consider epistemological obstacles as genuine knowings ("*connaissances",* Glaeser 1984).
3. *Editors' note:* The reader should bear in mind that this is written in the mid-seventies.
4. *Editors' note:* all these texts are published in the Proceeding from which the present Brousseau text is taken.
5. Studies on this subject are currently in progress (Viennot, 1979).
6. *Editors' note:* Brousseau (personal communication) explains that his point here is that since the mathematics required for an actual proof of this equality is at too deep a level

for these students, they will at best simply accept the result, without enough understanding for it to stick with them.

7. *Editors' note:* Brousseau refers to the French curriculum of the early seventies.
8. *Editors' note:* In the course of 1795, the Convention (successor to the National Assembly of the French Revolution) enacted a wide range of social legislation with a considerable long-term impact. Among its issues was that of educational reform, both in terms of modifying the forms of instruction and of establishing schools.
9. More generally, all appropriate "over-teaching" has a tendency to create such obstacles; but are they avoidable?
10. We know today, thanks to A. S. Saydan (1966), that it was al-Uqlidisi who, around 952, was the first to propose the use of decimal fractions and who wrote them as we do today.
11. *Editors' note:* Such a simple thing that it doesn't merit the name "invention" [...] it easily teaches how to expedite with whole numbers without fractions all accounts encountered in the affairs of man (*English free translation*).
12. *Editors' note:* it be regulated more legitimately by the higher authorities, the aforesaid tenth division, so that everyone who wants to can use it (*English free translation*).
13. *Editors' note:* toise: vertical, graduated rod for measuring people's heights; the pied was equal to 32.5 cm according to the *Larousse dictionnaire de français,* 1989.
14. *Editors' note:* Brousseau adds here: "see, for example, later chapters in this book". He is referring to texts of other authors published in the CIAEM proceedings, e.g. Ciosek, Glaeser, Janvier, Wermus, etc.
15. *Editors' note:* Brousseau, in the original version, refers the reader to a text "which is now in preparation"... see Chapter 3 and Chapter 4.
16. *Editors' note:* see Chapter 4, section 3.
17. *Editors' note:* see Chapter 4, section 2.3.
18. *Editors' note:* these examples are presented in detail in Chapter 4.
19. (Note, 1983) The second part of this remark has been shown to be false. The dialectics of the formulation and validation have been shown to be insufficient to result in teaching. A phase involving institutionalization is required.
20. *Editors' note:* see Chapter 4, section 2.2.
21. *Editors' note:* This last section is the "1983 conclusion and comments", published together with the original 1976 version in *Recherches en didactique des mathématiques* 4(2) 189–198. It follows a rather strong discussion between Georges Glaeser and Guy Brousscau about the nature of epistemological obstacles. We omit from the present translation polemical aspects too closely related to the debate of this time. See also Prelude to Chapter 2, note 2.
22. Glaeser (1981) defends the naïve use of the term "obstacle" but nevertheless refers to Bachelard. It is thus a question of a deliberately different point of view.
23. *Editors' note:* Glaeser (1981, p. 308).
24. *Editors' note:* This definition, first published in a work of a Brousseau's student, Duroux (1982), is reproduced in section 2.2. of this Chapter.
25. *Editors' note:* In his autobiography entitled *"La vie d'Henri Brulard"* (1835), Stendhal recalls his problems in making sense of the "rule of signs". His objection is that if one divides the world into two parts, one for positive numbers and the other for negative numbers, he does not see how if A and B are in the negative part, taking B A times gets you into the other part ... But if Stendhal is bothered by the extension of multiplication to integers, the question of d'Alembert is of another nature: should one even consider negative numbers to be numbers? See Glaeser (1981, pp. 310–311 and 323–325).

26. *Editors' note:* Chevallard 1985, p. 50.
27. Brousseau G. (1989) Les obstacles épistémologiques et la didactique des mathématiques. In: N. Bednarz and C. Garnier (eds.): *Construction des savoirs—Obstacles et conflits* (41–63). Montréal: *Centre interdisciplinaire de recherche sur l'apprentissage et le développement en Education* (CIRADE).
28. *Editors' note:* By Horus's eye, Brousseau means the sign/word  which means "part" and which was represented over the writing of a whole number in order to indicate the corresponding unit fraction.
29. *Editors' note:* Brousseau is referring here to the fact that in the steps of a multi-digit multiplication where this product might occur, as for instance in multiplying 321 by 201, rather than carrying out the actual operation 0 x321, the student will simply shift the 642 a column to the left.
30. *Editors' note:* Concerning the teaching of Decimals see Chapter 4.

CHAPTER 3 PRELUDE

The didactical study of the teaching of decimal numbers is both the crucible where most of the concepts of the Theory of Didactical Situations have been shaped, and also the more obvious evidence of the power of the theory. As Brousseau wrote, this research considers "teaching of rational and decimal numbers as an object of study and source of didactical questions".

This research was undertaken by the group assembled around Brousseau at Bordeaux almost from the moment of its creation in 1969. It was followed up by works on number and numerical operations, in particular multiplication and division which augmented or led to DEA memoires[1], documents for the training of teachers and doctoral theses in the years 1972–94. It also led to work on the teaching of probability and statistics in fourth grade (children of age 10–11) from 1972 to 74, where the first genesis of decimals appeared'. The subject was comparison of the results of drawing with replacement from several urns whose composition was unknown ("English bottles"), and the pair, (result of the draw, result of the events considered) appeared with a "semantic" equivalence: frequency. When the number of draws increased, these pairs appeared to regroup themselves in the direction of the natural ratio which represented the proportion of black or white balls. In fact, this notion of frequency is introduced directly by a division. It is the repeated use which gives a student a familiar knowing about decimals. At the end of the year, the research team demonstrated a great success on tests about the use of decimals (operations, order, etc.) but not mathematical knowledge about the instrument the students were using. They therefore sought a different route: Measure of a familiar size, fundamental "situations" with good informational characteristics and above all explicit construction, following the direct axiomatic introduction of rationals as classes of fractions. This construction was in style at that period for symmetrization of monoids **(Z, Q** and **C** were constructed on the same pattern). Salin, a nuclear physicist, then suggested to his wife, a member of Brousseau's group, that their set-up made him think of piles of paper. That was in 1975. The idea was taken up and experimented on.

In 1976, they presented before the cameras of OFRATEME for the training of teachers a lesson, still in very primitive form, on decimals as an approach to fractions.

The study of the dual, linear rational mapping, the puzzle, lessons on reproduction of "the Optimist" saw the light of day and students in April, 1976.

The corresponding lessons were put into experimentation at the Michelet School the same year. Each lesson required some thirty hours of study before being "enacted" with students. But they were reworked every year from 1976 to 1986 using observations of classes and critiques from the previous year. Nadine Brousseau recorded and set up the successive versions in her class preparations. The oldest collection of these first versions is in the “grey notebook”, manuscript of 1978. The volume published by the IREM in 1987 is the text which Brousseau put together for his *Thèse d’état* in 1986 using all this material. Brousseau conserves to whatever extent possible the “preparations” which contain the actual results of the experimental interaction and which make it possible to continue to realize the lessons and observe their spontaneous transformations by Nadine’s successors. (All the publications of Brousseau’s group at the IREM are called “provisional”.)

Chapter 3 reproduces the first of two articles published in *Recherches en didactique des mathématiques,* respectively in 1980 and in 1981, which present the essential of this research. This article shows the rôle played by the analysis of the content to be taught in the understanding of the problems of teaching. This is the first example of the analysis of the process of didactical transposition which Yves Chevallard introduced in the late seventies and to which he gave its full theoretical development in *didactique* (Chevallard 1985)[3]. Brousseau says himself about this study that it may appear “detailed but slightly naïve”; actually the examination it provides of practices in France in the sixties and the seventies is developed enough to make it clear that the difficulty of teaching cannot be reduced to students’ difficulty in learning.

Chapter 3 shows how the mathematical characteristics of the situations in which students will have to construct the meaning of what they are supposed to learn is crucial, and how these situations depend on the didactical transposition and the resulting characteristics of the content to be taught.

Brousseau, conscious of the fact that this study might appear to be dated to the eye of the 1980 reader added a short section about the situation in 1979 in France. We have skipped it here, since the main interest of this text is the methodology and the *problématique* it presents, and obviously it is out of question to update this study seriously for the more recent years.

The Editors.

CHAPTER 3

PROBLEMS WITH TEACHING DECIMAL NUMBERS*

1. INTRODUCTION

Comparing, measuring and reproducing lengths, masses or volumes are activities considered absolutely fundamental, and, consequently, they must be learned in the first years of schooling, immediately after those which consist of classifying, ranking, counting or reproducing finite collections. These activities bring some mathematical notions into play: the rational numbers or the decimal rational numbers. In taking into consideration the mathematical structure that defines them and governs their use, it is possible to reorganize these learning activities around a theoretical project in which students' acquisitions are identified by the knowledge revealed in them, ordered and justified by the place that this knowledge now holds in the body of scientific knowledge.

Now, it is because there exist corresponding, familiar, social practices (whose origins are lost in the darkness of time) that one can undertake such a project. For though the results appear clear, the actions assured, the methods evident, and the characteristics useful, these activities are often in reality, on closer examination, of very great complexity as are also, indeed, the so-called elementary concepts that are involved here. The relationships between these theories and these practices remain very mysterious from the moment that one attempts to translate them into *behaviours* of the subjects who accomplish them, into *modifications* of those who learn them and into *decisions* of those who teach them.

That is why the objective of this study will appear to certain people to be excessively presumptuous. The aim, in fact, is to study the conditions under which these behaviours or these adaptations can appear, along with the relationships between mathematical conceptions —of which these behavious are indicators—and certain characteristics of the *situations* which accompany them.

The number of variables of which one must take account would justify all the pessimism, for such a project cannot do without experimentation to assess these relationships, with a possibly specific methodology, any more than it can do without an initial theoretical investment. Moreover the aim of these observations is, of course, to allow an organization and a control of these situations with teaching as their objective, which leads us to try to locate the field of possible choices rather than to limit ourselves to the observation and comparison of current practices of the teaching of rational and decimal numbers.

---

* Brousseau, G. (1980) Problèmes de I'enseignement des décimaux. *Recherches en Didactique des mathématiques.* 7(2) 33–1 15.

The sources of our text include a large range of activities and research in the field described above, which took place in 1974 with the IREM of Bordeaux[4]. Our text does not constitute a research report, in particular not a report on some specific piece of research. Morever, it seems necessary to us to introduce the reader to discussions about *didactique* by a route that is different from the usual academic presentation of scientific works.

We have started with an analysis of curricula typical of the 60s and 70s and of the epistemological effect of the reform of 1970 on the students' and teachers' conceptions of decimals. We want to make this first study in the language and spirit which were current during that period, in such a way as to arrange a familiar access for the reader simultaneously to the observed object, to the phenomena and to the discussion; that is to say: to the current situation of the problems being addressed by the *didactique* of mathematics. This analysis will be followed by the examination of results and of alternatives to this method, by various theories of teaching and learning or of development. This introductory example will enable us to attest to phenomena some of which will be studied experimentally in the rest of the text.

At the moment when some countries are preparing to adopt the metric system, it seemed to me most opportune to examine in some detail the type of teaching which has arisen in the country which was the first to adopt this system. This presentation is not without risks, in particular that of confusion with the classical pedagogical discourse in use in professional exchanges or with polemic articles based only on opinions. I am certain to be reproached for returning to this subject at the moment when our desire to install *didactique* as a scientific field is manifest. But this mode of presentation must not be misunderstood: such an analysis has been made possible only by the existence of the experimental work mentioned above, which will be presented in more detail in the second part of this study[5]. The informed reader will undoubtly spot certain anachronistic terms and ideas.

It is necessary that it be clear, for example, that our rather brief analysis of the position of Diénès, even if supported by rather lengthy observations, is warranted above all by the existence of another theory about the process of production of mathematical concepts. This active and dialectic internalization replaces the immediate internalization of Diénès in order to bring in the interactive character of knowledge, the activating and observing of which is the goal of our experimental part.

"In the sciences more than elsewhere, one is led to identify the knowledge that one transmits with the knowledge that one creates." This opinion of Bachelard[6] makes the historical and epistemological analysis of the notion of rational and decimal numbers indispensible. This will provide us with examples of the working of concepts in various situations and will allow us to examine some of their characteristics and to study those which gave the notions their significance. We shall identify principally the nature of the obstacles which oppose the evolution of this knowledge.

It will then be proper to try to identify those of these obstacles which help build the concept and the types of situations which are associated with them[7].

[...]

2. THE TEACHING OF DECIMALS IN THE 1960s IN FRANCE

At the beginning of the 1960s, very simple "decimal numbers" as well as "ordinary fractions" were written into the curriculum of the 4th and 5th years of elementary school[8].

2.1. *Description of a curriculum*

Here is the summary of a method presented in a textbook which has been very widely distributed since 1936 and which gives a clear presentation of the stable and current practices of the teachers of this time (*Arithmétique nouvelle au cours moyen* by R. Jolly. Publisher, Fernand Nathan, Paris).

2.1.1. *Introductory Lesson*[9]

The textbook reviews measurements of length with a folding rule. The teacher is to carry them out—with the possible exception of having the text read out aloud or pictures commented upon. There is no commentary whatever on the practical difficulties of this activity nor on the characteristics of counting units with the aid of a graduated scale.

It defines the "submultiples" of the metre—which are in fact "multiples" of the smallest unit presented: the millimetre.

It has the students write the result of this operation in the form of a "decimal number". On this occasion, the word "whole" appears for the first time in the syllabus in order to distinguish a "number" without a decimal part from a "decimal number", which has a whole part, a decimal part and the indication of a unit—thus whole numbers are not decimals.

The analysis of this notation—placed in a table—is nothing other than a familiar exercise of numeration with a change of name in the column headings.

Nevertheless, this change of vocabulary leads to an innovation—for the justification of the name of these subunits conies about by reference to the principal unit. Thus, the "grouping in tens" of studies of enumeration is replaced by a "division by ten" used in the construction of the graduations of the metre. One cannot know if this "division" comes about geometrically, nor how. No doubt it is enough that the result coincides with division in the realm of the natural numbers and this part gives rise to no question, no problem, no action, for the student, who neither partitions nor divides anything.

The textbook gives some examples of the use of the word "unit"—which here means the material object referred to—but indicates only by a title, "change of units", that it is necessary to explain to the children how to accomplish conversion of units.

In fact, the true content of the lesson is indicated by the written exercises: to copy the numeration table and to enter numbers into it, to express the given lengths in metres and submultiples of metres and to "draw a line" of a given length!

2.1.2. *Metric system. Problems*

This lesson is followed by seven others—where one learns, always specifically with reference to lengths and by way of various practical problems, to do addition, subtraction, and multiplication of decimal numbers by a whole number—and by one review lesson presenting the table of multiples and submultiples of the metre, entitled "numeration of lengths".

For each of the other quantities, except for angles—capacities (2 lessons), weights (5 lessons), money (3 lessons)—the introductory lessons are based on the same model.

Eleven lessons are devoted to the study of other quantities: areas, volumes, densities, speeds, and of the corresponding "system of measurement".

All these lessons are followed by applications where it is most often a matter of the calculation of costs, lengths, areas, volumes etc., and where decimals are used in everyday settings.

2.1.3. *Operations with decimal numbers*

The textbook suggests the learning of multiplication of a decimal number by 10, then by 100, then by 1000, then by a natural number, then multiplication of a whole number by a decimal number and finally multiplication of two decimal numbers.

It should be noted that in finding the product of a decimal number and a whole number, changing the unit allows the validity of the rule to be demonstrated. On the other hand, in the multiplication of a whole number by a decimal number, the latter loses the indication of the unit which can no longer be changed. The rule is given without proof. The verification—established in a particular case, with the aid of a fraction: ½ replacing 0.5—is taken from current practice, for no theory has yet been advanced on this subject.

The study of division—defined as a concrete operation: the division of a cake—follows a plan which testifies once again to a concern for sticking simultaneously to mental representations which allow its understanding and to the execution of tasks which are progressively more complex. The order of the lessons—division of a natural number, then of a decimal number (measure), by 10, by 100, by 1000; division of natural numbers with a decimal quotient; division with the dividend, then the divisor being a decimal number; and finally division of decimal numbers—suggests that each step of the algorithm is to be explained by the preceding ones, but these explanations never appear.

The justification of division by a decimal number, which loses anew the indication of the unit and its character of "measure"[10], is based on the property established for natural numbers, and extended without commentary to decimal numbers, of the invariance of the quotient under multiplication of the dividend and the divisor by the same number. A special lesson is devoted to the division of a dividend by a divisor greater than itself.

2.1.4. *Decimal fractions*

Decimal fractions are introduced, just before ordinary fractions, as a new notation for the decimal number already studied. The five lessons which are devoted to them consist of reformulating the rules of calculation of decimals in terms of operations on fractions.

It becomes clear at this point that it is the theory of writing and not of mathematical facts which is being taught. Thus, $\frac{3}{10}$ is a decimal fraction, but $\frac{1}{2}$, $\frac{2}{5}$, $\frac{3}{5}$, etc., are "ordinary fractions"!

2.1.5. *Justifications and proofs*

Contrary to other textbooks of the same period[11], which content themselves with stating and applying the rules, this textbook attempts to justify them either by means of a real proof, or by an example, or by a verification. Even in cases where proof is not possible, the presentation of the text suggests that the student can and should understand. "The quotient" of two arbitrary numbers is always either an "exact decimal" or an "approximate decimal".

No study is made of a non-decimal rational number. The student should carry out division until she has obtained a reasonable precision depending on the context. This convention is not even stated; it used *de facto.*

2.2. *Analysis of characteristic choices of this curriculum and of their consequences*

2.2.1. *Dominant conception of the school decimal in 1960*

This method can be considered as typical of this period at least in the following characteristics:

a) The decimal number is always the expression of a "measure " (in the non-mathematical sense).
b) These measures take place in a metric system.
c) The decimal is defined as a natural number equipped with an indication of the unit and with a decimal point which indicates the place of the ones (the chosen unit).
d) The calculation algorithms are presented as being the same as those for the natural numbers, simply completed by a procedure for dealing with the decimal point.

2.2.2. *Consequences for the multiplication of decimals*

These choices have consequences which can be perceived:

a) No decimal operator:
The notation 3.25 has no meaning if the type of measure used is not accompanied by the appropriate unit. The use of decimals will thus be limited to cases which have previously been designated “concrete numbers”.

b) The product has no meaning:
It will be almost impossible for a student to give a meaning directly to an operation which consists of multiplying something by a decimal: “3.25 m x 4” can be looked upon as a repeated addition, as in the case of natural numbers; but not “4 m x 3.25” if the 3.25 was not obtained by the evaporation of the unit. These are not tricks like those we have already indicated, which can reduce the obstacle. Equivalences of the kind:

$$\text{“}4 \text{ m} \times 3.25 = 3.25 \text{ m} \times 4\text{”}$$

are inconceivable; likewise:

$$\begin{aligned}\text{“}7.25 \text{ m} \times 4.38 &= 7.25 \times (4 + 0.3 + 0.08) \\ &= 7.25 \times 4 + 7.25 \times 0.3 + 7.2.5 \times 0.08\text{”}.\end{aligned}$$

The product can no longer be interpreted with the aid of the representation in natural numbers which provided the definition. In order to continue to give the product of two decimals a “concrete” representation—that is to say: conforming to the conception described in the preceding section (§ 2.2.1.)—it is necessary to restrict its use either to cases of “product measures” (for example, the area of rectangle) or to that of isomorphisms of measures (for example, the price of a non-discrete quantity, the weight of a given volume of a homogeneous material, etc.).

c) Product measures and dimensions:
In fact, in the case of product measures (for example the area of a rectangle), one writes “2.5 × 3.25 = 8.125”, only if one has suitably chosen the unit of area (the rectangle having for length the unit chosen for the first number and for width that chosen for the second number). There is no way of justifying *a priori* this formula rather than 2.5 × 3.25 = 812.5, or that formed with any other value.

d) Proportions:
The case of isomorphisms of measures is more favourable. One measure acts on the other as a set of multiplicative operators while remaining a measure. What is required, however, is that a new interpretation of the product of two numbers be constructed, if possible encompassing the older interpretation used by al-Khowârizmî in the ninth century, in order to unify the notion of number: a × b is the number which is to *b* as *a* is to 1.

Faint-hearted attempts to detach the decimal from its function of being the result of a measure remain entirely formal:

– substitution of more general names (ones, tenths, hundredths) for those specific to the metric system and extending the denominations in the natural numbers (tens, hundreds, etc.);

– putting the name of the unit before or after the number—3.25 m or m:3.25 instead of 3m.25 (since 1945).

The pure and simple omission of the mention of the unit, starting with some lesson and without warning—the evaporation—as one saw done systematically after the reform of 1970 is frankly abusive but only furtively produced, the pressure of mathematicians to isolate the objects about whose structure they theorize not being very strong at this time.

2.2.3. *The two representations of decimals*

This is why one can observe that ratios and proportions (within the same dimension—abstract numbers) are always expressed, if they are not whole numbers, in fractions (ordinary or decimal) or in percentages. One might suppose that this specialization of vocabulary is only a vestige of their history, but it probably corresponds to distinct representations: on the one hand, decimal numbers representing measures with an addition and a product by a natural number, and on the other hand, fractions, even though they are also apparently defined from measure (cut a pie, then take a number of parts), used as ratios.

The *effectiveness of the reasoning process will therefore depend on how easily students are able to cope*, in a problem-solving situation, *with passing from one to the other of the representations*. The capacity to substitute one of the formulations for another in the course of an enquiry is just an indication that the student is able to make these changes of point of view. Further on, we shall give a historical account of these relationships.

2.2.4. *The order of decimal numbers*

But, in fact, these school decimal numbers are really just whole natural numbers. In every measure there exists an indivisible submultiple, an atom, below which no further distinctions are made. Even if the definition claims that all units of size can be divided by ten, these divisions are never—in elementary teaching—pursued with impunity beyond what is useful or reasonable, even through the convenient fiction of the calculation of a division. This tendency was to be clearly expressed in the commentaries of the programmes of 1970, where the decimal number is introduced as a measure of the cardinality of finite sets: "the thousand being chosen as the unit, the population of a town of 10 850 inhabitants is expressed by the decimal number 10.850".

Under these conditions, decimal numbers retain a discrete order, that of the natural numbers; many students using this definition will have difficulty in imagining a number between 10.849 and 10.850. Morever, this type of question was never considered during this period and can be solved only by means of learning an algorithm or by "imagining" a new sub-unit. But how?

Comparisons and sums of decimal numbers are not often correct unless they are written with the same number n of digits after the decimal point; that is to say: if they are presented in the same $\mathbf{D}_n$ (set of decimal numbers such that $10^n \times \mathbf{D}_n \in \mathrm{N}$) and only then if they are interpreted as natural numbers[12].

The embeddings of these $\mathbf{D}_n$ ($\mathbf{D}_0 \subset \mathbf{D}_1 \subset \mathbf{D}_2 .. \subset \mathbf{D}_n$) are difficult when the situation becomes more complicated, particularly for $\mathbf{D}_0$, the set of natural numbers, for we have seen that even though decimal numbers are basically natural numbers, whole numbers have been declared not to be decimals!

2.2.5. *Approximation*

The distinction between whole part and decimal part derives from methods of measurement and evaluation especially in calculations where the operation on the whole parts gives the order of magnitude. This distinction is quite useful, since 0.31, 3.1, 31... have a tendency to become identified (it is sufficient to choose its unit).

But then, the decimal number thus cut in two produces a temptation to follow certain rules for the whole part and other rules for the decimal part[13]. For example 3.9 will be smaller than 3.12 (because 9 is less than 12)[14] or, again, 2.3×2.3 will mentally give 4.9.

Decimal numbers will be identified with the decimal parts and thus be smaller than unity. Students hesitate when asked to find a decimal number having a single digit after the decimal point and larger than 0.9.

This idea that the measure leads to the provision of a number, “approximate” but “which counts,” and a negligible remainder takes strong root, mainly in division, which provides a visibly infinite sequence of decimal places which must be hidden without more ado because they have no possible status.

And when, later, in a problem, a student considers a series $\sum a_n$ of positive terms, whose general term tends to zero, and thinks that it converges, is this because she is going back to the implication of a known theorem, or rather because somewhere there still exists the idea that ($\forall \varepsilon\ \exists n$ st $p>n \Rightarrow a_p<\varepsilon$) means that ($\exists n$ st $p>n \Rightarrow a_p = 0$) by reference to the representation of decimal numbers which has been learned since childhood?

2.3. *Influence of pedagogical ideas on this conception*

2.3.1. *Evaluation of the results*

During this period, these difficulties did not appear or were held to be unimportant. The teaching of arithmetic presented no difficulty for the teacher and not much for the majority of the children. And if in mathematics there was one subject which led to no discussion and presented no difficulties it was that of decimal numbers. It was moreover so strongly associated with the use of the metric system and with measures employed in school problems that no-one could see how it could be taught otherwise or how one could not be successful in teaching it.

This opinion must be explained if one is really to understand the conditions of teaching decimal numbers in this period.

2.3.2. *Classical methods*

From the very first, none of the pedagogical theories of these days either claimed, or could provide, variants which would give an alternative to the dominant conception of the school decimal number of 1960. The contents were assumed to be constituted and organized by the rules of their discipline, and pedagogy was only the art of communicating them. It was supposed to have neither hold nor effect on them.

Let us first consider the oldest:

- Dogmatic methods led first to the establishment of rules and then to the application of them.
- Maieutics (Socratic) methods proceeded by means of questions and answers and simulated a rediscovery of the rule.
- Active methods emphasized the importance of the time devoted to the student's activity, above all manual activity, as compared to listening time or formal learning; but these activities were either manipulations illustrating an introductory statement, or exercises. Prisoners of the ideology of the hand teaching the brain, they only accidentally found effective problem-situations.
- The sensualist–empiricist move influenced the textbooks (or maybe vice versa) and these were swollen with coloured illustrations.

All these methods adhered to the following principles:

a) No knowledge is introduced which cannot be either accepted immediately or defined or explained with the aid of knowledge previously acquired.
b) Each definition or explanation must be accessible to formulation and justifiable—if not formulated—in the language of the science being taught or at least in its most widely accepted cultural expression.
c) Each "teaching" makes way for an identifiable and controllable learning, and, in the limit, only that which can be learned is taught.
d) The acquisitions undertaken continue and justify themselves by the use—importance and frequency—which is made of them in future lessons.

2.3.3. *Optimization*

These principles lead naturally to a conception of the optimization of teaching which implies principally that:

a) Each lesson must aim for the maximum number of acquisitions (the flow) compatible with the learning capacity of the students.
b) The learning of a concept should take place in a form which best uses earlier knowledge and modifies that knowledge least.
c) It must require the minimum time and must be consolidated by means of frequent subsequent use in applications of practical interest.

The introductory lesson that we have spelled out in detail does not seem to satisfy condition (a) above; but this is because, in the main, the majority of the facts are stated there only to justify the subsequent daily use and not to be learned immediately.

2.3.4. *Other methods*

Other methods having the aim of finding better motivation for students lead to a more or less profound reorganization of teaching. However, again, except by accident, these motivations, principally centred on the *milieu* (modern school of Freinet), or on centres of interest for the child (Decroly), or even purely arbitrary, are exogenous in relation to knowledge and without relevance to it.

Order of acquisition appears susceptible to total upset relative to the one which we have explained and which all the others follow fairly closely.

But, in reality, the breaking of the order of acquisition and the choice of familiar and stimulating situations hardly affects acquisition itself nor its significance. Actually, whenever the student faces a new problem:

– either the context allows him or her to construct a solution without having to refer to an earlier acquisition;
– or, otherwise, after failure and consultation with adults, it seems to the student that there does exist a technique for solving the problem.

In the first case, teaching must establish, without motivation, a process of identifcation of the question which has been posed and of what has been discovered and needs to be organized after the fact, as a piece of knowledge to be reapplied, and therefore to be learned. In the second case, the technique must be learned and it is necessary to worry about recognizing the conditions under which one can re-use it.

In both cases, the information, the knowledge, the use and the justifications are those of the dominant conception.

In addition, beyond the moment of discovery which holds the principal attention of the users of these methods (perhaps because it implicates the teacher more directly), learning continues by a solitary empirical effort of the student, who must respond to "auto-corrective" worksheets[15] of a nature close to programmed learning.

The principle problem of *didactique* consists of finding really specific situations of different conceptions of decimal numbers and of organizing these situations and these conceptions from the start in order to make possible an artificial genesis of knowing what to do, knowing what to say and knowledge.

2.4. *Learning of "mechanisms" and "meaning"*

2.4.1. *Separation of this learning and what causes it*

The short analysis given above described the conception which prevailed at this stage of the investigation of knowledge, and principally of algorithms in the construction of knowledge—a conception which itself plays a rôle in the creation and the functioning of representations of decimal numbers.

This learning was conceived in two parts that can be looked at separately:

– the learning of the algorithm that teachers called the "mechanism" of the operation, and
– what is called the "meaning" of this mechanism; that is to say, knowledge about when to apply it.

The first involves classical techniques of teaching and, in the limit, of conditioning; the other can be learned, by means of repetition of examples and applications in problems, only thanks to mysterious transfers which "the student accomplishes if, and only if, she has sufficient intelligence".

Certain textbooks make attempts to reduce the teaching of meaning to that of a mechanism by means of clever classifications of situations (search for the number of parts, search for the value of a part for division, for example) by the identification of special steps (the rule of three[16]), and indeed by the search for linguistic tips indicating the operation to be carried out ("it lacks", "there remains", "you take away",..., to recognize subtraction).

These attempts lead to the rejection of certain problems, certain formulations, and thus contribute to the modelling of the dominant conception.

This separation between what can bc taught formally in order to be applied "mechanically" and what cannot be—between form and "meaning"—plays a fundamental rôle in teaching. It results in what appears at first sight to be the extreme result of the didactic negotiation studied by M. Verret (1975) by which the taught knowledge consists of a didactical transposition of practical knowledge.

Actually, this didactical negotiation leads to the distinction, in what has been experienced naturally by the student as a normal response to intentional situations, of what is the object of knowledge and what is not, between what was a "problem" and what was a stupid hesitation, between what was a trivial adaptation and what was learning, between what must be learned and what can be forgotten. Later, we shall deal with the theoretical study of this important phenomenon, but we can observe immediately how it works in this precise case[17].

2.4.2. *Algorithms*

An algorithm is a finite sequence of executable instructions which allows one to find a definite result for a given class of problems. At the time of its use, it is a decision procedure, but the importance resides in the fact that the algorithm can be determined in advance, the execution of the nth instruction doesn't depend on any circumstance not foreseen in the n–1st instruction; it doesn't depend on the finding of new information, on any new decision, any interpretation, and therefore on any meaning that one could attribute to them. This is why the execution of an algorithm can attain a high reliability and a high speed of execution.

The determination of an algorithm allows the complex tasks in which it appears to be organised in a hierarchy and makes it possible to reason about them. A large number of activities and equilibria can be described as algorithms and thus appear to he the product of a mechanical execution or an "automatism" according to an expression which is erroneous, but current among teachers.

Very useful and well known in the sciences and technology (all the more as they get more complex), algorithms are tempting for teachers because:

- they can be learned either directly in the form in which they are used or by a natural "complexitication" (insertion of sub-algorithms);
- they can be taught without recourse to meaning, which they are, after all, designed to avoid, and therefore by formal methods: repeated application, recitation,...;
- their acquisition can be monitored; non-acquisition allows simple didactical decisions, in a clear contract, where the student's responsibility can be engaged *a priori;*
- their utility can be proven since their use is identifiable in their applications;
- after being learned, algorithms assure in the execution of tasks a phase of high reliability, great speed and confidence.

Moreover, they play a large rôle in the justification of the teaching model of learning by conditioning to which teachers refer either implicitly or explicitly.

In fact, the use of an algorithm is, in relation to mental activity, like the visible part of an iceberg. In numerous cases which serve as a metaphorical reference to pedagogical theories which advocate them, algorithms are acquired by a totally different procedure from formal learning: they are the result of an adaptation of the subjects, either continuous or by stages, where meaning and the character of the situations play a very important rôle In a certain manner, to learn calculation algorithms and the conditions of their use separately is an activity which is comparable to one consisting of learning quotations and when to produce them. It is conceivable in academic literature, but does not permit the learning of a language.

This method leads to making explanations and understanding useless—non-functional, in the sense that in case of error or uncertainty they are never used to rectify the algorithm. The only help is mnemotechnical aids, that is to say, ones which are arbitrary in relation to content. Motivation to carry out a task can thus become irrelevant, and therefore give way to more pedagogical requirements.

We therefore see it disappear in certain textbooks (Bodard, cited above), or playing only a secondary rôle such that explanation becomes a new piece of knowledge to be put beside the others. And if only the best can learn it, that is because it is not indispensible. It must therefore be considered as another knowledge, independent of the know-how..., a luxury which becomes superfluous at the first emergency.

Thus, the pedagogical preoccupations of the Sixties didn't take content into consideration in the design of teaching situations and even tended to reject it as organized knowledge in order to keep only the aspects of it which were related to the knowing of facts and algorithms[18].

At the conclusion of this study, one can assume that the method which we have presented is highly typical of those which were used in the 1960s. We will do without any verification, either historical, by the analysis of manuals, or experimental. by the analysis of the results of students in classes which conserve these methods (they are numerous). Indeed, these methods, obtained by the effect of "pedagogical variables", are not naturally exclusive; every textbook, every teacher combines them according to formulations which change from one lesson to another.

Only the uniformity of the results and their small dispersion can attest to the exactitude of our initial statement: the pedagogical variables mentioned above have no appreciable effect on conceptions which the students have of a concept such as decimals. We shall not make these observations until we have examined the reforms of the 1970s.

3. THE TEACHING OF DECIMALS IN THE 1970s

The reforms of the 1970s, in France in particular, can be seen to some extent as a reaction against the situation we have just described, but in order to express themselves or justify themselves they had to rely on some of the foregoing conceptions. The teaching of decimals was not at the centre of the struggle in which innovators and occasionally their adversaries were involved, but that makes it all the more interesting to examine them for the trace of the new ideas—which we will not list here.

3.1. *Description of a curriculum*

Let us examine the cxercises proposed by N. Picard in the *Journal de Mathématique*[19], some of which are commented on in "*Agir pour abstraire*" by the same author[20].

3.1.1. *Introductory lesson*

The student measures the area of tiled surfaces by counting the tiles. The unit of arca is the name given to the tiles ("the unit of area is a little triangle" mes. 3)[21]. She then has to replace this numbering by covering the figures with "pieces" from a system inspired by Diénès's multi-bases:

A covers 1 tile, B covers 2, C covers 2^2, D covers 2^3,
E covers 2^4, F covers 2^5. G covers 2^6.

After exchanges so as to have the minimum number of pieces of each sort, the result is transcribed into a table where the area—a natural number—is written in base 2. The student is then invited to draw a surface M, of area 11011 "when A is the unit", then measure this area by taking E "as a unit". The unit has become one of the pieces and by the game of exchanges, the class of equivalent figures of the same area and the same type (these terms and remarks do not appear in the text). In order to indicate that in this material operation she has changed the unit, the student must, in the table, put a hat on the name of the piece chosen and a point after the number of such pieces.

After exchanges have been made, one obtains the same table as appears in the statement—with the point and the hat in addition. The changes of unit take place within the same system; a square tiling is not replaced by an hexagonal one, for example.

3.1.2. *Other bases. Decomposition*

The same kind of work is done in base 3 and then in base 10. Exercises remind the students that to multiply a number by the base, it is enough to shift the digits one place to the left.

It appears in the account of the teacher's activity that accompanies these exercises, that in a very maicutique dialogue, the children have been induced to transcribe the following table:

G	F̂	E	D	C	B
1	1	0	0	1	1

Table 1

into the identity: S = 2f + 1f + f/8 + f/16.

Mme Picard's commentary about the situation is insufficient to allow us to discern the exact significance of this behaviour. The terms f/8 and f/16 were "found spontaneously". But this is the first time that a writing decomposed in this way has appeared. The teacher gives an example by saying, "Let us write in mathematics what he said". Straight away, the "f" can disappear:

$$S = 2 + 1 + 1/8 + 1/16.$$

The author indicates, however, that decimal notation is understood only when the children place the number in a table. The exercises are written in different bases, but the students do not make changes of base. They do not rewrite an area given in base 2, in base 3.

3.1.3. *Operations*

Operations involving addition and multiplication by a natural number on "numbers with a decimal point " (base 2) are studied in the same type of situation (measurement of areas with the aid of pieces), then, after exchanges, direct counting in the base.

3.1.4. *Order*

The study of the order of decimal numbers uses an entirely different representation. It is a matter of "numbering" library books. The books are coded with the letters A, B, C,..., but for reasons which are not explained, the librarian shelves each new book in a selected place between two other books that have been shelved already. For example, F being placed between B and C, "it is a question of finding a code for F", which is understood to mean:

1) in order to indicate F's position in relationship to the already shelved books,
2) but, such that it remains the same when new books are shelved, still keeping property 1).

Certainly, this "numbering" does not permit the counting of the books on the shelf.

In fact, the problem is not posed; the solution is given immediately, ("here is a method") not in the form of a practical method which would generate only a few codes (those corresponding to random introductions) but in that of an algorithm which will generate the whole scale; that is to say: all codes having one digit after the point, then two..., conforming to the model mentioned above (§ 2.2.4.)

3.1.5. *Operators. Problems*

Immediately afterwards comes the classical preparation for the product of two decimal numbers without indication of the unit. The author introduces a ranking of digits using whole numbers (**Z**) ... 3,2,1,0,$\bar{1}$,$\bar{2}$,$\bar{3}$ ... Multiplication by 10, 100, 1000 consists of shifting the digits in the table one place towards the left. The operation is justified by the exchange of pieces—the model remains necessary—of one place for that of a place immediately above.

Division (of what? Numbers, measure numbers,... units?) consists of an inverse algorithm: 123.35 = 12335 ÷ 100.

Then the product of two decimal numbers—when this is going to have to be carried out is unknown—is interpreted in terms of "operators":

$$\begin{aligned} 123.35 \text{ x } 4.3 &= [12335 + 100] \times [43 + 10]^{22} \\ &= [12335]\ ((\div 100).(\times 43).(\div 10)) \ldots\text{etc.} \\ &= [12335]\ ((\times 43).(\div 100).(\div 10)) \end{aligned}$$

Here, we translate as numbers in the square brackets those expressing measures and as numbers between parentheses those associated with the "operators" represented in the text by arrows. (The dot represents the "composition" of these arrows, and we have therefore distinguished three "products": the internal operation ×, the external operation [] (), and the composition of mappings, that are mixed in the text.)

This interpretation of the "concrete number" [43] as the operator (× 43) was prepared for in the study of natural numbers by numerous exercises on what the author called "machines". No problem-situation is offered in which the multiplication of two decimal numbers could take on a meaning.

Division is presented according to the old scheme: it is pointed out that in the natural numbers and for Euclidian division: $a \div b = ka \div kb = \frac{a}{h} \div \frac{b}{h}$ (if h divides a and b).

The property is extended to decimal numbers.

Apart from these introductory lessons and studies to which 32 worksheets out of 131 for the fourth grade and 58 out of 149 for the fifth grade[23] are devoted, no other allusion is made to decimal numbers (nor applications nor uses) except in the study of the bounding of an area (mes. 44 to 51).

3.1.6. *Approximation*

In a first exercise, students must verify that the bounding of an area of a surface which is connected, but not formed by the juxtaposition of atoms, depends on the position of the grid which is used for the estimation and which replaces the pieces. But does this area exist in this case?

Then they calculate the area of a surface bounded by the lines of the grid, first with one unit, then with another that is six times smaller. The partition of the unit is thus suggested, but the two grids are presented simultaneously and the situation doesn't formally differ from earlier ones (counting with groupings).

In exercises which follow (mes. 50 and 51), bounding of the area (?) of a surface with the aid of finer grids is supposed to suggest the search for precision.

This presentation calls for a few remarks.

3.2. *Analysis of this curriculum*

3.2.1. *Areas*

The introduction conforms to earlier conceptions, but the recourse to surfaces is clever because it clarifies the differences between different objects of the same class to which measurement assigns the same value. However, the use of material objects and the rule of exchange takes on a useless exercise of numeration that reinforces the counting aspect imposed by restriction to certain figures, a sacrifice to the ideology of manipulation.

3.2.2. *The decimal point*

The decimal point and the hat have no meaning when the numeral is written down in a table, as is always done in the first part of the work. The "mathematical" notation decomposed in polynomial form could truly have served as a basis for a correct study of the decimal number. But this episode appears rather casual and without follow-up; the author seems interested only in transcription.

3.2.3. *Order*

The study of the order of decimal numbers really is an innovation in response, as we have seen, to an insufficiency of earlier methods, but how do the reasoning carried out or the properties stated about numbering connect with the decimal-measurement already introduced? Or if these are new "numbers", how do we conceive of their sum for example?

3.2.4. *Identification and evaporation*

All that these entities have in common with the each other is the way in which they are written. Their representations are distinct, isolated and live together without any connection—other than a formal one. In order to reason about the order of the "numbers with a decimal point"—measure, one has to remember the algorithm for arranging books. No activity makes it possible for a new representation to include the two prcccding ones (one could, for example, have defined a distance between the marks and then a measurement of these "segments").

Meanwhile, from these lessons on the decimal loses all reference to a given unit. The decimal point retains a meaning, however, when one has to compare or add up decimal numbers : it shows the digit of the unit in each of them which is supposed to be the *same,* evcn if one does not know which it is (this explanation is neither asked for nor given to the students).

When the proposed numbers have nothing to do with each other and there is no reason to suppose that it is the same unit which has been used, the unexpected disappearance of the indication of unit means that at the moment when the decimal point could effectively and usefully furnish the intended information, it conveys another piece of information, and one doesn't know what it is.

Now, the identification between the measure—decimal and the operator—decimal hasn't been constructed. On the contrary, the insistence on their difference of status, reinforced by the use of diagrams, doesn't make it easy. Thus, it can only be done in a completely formal manner. Each conception inherits properties and rules established in the other by the magic of their identical notation. This is why it was necessary to use the decimal—neither as measure nor as operator—as a formal bridge: every representation in a concrete example would not only fail to bring any proof' of the legitimacy of the inheritance, but would furthermore obstruct its functioning (as a joke we've called this process the evaporation of the unit).

This identification is one of the fundamental problems in the creation of meaning in decimals. It has been totally neglected in studies about teaching. We will study it in the case of rationals where it has a much closer resemblance to the identification of a vector space with its dual.

3.2.5. *Product*

For the notion of product there is also nothing new, but it appears clear that the comprehension—not to mention the meaning—of the calculation of the product lies in the capacity to identify a "concrete" number, [43 ÷ 100], with an operator, (×43).(÷100).

3.2.6. *Conclusion*

The decimal is always introduced to represent a measurement made with a rather small unit and expressed with the help of another multiple.

That decimal is always a "concrete" natural number. On the other hand the metric system has been rejected as the primary domain of significance for the decimal. Its study occupies far too modest a place (one introductory exercise for each dimension: lengths, surfaces, volumes, masses). The algorithms of calculation are presented in a classical way, although the justification of the calculation plays a much more important role and rests on a quite heavy formalism (not mathematical but the subject of a prolonged teaching process: machines).

Thus the characteristics shown to be typical of the methods of the Sixties are for the most part conserved in these textbooks which are presented as being very innovative.

3.3. *Study of a typical curriculum of the '70s*

The most widespread textbook following the reform of 1970 was probably that by Mme Touyarot (*Itinéraire Mathématique.* Paris: Editions Fernand Nathan. 1976).

3.3.1. *The choices*

The choices of the questions which interest us are the same: introduction as a measure of finite cardinals with changes of unit then "evaporation" of the indication of unit, which makes it possible, when one is comparing two natural numbers or two decimal numbers, to have two different algorithms—provided without any justification—for the same ordering of numbers which continue to be treated as naturals.

3.3.2. *Properties of the operations*

While the properties of operations in naturals are explicitly founded on the examination of a representation (operations relating to sets—but the terms "union" and "intersection" are not pronounced), those of operations on decimals are simply borrowed from the preceding ones, certifying that with this representation there is no longer a problem, nor a debate, nor even any novelty. The decimal no longer exists as a mathematical entity but only as a transcription of an entity which is already known.

3.3.3. *Product*

For the product of two decimals, the student is asked to note that "one packet of sugar contains 168 pieces of sugar and so 1.68 hundreds of pieces, 28 packets of sugar contain 1.68×28 hundreds of pieces". Elsewhere these "2.8 tens of packets of sugar contain 16.8×2.8 hundreds of pieces of sugar since a group of ten packets of sugar contains 16.8 hundreds of pieces. So (evaporation): $16.8 \times 2.8 = 1.68 \times 28$.

3.3.4. *Operators*

Much later, the decimal operating in naturals then in decimals is defined as a mapping: ($\times$ 2.5), which allows one to present the givens for a rule-of-three problem.

3.3.5. *Fractions*

These are then defined as an operator reduced from a chain of operators itself presented without reference to any type of problem-situation whatever. They will "be useful" further on.

So the textbook falls back on the notion of decimal to show that a fraction (operator) can sometimes, by decomposition and recomposition, be "equal" to a decimal operator. Curiously, in this textbook, although there are very few exercises in non-decimal bases, the expression "number with a decimal point" is substituted for the word "decimal" frequently and without apparent reason.

For example, to establish that $\frac{21}{28} = \frac{75}{100}$, one undertakes a tedious proof, illustrated with a new type of diagram whose conventions have not been studied.

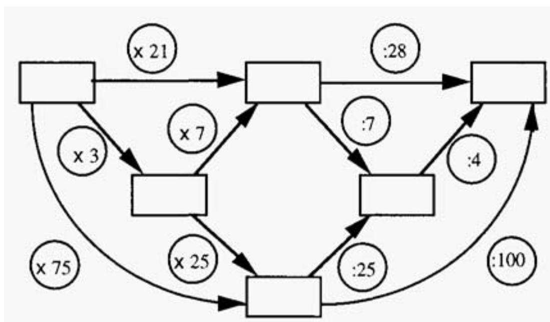


Figure 1

But we don't know why it is that: $\frac{75}{100} = 0.75$[24], nor why, to obtain the same "result", it suffices to divide 21 by 28 while writing as many zeros on the right of the dividend as are needed in order to finally obtain a zero remainder. (For the occasion the discourse is peppered with unknowns: x, y, z for the first time in the course.) Only one example of approximation is provided to explain and take account of those cases where division "never ends" ($\frac{10}{11}$).

3.3.6. *Conclusion*

This eclectic textbook preserves the old outline and illustrates it with all the new contributions of the time. The problems are numerous and very varied. The teacher doesn't know which ones to chose and doesn't clearly see how the solution of one will help the student solve another—which means that she will not recognize there the learning contents which she is used to. The vocabulary, too, is the result of a negotiation of circumstances; a few new terms, tied on to old objects. But let just one obstacle appear, as in the proof of the preceding paragraph, and the mathematical language reappears with the full arsenal of didactic innovations of this period. It bespeaks a difficult negotiation, the conditions of which we will now examine.

3.4. *Pedagogical ideas of the reform*

3.4.1. *The reform targets content*

In contrast to to earlier pedagogical movements, the reform of the Seventies aimed principally at modifying the content, the formulation, the organisation and the order of introduction of the mathematical knowledge taught. Although it often took pedagogical positions, none was specific to it. Having noted that, apart from some complements and changes of vocabulary, the concept of the teaching of decimals hadn't fundamentally changed, one would be tempted to infer that the difficulties of the students must remain the same.

Now we shall see that on the contrary the modification of the content led to that of the conception of learning, and to the reorganisation of the students' activities and to that of the teaching of mathematics. This reform claimed above all to be one of content:

- the mathematical knowledge had been re-organized and unified;
- the vocabulary had thus changed, there were new demands as to rigour, the scope of application of mathematics had enlarged in part thanks to the fruitfulness of this re-organization;
- the unification was expected to allow a better effectiveness of teaching by permitting learning to aim less at the solving of problems, and more at learning how to learn.

This is why it was asked for, promoted and masterminded by mathematicians. But the very concept of this unification carried in it various hypotheses about the functioning of mathematics as a "theory of structure"[25]. These structures, detachable from the content "can be conceived of independently of the nature of the sets they are based on."[26]

Thus, since mathematics is a means of making knowledge, learning mathematics is learning to learn. For many people, new mathematics is thus organized as a formal *language* of which the *semantics* are the theory of structures, of which the synthesis can be displayed, and which is applied in different fields by undergoing adaptations dependent on the *pragmatics* (a structuralist concept). This concept held out hope for learnings separated in a way contrary to what Piaget's work would allow one to think.

3.4.2. *Teaching structures*

If one wants to accept the wager of organising the acquisition by children of this type of mathematics it is thus necessary to say how these structures can be appropriated straightaway. In fact the economy of the structuralist plan consists of refusing to let the meaning of the structures lie within a history of the subject interacting with situations which are too rich or too numerous, where these structures could get stuck down and could take on meanings and limitations which are too particular. Furthermore, educational ideas of the period led to the rejection of a scholastic study of mathematical discourse.

The problem has been the subject of several attempted solutions. We will retain only that attempt proposed by Diénès which we consider to be the most representative and the most explicit. We will indicate, however, a frequently observed means of slipping away from it.

It consisted

- using the support of Piaget's work, of thinking that in fact genetic epistemology demonstrated the appearance not of elementary knowledge but of entire structures, and this in a genesis which went from the general to the particular;
- and of concluding, with a dubious reference to the ideas of Rogers, that the activity of the subjects and their natural development led to the appropriation of the basic knowledge that had been aimed for, on the condition that no obstacles to this be created by an untimely normative *didactique.* This position did not provide "communicable" teaching practices but justified many "wildcard pedagogical" experiments[27] which claimed to reject "traditional methods".

On the other hand, Diénès explicitly proposes a solution which he illustrates with numerous lessons and new material[28].

The process, which he calls "psychodynamics" proceeds generally in six stages:

- the playing stage;
- structured games;
- isomorphic games and abstraction;
- schematization and formulation;
- symbolization and formalization;
- axiomatization;

but is accompanied by phenomena like, for example, that of generalization.

3.4.3. *Diénè's psychodynamic process*

The interpretation which we are going to give of the Diénè's psychodynamic process does not conform entirely to the author's writings. It is an attempt to approach the interpretation that teachers were able to give it in the context of the ideas of the time. It corresponds, in any case, to the majority of the educational exercises and situations which refer to it and which I've been able to observe.

Structured game

A structured game is a game in which the child acts according to the rules that are given her either in a formal manner, or by means of the context. More often than not, these rules are those of the structure which is to be discovered.

The typical case is that of groups. The teacher choses a representation of a group, for example "Klein group" as a group of transformations of a certain set (for example four positions of a doll which somersaults, turns and rolls). The teacher presents this set, invites the student to make these transformations, to look for all of them, to find which one applied to this position gives that position, which one can replace this other one, etc. These activities are carbon copies of mathematical activities and correspond to the solutions of equations, to the search for properties, etc.

The word "game" is used in the sense in which it was used at the time pedagogically. It means that the situation is unusual, detached from the obligations of learning, from scholastic demands, that one can take liberties with it, that it brings the imagination into play, that one can make discoveries, exchange them. Even though the student is faced with the problem of achieving an objective in the framework of rules which are given to her, the educational situation is not generally organized like an ordinary game, nor in such a way that the player has several chances to try out the diverse strategies that she could put in competition. The term "game" thus here covers truly open situations as well as trivial exercises of imitation and of task-reproduction. The mathematical situation is a framework within which certain activities insert themselves and not an intellectual means commanding the choice of an optimal strategy in a problem-situation of familiar usefulness.

In these conditions the motivations will still be produced from the outside, not putting the content in play as the solution to an interesting situation.

Isomorphic games

In the stage of isomorphic games, the teacher proposes successively to her students several games which achieve the same structure. The students must recognize on their own, in between these games, the correspondence of object to object, of relation to relation which has allowed them to transport behaviour, properties and methods from one to the other of these problem-situations. In the eyes of the didactician, these transfers are already taken for proof that the student has been able to establish between these realizations a relation which constitutes the still implicit apprehension of an isomorphism modulo the chosen structure, and of which one becomes aware at the next stage.

Abstraction

Abstraction consists thus of identifying as an object of knowledge the "structure common to" diverse isomorphic games. The structure is here the set of properties which independently of the particularities of each example govern them all. The didactician must thus produce a set of realizations presenting a "variability" adequate to limit the fineness of the abstract structure. The search for the finest structure should itself be set up as a permanent rule, the achievements forming the semantics of the structure in accordance with Carnap's definition. But the reasons for this search, for the choice of example, and for the use of this structure are not accessible to the student so that for her there soon appears a fairly clear contract; she must recognize what the teacher has hidden inside the games, decode her intention according to a uniform rule, look for the resemblances and the differences. It is important to note that this stage, contrary to the preceding ones, doesn't call for any new decision on the part of the teacher, it usually isn't composed of any specific problem-situation; it appears as an answer, entirely the student's responsibility, given in the earlier stages which are at the same time the necessary and sufficient condition for it.

In my opinion the existence of the didactical contract is what assures the functioning of the process, and not an arbitrary law of the genesis of knowledge. Abstraction is not inevitable. In fact, shopkeepers have never abstracted the structures of modules which regulate their financial exchanges because they haven't the motivation to do so.

Schematization and formulation

The schematization and the formulation of the structure follows the process of identification and of updating. Diénès does not plan any general situations specific to this stage (would something which is well conceived of spell itself out clearly?)[29] but the representation by a graph is often envisaged as a simplified but natural and direct expression of the thought of a child. Representing objects by points and operators by arrows is learned by the use of imitation, like a language.

Symbolization

Symbolization is the transcription in a new language of properties represented in the preceding stage.

Axiomatization

Axiomatization is the stage of reorganizing the acquired knowledge, the stage of choosing the fundamental properties necessary to set up the relationships among structures, of putting structures in place in relation to each other; this position being, of course, that position assigned by the current state of mathematical knowledge (which must therefore identify itself, freeze itself, as "Modern Mathematics" to become a didactic reference). This stage also develops following the preceding approach: free games, with isomorphic rules, formalization...

And thus appears the nature of this psychodynamic process. The knowledge isn't organized as a response which is adaptable, adequate, economical and personal to problem-situations, it is provided completely equipped by the culture which assures its validity and future usefulness and leaves the student no responsibility but to adhere to it. It is only slightly hidden, following a regular, conventional and simple fiction. What the student must do in order to discover it is assumed to be the functioning of the knowledge in all didactical or non-didactical situations. Diénès's method is a language of communication with students, a mode of didactic encoding of knowledge which is assumed not to modify its nature (in accordance with what Bruner asserts) and it is a didactic contract with the students, for the decoding of messages. Of course, this method of exposition promises to be very slow and to be suited only to very simple examples.

Generalization and particularization

That is why it is taken up again by the process of generalization and its antagonist, particularization. Once the axioms of a structure are known, it is possible to abandon certain of them, or to stop others and thus to admit the debate of which science is constituted. Diénès gives the following diagram as an example of this process:

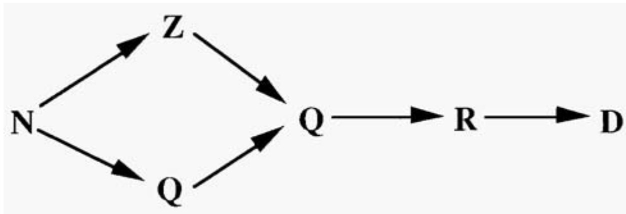


Figure 2

3.4.4. *The psychodynamic process and educational practice*

The contract which we were discussing earlier applies to an epistemological and psychological fiction which (at the time) was very widespread and which reflects well established professional practices:

a) Students' behaviour is assumed to be essentially empirical; their repeated relations with the "reality" of didactical situations give rise to structures which are included in the reality as the "gestalt" in the perception. This process of abstraction can be perfectly interpreted by the theory of memory paths[30].
b) The acceptance of the empiricist model in order to give an account of the student's functioning depends on a *traditional position* of teachers which is fundamentally realist and almost platonic: reality has structures which man discovers, the teacher is led to simulate nature by masking the structures which she wants to teach, discovery being a reading of the world.
c) The isomorphic games technique corresponds to the ancient practice of repeating problems in order to teach how to solve them (by classifying them into types which must be recognized). On this point, the new method can therefore oppose the old one, at the same time substituting itself for it without modifying the practices noticably.
d) The old position was supposed to separate the learning of know-how and that of meaning. The new position permits the hope that the two can be blended because it assures the teacher that the meaning of the structure which is finally learned and the algorithms which are attached to it is nothing more than the set of realizations which it describes and, conversely, abstraction is produced inevitably by the very same process which creates memorization. But in fact the "structure" substitutes itself for the know-how and the learning of even more neglected meaning. We even witness the disappearance of problems.
e) The triumphant structuralism of the 1970s guarantees the utility and fruitfulness of these acquisitions.
f) Diénès's *didactique* simultaneously satisfies conditions which previously appeared contradictory: for each structure, construction precedes analysis and the teacher understands that one proceeds from the concrete to the abstract, from manipulation and action to reflection, from play to study and then to application. But one can start by teaching of the most general, and thus the

simplest, structures so as to end up with the most particular and the most complex.

g) The reformulation in the language of mathematics (isomorphism, realization, etc.) or of the mathematics of psychological or didactical phenomena is consistent with the pan-mathematical project which Lichnérowicz expressed above and is consistent with a Piagetian approach as well. It plays a large rôle in the emergence of a first didactical "theory" whose value has no need to be rooted in experimentation, according to its own scheme, other than by a recognition, in situ, in the work of teaching; the theory is true because the teaching that it describes works.

h) A drawback of this conception was that it relieved teachers of the responsibility of providing certain steps by "guaranteeing" the results of the method on the basis of psychological laws. For example, problems in the classical sense disappeared as practices and as objectives; practically the only thing left was applications which the acquisition of the structures would make possible when the moment arose.

i) On the other hand, Diénès's position was not incompatible with most of the pedagogical theories that we have mentioned above. Moreover, he took some interesting positions, but independently of his didactical theory.

3.4.5. *Influence of the psychodynamic process on the teaching of decimals, critiques and comments*

Let us return to the teaching of decimals and return to our analysis of the modifications due to reform, in the light of these data.

Instead of a reference to the metric system and its related practices, the preference was to present different bases starting with base 2, where the number of different units increases most rapidly and the manipulations are simpler. The conditions for the reproduction of activities of numeration were facilitated, but not a single new problem appeared and these reifications are only repetitions.

And further, the real problems of measurement—the non-discrete character, the search for a unit and a means of comparison and reporting, bounding, errors—have all been scrubbed at the beginning and postponed, to be solemnly introduced at a later date. Activities no longer refer to a practical use, nor to an explicit theoretical justification or construction. It is revealed to the student that since she has done such-and-such a task, she has just discovered such-and-such a property. Structure which an observer judges to be present in the problem-situation is deemed to be contained in the student's acquisition by the very fact of her success in trying it, rather as if the labyrinth into which a rat is placed were known to it as soon as it had come out of it several times.

I do not believe that Diénès spoke or thought in this way, but nothing in his didactical "theory" allows the teacher to take the students' performance into account, to explain and predict it when they are "wrong" and thus to adapt teaching situations to it. For example, it is confidence in providence or people's foresight or that of the teacher which permits students to accept that measure-decimals inherit certain properties from decimals-marks. Nothing can be said about the meaning of the student's success or failure.

3.4.6. *Conceptions and situations*

This is not the place to carry out a systematic critique of Diénès's very interesting work, but it is certainly necessary to see the main obstacle on which it stumbles: the engagement of knowledge in the aimed-for action. A "mathematical structure" takes its meaning from the use which is made of it, from its function, from the constitution of others and above all from the problems whose solution it permits. It must be seen at the level of the concept. The analogy of function in a similar problem is not, very obviously, a guarantee that an isomorphism exists between the structures which present it but above all the observation of an isomorphism between two cases is certainly no guarantee of an analogous functioning of the two situations.

Structuralism, a fruitful instrument for research, becomes deceptive magic in teaching.

Actually, Diénès's work, although it successfully installed content in the centre of the debate about *teaching,* doesn't lead the didactician to question mathematics in order to search beyond the *structures* for the *concepts* and beyond the concepts possibly for the *conceptions* which could be built up by a subject in particular historical or didactical *situations.* The analysis of these conceptions, which the student would have to possess or avoid, is inseparable from that of the family of specific situations in which their function and use lies. Both are unavoidable in any enterprise which claims to provide both a theory equipped with methods of confrontation (probably specific as well) and didactical techniques which teachers can continuously control. Diénès's work does not lead, and we would be tempted to say that this is an inevitable consequence, to the questioning of students' behaviour, not only as a response to a didactical solicitation but, above all, as a source of information on theoretical questions of *didactique.*

And this is perhaps why, after these two long studies of teaching practice, we have the impression of knowing so little about decimals, about the various conditions of their genesis and even of their use, or about the observed or possible behaviour of students concerning them.

NOTES

1. *Editors' note:* see in particular Brousseau 1974a, 1976a, 1988; or Katembera 1986.
2. These texts were published in the notebooks of the IREM and in the Proceedings of Meeting of the CIEAEM, accompanied by various works carried out under the direction of P. L. Hennequin aand others directed by H. Rouanet. They formed the content of Rrousseau's second major research project. The first, under the direction of A. Lichnérowics and P. Gréco dealt with "Limit conditions of an experiment in

Mathematical Pedagogy" and was illustrated by the study of the teaching of natural numbers and their operations.

3. *Editors' note:* About the history of the concept of transposition in *didactique,* see Arsac (1992).
4. The group of people involved in our work consisted of postgraduate students and teachers studying *didactique* of mathematics, the teachers at the Ecole Michelet , established for observation, and teachers in the local School of Education (The *Ecole Normale).*
5. *Editor' note:* See chapter 4.
6. Bachalard (1968)
7. *Editors' note:* This will be made in the next chapter.
8. That is: CM1 (9–10 years old) and CM2 (10–11 years old).
9. page 12, 6th lesson: 1 hour duration.
10. The word "measure " will be used in the whole of this section not in a mathematical sense but in the current sense that is given to it in the textbook being studied.
11. For example, Bodart-Bréjaud published by Fernand Nathan, Paris.
12. This model was studied with precision by lzorche (1977).
13. *Editors' note:* See on these questions Léonard and Sackur-Grivard (1981, 1991), Sackur-Grivard and Leonard (1983).
14. 37% of the students of CM2, according to INRP 1979. *Editors' note:* Brousseau refers to an INRP inquiry in 1979 directed by Jacques Colomb.
15. *Editors' note:* self-correcting computation worksheet from C. Freinet, *Fichier ECL du cours moyen.* Editions de l'Ecole Moderne.
16. *Editors' note:* The rule of three is still taught but is often no longer given a special name in English speaking countries. It involves finding the fourth member of a proportion when the other three are known. See "*The words of Mathematics*" (Schwartzman, 1994, p. 191).
17. *Editors' note:* See Chapter 5 on this question.
18. Moreover, this tendency was established experimentally by E. Filloy in a research project on teaching counting where he showed that teachers didn't foresee their students' results and made them progress only on aims of low taxonomic level. *Editors' note:* See for example (Filloy *et al.* 1979).
19. (1972). CM1 fascicule II and CM2.
20. Mine Picard contributed to the spreading in France of the ideas of Z. Diénès, a number of which she borrowed. As a rcsearch worker in INRDP from 1966 to 1970, she represented for teachers the tendencies of the hoped-for reforms.
21. "mes. i" refers to the indexing of the tiles used by the author.
22. Which stands for: $[12335 \div 100] \times [43 \div 10]$
23. *Editors' note:* respectively CM1 and CM2, in the French school system.
24. $(\times \frac{75}{100}) = (\times 75)(\div 100) = 75 \div 100$, presents the same obstacle.
25. "*Remarques sur Ies mathématiques et la réalité*" by Lichnérowicz, 1967 (In: *Logique et connaissance scientifique,* pp. 474–485 / esp. 477). Encyclopédie la PIéiade. Paris: NRF, 1969.
26. In this article, inorcover, the author insists, through two examples, on the importance for a new science of placing itself in the strictly mathematical domain of total deduction, a position which breaks with the inductive method, extensive abstraction and accumulation of experimental results which predominate in sciences which are still young. He concludes: "Our modes of knowledge are truly mathematical, our powers are indissolubly

linked to them", which at once justifies the project of reform on a social level and gives it its psychological and scientific base (Lichnérowicz, op. cit.).

27. In the terminology of the INRDP: "expériences *'pédagogiques sauvages*"'.
28. Maudet C. (1979) *Etude et critique du processus psycho-dynamique selon Diénès.* Mémoire de DEA. Université de Bordeaux.
29. *Editors' note:* following the Boileau verse: "*Ce que l'on conçoit bien* s' *énonce clairement*".
30. *Editors' note:* Hull (1943).

CHAPTER 3 AND 4 INTERLUDE

Chapter 3 prepared the questions central to *didactique*—the ones which the Theory of Didactical Situations is designed to provide a means of answering. Brousseau discusses the questions as follows:

> "How can we elaborate situations which really make a notion function? That is to say that one cannot answer without putting this notion to work and giving it a meaning. On which parameters do the procedures which attest the use of this notion qualitatively depend?
> How can we make an assimilation (in the sense of Piaget) necessary? When is an accomodation (id) indispensible?
> These are a few specific questions which arise about the *didactique* of decimals and to which we shall try to bring some elements of response."

Chapter 4 answers some of these questions, presenting and using the fundamental didactical concepts. To some extent it is possible to consider it as the follow-up of Chapter 1, which presented the foundation and methods of *didactique,* despite the fact that some of the fundamental concepts are absent in this chapter. In particular, in 1981 the concepts of *devolution* and *adidactical situation* were not yet present. They appeared respectively, as Perrin-Glorian (1994) observes, on the occasion of the 1982 and 1986 Summer School of *Didactique des mathématiques.* Nevertheless the reader will recognize the beginnings of these concepts, and he or she might even consider as an exercise re-analysing what Brousseau presents here in terms of the Theory of Didactical Situations as it is comprehensively presented in Chapter 1.

With respect to the problem of the teaching and learning of decimal numbers, the study presented intends to show that conceptions of decimals exist which are real alternatives to the ones classically considered and thus didactical choices that are available to the teacher. The chapter presents both an epistemological analysis and an analysis of the main characteristics of the processes and situations which make students capable of acquiring these concepts.

Brousseau qualified this essay as one of "experimental epistemology", using an expression which at one time might have been used to designate our field of research itself.

The Editors.

CHAPTER 4

DIDACTICAL PROBLEMS WITH DECIMALS*

1. GENERAL DESIGN OF A PROCESS FOR TEACHING DECIMALS

1.1. *Conclusions from the mathematical study*

1.1.1. *Axioms and implicit didactical choices*

There exist many ways of defining or constructing decimals mathematically. They differ in the choice of what is considered known as a mathematical object and as a method of proof, but the result is the same in that there exists a way of showing the equivalence and the isomophism of the resulting structures. Each of these axiomatic constructions resides in the field of mathematics; on the other hand, the study of what underlies their differences, of the reasons for the choices, of what is or is not allowed, of what is or is not important, of what is or is not easy,..., does not fall within the province of mathematics. An axiomatic construction is implicitly loaded with epistemological options, and with didactic assumptions which one must be careful not to believe necessary at the same level as mathematical conclusions, but through which one must pass in order to obtain a discourse which allows the notion to be communicated. Two methods differ by the choice of axioms and rules for the production of theorems.

1.1.2. *Transformations of mathematical discourse*

There exist, also, formal procedures which transform one mathematical discourse into another referring to the same set of axioms. These procedures affect discourse more or less profoundly in the following ways:

- Logic permits changes in the order of statements, a variety of methods of presentation of the implications (necessary condition or sufficient condition brought out), regrouping into general statements, breaking up into lemmas and corollaries, etc.
- Rhetoric suggests its figures of thought, antithesis, comparison, repetition[1] point of hypotyposis and why not prosopopoeia[2], and we shall add illustration, example, commentary, etc.

---

* Brousseau G. (1981) Problemes de didactique des décimaux. Recherches en *didactique des mathématiques* 2(1) 37–127.

– Grammar and style allow reformulation in simpler language, replacement of terms by their definitions, the choice of synonyms, and so on, not to mention the procedures of presentation dear to Gagné (1980).

These varied procedures, applied in an almost automatic fashion, always provide a discourse, an account which one can hope is simpler, clearer, more redundant... and thus more intelligible, more assimilable than the original mathematical discourse. This point of view has nourished a number of research studies which, in spite of all their interest, have not succeeded in showing that these procedures were acting in a specific, differentiated way with respect to the notions and the acquisitions of the students. Like the pedagogical methods whose effects we studied in the preceding chapter, what these processes have in common is that they neither bring up nor question mathematical construction itself.

1.1.3. *Metamathematics and heuristics*

There exists a language which specializes in the description and comparison of these methods; it can be found in the discourse of mathematicians, in their commentaries, in their courses and in their confidences. Side by side with authentic mathematical terms which are sometimes used in a metaphorical way, essential mathematical terms (which describe the metamathematical language) and certain paramathematical terms (considered to be clear even though they are not defined), one finds essentially heuristic concepts: generalization, synthesis, analysis, problems such as those presented by Polya in his works[3]. This language, rather descriptive and classificatory as used by some of his followers, nevertheless allows the examination of different constructions of a notion starting from its motivations.

1.1.4. *Extensions and restrictions*

Let us take, for example, a direct construction of the decimals **D:**

Let us consider, in ZxN, the equivalence relation $\approx$:

- $(a,n) \approx (b,p) \iff a.10^p = b \cdot 10^n$, the class of (a,n) being written as $\frac{a}{10^n}$
- $\mathbf{D} = \mathbf{Z}\times\mathbf{N}/\approx$ is equipped with operations which remain stable when the quotient is taken:

  $$(a,n)+(b,p) = (a \cdot 10^p + b.10^n, n+p)$$
  $$(a,n) \times (b,p) = (a \cdot b, n+p)$$

  which extends operations in **N** (the set of natural numbers), identified with $(\mathbf{N}, 0) \subset \mathbf{D}$.
- **D** is ordered by $(a,n) \leq (b,p) \Leftrightarrow a \cdot 10^p \leq b \cdot 10^n$
- Then $(\mathbf{D}, +, \times, \leq)$ is a commutative unitary integrated ring and totally ordered.

This method is typical of a category of construction which consists of taking a known structure, here **Z,** and making *an extension* of it, which adds more elements to it. In the preceding chapter we outlined construction of **D**+, by an extension of **N** which proceeded by *the adjunction* of a single element d such that $10 \cdot d = 1$. All the powers of this element, their products and their sums with a finite number of others generate $\mathbf{D}^+$. Although producing the same result, this method allows us to "see" better that the smallest possible number of things has been added to **N,** and also what has been added. On the other hand, it requires an awareness of what represents all possible operations of one element with the others, that is to say, the semiring of polynomials with natural coefficients **N**[x]. It assumes knowledge of a more complex structure.

There is another category of constructions that proceeds, on the other hand, by restriction. A general construction has already been defined, for example **Q,** and one is limited to taking only part of its elements.

Example: a decimal is a rational number that can be expressed by a decimal fraction,

$$\mathbf{D} = \{x \in \mathbf{N} \text{ st } \exists p \text{ st } x \cdot 10^p \in \mathbf{Z}\}$$

There are many other methods based on other mathematical motivations (problems of order or of topology, for example). We shall not go into these methods in this chapter, but the reader may consult Ermel (1980) for details.

1.1.5. *Mathematical motivations*

One could also imagine many other extensions of **N** (the set of natural numbers). Why this one? What does it allow us to solve that others don't? It is too soon to look for a precise situation which could make the creation and the use of decimals necessary; but that is what we must arrive at.

Some problems have no solutions in certain sets because they are not rich enough. For example: no number exists in the set **N** such that (i) $a \times 3 = 2$, or even such that (ii) $7 + a = 5$.

Now, it can happen in particular domains or in some applications that such equations, obviously, always have solutions. For example, one can always divide a length into three equal parts even if it is 2 metres long. Thus, one would like to represent that particular length using a number. So, it is necessary to build an extension (here of **N**) in which these equations always have a solution, **Q** for (**i**), **Z** for (ii), or any other structure which would contain them.

One could have constructed **R** (the set of real numbers) so that every Cauchy sequence is convergent, or **C** (the set of complex numbers) so that any polynomial in one variable with coefficients in **C** would have at least one root.

And why a restriction? Following the immersion of a set A in an extension B, all the elements of set A have all the properties common to the elements of B, but the inverse is not true and the elements of B can "lose" some interesting properties. For example, "n is the successor of 17" has a solution in **N** but not in **D.** Thus **D** inherits facilities of calculation possessed by **N;** but **Q** has lost them, especially for subtraction, comparison, and calculation of intervals. It thus happens that one could, by some means or other, construct a set B satisfying the condition that the solution of a problem could be found in it, but such that the calculations or the reasoning could be tedious. One looks then for a sub-set, a restriction A of B, such that every element of set B can be represented by and approximated by an element of set A for which the calculations are easier. **D** approaches **Q** or **R** because, for every real number, whatever tolerance one sets, there is always a decimal whose distance from this real number is less than the chosen tolerance.

Problems of this kind occur very frequently in mathematics where one would often like to study all the objects created by a generating system or on the other hand to look for a convenient generating system for a given set.

Since the immersion of one set into an extension changes its own "properties", and those of its elements, which one can use from then on, we must expect huge difficulties and resistance to changes in usage when habit comes into play—whether they are psychological or cultural habits. This is one of the main epistemological obstacles met in mathematics.

This remark leads to the observation that heuristic methods of analysis cannot really handle situations of use or of creation of mathematics. Among other things, they ignore the subject, the social group and their history.

1.2. *Conclusion of the epistemological study*

In order to organize an experimental genesis which gives an acceptable meaning to the notion of decimal, an epistomological study is needed to shed some light on the forms in which the decimal is manifested, and the cognitive status of these forms. Such a study cannot be included in this book. We shall only extract a few conclusions from it.

1.2.1. *Different conceptions of decimals*

The "decimal" of antiquity which was used exclusively for measuring and representation of quantities.

For example: those which expressed decimal measures in China thirteen centuries before Christ. They worked a little like the hierophantic binaries of the Egyptians of 2500 BC and like the sexagesimals of the Babylonians of 1900 BC, in the sense that they solved similar problems in a similar way. It was a matter of directly employing the numeration system in use for counting as a means of describing division into fractions; some fractions could be equated, others simply approximated. They are well distinguished, by all sorts of formal, technical and even sociological characteristics, from other fractions with which the initiated tried to make exact calculations and then to define the notion of ratio, with which they overcame various obstacles... (passing to the form $\frac{1}{m}$, m being any natural number whatsoever; then to $\frac{n}{m}$, n being strictly smaller than m; then to $\frac{n}{m}$, n and m being any natural numbers whatsoever; etc.). Very few of these properties were recognized, even if they were used. I shall say—borrowing this term from Chevallard (1979)—that the decimal is a protomathetical notion; this structure is mobilized implicitly in uses and practices, its properties are used to solve certain problems, but it is not recognized, as a topic of study or even as a tool.

al-Khowârizmî (780–850), who unified natural number calculations with those of "geometric" ratio and who introduced the use of numeration based on decimal place value, made possible the emergence of the decimal—mathematical tool for approximation no longer of magnitude, hut of mathematical entities: first rational numbers, then radical numbers, and so on. These entities could bc counting numbers, measuring numbers, ratio, and finally, with Stevin (1585) authentic mappings.

The decimal then became a *paramathematical notion.* It was at first only a consciously used, recognized, designed tool, but one which its inventor, al-Uqlidisi [around 952]. didn't treat as an object of study (Abdeljaouad, 1978). The decimal was shown in its functioning (preconstructed) and appeared as a method of explaining fractions or as a curiosity. The writing of decimal fractions in al-Uqlidisi's work is identical to our own and yet the concept was not taken up by his comtemporaries.

On the other hand, its second inventor, al-Kashî (1427) recognized it as a mathematical discovery. But it was not yet under the control of a theory which fixed its definition, its properties and its epistemological position. It was the translation of the sexagesimal system used by astronomers into a more convenient system for calculations.

One can assume that for five centuries decimals have been potentially present in the culture and that it is their status which was in evolution (for example: in about 1350 Bonfils of Tarascon produced a rough outline of them).

It was after Simon Stevin (1585) that the decimal reached the status of *mathematical notion.* Stevin systematically introduced geometric numbers and multinomials—polyinomial functions—to unify the notion of number and the solutions of algebraic problems of his time. Decimals appear as a culminating product of this theory; they become an object of knowledge that can be taught and used in practical applications, in calculations, and in the construction of tables. Their conceptual rôle remains the most hidden. For Stevin, "rational, irregular, unexplainable, dull, absurd quantities" were (real) numbers because they could all be approximated by decimal numbers; he didn't actually write this sentence but everything happened as if he had thought it.

Decimals served as a heuristic model in the nascent Analysis and Newton used them to explain the approximation of functions and their fluxions with the help of polynomial functions and series, their derivatives and their primitives (Ovaert, 1976). This position was finally fixed and attested to only when real numbers had in their turn become mathematical objects and the procedures used by Stevin for the approximation of functions had in their turn received their mathematical identity.

1.2.2. *Dialectical relationships between D and Q*

Epistemological analysis provides other information. The progression of **D** fed on that of **Q**, then on that of **R**, in a dialectic that is hard to summarize. It will no doubt be essential to construct problems which would necessitate the construction of a super-structure such as **Q** or **R,** and perhaps to construct it effectively to give **D** its meaning (Douady, 1980).

For example: it seems to us difficult to conceive that the notion of ratio could be directly approached by means of decimal ratios.

1.2.3. *Types of realized objects*

The situations in which **D** or **Q** is used differ profoundly by the nature of the mathematical objects which realize these structures and which are currently being worked on. Originally $\mathbf{Q}^+$ and $\mathbf{D}^+$ were realized as a set of images in a measurement system; what was explained, written down, and had status was the ordered semi-group ($\mathbf{Q}^+$, +, $\leq$) with natural operators. The field properties of **Q** were used implicitly but were not recognized.

The translation of historical facts into modern language may lead us to some abuse, but it gains us some conciseness. Thus, alongside natural ratios (multiples, sub-multiples), the Pythagoreans used certain privileged ratios (*épimäre, épimune, émiole*...)[4] but they used them only in geometry, and calculations using them were painful; they surely did not assimilate them as fractions. At one time, Euclid rejected these barbaric distinctions, but that initiative got nowhere and even Archimedes did not really know the fractions which we call "Archimedian". The construction of L($\mathbf{Q}^+$) as a set of ratios or operators provided with group operations, and its identification with $\mathbf{Q}^+$ took over a thousand years (from 400 BC to 850 AD). At the end of this period, ratios functioned implicitly as *mappings,* but at least six hundred more years were needed (1585) before $\mathbf{Q}^+$ became explicitly a set of functions treated as numbers.

1.2.4. *Different meanings of the product of two rationals*

So, for example, the operation of multiplying two fractions can have different interpretations, many of which have appeared at often widely separated times, as presented in Table 1.

This table is to be looked at in the framework of the explanations given in Section 1.4 and the study of the levels of knowledge of the composition of two linear rational mappings in Section 2.1.7.

It can thus be expected that corresponding conceptions that it is possible to confuse in calculation today in fact arose in different situations and consequently were not conceived directly in the same way at the same time[5].

Natural operator $\times$	$n \times U$; $n \times L$; $n \times m \times M$ $\times$ means: $n \times L = \underbrace{L+L+\ldots+L}_{n}$	m, n variable naturals L, M size; U unit
Naturals operating on measure fractions $\dot{\times}$	$n \dot{\times} \frac{U}{a} = \underbrace{\frac{U}{a} + \frac{U}{a} + \ldots + \frac{U}{a}}_{n}$ $n \dot{\times} \frac{U}{a} = \frac{n \times U}{a}$; or $\frac{n \times m \times U}{a}$	$\frac{U}{a}$ is the natural fraction measure not necessarily random. n is a scalar which operates on fractions.
X	$\frac{n}{a} X U = n \dot{\times} (\frac{U}{a})$	$\frac{n}{a}$ is a "fraction measure" detachable from the almost scalar size
Ratio of naturals invariant in an implicit transformation $\bullet$	$\frac{a}{b} = \frac{L}{M}$ [$\frac{a}{b}$ measures L with a unit M] $\frac{a}{b} \bullet U = \frac{L}{M} \bullet U$ $\frac{a}{b} \bullet L$	$\frac{a}{b}$ is a ratio of sizes expressed by number $\frac{a}{b}$ relates the measure of L to that of M regardless of U 4th proportion of (nU; mL and L)
Linear rational applications $\ltimes$	$\frac{a}{b} \ltimes L = \frac{a}{b}(L)$ [designate the image of L by linear application $\frac{a}{b}$] $(\frac{a}{b} \ltimes \frac{n}{m})(U) = \frac{a}{b}(L) = \frac{a}{b}(\frac{n}{m} U)$	L+nU then $(\frac{a}{b} \ltimes n)$ is the number that measures L, extension of the appication of $\frac{a}{b}$ to measure fractions
Rational operators	$(\frac{n}{m} \cdot \frac{p}{q})(L) = \frac{n}{m}(\frac{p}{q} L)$ $\frac{n}{m} \cdot \frac{p}{q}$	application composed of two rational applications composition in £(Q)
Rationals	$\frac{n}{m} \cdot \frac{p}{q}$ ($\frac{n}{m}$ and $\frac{n}{m}$ "abstract" rationals)	after identification

Table 1

Note for the English edition (1996): The *following is the outcome of a request from the editors for clarification of Table 1 in this chapter. With characteristic generosity, Brousseau produced not a collection of typographical corrections but a description of the motivation, history and setting of the table and its contents. We present the essential content of his note below.* The Editors.

Before proposing a better presentation, I would like to show you, or rather reconstitute, what was behind this "curious" classification, which is badly explained in the text.

The object of the table was to "show" the different conceptions of multiplication which punctuated or might have punctuated a construction or a genesis (historical, ontogenetic or didactical) of different numerical structures.

Different conceptions of some particular mathematical notion are manners of understanding the same notion which differ from one another—different in the sense that for each one some interpretations, calculations and circumstances are made easier, while others are made more difficult. I had already used this idea in my book published by Dunod in 1965[6], The example which interested me at that point was the opposition between the conceptions of "commensuration" and "partitions of unity" for fractions. This interest manifested itself in the thesis of my student Ratsimbah-Rajohn (1981), in which we began to figure out means (theoretical and empirical) of identifying and distinguishing different conceptions of a mathematical notion. Table 1 was supposed to prepare this work, which was pursued in different theses, without ever making its way into a specific article. On can find an example of it (an attempt) in the list of different conceptions of division in (Brousseau and Brousseau 1987) about "Rationals and decimals", taken up and augmented in (Brousseau 1988) about the different meanings of division. The work was pursued thereafter, but it is still not in a satisfactory scientific state. Here is how I would justify these distinctions today:

In this Table 1, I was trying to apply a method (still not published, but reasonably clear) to discriminate between the different conceptions of multiplication, at least those that I suspected of playing a rôle in the relations with measures. I will try today first to make it explicit and precise. I could not do so at the time in the framework of this article which was already very long and had above all the goal of presenting a maximum number of research routes.

Principal elements for the differentiation of conceptions

We have a number of instruments of which we consider chiefly the following:

1. Mathematics itself, which is the most obvious. It can be used

– either directly because descriptions in terms of known structures permit us to distinguish various sorts of objects, for example
  - elements of a measurable space together with adequate set operations and compatible transformations
  - elements of the numerical set once they are provided with measure functions and their numerical structure (**N**, $\mathbf{Q}^+$, $\mathbf{D}^+$, **R**)
  - linear mappings and the structure of groups or associated (dual) linear semi-groups (Vergnaud later used this instrument in his own way)

– or as a complement to a study of situations. One could thus conceive of "new" structures appropriate to particular uses, these uses being further justified by the types of situations or different problems as for example for measures,
  - the set of "effective" measures, whose image is a pair (number, unit)

- the set of "concrete" measures, resulting from measurement, made up of triples (number, unit, interval of uncertainty (or of error, or distribution of probabilities, or confidence interval, etc.));
- the set of "practical" measures composed of a quadruple (number, unit, interval of uncertainty (or of error or distribution of probabilities or confidence interval, etc.), evaluation (measure of the measure: order of magnitude, rarity, position in a frequency distribution of values of this nature over the set of measures etc.));
- the set of "physical" measures, whose elements are measures of different sizes, structured by various mathematical procedures like product measure, derived measure, integration;
- in an accessory way, the set of values of a "size", that is the set of equivalence classes of objects—reciprocal images of a measure in the mathematical sense conceived in the manner of physicists as a sort of "absolute measure" (without a unit);

(I hazard the terms "effective", "concrete" and "practical", but the classification is important, and figures, except for the last point, in an article published much later in *"Grand N"*[7]—"The weight of a glass of water".)

2. Analysis in terms of situations (in particular of historical conditions) makes it possible to give a function to ancient mathematical objects, in particular the notion of ratios, and to put them into action. It makes it possible to distinguish uses as an implicit means of action, as a direct or meta-linguistic means of expression, or as an object of study or means of proof. It also makes it possible to distinguish between situations in which one is interested in an isolated element of a set and those in which it is the structure of the set which is the instrument or object of action.

3. The most convincing, though also the most delicate, instrument was the study of properties, in particular the ergonomic properties which the use of these mathematical structures gave to situations encountered more or less frequently in certain "*milieu*" (family of situations classified by their use and not just their logical structure).

Application to multiplication

How shall we combine the instruments above to identify the different conceptions which are *a priori* possible for multiplication?

1. For a start, let us consider finite sets as the objects of a "natural measure" function in the set **N** of natural numbers. There are already the following distinctions:

1.1 In the set of natural numbers, the internal operation "formal multiplication" ×, which one can imagine to be constructed from addition: $3 \times 4 = 3 + 3 + 3 + 3$.

1.2 In sets the operation × produces finite sets A × B. It is compatible with the natural measure mapping Cardinality (A): Card (A × B) = Card (A) × Card (B). This mapping and the operations attached to it can be naturally defined with children starting with the enumeration of finite sets (as we have shown with Mr. Briand in his thesis[8]).

1.3 Effective natural measure can be conceived as a (formal) operation which associates a natural number and an object (a "unit") or a set of objects: (3, cats) or (3, dozens of roses), which can be interpreted as an (external) multiplication 3*U.

1.4 A multiple proportion is not an operation, but one is associated with it: (3 ; 12) * 3 × 4.

There exist several types of ratios depending on the set or sets which supply the objects in the ratio. Certainly these types are easily confused, but each is associated with a slightly different type of multiplication.

1.5 The natural linear mapping of **N** into **N** is also not an operation, but it can be identified with a natural number operating multiplicatively in a set of measures (a semi-group) and that does have an associated operation.

Example: f(3) = 12, with f = 4. To mark the difference, one can write f = *4. Teachers in primary school call that an "operator", and they write *4(3) or *4 × 3 or, with an arrow, $3 \xrightarrow{x4} 4$ or simply 3 × 4. Here I will write this external operation as 4 * 3.

The use of the multiplicative signs * and "÷" as univalent symbols (as in *4) and at the same time as bivalent symbols (as in 3*4) requires differences in conceptions but produces the same difficulties as the simultaneous use of the signs + and – as univalent symbols and bivalent symbols in the whole numbers.

A natural linear mapping of **N** into **N** can also be identified with a class of equal natural ratios. Consequently, there exist as many types of linear mappings as of ratios and consequently as many slightly different multiplications.

1.6 Composition of mappings is an operation internal to the semi-group of linear mappings of **N** into **N.** It can be interpreted as a multiplication : *4 ∘*3 = *12.

2. We will now consider as an example rational measures, but any numerical structure could give rise to the same development.

2.1 There will thus be formal internal multiplication in the rationals. It is *a priori* different from the multiplication of naturals. The difference appears especially in the properties used by children to "understand" or check the use of these operations, for example: natural number mapping multiplication always "enlarges", but not rational. We will thus denote it differently: a/b * c/d = a × c/ b × d.

2.2 The (dense) measurable spaces on which a rational measure can be defined naturally possess a set product ×, which is formally different from the products of finite sets, but such that mes(A × B) = mes(A) * mes(B).

2.3 In the rational measures we again find the formal operation which associates a fraction with an object of a measure space or with a unit a/b * U.

2.4 Rational proportions. The conservation of rational proportions and their direct definition in measurable spaces (ratios of segments and of surfaces) permitted the Greeks to manipulate rationals and, in the case of Euclid, even reals.

Here as above there exist the same different types of ratios depending on the set or sets from which the objects in the ratio are taken.

2.5 Rational mappings can be given the same analysis as above.

The ordering of the conceptions

3.1 The important thing for understanding the genesis of rationals was to examine which conceptions could generate themselves and articulate themselves and how. The possibility that certain of the "Conceptions" envisaged above could really exist—that is, could be observable in isolation in history or in the development of knowledge in children—depends on these relationships. It is a few of the steps of a possible development that I attempt to sketch in Table 1.

Operations in measurable spaces, the intervention of numbers in these operations and their translations into numerical operations, then their transcriptions into new numbers and the long and painful emergence of new conceptions of the formal objects thus introduced in which the old identify and submerge themselves, not without confusion and contradictions. I have not published the epistemological analysis in which these evolutions are traced, I just give a resumé of some conclusions about them.

The first two lines of the table, which may raise a problem, are a resumé of a process more complex than that which I employed to carry out the process as a whole. I am trying there to represent the dialectic of transformations both of the space of objects measured and of the numbers used, a dialectic which makes it possible to pass from natural measures to rational measures (and decimals).

3.2 The first line presents the operation denoted above by a *. The natural operators serve to count the reiterations of a material operation * (putting end to end and alignment for example for segments "equal" to L) so n * L = L+L+...+L.

Naturally, these natural operators have their own multiplication, and we have for example: $(n \times m) * L = n * (m * L)$.

3.3 In the second line of the table, material operations on the objects in the measurable space are introduced. For example, a segment L can be partitioned geometrically into a equal parts (with a collection of equidistant parallels, for example). I call that operation L/*a.* The natural numbers operate on this new set as on the preceding one and there appear implicitly kinds of composite fractions with a measure in the numerator and a scalar natural number in the denominator. This is the case that turns up in my book. Fractions conceived by commensuration have an "enumerator" in place of the denominator and a "measurer" (it measures the number of objects counted) in place of the numerator.

Obviously, to keep the ideas clear, we have to distinguish this operation by a new sign rather than using any of the ones in the preceding paragraphs (in the article, a cross with a point over it; here I'll put a *) n * L/a = L/a * L/a *... * L/a (sums of material fractions of segments or areas are not as easy to conceive of as sums of primitive objects).

Then since L/a * L/a *... * L/a =(L + L +... + L)/a, it is clear that:

n * L/a = (n * L)/a and (n xm) * L/a = n * (m * L/a).

If n is a multiple of a, we are back to the case on the first line.

One finds reasoning similar to the ones specific to this conception in the manipulation of decimals and notably in calculations on changes of unit.

3.4 The third line presents the case where the operators are fractions engaged in a rational measure. The operation is the one we denoted above by *.

This step represents the outcome of a very important process which is much more complex than one would suppose given the formal proximity between n * L/a and n/a * L, because it has to be possible to define some numerical operations (sums, differences, products). (It would be better to say to "transfer" objects from the measurable space to the numerical system, but a solid reason for doing it is still needed.) There ought to be a whole bunch of steps between the second and third lines to represent the slow historical evolution or a plausible ontogenetic didactical evolution.

3.5 The fourth line presents various conceptions of a rational ratio like the ones indicated above. The difference between the third line and the fourth is that the fractional measure (third line) is applied to an object which plays the rôle of a sort of unit, which is not the case with ratios.

The rest of the entries seem clearer.

It is clear that though the table presents different conceptions of "multiplication", it does not show the genesis proposed in the didactical process presented in the present chapter.

1.2.5. *Need for the experimental epistemological study*

But it is improbable that all the distinctions, all the obstacles, all the particular situations actually continue to hold good. For example: to take the tenth part—to decimate—consisted, during antiquity, of ranking the elements, then of counting 1, 2, 3,..., and finally of retaining the tenth element. So there could only be one tenth, not two! It is therefore necessary to carry on a genetic, experimental study of epistemology by means of enquiries and experiments comparable with those that we are going to report.

There is an equilibrium to be found between a "historical" teaching, which would restore a forest of distinctions and obsolete points of view within which a child would be lost, and a direct teaching of what one knows today to be a unique and general structure, without bothering to unify the child's conceptions, which are necessarily and naturally different. Research into the conditions for such an equilibrium is one of the major problems which is presently of concern to *didactique.* One of the aims of this work is to foster reflection in this direction.

1.2.6. *Cultural obstacles*

We must also add that *didactique* of decimals has a long history. From Stevin, who envisioned it with a certain ingenuity and some well set principles, until the decision by the United States to adopt the metric system, and thus a decimal system of measure, via its first appearance in popular teaching in France after 1792, this *didactique* changed not only in form, but also, correlatively, in inspiration and even in political significance. This political and cultural significance always weighs on its teaching and sometimes constitutes a real obstacle (Brousseau, 1976).

1.3. *Conclusions of the didactical study*

Moreover, espistcmological study gives us a large number of variations of conditions, methods, and meaning. The difficulty is to group these variables and to form a hierarchy of their characteristics in relation to their assumed importance for the reproduction and supervision of situations which are able to provoke the apparition of knowledge.

1.3.1. *Principles*

We are going to leave this question open, and in accordance with our project, show how to produce a process. The artificial genesis which we envisage constructing must make the notion of a decimal function in such a way as to simulate the present different aspects of the concept. It is not a question of replicating the historical processes but of producing similar effects using other means. The epistemological phenomenotechnique consists of making choices about *certain* points that are very different from those suggested by history, and of restoring a nonetheless equivalent process by the use of rules and principles that it has been possible to discover.

The epistemological experiment bears on the effects of the corrections that these modifications produce on the system as a whole.

But this experiment is slipped into a teaching activity. It must be compatible with it, obey its existence, and undergo inevitable didactical transpositions.

From here on, we are therefore going to place our discussion in this didactical framework.

1.3.2. *The objectives of teaching decimals*

Let us examine the classical objectives of teaching decimals. They are to make students capable of solving classical, practical problems, implementing the operations and the order of decimals, which implies the use of decimal and hexadecimal measurement, the proper mastering of situations involving linear (and rational) mappings: scales, change of units, percentages, investment of money, speed, volumes, areas, densities,...

In most of these problems, children are invited to present or indicate their results in terms of the proposed situation. For example: "The sale-price in francs of the transistor radio is... ?" Then to express this result within $\mathbf{Q}^+$ by means of a formula (such as $\frac{280 \times 4}{3}$) and then to produce a decimal reasonably close to this result. This calculation consists essentially of passing from $\mathbf{Q}$ to $\mathbf{D}^+$. No explanation is written down; the justification consists of the decomposition of the final calculation into a series of intermediate, "simple" calculations, that is to say, ones belonging to an accepted repertoire of occasions on which this operation is used.

The curriculum is designed for children at least 9–10 years old and at most 12– 13 years old, who may have learned decimal operations with reference to the use of the metric system. It pursues fundamentally the same objectives.

This involves the possibility of making all the usual calculations with decimals (and with fractions). But it must favour the theoretical reprise which will lead children of 13–14 to reorganize their notion of decimals in a definitive way, and to use it in its present mathematical form (example: 1.394×10^{-4}), in particular the one used by (pocket) calculators.

1.3.3. *Consequences: types of situations*

If one wants children to be able not only to apply methods and produce solutions, but also to understand and discuss them, this reflective attitude must be made possible by giving them the use of a vocabulary, even a simplified one, and a theory, even an unsatisfactory one, about linear mappings and their properties.

Our epistemological study allows us to understand that, for a theory to be institutionalized, it must already have functioned as such in scientific debates and in students' discussions as a mean of establishing or rejecting proofs. This process corresponds to the third stage of our analysis, the one in which the notion is handled as a mathematical one. We call didactical situations which allow the simulation of this process *situations of validation and situations of institutionalization.*

But if these theories are to have meaning for those using them, they "must" previously have functioned as solutions to problems given to each student in conditions which allow her either to find the solution by herself—or more exactly to construct it (possibly progressively)—or to borrow it ready-made from among a number she could envisage, without any didactical intention or cultural pressure compelling her by substituting itself for her judgement. We then say that the theory works as an implicit model, and we use the term *situation of action* to describe a didactical situation that allows the apparition of this theory, whose status in the classroom is that of a protomathematical notion.

In order that the vocabulary may be acquired and the terms make sense, they "must" be used sufficiently to express and communicate information in situations which both justify their use and control it. Such so-called *situations of formulation* allow the acquisition of explicit models and languages which, in cases in which mathematical notions are not yet in existence, are therefore given the status of paramathematical notions (here, display and usage take the place of definition).

1.3.4. *New objectives*

The objectives will therefore include knowledge, know-how, a vocabulary and theoretical acquisitions. These objectives cannot be independent; a certain equilibrium is established between them as soon as one attempts to respect their reciprocal functions in an authentic genesis. Moreover, an unjustified theoretical knowledge would be lost and would have no meaning and an excessive practice without discussion would lead to learning by conditioning and prematurely which would create obstacles at later stages.

So, in the final phase, the student will have to calculate in the semi-field $\mathbf{Q}^+$, and in particular within the semi-group ($\mathbf{Q}^+ - \{0\}$, ×). The domains of application chosen lead to the consideration of all the types of realization which we brought up in the earlier section.

One could consider that the question of whether students can explain the product of two decimals both presented in the form of operators or linear mappings is a good test of acquisition of the target structure (Rouchier, 1980).

Examples:

1. What is the distance covered in 4.2.5 rotations by a rolling disk of perimeter 0.38 m? (1 *operator,* 1 *length*)

4.25 × 0.38, which is 4 times 0.38 plus 2 tenths times 0.38 plus 5 hundredths times 0.38

It is also $\frac{425}{100} \times 0.38$

2. It is estimated that the usual breakdown of a housing budget is:

rent: 0.68
maintenance: 0.18
heating: 0.14

The proportion of his or her income that a person has allocated for housing is 0.23. What proportion of this income has been allocated to heating? (2 *operations*)

But this hypothesis remains to be verified experimentally. It would be necessary for success with these exercises to dominate all the others heirarchically, that is to say, imply them.

Research on the relationship between these flowcharts of objectives, the hierarchy of knowledge and the implications of acquisitions has held the attention of many researchers for fifteen years (Gras 1980)[9].

In spite of the great interest in this work, no decisive conclusions have yet been drawn.

1.3.5. *Options*

Finally, we retained the following main options to which we will return later:

– a) Acquisition of measurement-decimals will follow a process distinct from that aimed at mapping-decimals. They will take place in that order.
– b) In both cases, decimals will be presented as rational numbers, a mere rewriting of decimal fractions. Rational numbers will thus be constructed as the first of the two steps. This is not very original for operators. But on the other hand, for measurements, it goes against the best established cultural habits.
– c) Students will choose measurement-decimal fractions in order to approximate rational numbers because of the ease they offer for calculation. Topological problems require many precise comparisons and the calculation of intervals. They will furthermore demonstrate properties of the natural order of **Q** and **D** which are contrary to those of **N.**
– d) This topological approach will not be reproduced in the study of linear rational mappings. It is indeed a matter of option: we have demonstrated in another part of this research, and shall not report here, that such an approach is possible.
– e) We shall attempt to make students acquire implicit models, or to make them function if they have already been acquired, before eliciting their formulation or analysis. We shall acknowledge that children possess an implicit model of proportionality in **N**.
– f) Sums and differences of rational mappings, although encountered, will be neither theorized, nor institutionalized.
– g) We shall explain the other options during the presentation of the situations.

1.4. *Outline of the process*

1.4.1. *Notice to the reader*

This outline is formulated in mathematical terms which are manifestly out of the students' vocabulary and organized as a presentation in which definitions and theorems appear in a classical manner. This could lead one to believe that any presentation of this type, that is to say articulated as a mathematical discourse, could constitute an outline. Well, it could not. This represents, in fact, a succession of questions and problems which tend to constitute a genesis : the question of rank (n) arises from problems encountered in solutions found to the question of rank (n–1), or from the consequences and developments of these solutions. Such an outline cannot be automatically obtained as a consequence of mathematical and epistemological analysis. One has constantly to assure oneself of the capacity of the general design to allow the invention, organization and progress of local situations.

This going back and forth, this dialectic between the design of the processes and that of the situations is made inevitable by the very nature of *didactique.*

Articulation from knowledge alone is not sufficient to determine the meaning given to acquisitions by the specific situations chosen.

It is a classical procedure to analyze the "curricula" backwards so as to demonstrate the implications between the terminal objectives and the subordinate objectives. The reader will not be surprised to find us using it in this text. In spite of our reiterated doubts about the possibility of determining acquisitions independently of the situations which produce them, we shall use the same procedure in the representation of activities in the next section. This procedure will perhaps increase the difficulties of the reader in understanding what knowledge the students really have available for the lesson, but in this way we hope to make the reader conscious of the necessity of specifying the conditions for the progress of situations and of the rôle of the history of the subject in this acquisition. If our attempt fails, we would advise the reader to read the sections in their chronological order.

1.4.2. *Phase II: From measurement to the projections of* D^+

Following these options in the final phase, we shall plan an institutionalized *identification* that is to say one which is reasoned and agreed upon, of ($\mathbf{Q}^+$, ×) (measurement) and L($\mathbf{Q}^+$, 0), implying in particular the systematic use of inverse mappings in the calculation of the ratio between two decimals (Phase 11.7: two sessions).

For that, it will have been necessary to be able to use *composition* and *decomposition* of rational mappings while eliminating the rôle of object-image pairs so as to be able to provide various decompositions of the same mapping. We have chosen to present the introductory situations of this phase (11.6, three sessions, "composition of two linear mappings", in which the students use a pantograph). This study cannot develop satisfactorily if fractions and decimals have not been identified as rets of mappings operating on fractions and on measurement-decimals.

During Phase II.5 (two sessions), the children try to assign a meaning to the product of two fractions or two decimals. They do so by interpreting one as a linear mapping operating on the other. In this case, the students recover the traditional vocabulary describing the "product" of a rational number by a rational operator (for example: taking a fraction of a number, a percentage, etc.. .) and they formalize and institutionalize the calculation of images by the elements of $L(\mathbf{Q}^+)$ that they already have practised in the previous phase, with very diverse, sometimes informal methods, and even by means of trial and error. The relationships between multiplying, dividing, enlarging, shrinking are matters for discussion.

The introduction of these linear mappings occupies the three preceding phases, which it would be best to explain in their natural order.

Phase II.1 (2 sessions) consists of asking the students to "enlarge" a puzzle, piece by piece, without precisely defining "enlarge", in such a way that a side whose length was 4 cm must now measure 7 cm. We shall explain this situation in detail (Section 2.2). Students try hard to find ways of calculating the image lengths, but only that which (implicitly) makes the sum of the images correspond to the image of the sum allows a satisfactory completion of the puzzle. What the children construct "empirically" is a set of ordered pairs, and has no name. "The linear mapping $\frac{7}{4}$" only exists as part of the plan of action for the subject.

Already, nevertheless, one must find the images of decimal and fractional lengths.

Phase II.2 (1 session) reproduces a situation almost identical to the preceding one. The enlargement of a regular tiling generates the same problems; the sides have decimal lengths. In the discussion, the image of 1 emerges as the means of establishing the other images, as well as the division of a decimal by 10^n, $n \in \mathbf{N}$.

Phase II.3 (2 sessions) starts with an identical situation. A drawing of a boat and six photographs of the drawing obtained by different enlargements are considered. From her seat in the classroom, each student tries to predict the length of every segment reproduced on one of the photographs. They can go up and verify the results of their predictions and possibly reconsider them. (There are "enlargements" and "reductions".) Then new photographs appear and they have to find a way of designating and ordering all the photographs so as to win in a game of communication (fairly similar to the one we shall explain in Section 3.1 during the lesson called "thickness of a sheet of paper"). It is, of course, the image of 1 which serves to identify the photographs and order them. Thus, the students are led to identify and designate linear mappings with the aid of decimals. But these numbers remain attached to one of the photographs, to a set of values. The game continues, but the model is changed each time. The calculation of images becomes familiar, the vocabulary and the discussion focus on the enlargements and reductions (which involves the subsequent debate that we indicated above). In conclusion, the students announce that they know how to designate linear mappings (of $\mathbf{Q}^+$ in $\mathbf{Q}^x$ and of $\mathbf{D}^+$ in $\mathbf{D}^x$).

It is time to put forward a few situations in which non-linear mappings slip in as "necessary (?)" solutions to linear mappings (Phase II.4, 2 sessions).

Here, students meet practices and language in the domains of "scales" and commerce (taxes, percentages, etc...).

1.4.3. *Phase I: From rational measures to decimal measures*

In Phase II, instead of defining rational operators directly as being composed of natural operators (which are then not mappings), a method whose difficulties and contradictions we mentioned, we have accepted the existence of $\mathbf{Q}^+$ and $\mathbf{D}^+$ as sets into which the measures arc mapped. The object of Phase I is therefore to build such a set: children create and experiment with new numbers in order to measure a variety of sizes.

Phase I.1 allows students first to invent "rationals" by using a method of passing to the quotient on the set of pairs of rational numbers (activity 1, 4 sessions). We shall analyze at length this first activity, "measurement of the thickness of a sheet of paper", to show clearly the evolution of the status of these rational numbers (Section 3).

They appear as a solution to a favourable situation, without cognitive status. This solution presents some problems of identification because it can take many equivalent forms. In this way, the students are led into a discussion of the question "Are these new objects numbers?" This is the motivation for Activity 2 (five sessions), which brings the children to identify them, add them, subtract them, multiply them and divide them by a natural number, and to compare them and order them.

Fractions are then recognized as new numbers encompassing numbers that are already known, but with certain properties which are different.

In Phase 1.3, in order to measure other lengths, capacities, weights, the students use these same numbers and pass on from the conception of definition of fractions by *commensuration* to a *constructive definition* (this phase will be commented on in Section 4; it is not essential to the process).

Phase I.4, "**D** as a means of studying **Q**", takes seven sessions. The new properties which are sought in rational numbers in order to make measurements are mostly topological properties; between two rational numbers one wants always to be able to place another and one wants to be able to measure all the intervals so obtained. Thanks to a "naval battle'' game where they cast finer and finer nets (filters) to catch rational "fish", the children thus explore the topological structure of $\mathbf{Q}^+$ (this activity will be presented in Section 2.3). But it so happens that among all the operations that can be performed with rationals in their fractional form, the longest ones, the hardest ones, are precisely comparisons and sums or differences. So that the children themselves, for reasons of efficiency, very quickly choose from among the rational fractions certain ones—decimals—which simultaneously allow rapid calculations and a convenient representation (an approximation) of rational measurements.

Phase I.5, construction and study of **D** (6 sessions). These decimal fractions are open to a simplified notation which allows an extension of the rules of calculation (addition, subtraction, multiplication by a scalar) from **N** to **D** at the price of only minor modifications (Activity 6).

Phase I.6 (4 sessions). Density of **D** within **Q**— division— approximation of a rational by a decimal.

Obviously, not all rational numbers are decimals, but we can approximate any of them as closely as we like by means of a decimal.

This approach, organized, standardized and institutionalized, will allow the result of the division of a rational number by a natural number to be converted into a decimal and will implicitly provide the method of division of decimals by integers.

In this part I, the children manipulate numbers as if they were measurements. The construction therefore includes limitations of meaning that must be respected (cf. Section 2.3).

The only usable operators are the natural numbers; they know how to multiply or divide by 2, 3,.. but not by 2/7 nor 2.5 (learning this will be the object of the second part). And these operators are not introduced as mathematical objects. They work as an implicit model of linearity borrowed from **N.** They will be used at most to explain *natural scalar ratios* that will be used during calculations.

The method has been conceived in such a way that the question of whether or not the students have already learned to use decimals for measurements, and to carry out operations, doesn't modify the process appreciably. This does not mean that they will have to pretend not to know what they already know, nor that they will revive an already acquired piece of knowledge by rejecting its old meaning and substituting a "new" meaning for it.

2. ANALYSIS OF THE PROCESS AND ITS IMPLEMENTATION

2.1. *The pantograph*

2.1.1. *Introduction to pantographs: the realization of Phase 2.6*

The pantograph is an articulated parallelogram which is used on drawing tables for producing homothetical transformations of a given figure. The apparatus used in this activity consists of plastic toys that are very much less precise than the professional apparatus—and it is precisely this peculiarity that will be used. Depending on the scale markings, they can produce certain very precise decimal homothetical transformations: 1.5, 2, 2.5, 3,4 and their reciprocals (another model allows one to produce $1\frac{1}{2}$, $2\frac{1}{4}$, $2\frac{1}{2}$, $2\frac{3}{4}$, 3, $3\frac{1}{2}$, 4, 5, 6, 7, 8, 10).

During the introductory session, students learn simply how to use the apparatus: the anchor-point must not move, nor the paper, the pointer follows the model, the pen draws the image, and so on. They try to enlarge and shrink personal drawings and then share their observations and their hypotheses:

- One can "enlarge" or "shrink" by interchanging the pointer and the pencil;
- The image does not change form whichever way the pantograph is used;

– The enlargement varies depending on the pantograph or on the choice of mounting-hole;
– If the pantograph does not form a parallelogram then the transformation is deformed;
– The pantograph makes certain lengths correspond to others; it makes the sum of the images correspond to the sum of the lengths;
– The pantograph executes known enlargements; the students recognize the names of the enlargements above the holes; etc.

2.1.2. *Examples of different didactical situations based on this schema of a situation*

Ordinary presentation based on demonstration: the teacher shows the apparatus, shows its magical effects, teaches how it is used, presents the different possible calculations: calculation of image, model and enlargement, and has them reproduce exercises.

Heuristic method (of rediscovery), similar to the one just mentioned above. It gives quite rapid collective results and is suitable if the students are familiar with individual research and debating; but with no other purpose than "to say interesting things", such a method often runs a very short course.

An active, maieutic (Socratic) method: each student has her pantograph; a sequence of questions is put to her so as to make her successively enunciate the different properties or correct the mistakes she has made.

A more elaborate process made up of many phases:

– *an action phase:* the teacher tells the students that in one minute she will choose a length between 1 cm and 15 cm and that each student (or pair of students) will have to predict the corresponding length transformed by her pantograph. This prediction will be subject to a bet and then to a test; the apparatus will be used to verify whether or not the predicted value is exact. While waiting for this moment, the students can practice prediction: they look for images of a few numbers and verify them. They can make bets as soon as they think they have discovered the law and feel fairly confident.
– *a validation phase:* the concrete results are quite imprecise and indicate sometimes worrisome shortcomings in the model; it is also easy to set up a false pantograph. The game therefore puts many groups into competition: it is up to each group to guess whether or not the pantographs of the others (which are hidden) make correct enlargements and, if so, which ones. Each group asks another one to show it images of different segments or figures (which it chooses) produced by its apparatus. When a group thinks it has guessed the enlargement given by the pantograph, it then states its conclusion. If one of the other groups (which doesn't have that pantograph either) thinks that the conclusion is false, that group then takes up a position against the conclusion. A discussion then takes place, in which one group must prove to the others that its opponent's position is false and that its own is

correct, appealing only at the last possible moment to the contingent proof. (This proof consists of showing the hidden pantograph and what it produces.) Rules about who gets the floor and a point system leave each group to choose between decisions consisting of whether to accept or reject the proofs produced by its adversary or to ask for new information (new images) (Brousseau 1979a). In this way, some students become convinced that if the pantograph gives the same enlargement of any segment in all positions the mapping ought to be (necessarily is) linear because then the image of the sum is the sum of the images.

2.1.3. *Place of this situation in the process*

With very few modifications, this pantograph situation could be taken as an initial situation in the study of linear mappings. But then the hypotheses that the students make in order to interpret the enlargements of the puzzle (cf. Section 2.2) with very simple translations ($x \rightarrow x + a$) would be taken less seriously. They would be rejected almost without examination, because the device would give the right image. Instead of constructing the model and predicting the correct result, the desired conditions, it would be sufficient to discover it as a law of nature. Now, the additive model is an obstacle resisting the establishment of the multiplicative model and must be able to oppose it in open situations, this choice being based on *rational* and *intellectual* criteria.

2.1.4. *Composition of mappings* (*two sessions*)

First the students, in pairs, must construct the image of a simple drawing that would be obtained by two enlargements done in succession (example: × 2.5 followed by × 1.5). The "prettiest" and "most correct" picture will win after verification. The instructions make it clear that how this picture is obtained is not important; it is only the result that counts.

In fact, the real manipulation produces an enlargement so obviously incorrect that it is necessary to redraw the segments with a ruler and to measure them when calculating their lengths (Figure 1).

Then, with the point and the pen both being guided correctly, one can "prove" that the enlargement is correct. The students can therefore be convinced (implicitly) that the composition of two linear mappings conserves the shapes.

2.1.5. *Mathematical theory/practice relationships*

The last two situation-problems of Section 2.1.1 are a correct example of functions which must first satisfy a theory (a structure or a piece of mathematical knowledge) with respect to a manipulation. This theory must provide a model which allows prediction, in a simpler, more economical, more precise way than practice does, and is not a useless complication. And it is because one wants to make predictions that one must require Mathematics not to be contradictory. The theory receives from this practice a certain type of proof and justification and in turn, is used as (intellectual) proof for the practice.

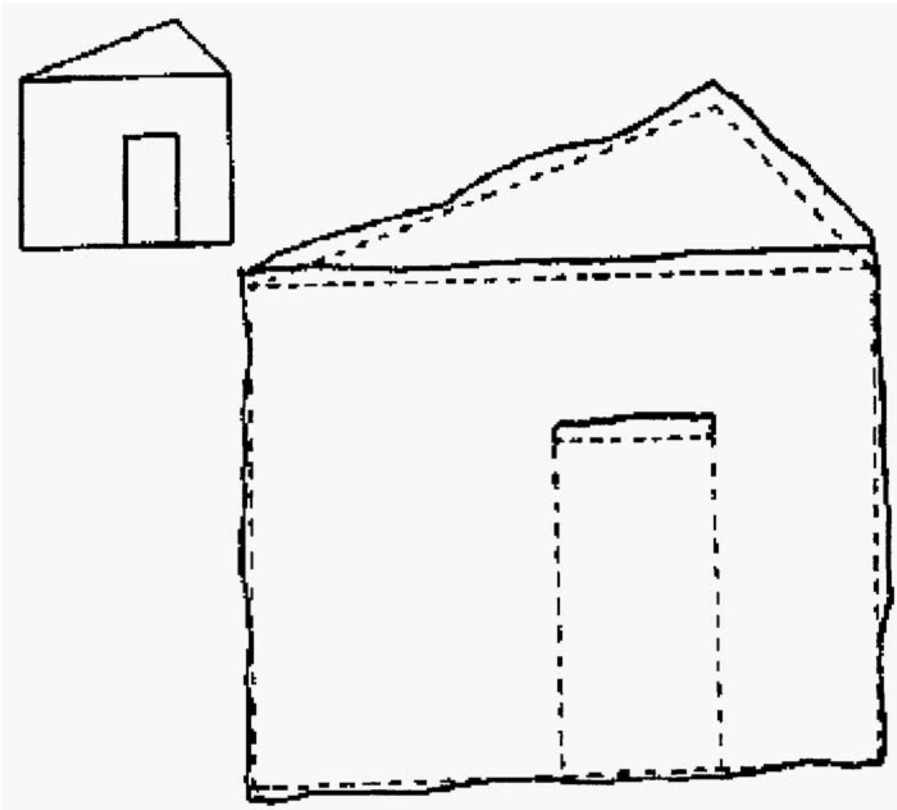


Figure 1 In the corner on the left is the model. On the right, the plain line image is the image obtained with a pantograph, the dot line is the homothetical image of the model (× 3)

The second session consists of suggesting that the students predict the lengths of several segments forming a picture (in the ordinary sense) obtained by a composition of homothetical transformations. They possess the dimensions of the model and if required either the lengths of the segments corresponding to each of the enlargements or the decimals designating the linear mappings (in fact the image of 1).

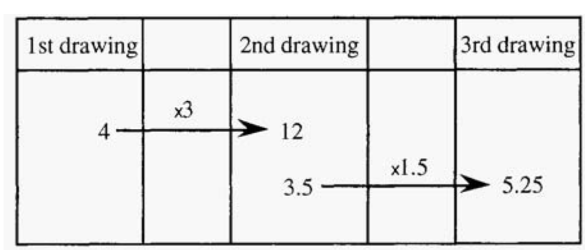


Figure 2

For the third drawing, the images of lengths 2.5, 6, 2, 5.1, 15.6,2.25 on the first must be calculated. The students use various procedures in the calculation of this image:

– P_0) The longest one, by linearity and calculation of the intermediate values, is a left-over from early discoveries: fewer and fewer students hang on to the initial definition which leads to the following calculations, where $\frac{3.75}{2.5}$ is pretty dissuasive:

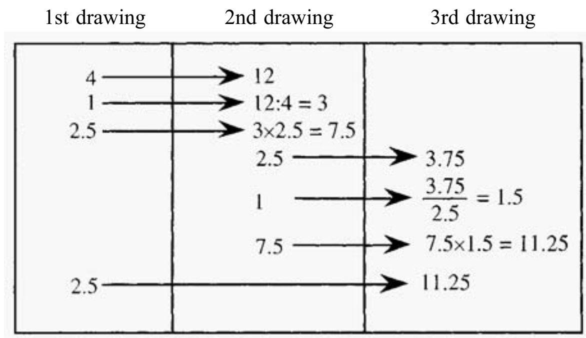


Figure 3

– P_1) By direct linearity, without intermediate calculations, the first and third drawings:

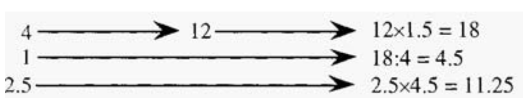


Figure 4

– P_2) By the composition of the two linear mappings with an intermediate calculation:

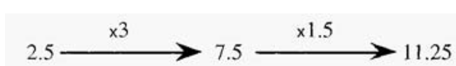


Figure 5

– P_3) By direct use of the composite of the two mappings:

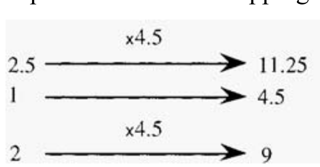


Figure 6

this composite being calculated once and for all, in some way or another.

In this situation, it is most advantageous implicitly to use the fact that the result of the composition of two homothetical transformations is itself a homothetical transformation, to calculate it and to use it to obtain the final image of each segment of the model.

As before, the images really are attached to the board; each student makes a prediction, goes up to check it herself, returns to her place, corrects it if it is wrong and tries to modify her calculation or her system of prediction.

The replacement of two homothetical transformations by a single one has appeared in the students' behaviour by way of a solution to the problem-situation. But the different procedures testify to different cognitive levels. In Procedure P_0, the student has made a composition of mappings, but there is no reason to believe that she was interested in the composite mapping. The latter is present for an observer, in the situation in the usual sense, but it isn't in Procedure P_0. It is in the problem-situation because it is the characteristic knowledge of passing from an inferior solution to a better solution from the point of view of the complexity of the calculations.

2.1.6. *Different "levels of knowledge" relative to the compositions of the linear mappings*

These levels depend simultaneously on the behaviours which manifest the knowledge and the types of situations which cause these behaviours. Classification into levels tends to allow *a priori* the explanation of their differences by the differences in complexity between the situations, between the procedures or between the cognitive models which characterize them. These differences in complexity are based on information with mathematical, epistemological, psychological origins which must justify itself in an intrinsic way in the analysis of the situation or of the didactical process.

Experimentally, the existence of such levels can manifest itself by clues that we shall discuss later (section 4.3.5) and in certain specific, important cases by what Vergnaud called "levels of psychogenetic complexity"[10].

– For the record, we could designate the one that procedure P_0 reveals as *Level 0*: the structure manifests itself in the actions of the student.

– *Level 1:* The one revealed by replacing the action of a sequence of similarity transformations by a single one (Procedure P_1). It can be assumed that this behaviour is possible only if the subject has a model, perhaps still implicit, that allows her to recognize this substitution. This can appear without its being possible for the subject to recognize it, to formulate it, to consider it as an object of study and a fortiori to justify this substitution any way other than by contingent proofs.

– *Level 2:* A first step is taken if a student says "I have the pantographs [× 2.5] and [× 0.251 so I shall calculate the images with a single mapping"; [× 0.625] itself is obtained after calculation of some particular lengths (Procedure P_2).

– *Level 3:* If she says "with the two pantographs that you gave me, [× 4.5] and [$\times\frac{1}{3}$], I predict that the enlargement will be [$\times 4.5 \times \frac{1}{3}$]", another step has been taken. Verification is possible by calculation. (Procedure P_3).

– *Level 4:* The student says: "We must always be able to replace the action of two or more pantographs by the action of just one, which can be calculated by multiplying the first two together".

– *Level 5:* To consider that the same properties could hold in other domains and acquire specialized formulations: "take a fraction of...", "a percentage ...", requires other types of situations, in which the concept can improve in familiarity, in extension, and receive a new cultural status without changing very much from the mathematical point of view. On the other hand, calculation with product-measures affects even the conception of the notion of fraction and must be treated separately at this stage (Vergnaud *et al.* 1979; Rouchier 1980).

– *Level 6:* Even though certain previous remarks may have caused the student to have difficulties, if not doubts, about the following model arising from practice with natural numbers:

to enlarge ——— to multiply (or to add)
to reduce ——— to divide (or to substract)

and even though she may have noticed that $[\times\frac{1}{3}]$ is the same operator as: $[\div 3]$ on integers (Phase II.3), the obstacle that this model constitutes, along with all those built on the use of **N,** is a long way from being overcome.

So the students do not directly know what $3.2 \xrightarrow{\div 3}$ means, nor how to calculate the corresponding image, even though they have the means of finding it out.

It is clear that the game that can lead the students to identify the different ways of designating "similarities" and to unify the methods of calculation requires that they no longer occupy themselves with only one or a few mappings, but with all of them. An intellectual discussion is needed, and only a suitable social game can promote such *problématiques* and lead one to consider the question "Does every pantograph have an inverse?". At Level 6, the student can take care of certain properties of operations to the extent to which these properties serve an action.

There is a very large jump in complexity between situations which, like that of the pantograph, lead to the use of the fact that this inverse exists and those which lead to posing the question, especially if one wants this question to be asked for mathematical reasons (this would be useful to know) rather than for didactical and formal ones, and these reasons to appear before the students' eyes.

– *Level 7:* The next stage consists of making this operation function as a means of analysis. For example: when taking into account that every rational-number mapping can be expressed as the composition of a whole-number mapping and the inverse of a whole-number mapping. Example: $(\times 3/4) = (\times 3)\cdot(\times\frac{1}{4})$.

The calculations in this system ensure the possibility of establishing genuine theorems about rational numbers, like the one Mme Touyarot requests in a somewhat abrupt manner (Brousseau 1980, p.36)[11]. Example of a question: "In order to pass from a model to its image, I used the linear mapping × 4/7; what is the linear mapping that allows one to pass from the image to the model? Can you prove it to a friend without calculating any image?"

This stage allows the search for and the discovery of how to solve all cases of division of decimals in the most general sense. It is possible for students to settle the technique of division in one of the earlier levels of knowledge without having recourse to the composition of linear mappings, but with different meanings: implicitly, from the bounding of the measure-rationals (for example in the phase 1.5, then 1.7) in order to calculate the image for certain homothetical transformations ($\times\frac{1}{4}$ for example, Phase II.3), by looking for the image using the reciprocal mapping or by the composition of two mappings made exceptionally obvious by the context. For example, in order to calculate 3.25×1.25 without having recourse to composition, the student looks for the mapping equivalent to $\times$ 1.25, which she knows corresponds to $1.25 \to 1$—what leads her to search laboriously for a decimal fraction equivalent to $\frac{100}{125}$, that is to say, $\frac{800}{1000} = 0.8$ The calculation 3.25×0.8 is therefore carried out.

It is clear that these processes, even though they have appeared once or twice in the behaviour, do not at this time deserve to be given the status of methods and made the object of learning. The students can produce them once, in the proper place, thanks to an exceptional investment and to the semantic support which ensues from it. It would be completely despicable to want to detach this "procedure" from its context and to set it up directly as a method by means of drill exercises.

It is Level 7 that allows the reasonable, controlled solution of these problems.

– *Level 8* consists of becoming aware of Level 7, of taking it as an object of study, and of describing it in a way which will formalize it, using an algebraic language and an axiomatized mathematical theory. The interest of such an adventure could be a classification of numerical structures. Such a construction did indeed enter into the intentions of the reformers of the 1970s for the 8th grade (13–14 year old) students. But in the absence of a clear didactical contract with the students about what the stakes were in the mathematical activity, all the epistemological levels that we have just detailed were confused and the attempt was doomed to failure in the majority of cases.

A current comparable practice in elementary teaching consists of the teacher "exploiting" the kind of didactical situations presented above by immediately institutionalizing a student's discovery: "You have discovered such-and-such an object (implicit affirmation of the fact that it has a general characteristic); it is called 'composition of two mappings'" (so suggesting that it has a cognitive and cultural status, and inviting students to recognize it and use it in the following exercises).

This practice short-circuits all the mathematical work and even negates it; it comes down to affirming implicitly that it is sufficient to think of it to transform a protomathematical concept into a mathematical notion. Of course, this didactical procedure works and one can legitimately use it locally. But we are justified in thinking that the rejection of its systematic use is necessary in order to change fundamentally the relationship between the "learning subject " and knowledge.

2.1.7. *About research on* didactique

Among the conditions that give the problem-solution its didactical properties (see Section 5.2.2) are:

- the possibility for each student to have a sufficient number of opportunities to change her knowledge level;
- implicit penalties for the most complex procedures—the ones which result from simpler knowledge—in the form of cost of execution and a greater probability of errors.

The situation in which the student must predict the given image by the composition of two mappings possesses these properties.

The problem that arises for the didactician is then a problem of feasibility; in the situation of the pantograph, there is a minimum number of drawings and segments of drawings which must be carried out, and there is a maximum number of operations possible given the available time and the students' motivation. Is the intersection empty? A more rapid evolution must be obtained, generally by reducing the student's uncertainty, either by means of an input of information or the choice of a more closed situation. To what extent is the meaning of acquisitions affected by this? Only experimentation can answer. But it can provide much more information on the effect of these conditions. In certain cases, it is possible to make use of studies *a priori,* of "calculations of situations". For example, one can compare the costs of the various strategies in terms of the number of operations to be performed (either mental or written).

	P_0	P_1	P_2	P_3
Cost of a result	2D+2M	D+M	$2M+\frac{2D}{n}$	$\frac{D}{n}+M$
Reliability of a result	$d^2 \cdot m^2$	$d \cdot m$	$d^2 \cdot m^2$	$d \cdot m$

D: cost of a division
M: cost of a multiplication
d: probability (exact division)
m: probability (exact multiplication)
n: number of segments

Table 2

One can try to predict the effects of those of the variables that one can vary (D, M by choosing the sizes of the numbers, n). We shall use the term "didactical variables" to designate the command variables which will be shown to have an important qualitative effect on the evolution of the procedures.

By taking other procedures into account, one can formulate certain hypotheses; for example: Method P_1 leads to Method P_3 more rapidly than Method P_2 does.

Observation of two classes (fifty students), such as we regularly practice at the observation centre at the Ecole Jules Michelet de Talence, allows the testing of hypotheses of this kind by means of simple Chi-squared procedures, provided that collection of information takes place at the moment when the number of available choices is sufficient for each method. We shall give examples of this in Section 4.

2.1.8. *Summary of the remainder of the process (2 sessions)*

The activities which follow the two that we have just described have the objective of provoking modifications to the meaning detailed in Levels 1 to 7 above.

This process is set up to lead students to ask themselves mathematical questions about the properties of composition : "Is $\times 2 \cdot ((\times 1.5) \cdot (2.5))$ the same mapping as $\times (2 \text{ x } (1.5 \times 2.5))$?" "What does $\div 3$ signify; what other names can it be given?" (The definition of the equality of two compositions works here as a means of proof.) "What enlargement can be obtained by the composition of two pantographs (then of three, etc.), taken within a certain set, for example: ($\times 2$; $\times 3$; $\times 4$; $\times 7$; $\times 10$)?" "What is the reciprocal of $\times\frac{4}{7}$, etc.?" "What does $\times\frac{4}{7} \text{ x } \frac{5}{3}$ produce?" One of the goals of these activities is to interest the students in researching the properties of objects which they are occupied with, or in becoming interested in curiosities such as, for example, situations in which one has to choose between two models of augmentation of price by sum or else by composition. For example: $\times (1 + 0.03 + 0.05)$ and $\times (1 + 0.03)(1 + 0.05)$.

Of course, each stage poses a specific problem which invokes the choice of situations that are also specific. Nevertheless, a certain number of didactical concepts and a certain dexterity which we shall talk about in the next section allow one to produce some and to estimate their adequacy. Most of them fall within a validation format. Habits formed in this type of situation, which starts off following precise rules, allow students to set up and conduct debates without the rules being made explicit every time, and to choose subjects from among them.

The final activity consists of identifying measure-fractions, linear mapping-fractions and ratios, of unifying the expressions and the particular explanations produced during the course of the process, and of reformulating them into the new common language.

2.1.9. *Limits of the process of reprise*

This "after the fact" reorganization of knowledge is a fundamental activity in which one could be tempted to see a good training in mathematics for the students (Brousseau 1976); but it has not been proved that this transposition from historical genesis to didactical genesis is of much interest. Ratsimba-Rajohn (1981) has shown, contrary to what could be expected and despite a suitable choice of situations, that in certain cases the oldest model, on which other models base their source and their justification, winds up reinforced by a process which had been destined to replace it by another, even if the student becomes convinced that the new model is better and even if the situations give it a great superiority. Thus, the study of the characteristics of didactical situations can no more be reduced to epistemology than to psychology.

2.2. *The puzzle*

2.2.1. *The problem-situation*

The first situation for study of linear mappings put to students is the following.

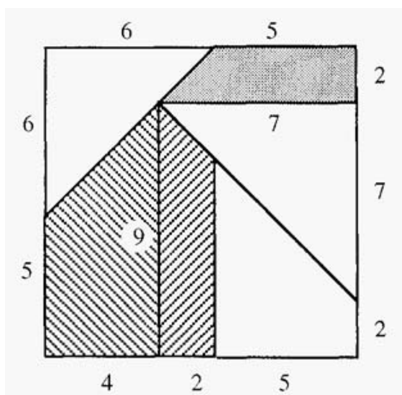


Figure 7

Instructions:
Here are some puzzles (Example: "tangram", Figure 7). You are going to make some similar ones, larger than the models, according to the following rule: the segment that measures 4 cm on the model will measure 7 cm on your reproduction. I shall give a puzzle to each group of four or five students, but every student will do at least one piece or a group of two will do two. When you have finished, you must be able to reconstruct figures that are exactly the same as the model.

Development:
After a brief planning phase in each group, the students separate. The teacher has put an enlarged representation of the complete puzzle on the chalkboard.

Almost all students think that the thing to do is to add 3 cm to every dimension. Even if a few doubt this model, they rarely succeed in explaining themselves to their partners and never succeed in convincing them at this point. The result, obviously, is that the pieces are not compatible. Discussions, diagnostics; the leaders reproach their comrades for being careless. It is not the model, it is its realization that is put into question; verification, some students re-do all the pieces.

It is necessary to give in to the evidence; this is not easy! The teacher intervenes only to give encouragement and to verify facts, without particular requirements. By means of successive rearrangements, some students produce a puzzle that roughly reproduces the form of the model. Others withdraw from the affair by cutting out a large square; the lining up is impeccable. The teacher, invited along with the other groups of students to confirm success, in this case suggests that the competitors use the model to form a figure (see Figure 8) that cannot be reproduced with the image (Figure 9). It is generally quite easy to find three sides, a, b, and c, such that $a = b + c$ and $f(a) + f(b) \neq f(c)$. This often leads students to observe the need to fulfill the characteristic condition of linearity.

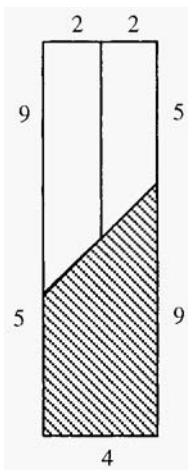


Figure 8

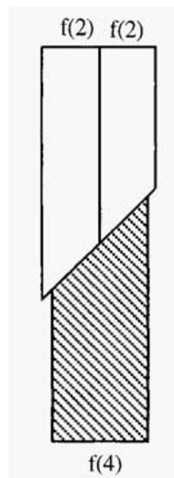


Figure 9

Various social and intellectual processes join together to make it difficult to reconsider the model. One can find oneself in the midst of extremely lively affective manifestations: disputes, fury, tears, threats.

When the children accept that there must be another law and set about searching for it, things move along much more quickly, especially if one of them, or the teacher, displays the lengths on the blackboard (Figure 10).

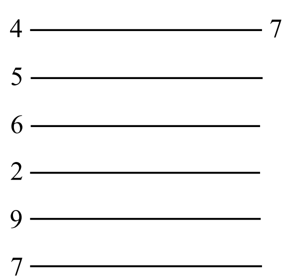


Figure 10

They first find the image of 8: if 4 $\rightarrow$ 7 then 8$\rightarrow$ 14, which does not help and, curiously, this idea is not contested, as if, as soon as the other model is rejected, this one replaces it. "We would need the image of 1." "Yes, then we could find all the others." "For that, 4 will have to be divided into four parts, and 7 will have to be divided into four parts as well." The model of commensuration that they are taught allows them to write directly: 4 times the image of 1 measures 7; the image of 1 is therefore 7/4 (see Section 3.2.6). They do not use it spontaneously but calculate using procedures of the following type:

$$7=\frac{70}{40}=\frac{280}{40} \quad \frac{280}{40} \div 4 = \frac{70}{40} = \frac{35}{20} = \frac{175}{100} = 1.75$$

then $\frac{350}{100} \div 2 = \frac{175}{100} = 1.75$ etc. Clearly, these calculations can take place only because the measure-fraction is known. At this stage, very few remarks are observed as regards the need to have segments such that f(a) + f(b) = f(c) because a + b = c. These calculations now allow success duly verified by effective construction.

2.2.2. *Summary of the rest of the process*

The following lesson consists of finding the image—in the same situation—of a fractional length. The children can verify their predictions for simple fractions by using constructed puzzles.

In order to teach the calculation of the images of decimals, the teacher introduces again a situation similar to that of the puzzle; it is based on the extension and reproduction of a T-shaped piece which, by juxtaposition, will produce a tiling. This time, there are lots of remarks about the fundamental relationship and the linear model is adopted at the outset, but it would be wrong to assume that it is definitively absorbed, because, in a later activity (six sessions later) the students will accept with no problem at all the idea that the mappings $\times$ 2.2, x + 5 and 2x + 3 are all linear mappings ("enlargements", like the others). And it will be necessary, as well, to distinguish them from quadratic functions (for example: side of square $\rightarrow$area of square) in order for the students systematically to use the fundamental criterion to assure themselves, in each case, of the correct basis of the use of linearity.

The reader will have noticed that in this first session on the puzzle, enlargement did not need to have a name. The activities that follow will multiply these enlargements and, when it becomes necessary to compare them, find those which are equivalent. rank them; domination of the image of 1 will be chosen consciously. Then the question "Are enlargements numbers?" will be asked as it was for measure-fractions, and left without an official answer.

2.2.3. *Affective and social foundations of mathematical proof*

The situation of the puzzle is of the same didactical type as that of the pantograph, but the correct prediction clashes, in a very dramatic way, with an epistemological obstacle: the pre-emptive nature of the additive model.

It is essential that the teacher has previously been able to teach her students the habit of searching for their solutions in the problem-situation, rather than trying to interpret the signals that she could provide them with. She will need all her “cognitive neutrality” in order to be able to sustain the students at the affective level without interrupting the social and psychologcal processes that must be accomplished. At first, the situation appears to the students to be perfectly innocent, familiar, and without mystery. Each student has the time to form an idea and to invest personally in a material task that will engage her responsibility with regard to the group. The suspense is quite moderate, but it exists nonetheless. Scandal bursts out of a clear, blue sky: this doesn’t work! It must work! Convictions clash and are expressed according to the characters and social positions within the group. It is then that the scientific process starts; they have to look for the cause, to persist. Nothing is gained by seduction or intimidation of the opponent; it is necessary to convince oneself, to convice others, to prove. The group splits into subgroups; some check the work done, others want to enlarge the square, to double it and cut off a bit. Friendships undergo severe trial, doubts are taken as treachery, the “weak” place the competence of the “strong” in doubt. It cannot be helped, rhetoric must give way to scientific and intellectual proof, the only honourable ways of surrendering to the “adversary”. When, the obstacle overcome, the solution appears to everyone to be demystifying, accidental, commonplace, and even a little disenchanting, each student will have appropriated it, will have triumphed, not over her comrades, but over herself; everyone will have won without the teacher needing to conduct any lesson. The merit of abandoning an idea that is found to be false is as great as that of directly finding a true one. Stubbornness is as necessary as the renunciation of stubbornness. The children have been led to take the part of truth more passionately.

We have continually observed that they also take great pleasure in these games, which are comparable on many points with sporting games. This pleasure leads them to like mathematics and to give it a human and philosophical significance that it could not otherwise have. The teacher must manage the affective investment, her students’ desires and lessons of this type are of a nature to make them not only accept, but possibly even demand exercises or training. We have not described the very complex system that governs her interventions, no longer strictly didactical, whose importance will not escape the reader. A. N. Perret Clermont (1979, 1980)[12] studied this construction of intelligence in social integration and N. Balacheff (1980) observed in a detailed manner how students at various levels approach the problem of proof[13].

2.3. *Decimal approach to rational numbers (five sessions)*

These situations form the part of Phase II.4 which occurs as soon as the students have constructed the set of rational numbers equipped with the operations of addition, subtraction, multiplication and division by a natural number, and order.

2.3.1. *Location of a rational number within a natural-number interval*

Player A chooses a fraction between 0 and 10 (without saying it out loud). She writes it down on a piece of paper which she puts into her pocket.

Player B tries to guess which interval bounded by consecutive natural numbers the fraction falls into. In order to do this, she is allowed to ask questions; for example: "Is your fraction between 7 and 9?" Player A can answer only "yes" or "no". Player B asks questions until she has discovered the two consecutive whole numbers which the fraction lies between. The teacher intervenes in this phase only if the students appeal to her, either to resolve a conflict or to provide details or information.

The game starts as a competition between groups, but very soon it is pairs of students who simultaneously choose two fractions and ask each other questions, turn and turn about. The game is followed by a sharing of observations.

The game leads the children to handle the language designating intervals of **N** and, later, their intersections. Quite quickly, the search for a minimum number of questions leads to a strategy of successive linear partitions into intervals of roughly equal size.

Example:

[0, 5[? A: Yes; [0, 3[? A: No; [3, 4[? A: No.
I know: [4, 5[! Many ten-year-old students need, after "[0, 5[? A: Yes; [0, 3[? A: No", to reassure themselves about [3, 5[? Group games lead to some interesting explanations.

In the earlier phases, the students noticed that whole numbers correspond to certain fractions but this activity requires the systematic use of the immersion of **N** into **Q.**

2.3.2. *Rational-number intervals*

The teacher asks the students to look for fractions, for example between 3 and 4. Comparison provides them with the means of finding quite a few: $3 = \frac{12}{4}$, $4 = \frac{16}{4}$, and then $\frac{13}{4}, \frac{14}{4}, \frac{15}{4}$ answer the question. She then introduces new rules for the earlier game: we shall try to continue the game by attempting to bound our opponent's fraction within the smallest possible interval (without defining it at first).

Remark: One year, this game was called "the Radar Game" (known also by the title "Explorer's Game") because the opponent's fraction was an aircraft that had to be located in the best possible way with a narrower and narrower radar beam; but this "justification" was used once only and didn't help the children very much.

The difficulty consists as much of finding fractions between two others as of comparing the chosen one with those in the opponent's "net".

After a number of trials, the method of the game becomes a bit clearer, but it is necessary to agree about what will be called the smallest interval. The students discuss and formulate a law. Difficult session: the rule of the game is difficult to learn and necessitates numerous calculations—the stake is not evident; this activity consumes “pleasure-capital” right from the first session. The children stop as early as the first interval less than unity and compare the sizes of the intervals. In the following session, some of them can pursue and refine their partition once. The children have the time to play only two rounds. Moreover, intervals bounded by decimals are already appearing, even though the children see that the opponent’s calculations are facilitated by this. This is a weakness of the game, because the students do not understand by themselves that complicating their opponents’ calculations has no merit. As soon as the question is settled by a collective discussion, the rounds work out very much more quickly and the denominators of the fractions become very much larger: 99/10 is expressed in the form 9900/1000. Bounded by a very small interval of magnitude 1/100000, the fraction 22/7 is not yet trapped. An animated discussion results: “It will never be possible to trap it!”, say some... “Yes! Why?” Some give the reason: “Because 10, 100, 1000,..., are not multiples of 7”, but very few are convinced and, the teacher appearing to be interested but ignorant or neutral, the problem remains open for the time being.

2.3.3. *Remainder of the process*

The decimal fractions which we used in the earlier game can very quickly he placed on a graduated line. This activity allows the establishment of a relationship with numbers that are written with a decimal point (the decimals) if the students already know them, and with lengths. If not, the teacher introduces decimal notation. Are these numbers?

Discussion are very short. Students deduce calculations with decimals from those with fractions.

The game of location re-appears at the end of this sequence of lessons; “Where is $\frac{221}{35}$ located?’ But it also takes the form “We want to share 4319 among 29”. The students “rediscover” division with whole numbers, and extend it. In Table 3, we present the final table, laid out by class, in which the work accomplished and the consequent improvement arc shown. The rules of division have been perfected.

2.4. *Experimentation with the process*

2.4.1. *Methodological observations*

The method which we have used for identifying teaching problems with decimals and observing the functioning of didactical concepts is well known in physics and empirical technology; it is that of “comparison under constant results”.

Having identified a certain number of variables or options which one believes to be capable of acting in a significant way, one sets up experiments based on the different choices with the intention of comparing them, independently of the ideas one has formed about the possible interest in these choices for teaching.

What we have sought	What we have done		
1. I am looking for whole numbers I am trying to find out how many times 35/35 goes into 221/35 6 < 221/35 < 7	221 ÷ 35 = 6 remainder 11, which represents 11/35		A young girl suggests the replacement of successive subtractions by a division, that is at once done by the children. The same girl then asks whether all these divisions can be combined into a single one. **IV – Division**. Algorithm
2. I am looking for tenths I am trying to find out how many times 1/10 goes into 11/35 6.3 < 221/35 < 6.4	11/35–1/10 110/350–35/350 = 75/350 75/350–35/350 = 40/350 40/350–35/350 = 5/350 You can take it away 3 times	110 ÷ 35 = 3, remainder 5 which represents 5/350 →	6.31 35 | 221 35×6 = 210 → 210 011 | 0
3. I am looking for hundredths I am trying to find out how many times 1/100 goes into 5/350 6.31 < 221/35 < 6.32	5/350–1/100 50/3500–35/3500 = 15/3500 You can take it away once	50 ÷ 35 = 1, remainder 15 which represents 15/35000 →	35×3 = 105 10 | 5 00 | 50 35 15
4. I am looking for thousandths	15/3500–1/1000 150/35000–35/35000 = 115/35000 Here, the children suggest writing "etc." because they realise that "it is always the same"		

Table 3

The classical experimental method consists of organizing different choices into an experimental plan and carrying out an inferential statistical test on the data.

This does not fit our purpose:

- for one thing, it is not ethically admissible; no professional can agree *a priori* to teach “in order to see”, detaching herself from the result;
- for another thing, it is not possible; by nature, the educational system reacts to its own results by modifying its teaching conditions; thus, the basis of comparison, which is based on the identification of the conditions, is null and void and, in consequence, what is required is the analysis of the reactions of the educational system and therefore those of the teaching conditions.

These comparisons “under constant results” concern the conditions of obtaining, and the means put into place for verifying, the results and not the results themselves. We are trying to determine whether the same results are more costly or less costly to obtain depending on various choices. Very many reasons, which we shall not explain here, argue in favour of this procedure.

It is necessary, nevertheless, to break away from certain fundamental research routines. Thus, instead of comparing slightly different procedures to observe the effect of a modification of the conditions, holding all the others constant, it is preferable to produce very different processes by varying the conditions which are judged to be important. However, in order to make teaching possible—that is to say, so as to realize and conduct these concurrent situations—we discovered that we were induced to answer numerous questions and to make many choices. These questions and choices constitute the tissue which didactical concepts and theories have the responsibility of describing, while at the same time providing the means of analyzing the obtained realisation.

2.4.2. *The experimental situation*

Curriculum

The curriculum was conceived and pre-experimented during the 1974–1976 period (following attempts that started in 1971–1972) with the collaboration of Mme Llorens and Mme Brousseau, teachers, in two classes of 10–11 year-olds (5th grade) at the Ecole Jules Michelet de Talence, where the observation centre of the IREM of Bordeaux is located.

It was presented in the form of a series of texts comparable to [section 2 of this Chapter] and of a film made by OFRATEME in collaboration with M. C. Prouteau.

Its communicability was demonstrated with the same age-group at the Ecole normale de Pau (five classes), at the Ecole normale de Périgueux (three classes) and at Orléans, La Source (one class), and three times with 9–10 year-old children.

Population

We shall give an account of its reproduction in two classes of 10–11 year-olds at the Ecole Jules Michelet over a period of three years (from October 1977 to June

1980) by Mme N. Brousseau and Mme D. Greslard. The sample sizes were 54, 50 and 47 children.

Techniques
We take steps to ensure that the "reproduction" effected is that of the process as a whole. Within the process, we arrange some modifications of the conditions and compare the efforts needed to obtain the same results. The collection of a large number of observations under determined conditions allows the classical experimental study of certain didactical questions which we shall discuss in Section 4.

2.4.3. *School results*

The results which we are presenting have been chosen from among all those which were collected in a way that shows, as much as is possible, the students' end-of-year level on the different mathematical categories of exercises. In Table 4, we give all the results obtained during the course of the different CAS (standardized school year tests), all those figuring in the TAS (tests of school achievement) and some results of ordinary tests, given as late as possible, in categories where neither the CAS nor the TAS presents questions.

FRACTIONS				DECIMALS			
Statement	Results (%)			Statement	Results (%)		
Years: Number of students:	77–78 54	78–79 50	79–80 47	Years: Number of students:	77–78 54	78–79 50	79–80 47
1 – Ranking							
Rank the following from the smallest to the largest $\frac{1109}{1000}$, 0.802; 1.019				Circle the smallest of the following numbers; make a cross under the largest: 8.709, 8.09, 8.079			70
$\frac{41}{50} \div \frac{4}{50}$	78.5			8.90, 8.097			72
CAS 78				**CAS 80** $0.12<0.097<0.107$ $0.097<0.107<0.12$ $0.097<0.12<0.107$			
CAS: *Asessment of scholastic achievement passed at the end of the school year*				**TAS:** *test of scholastic achievement (check the correct answer)*	39.5	39	70

Table 4

2 – Locating							
Place the following fractions on a number line $\frac{75}{50}, \frac{160}{75}, \frac{450}{50}$ (Examination of 4/2/78 and 2/79)	62	84.5		Place the following numbers on a number line: 2.2, 2.63, 2.04. Place under each arrow the number that corresponds: (they were: 0.8, 3.7, 8.5) **CAS 80**			50 50 52 70 72 90
3 – Intervals							
Write three fractions between $\frac{5}{10}$ and $\frac{45}{25}$ (Examination of 1/2/79)	67.7	77.7		Write decimal number situated between 1.2 and 1.3 Can you write a decimal between 3.14<…<3.15 **CAS 79** Write five decimal numbers between 1.019 and 1.021 **CAS 78**	33	53	67
4– Transformations							
Fraction – decimal Write in the form of a numeral with a decimal point $\frac{123}{100}, \frac{12}{25}, \frac{2}{125}$ **CAS 78**	56			**Writing in letters –> to writing in numerals** Write in numerals: forty-eight units, seven-tenths; twelve units, three hundredths; thirty-five hundredths **CAS 80**			80

Table 4, continued

				Write in decimal number form forty-eight units, seven tenths; twelve units, three hundredths; two hundred units, eight thousandths; eighty-nine thousands; six hundred forty-eight hundredths **CAS 79**		79	
Equivalance Write three fractions equal to 15/25 **(comp 1, 78–79)**	65.5	82.5					
Which is the exact series of equivalent fractions: $\frac{2}{3}, \frac{4}{5}, \frac{6}{7}$; $\frac{2}{3}, \frac{4}{6}, \frac{6}{9}$; or $\frac{2}{3}, \frac{4}{6}, \frac{16}{81}$							
TAS	50.5	54	62.5				
5– Addition							
Calculate: $\frac{84}{10} + \frac{425}{100} + 7 + \frac{3}{5}$ (composition 1)	24	38	52	*Work out:* 12.04 + 108.974 **CAS 78 to CAS 80**	82.5		90
				2763 + 544.65 **CAS 79**		89	
				Mark the correct answer: 0.07+0.05+0.01 =0.013 0.07+0.05+0.01 =1.3 0.07+0.05+0.01 =0.13			
				TAS	84.5	75	82

Table 4, continued

6 – Subtraction							
Calculate: $5/8 - \frac{24}{100}$	57.5	69	71	*Work out:* 8.043 – 7.95 **CAS 78**	65.5		
				273.08 – 67.5 **CAS 79**		78.5	
				8.14 – 7.956 **CAS 80**			87
				Mark the correct answer: 1 – 0.991 = 0.009 1 – 0.009 = 0.091 1 – 0.001 = 0.099 **TAS**	47	45	45
7 – Multiplication							
				Work out: 17.03 × 4.507 **CAS 78**	53.5		
				2.6 × 30.5 **CAS 79**		89	
				2.68 × 30.9 **CAS 80**			67
				Mark the correct answer: 10000 × 0.042 = 42 420 4200 **TAS**	52.5	66	62
8 – Division							
				Work out: 50.25 ÷ 33.5 **CAS 78**	58.5		
				491.4 ÷ 0.7 **CAS 79**		61	
				50.25 ÷ 33.5 **CAS 80**			62

Table 4, continued

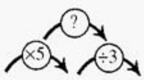
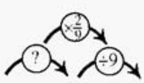
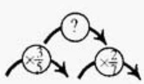
				Mark the correct answer: 0.17342 ÷ 0.0017342 = 1000 100 10000 **TAS**	67.5	64	75
			PRODUCTS 1 – Operators				
 **TAS**	82.5	87	82.5				
 **TAS**	86.5	93.5	90				
	63.5	75	75.5				
			2 – Word problems				
A cheese contains 35% fat. How much fat is contained in 250 g of this cheese?	C[1]:64 R:45.5						
What distance is represented by 12 cm on a map of scale 1:5000000? The distance between two towns is 35 km. What is the distance between them on the map? **CAS 78**	C:75.5 R:53						

Table 4, continued

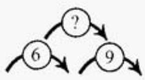
QUOTIENTS 1 – Operators							
	63.5	75	75.5				
2 – Word problems							
				A case of twelve bottles costs 56.40 F. What is the cost of one bottle? **CAS 80**			C: 95 R: 72
				A case of ten bottles costs 36.50 F. What is the cost of one bottle? **CAS 78–79**	79	C: 93 R: 82	
				Yves buys a 14 kg lot of oranges for 20.00 F. What did 1 kg cost? *Give your answer in form of a fraction, then to the nearest centime.* **CAS 78**	C: 79.5 R: 53		
II – Problems							
On a clock, the large hand turns 5/12 turn. How long does this take? 5 minutes 25 minutes a quarter-hour **TAS**	37.5	43	32	Gérard paid 25.20 F for a 900 g steak. What is the price per kilogram? How much would she have paid for a 350 g steak? **CAS 78**	C:55.5 R:28		
On a plan of scale 1:200, the				On a road map, 4.5 cm represents			

Table 4, continued

dimensions in centimeters of a room are 35 × 20. The real dimensions in meters are: 2.35m × 2.20m 70m × 40m 7m × 4m (Check the correct answer) **TAS**	26	31	27	a distance of 9 km. What is the scale of this map? **CAS 78**	C:31 R:18.5		
				At 40 F per kilo, what does a 1500 g roast cost? **CAS 78–79**	64	C:58 C:53.5	
				At 40 F a kilo, what does a 3550 g roast cost? **CAS 80**			
				Colette wants to buy some apples. She has the choice of four types, A, B, C and D, sold in bags of different weights: *weight* *price* A 1 kg 2.30F B 2 kg 5.00F C 3 kg 6.90F D 5 kg 10.50F What is the best offer? (Which is the cheapest per kilo?) **CAS 78–79**	49	C:43 R:43	

Table 4, continued

The TAS is a multiple choice test, standardized at the national level, which measures acquisition in French as well as in mathematics. It allows us to ascertain that the experimental school results do not deviate significantly from the national results.

The CAS is a list of exercises, corresponding to the objectives of the experiments, from among which are chosen those which make up the final test each year (the questions can therefore differ from year to year). Let us remember that these results are given as information only and that the curriculum was not chosen in such a way as to produce the maximum school effects. The sole ethical requirements of the experiment are that:

i) the teachers try to obtain the best results in the conditions that they are given;

ii) the results obtained be at least as good as those produced by other methods.

Table 5 shows that the requirements were satisfied. It is not a question of drawing scientific argument from differences between methods when one knows perfectly well that they could be produced by extremely varied conditions that we cannot control.

2.4.4. *Reproducibility—obsolescence*

Reproducibility must be envisaged at first across the stability of the results of comparable exercises used in each of the three years of the main experiment, under comparable conditions.

With a great deal of latitude being left to allow teachers to adjust their teaching, only 40 common exercises were presented over the three-year period (out of the 75 given each year); 21 were given during the process, 8 appearing in the end-of-year control and 11, which we shall treat separately, appearing in the TAS.

Years % of success	X 77–78	Y 78–79	Z 79–80	XY t, test of student r, correlation		XZ t r	YZ t r
number of students	54	50	47				
exercise r = 21 m	65.93	75.23	76.05	1	1.57 NS	1.98 NS	NS
σ	20.21	18.08	11.75	1	0.97 S.001	0.67 S.001	0.64 S.001
annual control r = 8 m	73.25	77.50	72.50	1	0.65 NS	NS	NS
σ	11.46	14.17	10.12	1	0.75 S.05	0.80 S.02	0.47 NS

Table 5

Examination of Table 5 indicates:

– quite a high stability of the *results of the annual* controls despite, in the teachers' view, major differences between levels of students of different classes; the results did not differ significantly from one year to another. On the other hand, a good correlation between these exercises shows that they are highly comparable.

– the results of the *exercises done during the course of the learning* appear less useful but neither is there on the other hand a significant difference among them. The correlations are excellent; at the price of a linear correction, the percentages could just about be switched. Examination of partial correlations allows the elimination, to some extent, of the influence of the "nature of the exercises". While the two earlier years remain strongly correlated, the third year is now relatively independent ($r_{xz/y} = 0.055$ (ns); $r_{xz/y} = 0.26$ (ns); $r_{xy/z} = 0.95$ (sig)).

Observation of the regression graph shows that a quadratic correction, in fact, would be better because the exercises which had the lowest percentages of success in the first year (five exercises having a less than 50% success rate) saw a major improvement in the following years (three below 50% in the second year, and only two between 50% and 60% in the third year). Only exercises given during the course of the process belong to this case, but they are not characterized by a type of mathematical operation.

The hypothesis of *reproduction* of the same process must be envisaged mainly against the two following:

1. That of an "improvement", at least a local one, like the one we have just mentioned, whose effect, visible here, is a contraction of the dispersion of the results. (The standard deviation diminishes strongly from one year to the next.)
2. That of an *obsolescence of didactical situations.* By obsolescence, we mean the following phenomenon: from one year to another, teachers have more and more trouble reproducing the conditions likely to lead their students to create, perhaps through different reactions, the same understanding of the notion taught. Instead of reproducing conditions which, while producing the same result leave the trajectories free, they reproduce on the contrary a "history", a development similar to that of previous years, by means of interventions that, even if discrete, denature the didactical conditions guaranteeing a correct meaning for the students' reactions; the obtained behaviour is apparently the same but the conditions under which it was obtained modify *the meaning.* It is closer to being a cultural behaviour.

The child then draws the information necessary for the establishment of her answers less from analysis of the situation and comprehension of the problem (which has been put to her) than from "pedagogical" indications which are provided from time to time according to an implicit didactical contract independent of the content (Brousseau 1979a). This process has the advantage of producing the institutionalization of knowledge: the student reads the questions that we want to ask her, the expected answers, their cultural status, etc.

The didactical situation initially envisaged as a situation of adaptation of the students in a problem-situation has, in fact, become a situation of communication of an *institutionalized piece of knowledge* with drawbacks for understanding and acquisition that are well known.

The work of Eugenio Filloy[14] allows us to predict that certain objectives of high taxonomical level evade the teachers' control and, therefore, didactical correction. It can be hoped that obsolescence, if it occurs, will cause on the one hand an evolution of the questions chosen by the teacher in the sense of an increase in the number of problems formed and, on the other hand, a decrease in the number of successes on the most open questions.

Tendencies in the direction indicated can be noted by observing, in Table 6 for example, the evolution of the percentages of success on operations and comparing them those on problem structures. These tendencies are not significant.

Without reporting at length on the means of the same order by which it was obtained, we shall accept the following conclusion: if the experiment reproduced the predicted conditions—which clinical and other statistical observations have the responsibility of establishing—it produced approximately the same results. This is why we shall attribute these results to the process which we are studying, whatever year provided the information studied in the table.

		78	79	80
operations	m	72.0	77.4	77.2
	σ	12.8	11.08	10.97
problems	m	73.8	73.7	67.2
	σ	8.13	17.43	5.87

Answers

	78	79	80
operations	19	15	15
problems	7	1	1

Number of controls

Table 6

2.4.5. *Brief commentary*

External comparisons

With the reservations of previous observations, placing the results of **D** alongside those in the tables of the first part of the preceding section, it can be observed that they are quite comparable, without being inferior, in terms of objectives which teachers consider fundamental (operations). They have the same order of success: between 60% and 90%.

Fluctuations can be attributed to differences among the questions asked.

Likewise, here, as there, one is in the presence of a considerable lessening of success when it comes to "Problems".

Internal comparisons: decimals

Factor analysis of the results indicates that the type of questioning (TAS or CAS) is the first dispersion factor of the students and the questions. Operations on decimals have been learned reasonably well. Differences of success in addition and subtraction tend to cancel out, except for TAS questions which present certain specific difficulties (1-0.09, etc.). Similarly, success rates are similar for multiplication and division and, on average, 10% to 15% lower than those for the preceding operations.

The relationship of order and its manifestations, ranking, locating, and bounding, have success rates that are of the same order as those for operations (contrary to what usually occurs).

The conception of decimals as a linear mapping is definitely acquired: use of the operations is well-advised (understanding rate from 80% to 95%) and very successful.

Situations and problems are less successful, and it is the understanding of the situations which is at issue.

Rational numbers
Ranking, locating, and bounding are a little more successful with rational numbers than with decimals, and the results are respectable. This has to be the effect of reinforcing the initial model of which we have already spoken: fractions serve to establish the rules for the comparison of decimals. Knowledge about them gets reinforced.

This phenomenon appears also for the subtraction of fractions which is used at the beginning of the approach to rational numbers (65%), while for addition it is very much smaller (40%); in the last exercise, however, it is the presence of 7 which lowered the results.

The use of operations and functional symbolism is effective in spite of a very short learning period for this subject (75% to 90%).

The fact that success in exercises using decimals is quite similar to that for corresponding exercises using rational numbers is a new, positive indicator of a better controlled functioning of the concept.

The analysis of the process will not be made here. The main question that interests us (see Section 4.4) would require the examination of the number of exercises and lessons, and of dependencies and implications among the students' results. This examination must look at their errors and not only, as here, at their successes related to situations of a higher taxonomical level.

3. ANALYSIS OF A SITUATION: THE THICKNESS OF A SHEET OF PAPER

3.1. *Description of the didactical situation* (*Session 1, Phase 1.1*)

3.1.1. *Preparation of the materials and the setting*

The teacher has arranged the following:

- on a table in front of the children, five piles of about 200 sheets of paper of the same size and colour but of different thickness (for instance, photocopy paper and card stock), placed in a quasi-random order. Some of the differences in thickness cannot be perceived merely by touch. The teacher does not need to "know" the thicknesses in advance; there is no "correct measurement" to be found;
- in the back of the classroom, on another table, five other piles of about 200 sheets of paper of the same type, in a different order, which will be used in Phase 2;
- ten plastic slide-calipers (two for each group of five children);
- a curtain or a screen that can be used to divide the classroom into two sections.

3.1.2. *First phase: search for a code* (*about 20–25 minutes*)

The teacher divides the children into groups of four or five.

Presentation of the situation

Instructions: "Look at the sheets of paper that I have placed in these boxes labelled A, B, C, D and E. In each pile, all the sheets have the same thickness, but from one pile to the other, perhaps the thickness is not the same. Can you feel these differences?"

Some sheets from each pile are passed around the class—the children touch and compare them. "What do they do in business to compare the different qualities of sheets of paper?" (Weigh them.)

Objective

"You are going to try to invent another way of denoting and recognizing these different types of paper, and of distinguishing them by their thickness alone. You are in competing groups. Each group is going to think of a way of denoting the thickness of the sheets. As soon as you have found a way, you will try it out in a communication game. "You can try things out with the sheets and with these instruments called 'slide-calipers'." (Rulers would suffice, but the children have already used slide-calipers.)

Development and remarks

Nearly all the children try to measure the thickness of a single sheet so as to obtain the required result immediately.

They make remarks like: "It is so thin that one sheet doesn't have a thickness" or "It is very much smaller than 1 mm" or "You can't measure one sheet".

At this stage, there is often a phase of disarray or even discouragement among the students. Then they ask the teacher whether they can take several sheets. Very soon then they try measuring five sheets, ten sheets, until they have obtained a thickness sufficiently large to be measured with slide-calipers or 20 cm rules. Then they exchange systems of designation, such as:

> 10 sheets 1 mm
> 60 sheets 7 mm
> or
> 31 = 2 mm (The teacher will get them to comment during the discussion that the latter use of the equality-sign is not correct.)

In one of the groups, the children refused to use the slide-calipers, and established the system of designation A = TG, B = TF, D = M; A, B and C being the names of the different types of paper and TG, TF and M standing for "very thick" ("*trés gros*"), "very thin" ("*trés fin*") and "average" ("*moyen*"). During this phase, the teacher intervenes to the smallest possible extent. She makes remarks only if she sees that, in the groups, the children are not paying attention to, or have simply forgotten the instructions. The children can get up, collect sheets, exchange them, etc.

When the majority of the groups have found a system of designation (and the five group members agree with this system or code), or if time has run out, the teacher passes on to the next phase, the communication game, even if not every group has yet found a system of designation.

3.1.3. *Second phase: communication game* (*10 to 15 minutes*)

Presentation of the situation; instruction

"In order to test the code you have just found, you are going to play a communication game. During this game, you will see whether the thickness-code you have established will let you identify the type of sheet designated.

The children in each group will divide into two sub-groups, one of senders and one of receivers (with two senders and 3 receivers depending whether they are 4 or 5 in a group).

– All the sender groups will be located on one side of the screen, all the receiver groups on the other.
– Each sender group will choose one of the types of paper on the first table (A, B, C, D or E), which the receiver-group cannot see because of the screen. They will send their receiver-group a message which must allow them to determine the type of paper chosen. The receivers will use the piles of paper on the second table at the back of the classroom to find the type of paper chosen by the senders.

When the receivers have succeeded in finding the type of paper, they will become the senders. Points will be assigned to groups whose receivers correctly find the type of paper chosen by the senders."

Development and remarks

Before the game starts, the teacher sets up the screen that separates the senders and the receivers.

The teacher

– passes messages from the senders to the receivers;
– receives the receivers' replies;
– after deciding whether the reply agrees with the senders' choice, takes note with the whole group of the success or failure.

All messages are written on a single sheet—which we could call the "message form" (see Figure 11)—that passes between the senders and the receivers of the same group; this sheet bears the number of the group. In addition, so that the teacher can tell whether the reply is right or wrong, the senders write down on another sheet, which they retain—which we could call the "control form"—the type of paper that they choose for each game.

Remarks

It is clear that the teacher did not in fact introduce superfluous vocabulary such as "message form" and "control form" nor formal requirements for the presentation of messages—which the children would have had to learn to respect. There was no general instruction on this subject, only assistance and particular corrections for "malinspired" children.

(1)

Group number	I	
First game: message sent	E: 10 = 1 mm	
Reply	R: D	success
Second game: message sent	E: 21 = 1 mm	
Reply	R: B	success
Third game: message sent	E: 8 = 2 mm	
Reply	R: A	success

Message form

	I
1st game	1 D
3rd game	3A

Control form

Figure 11

If some groups had not been able to construct efficient messages, the teacher would have organized a new phase of discussion, for each group, for the search of a code (with the same instructions as in the first phase).

But this situation never occured (in 8 identical experiments). The children succeeded in completing two or three rounds of the game.

During this game, three different tactics are observed among the children:

– some choose a number of sheets whose thickness they measure;
– some choose one thickness and count the number of sheets;
– some look for thickness and number of sheets at random.

It can be noted, also, that by preference children choose types of sheet of extreme size, the thinnest or the thickest, so as to facilitate the work of their comrades.

3.1.4. *Third phase: result of the games and the codes (20 to 25 minutes) [confrontation]*

Presentation of the situation and instructions

For this phase, the children come back into their groups of five, as for the first phase of the session. The teacher announces a comparison of the results and prepares on the blackboard a two-way table, (groups) x (type of paper), in which she will write the messages exchanged and the points obtained by the groups (see Table 8) as their reporting proceeds.

Development and remarks

Each group in turn provides a "representative", who reads the messages out loud, explains the chosen code, and indicates the result of the game.

The children compare and discuss the different messages. As they are often very different, the teacher requires them to adopt a common code.

Example:

10 =1 mm
TF
60 sheets 70 mm

After discussion, the whole class decides to write":

10f; 1 mm
60f; 7 mm

When all the messages are written down, the children look at the table and spontaneously make remarks such as "That doesn't work' or "Here, this is good", etc.

These remarks can be classified into four categories:

Category 1

If the sheets are of different types, the same number of sheets must correspond to different thicknesses.

Examples from Table 1 :

19f; 3 mm → A
19f; 3 mm → Type B } "That doesn't work"

19f; 2 mm → Type C
19f; 2 mm → Type D } "That doesn't work"

Category 2

For the same type of sheet, the same thickness corresponds to the same number of sheets.

Example from Table 1 :

30f; 2 mm → Type C
30f; 3 mm → Type C } "That doesn't work'

Category 3

If there are twice as many sheets, the thickness is twice as large.

Example from Table 1 :

30f; 3 mm → Type C
15f; 1 mm → Type C } "That doesn't work"

And the children go on, "We must have found it":

$$\begin{array}{lcll} 30f;\ 2\text{ mm} & & 15f,\ 1\text{ mm} & \\ & \text{because} & \times 2 \downarrow & \downarrow \times 2 \\ 15f;\ 1\text{ mm} & & 30f,\ 2\text{ mm} & \end{array}$$

Category 4
Differences in the number of sheets must not correspond to equal differences in measurement.

Example:

$$\left.\begin{array}{l} 19f;\ 3\text{ mm} \\ 20f;\ 4\text{ mm} \end{array}\right\} \text{"That doesn't work because one sheet can not measure 1 mm"}$$

At the end of the session, the teacher suggests that the children examine the table during the next session, as a class, to verify the measurements by manipulation and to correct them if necessary.

Remark: The presentation of operations on the numbers during the search for equivalent pairs using arrows is neither formal nor obligatory; it is a familiar "manifestation" of the use of natural operators which the children are used to.

3.1.5. *Results*

All the children know how to measure the thickness of a certain number of sheets of paper, to write the corresponding pair, and to reject a type of paper that does not correspond to a pair that they are given. The majority are therefore able to set up a comparison strategy and use it to accept a type of paper as corresponding to a measurement. Some of them have formulated this strategy. Most of the children can analyze a table of measurements and point out incompatibilities by implicit use of the linear model.

3.2. *Coniparison of thicknesses and equivalent pairs (Activity 1, Session 2)*

3.2.1. *Preparation of materials and scene*

The material is the same as for Session 1: piles of sheets set out in the same way, slide-calipers.

3.2.2. *First phase (25–30 minutes)*

Presentation of the situation—Instructions
The teacher asks the children to return to the examination of the table drawn up during the first session.

This observation is first done silently so that the children can spot the most obvious incompatibilities among the measurements. Then the teacher suggests that they fix up the errors that they have seen, line by line (for each type of paper).

Development and observations

After observation, and on request, the children come up one at a time to the board to indicate the "messages" which seem to them to be incorrect and possibly to propose a correction.

These corrections are discussed by the whole group of children. If they all agree, the correction is made, otherwise, they suggest verification by manipulation: they count out the number of sheets indicated by the message and measure them. After collective verification, the new message is adopted and written into the table. (This manipulation is made by groups: two groups verifying one message, two others verifying another message, and so on.) It has often happened that when the same type of paper is measured by different groups of children, these groups do not obtain compatible measurements. This is often due to reading errors or to the fact that the children have compressed the sheets to a greater or lesser extent. They notice it very quickly, and say so.

Likewise, for types of paper close in thickness it has happened that the measurements did not permit the recognition of the type of paper in question. It was during this phase that the children noticed that they had more chance of distinguishing between papers of similar thickness by taking a larger number of sheets. Thus, fifteen or twenty sheets of paper of almost the same thickness have measurements so close that the children cannot distinguish between them with precision. It is at this stage that they often suggest measuring the thickness of fifty, eighty, or a hundred sheets.

Examples of the children's observations (taken from Table 7):

10f; 2 mm
and 5f; 1 mm } for type B; "That's fine."

A child adds:

15f; 1 mm

×2 ↓ ↓ × 2

30f; 2 mm

The three measurements: 5f; 1 mm—10f; 2 mm and 15f; 3 mm, are thus conserved.

On the other hand, in the "Type C" column, the measurements

12f; 1 mm
and 8f; 1 mm are contested and rejected by the children, who suggest that they be re-done.

The first two conclusions do not correspond with the third; the children decide to retain the first conclusion because they see that for a hundred sheets there is 3 mm difference whereas with 8f, 14f, 12f there is no difference in measurement.

They say, also, that a difference of 3f (12–8) or of 2f (14–12) doesn't change the measurement. (They verify this by manipulation.)

<table>
<tr><th>Group</th><th>Type C</th><th>Type D</th><th>Children's conclusions</th></tr>
<tr><td>1</td><td>100 f; 8 mm</td><td>100 f; 11 mm</td><td>1) D>C
(D is thicker than C)</td></tr>
<tr><td>2</td><td>12 f; 1 mm</td><td>10 f; 1 mm</td><td>2) D>C
(D is thicker than C)</td></tr>
<tr><td>3</td><td>8 f; 1 mm</td><td></td><td rowspan="2">3) C>D
(C is thicker than D)</td></tr>
<tr><td>4</td><td></td><td>14 f; 1 mm</td></tr>
</table>

Table 7

3.2.3. *Second phase: Completion of table; search for missing values* (*20–25 minutes*)

Presentation of the situation—Instructions
The children notice that certain types of paper were not chosen during the communication game and that they have no measurements.

The teacher then suggests that the children complete the table by measuring the missing types of paper.

Development and remarks
The children carry out the work in groups of five which are no longer competitive but co-operative and not restricted to the same groupings as in the previous session. (Several groups take the same type of paper and later verify the compatibility of the measurements.)

Example for Type C:

$$\begin{array}{ccc} & 12f;\ 1\text{ mm} & \\ \times 8 \downarrow & & \downarrow \times 8 \\ & 96f;\ 8\text{ mm} & \end{array}$$

When all the children agree, the new measurements are written into the table (Table 8). At the end of this phase, the table is therefore entirely correct and complete. There are many compatible measurements for each type of paper.

3.2.4. *Third phase: Communication game (15 minutes)*

The teacher suggests that the children re-play the communication game from the first activity, taking into account all the observations and corrections that they have made (large number of sheets, compression of the paper, etc). The game is thus replayed to the great satisfaction of the children, who succeed in everything, even if one adds one or two new types of paper.

3.2.5. *Results*

These children know how to adapt the number of sheets chosen to the necessities for discrimination of their thickness (to augment the number of sheets when the thicknesses are very close). They know how to calculate pairs corresponding to the same type of paper. Everyone now knows how to use the linear model to analyze a table. A small group of the children is able to use relationships of closeness between pairs. A large number of the children have learned to judge declarations and to argue their own points of view.

EXAMPLES OF FIRST TABLE OF MESSAGES (uncorrected)

Type of sheet	Group 1	Group 2	Group 3	Group 4
A	19f, 3mm	10f, 2mm	20f, 4mm	
B	19f, 3mm		4f, 1mm	15f, 2mm
C	19f, 2mm	30f, 2mm	100f, 8mm	30f, 3mm 15f, 1mm 20f, 2mm
D	19f, 2mm 12f, 1mm		100f, 9mm	
E			9f, 4mm	13f, 5mm 7f, 3mm

Table 8

FIFTH GRADE (CM2) (1977)

Type of sheet	Group 1	Group 2	Group 3	Group 4
A	49f, 9mm			
B			10f, 2mm	5f, 1mm 15f, 3mm
C	100f, 8mm	12f, 1mm 96f, 8mm	8f, 1mm 64f, 8mm 12f, 1mm	
D	100f, 11mm	10f, 1mm 100f, 10mm		14f, 1mm 154f, 11mm
E	23f, 10mm	10f, 4mm 8f, 3mm	6f, 2mm	

Table 8, continued

3.2.6. *Summary of the rest of the sequence* (*Session 3*)

After verifying that the children know how to recognize sheets designated by their thicknesses, as for example (48; 9), the teacher asks them to find other ways of writing each thickness, and then to rank the sheets from the thinnest to the thickest. Then she gives them (25f; 7 mm) to rank with the others.

By the end of the session, the teacher tells the children a method of writing the thickness of a sheet; (50; 4) designates a pile of 50 sheets which is 4 mm thick; the thickness of one of these sheets is written $\frac{4}{50}$, read as "four fiftieths of a millimetre". She verifies by means of exercises that this information has been transmitted.

The distinction between the thickness of a sheet and the designation of a pile of sheets is essential but difficult, and will be learned only progressively.

3.2.7. *Results*

The children know how to find equivalent pairs. They know how to compare the thickness of sheets of paper (many of them using two methods). Using these comparisons, they have a strategy for ranking pairs. They know how to designate the thickness of a sheet of paper by means of a fraction and how to find equivalent fractions.

They do not know how to find the equality of two fractions in the general case.

Remark: This know-how occurs within situations. It is not yet possible to take a question out of context and ask it independently. The results can not yet be depended upon as "acquired" knowledge, nor do the children identify them as such.

3.3. *Analysis of the situation—The game*

3.3.1. *The problem-situation*

Analysis of the students' task—senders

The task of the sender is much less arduous than that of the receiver; senders have only to count the sheets of the chosen pile and be careful to measure them correctly. The desire to have simple, correct numbers leads to the choice of a number of sheets such that:

– the thickness is as close as possible to a whole number of millimetres;
– errors due to the compression of the sheets remain small;
– the number of sheets to be counted remains fairly small.

A more thorough analysis or repeated practice would lead the sender to choose:

– a large enough number of sheets so that the difference between their thickness and the thickness of the same number of sheets of the next thickness of paper is quite distinct; for example: more than 1 mm (compression eliminated);
– a number of sheets such that the cost of the operation is minimal. (The cost of the operation is evaluated in terms of the cost of counting the sheets and the

price of errors of trial. The cost of errors depends on the relative error by way of the probability of error it generates.)

This point gave rise to a mathematical study not reported here.

Analysis of the students' task—receivers

The task of the receivers is much heavier. If they were to "apply" a rational algorithm, in order to interpret the message (60f; 7 mm) they would:

– count 60 sheets of each type of paper;
– measure the thicknesses of these five piles of sixty sheets;
– compare: the pile whose measurement was closest to 7 mm would be selected.

In fact, if it were systematically applied, this procedure would lose the essential benefit of the situation; the children can (must) eliminate certain hypotheses by using the experience that they acquired in the first phase and by working out a representation (an implicit model) for the thickness. In this phase of action, relationships, estimations, operations—still hazy, incomplete, approximate—are tested, clarified, made more complex, before even being formulable.

Example:

> "60f; 7 mm, that is a thin paper; it is not from pile A, we have found the pair (3f; 1 mm) for A"—it's being understood that the thickness of 60 sheets from pile A would be much more than 7 mm...

If the children's hypothesis proves to be wrong, they can become aware of it quite quickly and with no penalty other than a small waste of time. Representation allows the anticipation and economizing of attempts; it is verified by the action without conditioning it completely, nor being conditioned by it. However, it governs these decisions for action, because once unlikely "papers" have been separated out, the others must be retained. "In C, 30 sheets are 2 mm thick, the same for B; you can't tell. So, measure again... If 30 sheets from C measure a little less than 2mm, it must be D. We need more sheets,..."

For the receivers, two messages tend to be equivalent if they allow the same discrimination between the papers; thus they are in the same topological neighbourhood.

The children must be senders and receivers, turn and turn about. Even if not all the children acquire an equal competence in reasoning about relative thicknesses, they all acquire a sound knowledge of the conditions which the model they will choose must satisfy; what it ought to allow.

3.3.2. *The didactical situation*

Analysis of the teachers' task—Phases 1 & 2

In the two first phases, the teacher's most important task does not consist of controlling the content and the development of the children's reflections; she must not intervene whether she is hearing an interesting proposition or a false declaration. It is the situation which must provide the necessary feedback. She is no longer—for the time being—the guardian of truth, the guarantor, the refuge, the necessary and final recipient of all the children's interventions.

She must—by her attitude—convince the children of her neutrality as regards their comprehension of the situation, so that they cease to rely on her for information and assistance which they can draw only from themselves. Neutrality but not indifference; she receives suggestions with an equal interest and returns them with conviction without modifying their content, but she tries to make them usable.

The teacher sends the children back to the situation but concentrates on increasing their investment, their desire to succeed. She facilitates the solving of subordinate problems, keeps an eye on their following of the rules and the assignments, which she identifies and repeats on occasion, resolves organizational problems, helps to settle conflicts within the groups. She makes sure that everyone invests in and contributes to the result that is being sought. In order to do this, she takes great interest in the final result which she records, participating in successes, as well as in deceptions, rejoicing with some, encouraging others, in a sort of "sporting" spirit.

Here again, it is the efforts which are educational, and not the stated goals.

In particular, she makes no comment on the loss of points by the groups; the groups re-form from one lesson to the next. The points serve only to determine the strategies during the game, not to classify people after the game.

Phase 3

Phase 3 of the first activity is a phase of confrontation (this is not a situation of formal validation). There again, the teacher is a director of the game, directing the play but not playing herself. For all the children, the problems that she manages to get them to resolve: clear formulation, precise information, and so on, are those which would hinder the functioning of the situation. There are no "external" requirements. She lets false or absurd declarations come to their correct formulation, she leaves the others the time to formulate their judgement. She doesn't confirm a correct statement before everyone has agreed. She encourages minorities to express their reservations, and clarifies debates, even if she cannot resolve them.

3.3.3. The maintenance of conditions of opening and their relationship with the meaning of the knowledge.

In situations of investigation, classical *pedagogy* leads the teacher to "exploit" a "good" statement almost immediately. She speaks to the first (or one of the first) children who "finds it". In the end, the exchanges involve 20% of the children (the most "alert"). In order for these exchanges to be understood by the others, the questions asked must be such that 80% of the children would be able to respond to them almost directly if they were given the time needed; the questions will therefore be quite closed and the investigation will consist of a sort of speed test devoted to algorithms. Thus, 60% of the children participate by proxy. As for the 20% who do not know the answer, for them it is never a question of investigating, but of learning prefabricated knowledge, that they ought to possess, that it is even "disgraceful" not to possess, since in the end it is only a question here of knowing who "has it" or who does not "have it". Thus, for 80% of the children, the investigation is a negatively sanctioned situation where the emphasis is put on knowledge; to have to search is the recognition of an unpardonable weakness. To have to learn is the fate of a disadvantaged, scorned minority.

The situations that we propose must not be conducted in the same way.

The questions are often more open, in the sense that the number of children capable of responding directly to them is very small; nearly everyone has to search. The exchanges will involve 80% of the students in problems that less than 20% of them can solve directly. The teacher keeps the situation open by not exploiting ideas. She is content to take them into consideration. The fact of being the first to have an idea loses some of its importance.

Children who have found the answer most quickly must learn to share their conviction without the security of validation by an adult. Thus, investigation and learning become the principle concern of most of the children.

These situations are delicate to conduct; the way is open to a field of very diverse social attitudes. The teacher has learned to counteract the action of diversions; some children lead others astray in order to have the time to search themselves, monopolize attention, fail to respect the research of others, etc. It is the entire function and use of knowledge and truth that are being learned at this moment. Some children, used to a status within the class and to a classical didactical contract, find it hard to accept the sudden changes in the teacher's modes of didactical action, such as the absence of security or of immediate evaluation, the dependence on the action or the opinions of classmates, etc. For the children and even for the parents, the teacher has had to be temporarily the mediator of changes, often profound ones, that she has caused. In any case, frequent changes in the type of intervention used favour smooth evolution and adaptation of attitudes in spite of their diversity.

The evolution occurs correctly to the extent that one succeeds in seeing to it that relationships with the object of study and relationships with the other learners become sources of pleasure and desirable stakes. The taking of decisions, actions, exchanges, judgements must first become occasions for a fundamental jubilation of the class. The teacher is led to accede to the children's chief desire in life.

Knowledge, the symbolic alibi of a contest among the children, must be able to become the object of a lusty activity.

The teacher-pupil relationship gives way to child-knowledge, child-*milieu*-knowledge and child-others-knowledge relationships.

3.3.4. *The didactical contract*

The teacher is the class's reference memory[16]. She remembers relevant conventions, agreements and facts. She recalls them advisedly. It is in this rôle that she directs and controls learning.

Finally, the teacher represents the knowledge of adults when she provides information about this subject (for example: Activity 1 – Session 3 – Phase 3). The children, who have their own system, can understand the current notation and its advantages over—or its equivalence to—the system that they have constructed, and adopt it, not freely but consciously (situations of institutionalization).

Indeed, possible changes of notation or vocabulary cost both children and teachers. This is why teaching conditions have a critical character and depend in an essential way on the didactical contract.

At no time during this first activity does the teacher indicate that something must be learned. At no time does she ask that an activity be repeated having agreed with the children that it must be known, understood, learned (this will come later, perhaps), and recognizing that a phase was not justified by other reasons.

In fact, Phase 1 of the first session, "search for equivalent pairs", must allow all the children to search for at least two pairs that are equivalent to a given pair, to compare two pairs at least a dozen times, to explain an equivalence at least once, to hear many methods of comparison. All this within a situation of collective control. But the contract agreed with them is not "you must learn and *know how to do* this in order to be able to answer the following question in the way I want it", but "you must *do* this in order to be able to take up the following challenge". Learning to search for equivalent pairs continues in Phase 2 and later on; it is not necessary to "achieve" this during Phase 1. Further, this would generally require the *formulation* of a search-method such as the following, for example: the two terms in a pair can be multiplied or divided by the same number, or, "worse", two pairs are equivalent if... "infaillibly", at this point, one would have to state either a nonsense phrase or an incomprehensible sentence. Moreover, these formulations of methods are only rarely economical in the long term. We reject them categorically here.

The solution of the situation of Phase 2, ranking the thicknesses, is no more complicated if one doesn't know to search for equivalents, since the child considers the representation of the situation—the number of sheets and thicknesses of the piles—rather than applying an algorithm to record the thickness of a sheet.

Any *formal* (conventional, institutional) learning at this moment would carry a heavy mortgage of nonsense. On the other hand, the teacher must verify that every student has correctly *reflected, chosen* and *calculated* a new pair, using the table if necessary, or with the help of the others.

This is not to say that there is neither learning nor supply of information. Phase 3 of the third session demonstrates an example of the supply of information. It is a question of conventions of recording and use. These conventions will henceforth be regulated and the teacher will "require" their use in the children's "public" communications. In fact, as the work goes on she will exercise an increasing pressure for the standardization of the convention, without refusing to understand other formulations, particularly at the beginning.

3.4. *Analysis of didactical variables. Choice of game*

3.4.1. *The type of situation*

We shall not come back to the fundamental *cybernetic* character of communication. The most economical coding, if one agrees that one can augment the number of types of paper at will, consists of designating them by means of a pair, (number of sheets, thickness in mm). Note, however, that the situation of searching for a code is not repeated. There is no dialectic of formulation of pairs. The code is set up almost immediately. The situation of communication does not function very much as a learning situation; it does nothing more than give "meaning" to the action phase of senders and receivers. The "optimal" character of the messages has provided no opportunity for discussion with the children. It assures us only of a satisfactory acceptance on their part.

3.4.2. *The choice of thicknesses: implicit model*

On the other hand, the "action" component of the situation is the opportunity for the creation of an implicit model for the system of thicknesses.

Why, given that it is actually a question of measuring lengths, were objects of appreciable length not chosen? At this age, children possess an effective representation of "lengths" between 1 mm and 50 mm. That is to say, they can compare thcm, put thcm end-to-end, partition them and if necessary evaluate them. (That is to say, they can make them correspond approximately to a whole number with the help of the systems of units being used.)

But in fact, this representation is efficient enough to allow the solution of most practical situations *without* its being necessary to use a new numerical system: perception, whole-number measurement, etc.

The child can imagine or carry out operations on "lengths", addition and subtraction, without using any model of representation. For example, she can indicate the desired length of a stick by sending a thread of equal "length". The mathematical activity, then, can at best translate, transcribe, the "reality" that is assumed to preexist.

As in geometry, the implicit model that children have available is so rich that it governs all decisions. By the time it fails to give the solution of a situation, the mathematical theory that could supply it is too complex to be constructed. If one has recourse to this implicit model, the constructed mathematical theory (here measurement) would never appear except as a complication, a supplementary or superfluous requirement. The theory would appear to need to be guided, controlled by "intuition"; at best, it provides more precise information, etc.

Thus, we have chosen "size of objects" as the didactical variable, a domain which foils the natural implicit model without, however, excluding manipulations.

With lengths of some hundredths of millimetres, one can "conceive" of the requisite operations, but certainly neither perceive nor carry them out directly. Some are even entirely inconceivable, such as partitioning one of these lengths (a hair into four!... lengthwise!). We shall see that this impossibility is wanted for other reasons. The number that measures the thickness then becomes the means—the only "concrete way"—of capturing this thickness, of constructing comparison experiences, of predicting the sum, etc. Mathematical representation has been reintegrated in its fundamental rôle of theory under construction, within its dialectical relationship with the constructor and the situation.

Moreover, it will be possible to control the enrichment of this representation if, initially, one favours the development of the implicit model by making it useless to explain methods (of comparison, for example). One can provoke a formulation phase (Activity 1, Session 2), then one of progressive formalization in Module 4, at the opportune moment for each notion, before the implicit model is too effective; soon enough to be useful but late enough to have meaning; soon enough for the meaning attributed to language to be essential to action and its formulation, late enough for the formulated concepts not to be isolated from each other, but to work together.

3.4.3. *From implicit model to explanation*

The implicit model developed by the children in the first session includes an approach to the relationship of fundamental algebraic equivalence (number of sheets from Pile 1 × thickness of Pile 2 = number of sheets from Pile 2 × thickness of Pile 1, but certainly not in this form), in the form of a set of linear mappings of **N** to **N.**

This mapping is approached by way of its properties:

– The children's first observation indicates that they want different mappings for sheets that are perceived as different (projection of the set of mappings onto the set of types of paper).

  Let us observe that it is not realistic on the part of the children to want different mappings to indicate different types of paper. They have right before their eyes examples where for the same type of paper and for the same number of sheets they get different thickness, where, however, the message worked. It is enough to have results close enough to each other and far enough from other results.
– Nevertheless, this requirement is formulated: it seems that they want one number per type and one type per number. In fact, it is only the shocking results that are rejected: "30f → 2 mm; 30f → 3 mm; someone made a mistake". (These results are not algebraically equivalent, nor topologically close.)
– It is only afterwards that linearity is formulated, by its characteristic property of conserving ratios. The children reconsider pairs that contradict this property and then formulate it.

Remark: If they had not made slightly varied attempts, with the freedom of taking the number of sheets that they wanted, simple ratios would not have appeared. If the game had not been rapid, and a little anarchic, good natural contradictions would not have appeared either.

One should observe that the implicit model does not appear as a positive formulation of known properties on the occasion of their existence. Pairs which obey the implicit law give rise to no commentary. It is the pairs which do not obey it, which, by the trouble that they cause, make formulation necessary; like a theory, the model is revealed by its contradictions—apparent or real—with experience, and not by its agreements.

This fact shows clearly a phenomenon that is entirely general and important which formally contradicts certain empirical didactical theories such as, for example, the psychodynamical process of Diénès; it is not similarities (isomorphisms) between encountered situations which are the principal motive of "abstraction" any more than are the transcription or schematization of structures the key to formulation. Simple familiarity, even active familiarity, with well structured situations never suffices to provoke a mathematization. On the contrary, problems posed by a situation at the time of putting a pre-existing model (implicit or explicit) to work, or by a theory at the time of making of a decision provoke the evolution, the modification or the rejection and the formulation of theories.

This dialectic characterizes most of mathematics.

Here, the initial situation (first session, Phases 1 and 2) can be envisaged without a particular initial model (enumeration and whole measurements—homothetical transformations on **N**). Its solution leads to a language and to methods that are not very satisfactory from the logical point of view. The resolution of this theoretical problem (Phase 3) requires new "experiments" and confrontations (second session) which lead to the reduction of contradictions and the use of a partly explicit model, especially when it comes to the technique of measuring sheets of paper and to some extent, also, the relationship of fundamental equivalence between writings.

Another stage starts with the third session: putting the system of measurement on trial. Does it allow recognition of the notation for the thickness of one of the known sheets, its discrimination from the notation for another thickness? Here, coding again becomes the children's object of study with the formulation of equivalence, the creation of equivalence classes and the study of the compatibility of the order relationship on pairs with the relationship of equivalence; the conclusions must be the same, whatever the pairs that furnish the equivalence.

The designation of each class by a "fraction" (the word does not have to be articulated), the writing of the equality of the classes, is information introduced by the teacher as social convention (so as to be understandable to others) and as a didactical conclusion, an implicit confirmation of the validity of the proceedings of the class.

The children think that the types of sheets can be ranked in increasing order of thickness. The theory that they have constructed allows this ranking whereas their senses do not, and all the same, so they can have a quite concrete confirmation of their inferences—their theory functions as a real theory.

She poses new problems, for the comparison method does not appear valid for every pair and she has recourse at the same time:

- to algebraic relationships which allow recall of the comparison of (40; 6) to (20; 4), with that of (40; 6) to (40; 8) or, furthermore, recall of the comparison of (30; 4) and (20; 2) with that of (30; 4) and (40; 4);
- to topological relationships that allow the comparison, for example, of (19; 3) and (10; 2) because (19; 3) is thinner than (20; 4) (Children's commentary: "With one sheet more, the pile is 1 mm thicker, and as the sheet doesn't measure 1 mm...")

It turns out that the general construction of this sequence follows a model of an axiomatic construction of rational numbers which is quite modern (pairs, taking quotients). It cannot be denied that knowledge of such an axiomatic has made it possible to envisage it as a solution to the didactical problems of decimals that we posed ourselves.

It would be wrong to believe that this choice preceded didactical analysis. If we had chosen another domain of didactical variables and another genesis, the pairs could have lost all significance, and equivalence all interest; constructions then might or might not have coincided *locally* with another classical axiom system.

In any case, to let the trace of this architecture appear within the children's activity in any form whatever would be without interest for them.

4. QUESTIONS ABOUT *DIDACTIQUE* OF DECIMALS

4.1. *The objects of didactical discourse*

Didactical discourse focuses on four levels of objectives:

1. *The level of contingent facts.* This is the description of the actual productions of the teacher or the educational system: declared objectives, information provided, assignments, etc. The students' actual productions, behaviours, results, the reactions of the milieu, etc.
2. *The level of the didactical situation.* Here it is a question of interpreting the teacher's decisions, the student's behaviours, her motivations, etc., while discerning over which other behaviours those that are attested to have been chosen, and of collecting them into strategies, into models of behaviour, of errors (if possible, ones that explain the "games" and the respective and reciprocal contracts of the partners). This analysis brings to light variables that ensure the reproducibility of the situation across some differences among the observed facts.
3. *The level of analysis of the concept and of its geneses.* It is then a question of determining the set of situations that is likely to make a notion work, giving it the different meanings that determine the corresponding concept. Only differences between situations that affect the concept lie in the field of didactics; they are the effect of variables which are to be determined in each case.
4. *The level of events and didactical debates* where the concepts, methods, and means of analysis necessary for the above levels are produced.

Each level has its own methods of study and proof—and its methodological problems—which cause a didactical experiment to be considered an experimental epistemological experiment and distinguished from a teaching experiment.

The interactions of these four levels are evident, but the way in which teaching problems and didactical questions connect is not yet clear to many people, any more than the way that the scientific and the didactical phenomenotechnical points of view oppose and complement each other. This is why we must come back to the process that we have just described in a very detailed manner, so as to show how it can be the place where the questions that interest us can be tested and actualized.

4.2. *Some didactical concepts*

4.2.1. *The components of meaning*

The definition of the meaning of a notion is, as we have said, one of the central problems of *didactique.* What has already been said now allows us to get a glimpse of how we propose to solve it. It will be a question of locating and classifying every situation where this notion appears, either as a solution—whether necessary or not, optimal or not—or in the statement, or in the behaviours of the opposing players in the didactical game. Thus, the notion appears in its functioning and in its relationships with the different sectors of mathematics. One can identify various particular conceptions which permit the solution of one class of situations, even though they suggest incorrect answers to another situation, and which as a collection constitute the concept.

Let us gather all the criteria that we have used in choosing the procedure being studied. They serve as a first approximation to the components of the meaning of decimals and allow the production of most of the situations we are looking for.

1. The mathematical type of the problems: algebraic, topological, or order.
2. The type of mathematical objects that realize the notion and in connection with which the problem arises explicitly: the decimal as image of a *measurement,* the decimal as a *scalar* or a ratio operating within a set of measurements, the decimal-proportion, the linear decimal-operator in a vector space, etc. [cf. Table 11.
3. The type of mathematical structure competing with **D** in the proposed situation (this can be **N**, **Q** or **R**) as the only, or the best, solution.
4. The type of didactical situation on the basis of the knowledge manifested, situations of action, formulation, validation, institutionalization. Here, the decimal appears respectively as a rule of action, as a language, as a system of proof, or as cultural knowledge. This classification produces functions of **Q** also: calculations, algorithms, designation, representation, object of study, means of proof, theory...
5. The domain of realization of the situation (bank, commerce, physics,..., etc) and scientific status (mathematical necessity, physical law, social convention).
6. The type of formal presentation of the language expressing decimals and the informational characteristics that set it up against neighbouring solutions: decimals versus sexagecimals or versus binaries (characteristics of numeration).
7. The frequency with which the situation is likely to occur.
8. How **D** or the subset used was generated; type of operations used and characteristics of the frequency distribution of the use of the decimals employed.

9. The cognitive or didactical status of the notion in the subject's behaviour in the situation. (See what we have called levels of knowledge in [Section 2.1.6])
10. The type of representation or definition, commensuration or partitioning, for example. (See [Section 4.3.1 .])
11. The type of axiomatic organization of the implicit and explicit knowledge of the subject.
12. The type of mathematical operation or relationship solving the situation: ranking, bounding (between two whole numbers, between two fractions, between two decimals of the same order of magnitude), locating (placing or finding a number within an interval), transformation (fractions⇔decimals ⇔ sexagecimals, change of units, scientific notation, polynomial decomposition, numerals ⇔letters), sums (of measurement, addition, translation, composition of translations), differences (*idem,* negative decimals), products (measurement × natural or decimal operator, measurement x measurement, operator × operator, derived measurement, etc. See Table 2), division (partition, division, multiplication by an inverse, composition, measurement by a ratio, etc.)

These distinctions are not all of equal importance and the list is not exhaustive. This set of situations should still be reorganized so as to favour a less heavy analysis.

4.2.2. *The didactical properties of a problem-situation*

The term "situation" designates the set of circumstances in which the student finds herself, the relationships that unify her with her *milieu,* the set of "givens" that characterize an action or an evolution. A situation is a problem-situation that necessitates an adaption, a response, by the student. In particular, if the need for this response has been the basis for a precise instruction, if the student has a project, a declared objective, we have a "strict problem-situation" (or a formal one), and even a "problem" if the *milieu* is reduced to a statement and provided that no material constraint due to certain physical aspects of the situation, nor any psychological or social condition, modifies its interpretation. A didactical situation is a situation in which there is a direct or indirect manifestation of a will to teach—a teacher. In general, in a *didactical situation* one can identify at least one problem-situation and a didactical contract.

What conditions make a problem-situation a learning or a teaching situation? Régine Douady (1980) proposed a list which corresponded very well as a first approximation to the one that I wanted to satisfy in the process presented above and that I cannot describe here. Analysis and theorization of situations leads to the refinement and extension of this list, to the construction of indices that allow one to study and make objective the too intuitive criteria, as Bessot and Richard (1979) have done. This is a central problem for *didactique,* as much for analysis as for the realization of such processes, and it has not yet received the place that it deserves in spite of a few old but nearly confidential publications (eg. Brousseau, 1970).

4.2.3. *Situations, knowledge, behaviour*

Each problem-situation demands on the part of the student behaviours that are indications of knowledge. This fundamental correspondence, established case by case, is justified by the interpretation of problem-situations in terms of games, and of behaviour, in terms of indication of strategies the adapted nature of which must be demonstrated in the model or representation attributed to the student. The object of Table 9 is to show that different types of situation, produced by the study of didactical conditions of learning, produce distinct behaviours depending on the form of knowledge. We have since (Brousseau, 1970) used a four-class model in which the names have varied according to the nuances which distinguished among the cases studied: (1) Procedure (or implicit action model, or pattern of action), (2) implicit knowledge equivalent to a statement, property or relationship, (3) explicit knowledge, (4) language (code, formal system...). Now and then, (2) and (3) are merged.

This system could be compared, to a certain extent, with those of Skemp (1976) and Byers and Herscovics (1977): (1) evidence of instrumental understanding, (2) of intuitive understanding, (3) of relational understanding, (4) perhaps of formal understanding[17].

We shall not repeat here the well known characteristics of situations of action, formulation and validation (Brousseau 1970).

Situations of institutionalization are those by which the cognitive status of knowledge or a piece of knowledge is fixed conventionally and explicitly. Institutionalization is internal if a group fixes its conventions freely, according to an arbitrary process which makes it a quasi-isolated system. It is external if it takes its conventions from a culture. This is the most common situation in classical *didactique.*

4.3. *Return to certain characteristics of the process*

4.3.1. *Inadequacies of the process*

We have emphasized on many occasions that we would not consider the curriculum as a method to offer to teachers. We should explain why.

The main reason is the difficulty of communicating all the necessary information, in particular that relating to the rules with which the teacher must force herself to comply. They are very different from those to which teachers are accustomed and are based on a very "new" didactical concept, on modes of evaluation, and even on a teacher-student sensibility which is new in teacher-student relationships. It would probably be harmful to students to teach them every step in this long genesis in the classical way, and to institutionalize temporary behaviour.

Besides, the needs of epistemological experiment have led to choices that it would be, at the least, premature to put to teachers. For example, we do not know how to provoke and control the overcoming of epistemological obstacles, or simply reprises, that is indispensible if we want to replace the model of regular inflation of compatible knowledge. Or again, we have chosen a representation of decimal-measurement very far from what history suggests.

Observed behaviours classified by type of situation and type of knowing

Types of situations **Types of knowings**	**Situation of action**	**Situation of formulation controlled by a situation of action**	**Situation of proof (or validation)**	**Situation of institutionalization**
Procedure	Know-how. Implement the procedure; chose it in preference to another	Detailed description Designation	Justification of the relevant procedure (it can be applied) which is adequate, correct and optimal	Canonization of the procedure into an algorithm (term invented by A. Rouchier)
Implicit model **Property** **Relation** **Representation**	Make choices, make decisions motivated by the related knowing (without being able to "formulate" it)		Contingent proofs, experimental proofs, proofs by exhaustion	
Knowings **Statement** **Theory**	Apply a knowing (the knowing could be formulated)	Statement of the property or of the relationship. More "correct" reformulation	Proofs Mathematical proofs More convincing translation Organization Axiomatization	Canonization of a theory, of a knowing Didactical transposition (Chevallard 1980)[1]
Language	Use of a language for explaining. Behaviour shows a division into objects corresponding to signs and words	Use of a language, of a formal system, of a formulation for communicating, speaking know-how	Justification of a word, of a language, of a formal model (relevance, adequacy, optimization) definitions. Metalinguistic activities.	Choice of definitions, Linguistic and grammatical conventions

[1] *Editors' note:* Brousseau refers here to the original lecture of Yves Chevallard which he presented to the first Summer School of *Didactique des mathematiques.* The corresponding text is published as (Chevallard 1985).

Table 9

Finally, the process presents notorious imperfections of just the kind for which we reproach classical methods" and which are the object of studies. Definition by commensuration hardly gives the student a more available notion of fraction than the old definition.

For example: in Phase II-1, students experience difficulties of writing the image of 1 in the linear mapping which to 4 associates 7, even though they know that this image, repeated four times, forms a segment 7 cm long: the image is $\frac{7}{4}$.

The effort of definition of fraction-measurements is probably quite sterile from the cultural point of view. We are now going to examine some of these questions.

4.3.2. *Return to decimal-measurement*

An epistemological study of measurement and its functions has shown that it played an important rôle in the emergence of decimals (through the intermediary of sexagecimals). This rôle is related to various situations in which it is a question of representing, communicating or forseeing the result of certain manipulations of everyday life: control of conservation, comparison, reproduction of equal quantities, partition. In these situations, the values of the physical variables and their relationships with human characteristics are essential. Thus, in the majority of civilisations, one identifies three "unit" functions:

– the "counting" unit, (C). Any larger quantities must be split up, in any case, in order to be transported or for their evaluation, into equal parts which can be counted. The unit (C) is the largest among those that humans can handle easily.
– the "fractional" unit, (f). Quantities that are smaller than it can be manipulated "all at once". Their division into a large number of equal parts that can be counted is less economical than a direct partition.
– the "precision" unit, (S). This is the smallest that can be taken into consideration.

In general, S « f « C and $10 \leq \frac{C}{f} \leq 60$.

One may observe at this point the quite efficient rôle in practice of "deca" and "deci" in comparison with "hecto" or "kilo".

The use of quantities in the interval [f, C] serves as a model for identification of enumeration and of certain fractions.

The system of writing has also as an objective the bringing of the "measurements" of the most familiar quantities into the best known natural interval.

The historical negotiations between these requirements have produced various systems of which some, too well adapted, have not been able to evolve. One can hardly hope to go beyond the use that the Babylonians, and then the astronomers, made of the sexagecimal system.

But this study lets us see all the options available from this *problématique* in a dialectical perspective where the practice of measuring, the use of the decimal system of measurement, and the introduction of decimals would be disassociated and not completely mixed, as is the present case in France. For that reason, it would be very useful to reopen the studies and observations of F. Colmez (1974) and to examine them within this "economical" perspective which the theory of situations allows.

4.3.3. *Remarks about the number of elements that allow the generation of a set*

It is possible to generate all the operations of binary logic with the aid of systems consisting of from one operation (the incompatibility of Sheffer[19]) to sixteen operations. Systems that are used commonly consist of four or five. It is frequently advantageous, to understand a set, to generate it with a restricted number of elements which become particularly familiar. This number must be small enough so that all the generating elements can be considered together (less than ten), and large enough so that the generation of rarer, but current, elements is not too long, and so that the system can support a sufficient number of distinct relationships.

This, according to Youschkevitch (1976), is why Abû'l-Wefâ, around 961–976 AD, "generates" fractions with the help of a system of fractions "pronounceable" in Arabic, itself consisting of the principal fractions: unit nominator and demoninator from 2 to 10, fractions of the form m/n, $m<n\leq 10$, unified fractions, products of the form $\frac{1}{n}\cdot\frac{1}{m}\dots\frac{1}{P}$ —those whose denominators have prime factors greater than 10 are unpronounceable. (He chose this system, which goes back to the antiquity of Baghdad although al-Uqlidisi invented decimals at Damascus ten years earlier.)

But, after all, that is how first-graders handle natural numbers.

Now, here we have chosen to introduce all at once a rather large set of fractions; all denominators up to 50, all numerators from 1 to 20, have similar probabilities of occurrence. No fraction has a dominant rôle.

It doesn't seem that this method, which conflicts with cultural habits (which always favour $\frac{1}{2}, \frac{1}{4}, \frac{1}{3}, \frac{2}{3}$ and $\frac{1}{5}$), has met with special difficulty; on the contrary, it seems to have favoured reasoning about any fraction whatever. It seems likely that the notion of ratio could not be introduced in the same way.

4.3.4. *Partitioning and proportioning*

The present cultural system provides the following general definitions for rational numbers:

Definition 1 (by partitioning)

A fraction has remained as in the seventeenth century (d'Alembert) "*one or many parts of a whole partitioned into many equal parts such as one-half; one-third, two-thirds, ...*". This "a" is unfortunate and makes the conception of fractions greater than unity difficult; $\frac{a}{b}$ is thus the result of a material operation which consists of partitioning a number of wholes and not only one whole into b parts that can be compared and declared equal, and then taking a number of these parts.

It is a constructive definition. It refers to the manner in which the defined object is constructed.

It is well adapted to *the construction of a magnitude corresponding to a given number* (a unit being given), except in the case in which the practical way of carrying out the necessary operations effectively is not known.

For example, it is not possible to make a gold bar which is two-thirds the weight of a given ring unless one is prepared to destroy the ring (but one can partition an equal weight of modelling clay into thirds by trial and error). In these cases where either division is not conceivable or comparison is not possible (or the result is not defined, or counting isn't possible), the definition works not as a representation of reality or as a theory, but as a symbolic system in which it is useless to seek a concrete meaning. For example, this definition is of no direct help if one must assign a number to a given magnitude, unless the method of construction is allowed to appear.

The definition used in the curriculum is very different.

Definition 2 (by commensuration)
A quantity (if it exists) will be the fraction b/c of a whole if by repeating it c times (taking c identical parts) one obtains b wholes.

This definition supposes at the outset that the size exists, and that the operations that it calls forth are much more frequently and easily performable (multiplication). It doesn't tell us how to construct 3/4. It isn't constructive, but it does provide an algorithm for recognition. It allows us to say whether of not such-and-such a size is three-quarters of another one. One must also be able to "repeat many times" or to lay out many samples of the size in question.

These two mathematically equivalent definitions correspond in fact to different conceptions of fractions and decimals in the sense that they are not relevant, or more efficient, or better adapted, to the same problem-situations. Depending on the value given to the unit, the ratio of the dimensions of the object to that of the chosen unit, the presence or absence of a partition or the measurement of a subdivision, it is possible to block the use of one or the other, or to make them more or less costly in terms of the number of elementary steps, or to counter the control of the uncertainty of the result, or to augment the probability of error.

Table 10 summarizes a theoretical study that can be found in Ratsimba-Rajohn (1981). It indicates which of the two conceptions the values of these variables make more efficient. It seems that they are quite complementary and that the constructive definition is much the most frequently used. There exist other conceptions that we shall not study here.

Ratsimba-Rajohn demonstrated that the stability and the homogeneity of behaviours on quite a varied set of exercises allows one to speak of two different, and to some extent incompatible, representations. He exposed a sample of middle school students to situations successively blocking one or the other. He observed the percentage of students who were capable of practising the two definitions to be very much smaller on each test than the expected probability of success on each problem would have led one to expect.

VARIABLES OF A DIDACTICAL SITUATION OF MEASUREMENT FOR THE EVALUATION OF THE EXPECTED COST

Command variables

Problem situations

1. Size, U, of the unit chosen in relation to reference values;
   S_1: threshold of perception **or** S_2: threshold of useful precision
   f: units "to be divided": (easily and frequently)
   p: unit "to report" or serving for counting
2. Ratio of the size ∇ of the object that of U
   – $\frac{\nabla}{U} \ll 1$ $\frac{\nabla}{U} \approx 1$ $\frac{\nabla}{U} \gg 1$
   – "better" attracter of the interval $\frac{\nabla}{U} - S_2$, $\frac{\nabla}{U} + S_2$;
   ex: $\frac{3}{5}$ or $\frac{59}{57}$
3. Conditions of "comparison":
   – 1 to 1 (Roberval), 1 to n (graduation), direct to indirect (calibration)
4. Conditions of adjustment:
   – possibility of material partitioning
   – possibility of repetition
   – maximum number of repetitions of the unit, etc. (available space)
5. Type of activity:
   – realization of an object of given size or measurement of an object

Didactical contract

1. Conventions about requirements on the measurement
   – useful (action) or conventional (communication) precision, usual fractions, decimals, etc.
2. Value of success, cost of error or failure…
3. Number of attempts allowed – types of feedback – opening of situation
4. Variables of the contract relative to learning: time, frequency, etc.

Dependent variables

1. Number of repetitions of the unit for a given precision
2. Number of actual divisions
3. Number of comparisons
4. Relative density of easy or complex measurement points

Cultural variables

Frequency distribution of values of variables of the problem situation
– in the school *milieu* (variable more or less adjustable)
– in the cultural *milieu* (free)

Table 10

4.4 *Questions about methodology of research on didactique (on decimals)*

Classifying situations by models, conceptions, representations, levels, etc. poses the problem of knowing what is equivalent and what is not, what is different and what is not, and then, having made these distinctions, of possibly observing the relationships internal to the model (synchronic) and the relationships between models (diachronic).

4.4.1. *Models of errors*

M. L. Izorche (1977) demonstrated that students' mistakes and successes are explained if they are put into the following model: given many decimals (d_i), one looks for the smallest n, for example, such that, for every d_i, $d_i \times 10^n \in \mathbf{N}$, and one identifies with **N** the set $\mathbf{D}_n$ of decimals such that $10^n \cdot \mathbf{D}_n \subset \mathbf{N}$ The child reasons and operates on $\mathbf{D}_n$ as on **N**. Léonard and Grisvard (1981) produced, about a problem of order, a rather more precise model of behaviour taking account of the independent working of the integer part and the decimal part. These studies are based on the patterns of high percentage failure on specific problems within an area of general success, then on a hierarchy of individual behaviours; they show that the model sustains few exceptions. Nevertheless, when the explanation bears on modification of the probability of error and not on deterministic models (assuming that the same student always makes the same error), the method is rarely satisfactory.

One could hope that factor analysis of correlations or hierarchical analysis will produce classes of questions and classes of students that correspond to these models. This method is systematically used by many researchers, but it is a delicate procedure and interpretation commonly stops at the very first principal factors.

4.4.2. *Levels of complexity*

If a sufficient "discrepancy" is observed among the students, in the time between the mastery of two questions belonging at first sight to the same concept, it can be inferred that there exist, between their solutions or the conceptions that command them, sufficient differences of complexity to form different psychogenetic levels. The genetic analysis that derives from this principle produced many invaluable observations on the part of Vergnaud and Durand (1976), Ricco (1978) and Rouchier (1980), and their colleagues. It is necessary, however, either to give up working too closely with didactical phenomena which can introduce discrepancies of arbitrary and didactical origin, or to redefine a specific methodology. In particular, the formalization of this notion of level, although attempted on many occasions, presents difficulties.

4.4.3. *Dependencies and implications*

Systematic observation of differences in behaviour related to different didactical modalities or conditions have been made possible by the astute use of the analysis of correspondences using the method of modality questionnaires by Pluvinage (1977) and have produced interesting observations on proportionality.

But in *didactique,* it is less the proximities or distances that are interesting than asymmetical dependencies that suggest implication. Regis Gras (1979) thus studied heirarchical and factor analysis methods based on an index of implication improving on that of J. Loevinger. But the implication itself suggests a transitivity, quite often absent in fact, and a more refined specific concept must be produced before a satisfactory statistical index can be drawn from it.

Perhaps the analysis of diachronic dependencies performed on processes like those which we have available could lead to this result.

NOTES

1. Here the obliging reader would probably like to insert the forty-three main figures which he will find listed, for example, under the word "figure" in Robert's Alphabetical and Analogical Dictionary of the French Language.
2. cf. Lucienne Félix. *Editors' note:* Félix (1971). Lucienne Felix imagines a dialogue among three people, Drawy, Mati and Logi, who combine their talents in drawing, mathematics and logic to study some geometry problems. Prosopopea is a rhetorical device in which speeches are delivered by the absent, the dead, gods, supernatural beings, abstract entities or allegories.
3. *Editors' note:* Polya (1957).
4. *Editors' note:* the interested reader may refer to Sir Thomas Heath's *History of Greek* Mathematics (1921, Dover reprint 1991, especially pp. 101–105) to learn more about επιμερης, ημιολιος and other επιτεταρτος.
5. We will come back to this question later on, sections 2.1.4 and 5.3.5. See also (Vergnaud and Durand 1976).
6. (Brousseau 1965). *Editors' note:* see also the Brousseau biography in this book for indications about the context of the publication of this book.
7. *Editor's note: "Grand N"* is a journal for elementary school teachers published by the IREM of Grenoble. For the article Brousseau refers to see (Brousseau and Brousseau 1987).
8. *Editors' note:* see Briand (1993).
9. *Editors' note:* see also (Gras 1996).
10. *Editors' note:* see Vergnaud and Durand (1976).
11. *Editors' note:* see Chapter 3, section 3.
12. *Editors' note:* the second reference is to (Schubauer-Leoni and Perret-Clermont 1980).
13. *Editors' note:* for further development see (Balacheff 1987, 1991).
14. *Editors' note:* see (Filloy *et al.* 1979).
15. *Editors' note:* Where "f" stands for "sheets", in French: *feuilles.*
16. *Editors' note:* see (Brousseau and Centeno 1991).
17. It is a pity that Herscovics, who had a long conversation with Skemp and myself on this subject in Karlsruhe in 1976, has not examined this correspondence.
18. For example: the evaporation of the unit in the first phase (see Brousseau, 1980, page 33). *Editors' note:* see this book, Chapter 3, section 3.2.4.
19. *Editors' note:* Sheffer (1913).

CHAPTER 3 AND 4 POSTLUDE

DIDACTIQUE AND TEACHING PROBLEMS

Editors note: *The present postlude reproduces a text written by Brousseau as a conclusion of the article on the didactical problems of decimals. It is in fact a postlude to both papers, aiming essentially at delivering a message about what this study could tell about the usefulness of the research in didactique. This question will reappear as the center of Chapter 6.*

In the two preceding chapters, we have tried to present the problems of teaching decimals without destroying the fabric into which they are woven. And that has led us gradually to translate these teaching problems into more or less general didactical problems, and to tie the questions together into a research methodology. Have we perhaps strayed along the way? We have had to summarize or exclude numerous studies which were too particular or too centrifugal... then what unity can *didactique* boast, what is its field, its relationship with teaching? And how have we advanced in terms of our first observations?

The teaching of decimals, like that of counting, poses in my eyes a difficult didactical problem of paramount importance.

On the one hand, their use is so extensive, so convenient and so commonplace that children meet up with them very early in their developed, present form. Whatever the form of learning, it is familiar habit and use that will govern the meaning of the concept.

We have seen the inadequacies of this “mechanical” conception and the necessity of placing this use under the control of an understanding and even, as early as possible, of rational knowledge. Only the resolution of certain problem-situations can give this clear conception, this understanding and this knowledge. But these problems are highly complex and it seems hardly possible to give them to students at the desired moment, too prematurely.

On the other hand, it is unthinkable to delay the teaching of decimals, or to approach them by means of a genesis which is too long, or which would lead to strange practices. And then again, at the moment when the necessary renewal and accomodation can occur, in secondary teaching, there is no longer time to devote to this subordinate question.

We thus have a teaching problem without a solution. There are others which lead to similar contradictions.

In a general way, such problems are, if not resolved, at least treated by a didactical negotiation about which one knows little as yet, but which follows classical discourse only at a considerable distance. In this discourse, every success is perceived as a monotone function of variables; what is good at one extreme, what is bad at the other. Potentially, the teacher is all powerful.

What are the reciprocal rôles of habit and understanding in the acquisition and functioning of a notion? How do these rôles evolve? What results can one count on seeing changed if a better compromise is obtained? The translation of this debate into a list of hypotheses that can be proved wrong, that can be studied experimentally, demands the creation of new notions. This is the price that *didactique* must pay in order to rise to a scientific status without ceasing to be a theory of didactical fact and to maintain control of its technique. Problems of teaching are problems of decisions. They must receive a response, even a provisional one, whatever the cost. The response cannot be a response "just to see". The teacher cannot commit herself to drawing all the logical consequences from a choice, and to sticking with it no matter what happens. Moreover, problems do not give much reward to experimental method, they are indices to be interpreted. To the extent that they are content-specific, some of the teacher's questions can be questions in a "didactial science" that would defer the need for decisions. What are the students' results? What decisions can improve them? What are the alternatives? How to produce other methods, how to choose one of them? How to manage it, conduct it, communicate it? What indices must be looked after? These decisions extend to other cases on condition, however, of supporting a subtle modification from the point of view of vocabulary and methods; what situations and what behaviour correspond to a suitable appropriation of a concept? What are the erroneous behaviours that appear, and their meaning? What are the variables that may act upon the appropriation, what hypotheses can explain the good or bad results?

In any case, the majority of choices that appear critical for the teacher do not provide good didactical questions. (For example: should one adopt textbook X or textbook Y?) Can research in *didactique* be content to provide answers only where it can support them by a scientific study, leaving it the responsibility of the teacher alone to decide the importance or the relevance of the information and the use she can make of it'? Someone must always compare the conclusions to a precise teaching process and choose an optimal response. At any rate, every "in vivo" study of learning in the school situation ought to be part of a teaching activity whose variables must be carefully controlled. In consequence, scientific *didactique* will not escape phenomenotechnical *didactique.* It must take charge of the totality of interactions between the systems present and handle what can be isolated and is content-specific (without being reducible to it). It must thus produce processes, not as optimal suggestions to teachers, but as objects of study.

CHAPTER 5 PRELUDE

The notion of didactical contract deserves a specific chapter. It is one of the major constituants of the Theory of Didactical Situations, and plays, as Brousseau wrote, a central rôle in the analysis and the construction of situations for the teaching and the learning of mathematics.

The origin of the notion of didactical contract lies in the case studies of selective failure in mathematics, especially the case of Gaël[1] which Brousseau followed and studied in detail. In a paper published in 1980, entitled *L'échec et le contrat* (Failure and the Contract), Brousseau wrote:

> "So, we are going to examine other causes, those which concern the child's relationship with knowledge and with didactical situations, and no longer those linked to her aptitude or other individual characteristics. [...] The subject's acquisitions in a learning situation are regulated as much by the relationships with knowledge and with the content to be learned which results from these acquisitions as by factors of the individual herself. To question the characteristics of the student, and only the student, seems to me to be an attitude analagous to (and just as ineffectual as) trying to explain why water flows out of a bucket with a hole in it by analyzing the difference between the quality of the water which escapes and that which remains, as though the reasons for its having escaped lay in the qualities of the water itself.
>
> In a teaching situation, prepared and delivered by a teacher, the student generally has the task of solving the (mathematical) problem she is given, but access to this task is made through interpretation of the questions asked, the information provided and the constraints that have been imposed, which are all constants in the teacher's method of instruction. These (specific) habits of the teacher are expected by the student and the behaviour of the student is expected by the teacher; this is the *didactical contract."* (Brousseau 1980a, p. 180)

So the notion of didactical contract appeared as a theoretical necessity, imposed by the effort of understanding deep dysfunction of students' learning. It bridges adidactical learning situations and didactical situations, acknowledges the different rôles of the student and the teacher, and brings to light the process of devolution as well as providing the ground for situations of institutionalization. It also gives its specificity to the concept of *milieu*, as Brousseau wrote in the article published in 1990 under the title *Le contrat didactique: le milieu* (The Didactical Contract: The *Milieu*):

> "To consider the *milieu* as the source or the image of the game of each actor in not sufficient; it must be shown that it is a necessity of the didactical contract.

The intervention of the teacher modifies the conditions under which knowledge functions, conditions which are also part of what the student should learn. The final aim of learning is that the student be able to put this knowledge to use in situations from which the teacher has disappeared which is why we distinguish between a *didactical* functioning and an *adidactical* functioning of the student in the classroom.

The objects of teaching and the knowledge communicated must allow the student to engage in all non-didactical situations and social practices as a responsible subject and not as a student. This involves, on the one hand, the teacher progressively freeing from their didactical presupposition, the situation which she proposes to the student in relation to a notion, and on the other hand, her acknowledging this adidactical *milieu* as a territory of cultural reference and of functioning of the knowledge taught." (Brousseau 1990, pp. 322–323).

In Chapter 1 as well as in Chapter 4, the reader has had the opportunity to read a definition of the didactical contract and to examine its use in the design and the analysis of didactical situations and processes. But, as the reader may remember, Brousseau emphasized in the presentation of the foundation of the Theory of Didactical Situations that the theoretical concept in *didactique* is therefore not the contract (the good, the bad, the true, or the false contract), but the hypothetical *process of finding a contract.* It is this process which represents the observations and must model and explain them.

Thus, in Chapter 5, we have chosen to present a paper which allows the more precise location of the didactical contract within the theory. By analysing the different rôles of the teacher, Brousseau shows clearly the articulation of the didactical contract and the concepts of devolution and adidactical situation. Since the article contains an example already presented in Chapter 1 (the situation for the learning of enumeration) we have instead included an example of didactical engineering analysed in the 1990 article about the didactical contract and the *milieu.* We have also kept some other parts of this paper which we thought helpful for the purposes of this chapter, including a more formal definition of devolution.

The Editors.

CHAPTER 5

THE DIDACTICAL CONTRACT*: THE TEACHER, THE STUDENT AND THE MILIEU

1. CONTEXTUALIZATION AND DECONTEXTUALIZATION OF KNOWLEDGE

Mathematicians don't communicate their results in the form in which they discover them; they re-organize them, they give them as general a form as possible. Mathematicians perform a "didactical practice" which consists of putting knowledge into a communicable, decontextualized, depersonalized, detemporalized form.

The teacher first undertakes the opposite action; a recontextualization and a repersonalization of knowledge. She looks for situations which can give meaning to the knowledge to be taught. But when the student has responded to the proposed situation, if the personalization phase has gone well she does not know that she has "produced" a piece of knowledge that she will be able to use on other occasions. In order to transform her answers and knowings into a body of knowledge, she will, with the assistance of the teacher, have to redepersonalize and redecontextualize the knowledge which she has produced so that she can see that it has a universal character, and that it is a re-usable cultural knowledge.

One can easily see two aspects of the teacher's rôle which are rather contradictory: to bring knowledge alive, allowing students to produce it as a reasonable response to a familiar situation, and, in addition, to transform this "reasonable response" into an identified, unusual cognitive "outcome" recognized from outside.

There is a strong temptation for the teacher to short-circuit these two phases and to teach knowledge directly as if it were a cultural fact, thus saving the cost of this double manoeuvre. The knowledge is presented and students make it their own as best they can.

2. DEVOLUTION OF THE PROBLEM AND "DEDIDACTIFICATION"

2.1. *The problem of meaning of intentional knowledge*

If one accepts that learning is a modification of a student's knowing which she must produce herself and which the teacher must only instigate, one is led to the

---

* Editors' note: Sections 1, 2.1, 4 and 5 of this chapter come mainly from a talk given by Guy Brousseau at the *Université du Québec à Montréal* on Thursday, 21 January, 1988, published as: Brousseau G. (1 988a) Les différents rôles du maître. *Bulletin de l'Association Mathématique du Québec* 2/23, 14-24. Sections 2.2, 2.3 and 3.1-6 come from the article: Brousseau G. (1990) Le contrat didactique: le *milieu*. *Recherches en didactique des mathématiques* 9(3) pp. 309–336. These sections are indicated by a star by their title.

following reasoning. In order to make the student learn, the teacher looks for an appropriate situation. In order for this to be a learning situation, the initial answer the student considers in response to the posed question must not be the one that one wants to teach; if one had to possess the knowledge already in order to be able to answer the question, it would not be a learning situation. The "initial answer" must only allow the student to put into action a basic strategy with the aid of her existing knowledge; but this strategy must very quickly prove itself sufficiently ineffective to make the student feel obliged to make some accomodation, that is to say, some modification of her system of knowledge, in order to cope with the proposed situation. The deeper these modifications of knowledge, the more "the game must be worth the candle", so the more the system must allow a long interaction and be clearly general or symbolic.

The teacher's work therefore consists of proposing a learning situation to the student in such a way that she produces her knowing as a personal answer to a question and uses it or modifies it in order to satisfy the constraints of the *milieu* and not just the teacher's expectation. There is a great difference between adapting to a problem raised by the *milieu,* an unavoidable problem, and adapting to the teacher's expectations. The meaning of the piece of knowledge is completely different. A learning situation is one in which what one does appears to be necessary with respect to obligations which are neither arbitrary nor didactical. Now every didactical situation contains a component of intention and expectation on the part of the teacher.

The teacher must make sure that the student removes any didactical presuppositions from the situation. Otherwise, the student sees the situation as being only justified by the teacher's expectations; in fact, in any case, this view always exists.

We all have a tendency to read everything that comes to us in life as something that is organized for us or intended to teach us a lesson. In order to get a child to read a situation as being a necessity that is independent of the teacher's will, an intentional cognitive epistemological construction is necessary. The resolution of the problem is then the responsibility of the student, who has to obtain a certain result. This is not so easy. The student should have a project and she should accept her responsibility.

Let us observe that it is not sufficient to "communicate" a problem to a student for this problem to become *her* problem and for her to feel solely responsible for solving it. Nor is it sufficient for the student to accept this reponsibility in order for the problem she is solving to be, for her, a "universal" problem unattached to any subjective presuppositions.

We use the term "devolution"[2] to describe the activity by which the teacher seeks to obtain these two results.

2.2. *Teaching and learning**

Let us now reformulate the domain that the Theory of Didactical Situations must model.

Teaching consists of inducing students to assimilate the projected learning by placing them in appropriate situations to which they will respond “spontaneously” by adaptations. It is therefore a question of determining which adaptations correspond to the target knowledge, and which circumstances they respond to. One of the fundamental contributions of modern *didactique* consists of showing the importance of the rôle played in the teaching process by the learning phases in which the student works almost alone on a problem or in a situation for which she assumes the maximum responsibility. The contribution is based on an evolution of conceptions about learning theories; for the student, the meaning of the knowledge whose acquisition is formally observed is in fact built up from the set of rules for action into which these items of knowledge enter. But learning these rules seems more costly (or, indeed, impossible) to obtain by the communication of knowledge than by the autonomous and adaptive productions of the subject herself. This orientation has led to the contrasting, sometimes in a rather artificial and excessive way, of “teaching situations” in which the teacher brings in all the information and doesn’t delegate any responsibility and “learning situations” in which the opposite occurs. In the face of objections which have suddenly appeared to the former, particularly after certain attempts at the reforms described by formalists, the transformation of the greatest number of teaching situations into learning situations was envisaged.

It is within the framework of this interesting orientation that research is being carried out on the teacher’s action, which consists of transferring all or part of her responsibility to the student.

2.3. *The concept of devolution**

Statement 1

The main objective of teaching is the functioning of knowledge as a free production of the student within her relationship with an adidactical *milieu.*

Free production: The response to a milieu which is managed by the meaning—that is, by whatever the student is able to insert between her internal or external conditioning and her decisions. This implies an actual—not just a potential— possibility of choosing one of many paths for “intellectual” reasons; it also implies a personal production.

Adidactical milieu: The image, within the didactical relationship, of the *milieu* which is “external” to the teaching itself; that is to say, stripped of didactical intentions and presuppositions.

Statement 2

The student acquires knowledge by various forms of adaptation to the constraints of her environment. In the school setting, the teacher organizes and sets up a *milieu* (for example, a problem), which reveals more or less clearly her intention of teaching certain knowledge to the student. The teacher conceals enough of this knowledge and the expected answer so that the student can obtain them only by a personal adaptation to the problem. The value of acquired knowledge therefore depends on the quality of the *milieu* as an instigator of a "real", cultural functioning of knowledge, and thus on the extent to which adidactical suppression is obtained.

The child doesn't spontaneously view the world as a system stripped of intention with regard to herself, and is interested in unearthing and using to her advantage the didactical machinery because of what it can offer her. It is therefore essential that the teacher prepare the student for this adidactical functioning by integrating it into the didactical phases; the student can learn only by producing, by making her knowledge work and evolve—if not all the time, then quite frequently—in conditions "similar" or asymptotically similar to those which she will meet in the future.

Corollary 1
In order to allow this functioning, the teacher cannot tell the child in advance exactly what response is expected from her. The teacher must therefore make her take responsibility for finding a solution to problems or exercises whose answers she doesn't know.

Definition
Devolution is the act by which the teacher makes the student accept the responsibility for an (adidactical) learning situation or for a problem, and accepts the consequences of this transfer of this responsibility.

First paradox of devolution
The teacher wants the student to find the answer entirely by herself but at the same time she wants—she has the social responsibility of wanting—the student to find the correct answer.

Hypotheses: Devolution presents major difficulties which are traditionally analyzed in terms of the student's motivation; the recommended solutions are therefore psychological, psychoaffective or pedagogical in nature. Now the meaning of knowledge and that of the situation play an important rôle and *didactique* consequently offers specific means.

Method of study: To model in the form of a game the conditions of functioning, production and genesis of such knowledge, set out in advance or whose appearance is observed. The goal of this modelling can be to provide an engineering or even to explain or predict the behaviour of the protagonists in the didactical relationship.

3. ENGINEERING DEVOLUTION: SUBTRACTION*

3.1. *The search for the unknown term of a sum**

Teaching an arithmetic operation is often based essentially on the communication of a computational procedure[3] associated with a small universe of problems which are supposed to present the meaning of it. The problems of devolution arise in a more pressing and more obvious way if teaching is based on the study of a relationship[4]. We shall look at introducing subtraction to seven-to-eight year-olds in this way.

Usually, teachers present knowledge that they want to teach as answers to *questions,* perhaps to avoid being dogmatic. But they usually focus on the teaching of the *answers,* the questions being there only to introduce and justify them. Moreover, these answers are rarely relationships or assertions which could have any meaning if they were to be isolated; they are essentially procedures to whose progressive acquisition the introductory questions are tightly constrained to lead. Detached from their context, algorithms become answers acquired for future questions about which nothing very much is known.

The purpose of the teaching sequence described below is to make questions pass from the teacher's domain to the student's domain, to teach questions as well as answers, and as much as possible to teach knowledge together with meaning.

All learning will be organized around the same *basic situation* which will be repeated, evolving in the process: "the box game". On the teacher's desk is quite a large cubical box made of opaque plastic, containing between a dozen and a hundred objects of the "Diénès blocks" variety, many of which may be identical. The students have to say how many objects of a given type this box contains, but sometimes it is not possible to know this number without counting, while at other times it is possible to predict it by means of a calculation based on known information. Certainly, most of the time students do not know which of these situations they are in. The mark of a certain amount of knowledge of subtraction will be knowing precisely *when* and *how* "one" can determine these numbers, and spotting situations which can be modelled by *the box game.*

3.2. *First stage: devolution of the riddle**

The teacher presents the box for the first time. She asks, "How many objects do you think are in this box?", and then, "How many round objects?", "How many that are not round?". The game proceeds with the repetition of questions such as, "I am putting everything back into the box; how many are there now?", "I am taking out a handful; how many remain?", "How many in the handful?", "How many red objects?", "How many objects that are not red?". The students write down an *answer* to every *question* in their notebooks, then one of them comes up to count the actual number each time so that the *solution* to the riddle is known. Those who guessed correctly have *won,* those who guessed wrongly have *lost.*

As long as the student doesn't think of the possibility of predicting the solution, thus imagining how to make such a prediction, the teacher cannot get her to understand that she is giving her a problem in which there is something to understand and learn. The situation is therefore presented as a situation of action whose basic strategy is answering at random.

But the situation "reproduces" itself, the children learn what they have to do, how to tell whether they have won, and who decides. *Every* student must and can produce an answer. Everybody, or nearly everybody, thinks that counting is necessary, otherwise "one cannot know". One of the pleasures of guessing lies in the fact that you have no real idea whether you are going to succeed or not. Repetition of the game allows the students to understand its instructions and minimal technical vocabulary.

But in order to pass on to a true problem, will it be necessary to teach a method of solution? or, rather, several, seeing that there are many possibilities?

Teachers find it difficult to accept the guessing game: "We already have so much trouble getting the students not to give any old answer!" Some students are inhibited and refuse to answer; realizing that they have a very high chance of losing, they dread being judged badly because of this. Those who win think "it's a trick". This situation runs completely counter to the customary didactical contract in which the answer must be obtained by doing or from an intermediate piece of knowledge. But this first, basic contract is necessary here, just to allow rational prediction to emerge by itself and to be defined *against* random response.

3.3. *Second stage: anticipation of the solution**

Some questions are so simple that the students think of only one answer. For example, all the objects have just been counted and put back into the box: 52. Then all the ones which are not large, red, thick disks have been taken out for counting: 50. The teacher asks, "How many pieces are there in the box now?". Many students think that the solution will be "2". Their answer is an anticipation of the solution, but it imposes itself on them in the mode of a contingency (of evidence).

By suggesting in this way small numbers or numbers whose difference is small (or very large), the teacher encounters intermediate cases where the conviction is not so great, but where the answers do not all seem equally likely to the students. Thus, they enter into a new position (that of a cognitive subject), more reflexive relative to the previous situation of action since, from their point of view, their answer can be the object of appreciation, or calculation, or reasoning.

The formulation of questions varies but always takes the character of an ordinary conversation; "what's in the box", "those things", "what's missing", "what's left", "the others", and so on. On the other hand, the "game ", its organization and its terms are now well identified and institutionalized. A student can play against friends and make sure it is played correctly.

3.4. *Third stage: the statement and the proof**

Meanwhile, before accepting counting, the teacher asks more and more often, before letting them count, "You think you are going to win?", "Why?", "Are you sure?"

In some rather simple cases, some students explain a method. For example, there were 37 cubes, 31 remain; how many does the teacher have hidden in her hand? "I counted the hidden cubes on my fingers; 32, 33, 34, 35, 36, 37." More rarely: "There were 21 cubes, two are left; the 21st and the 20th; therefore, she has taken them all up to the 19th."

The answer can thus subtly change its status. The ordered pair: "term: '33'/ result: 'win"' can become the ordered pair "assertion: 'I say there are 33 cubes in the box' / test: 'That is true"'.

Of course, the teacher remains entirely neutral and accepts statements such as, "I have looked carefully from where I am sitting and I could count five of them". She does not make judgements about the meanings of successes; the fact of reasoning and finding a solution does not prove that the reasoning is good, even if it produces a correct response! Counting is then the institutional way of testing the answer and, for some, already a way—a mental, private one—of anticipating the testing and obtaining the answer.

3.5. *Fourth stage: devolution and institutionalization of an adidactical learning situation**

At this point, the teacher can state that it is a question

- for everyone of' learning to respond, being sure of the answer or of knowing that one cannot be sure;
- for the class of finding, without the teacher's teaching it, and saying what methods can be used;
- for everyone of learning by trying and trying again, and by taking advantage of other people's ideas if one believes them to be good ones.

"Can predictions be improved?" By favouring descriptions of strategies and mini-discussions about answers or strategies and avoiding institutionalizing them prematurely, the teacher tries to nourish the hope that one can learn how to win, and the pleasure of doing so with a bit of difficulty (just enough difficulty to optimize the pleasure), so as to obtain the greatest number of transfers toward the level of control by internal knowledge, which best calls forth rather an exalted intellectual activity.

The purpose of the didactical behaviour of such a situation is to assure the following fundamental equilibria:

- equilibrium (oscillations around a position) between uncertainty and certainty, disorder and order, difficulty and ease, etc.;
- equilibrium between levels of control: the brain cannot simultaneously handle too many conditions which are too uncertain. Mastery of uncertainty comes about through a sharing of the burden among the different levels of adaptation. Sufficient knowledge well understood, a little knowledge on the way to acquisition, and a public and private cognitive activity sufficient for justification but allowing interactions as well, etc.;
- temporal and rhythmical equilibrium: if knowledge and algorithms do not develop quickly enough to relieve the implicit models and pieces of knowledge, by conversion, information or teaching, then personal investigation dries up (complicates itself, hardens and fails), and the didactical contract loses its

objective. If, on the other hand, knowledge and algorithms develop too quickly, then understanding will not have had enough time to give the knowledge meaning;

– equilibrium between the pleasure of defining oneself by means of one's intellectual activity and the pleasure of obtaining a recognized security in a rapid, effective way without excessive intellectual activity, by using received knowledge;
– equilibrium between desire consumed (difficult or ungratifying tasks, or simply the completion of a project) and desire produced (success, tests successfully surmounted, etc.);
– social and cultural equilibrium in the class between the producers and the users of ideas, successes and failures.

The management of these equilibria requires numerous pedagogical and psychological qualities but it depends first on didactical choices. In the example which we give below, the choice of the sequence of numerical values linked with the probable strategies to be suggested to the students at the opportune moment is decisive.

"The discovery and use of knowledge" is a drama produced by the teacher, in which each student chances herself in a quite limited rôle, but it is also a *milieu* which should leave her freedom at the place where she should express herself. The juxtaposition of these episodes constitutes *her* story.

3.6. *Fifth stage: anticipation of the proof**

This stage is recounted in "*le cas de Gael*"[5]. There are fifty-two objects in the box. Eighteen are taken out and visible, and the student answers that there are still thirty objects inside the box. At the moment when the student is going to check the result by counting, the teacher stops him:

Teacher: "You are sure? You want to bet? [...] Good. Before firming up the bet, you may count and test your result."

The teacher gets the student to count: "Thirty in the box", showing one of the eighteen objects outside the box, "thirty-one", another, "thirty-two",...".

The student finishes counting; "Forty-eight".

Teacher: You think there are forty-eight objects altogether?

Student: Yes.

Teacher: And it's true?

Student: No, there are fifty-two of them!

Teacher: Do you want to bet, then?

Student: No.

Teacher: Whew! You haven't lost, so you can try again with another number.

In this way, the teacher teaches a method of improving the chances of guessing correctly. It is not very useful, but it will allow many discoveries and, by means of various improvements, it will lead to the standard method surrounded by several other methods.

This stage marks the beginning of the student's passage from using a contingent truth to using a necessary truth. The student is led to predict the value of his answer by simulating verification by counting. This apagogic[6] reasoning is not spontaneous but becomes familiar with use and allows investigation while providing security for those who might start falling behind.

Soon it will be necessary to reject investigation by exhaustion and even trial and error, following an explicit moral of adaptation to the *milieu* : "In order to answer the question, the student responds by acting on the system, by adapting herself in order to improve her efficiency", etc. The process is made up of twenty-two stages in the course of which the relationship between the students and the knowledge evolves. Addition replaces counting as the method of anticipating the result and then it becomes so communally accepted and so certain that it eventually replaces counting as *proof,* thus making recourse to the physical materials useless. While subtraction methods increase and become perfected, explicit exploration of problems capable of being modelled by the "box game " will allow students to classify these problems according (implicitly for them but explicitly for the teacher) to the conceptions which are being mobilized.

4. INSTITUTIONALIZATION

4.1. *Knowing*

Let us recall our initial project: the choice of teaching conditions which we have just touched on is justified essentially by the necessity of giving meaning to knowledge.

The meaning of a piece of knowledge is formed:

- by the "fabric" of *reasoning and proofs* within which it is embedded along with, obviously, the traces of the situations of proof that motivated this reasoning;
- by the "fabric" of *reformulation and formalization* with the aid of which the student can manipulate it, along with some idea of the constraints on communication which accompany them;
- by the *implicit models* which are associated with it (whether they be products of knowing or whether they produce it) and traces of the situations of action which functionalize them or which, simply, contextualize them;
- and by *the more or less assumed relationships* among these different components, relationships which are essentially dialectical. The "question/answer" linkage, for instance; the questions tend to be interconnected independently of the answers received; and the answers, in their turn, tend to be linked in the same way. To match "good" answers with "good" questions leads to reformulation, alternately and pertinently (we say "dialectically"), of both the answers and the questions.

The objective of the different types of situation whose devolution we have touched upon is that the student herself give meaning to the pieces of knowledge that she manipulates by combining these different components.

We once thought that we had envisaged all the possible classes of situations. But, in the course of our studies at the *Ecole Jules Michelet,* we saw that after a while the teachers needed some more space; they did not want to go on from one lesson to the next, wanting to stop so as to "review what they had done" before continuing; "some students are lost, we can't go on, something has to be done about it". It took us some time to realize that they really needed to do some things, for reasons that had to be understood.

"Adidactical" situations are learning situations in which the teacher has successfully hidden her will and intervention as a determinant of what the student has to do; they are those which function without the intervention of the teacher at the level of knowledge. We had constructed all sorts of adidactical situations. The teacher was present in order to make the machine work, but her interventions in terms of the knowledge itself were practically nullified. We had there teaching situations in the psychological sense, and it could be thought that we had reduced teaching to a sequence of learning episodes. But then we had to ask ourselves what justified this resistance of teachers to the complete reduction of learning to the process that we had conceived. It wasn't a question of judging them or their methods, but of *understanding* what they legitimately needed to do and why they needed a degree of opacity in order to do it, faced with researchers.

This is how we "discovered" (!) what all teachers do all along their courses but which our method of systematization had made unacknowledgeable. They have to record what the students did, describe what went on and what was related to the knowledge in question, give a status to what happened in the class as a result of both the teacher's and the students' initiatives, take responsibility for an object of teaching and identify it, bring these products of knowledge closer to others (either cultural or linked with the curriculum), and indicate that they could be used again.

The teacher had to record what the students *were supposed to* or must not do (and re-do), what they had learned or had to learn.

This activity cannot be bypassed; teaching cannot be reduced to the organization of learning episodes.

The "official" taking into account of the object of knowing by the student, and of the student's learning by the teacher, is a very important social phenomenon and an essential phase of the didactical process. This double recognition is the object of *institutionalization.*

The rôle of the teacher is also to institutionalize! Institutionalization applies just as much to a situation of action—the value of a procedure which is going to become a means of reference is recognized—as to a situation for formulation. There are formulations which will be conserved ("That's the way we say it", "Those are worthwhile keeping"). And for proofs, in the same way, it is necessary to identify what we retain of the properties of objects that we have encountered. It is clear that everything can bc reduced to institutionalization. Classical situations are situations of institutionalizion without the teacher's undertaking the creation of meaning; one says that one wants the child to learn, one explains it to her, and one verifies that she has learned it. From the start, researchers have been rather obsessed by adidactical situations because they were what was missing from more classical teaching.

4.2. *Meaning*

There is another thing that we took a long time to notice. Initially, we implicitly thought that learning situations were almost the only means by which knowledge is passed on to students. This idea arises from a rather suspect epistemological conception, as an empiricist idea of the construction of knowledge: the student, placed in a well chosen situation, should, on contact with a certain type of reality, construct for herself knowledge identical to the human knowledge of her time (!) This reality can be a material reality in a situation of action or a social reality in a situation of communication or of proof. We know very well that it is the teacher who has chosen the situations because she was aiming at a certain piece of knowledge—but could it coincide with the "common" meaning? The student had "constructed a meaning" but was it institutionalizable? One could proceed to an institutionalization of knowings, but not to that of meaning. Meaning put into a situation cannot be recovered by the student; if the teacher is changed, the new teacher does not know what has been done before. If one wants to review what has been done, one must really have the relevant concepts for this purpose, they must be universal and they must be able to be mobilized along with the others.

Meaning ought also to be institutionalizable to some degree. Let us see how. This is the most difficult part of the teacher's rôle; to give meaning to pieces of knowledge, and, above all, to recognize this meaning. There is no canonical definition of meaning. For example, there are social reasons why teachers have stuck to the teaching of the division algorithm. All reforms try to operate on understanding and meaning, but generally they are not successful and the objective of the reform appears contradictory to the teaching of algorithms. Teachers have cut down to what is negotiable, that is to say, formal and dogmatic learning of knowledge, because the moment when this has been done can be identified by society.

There is the idea that knowledge can be taught but that understanding is in the student's hands. One can teach the algorithm and "good teachers" then try to give it meaning. This difference between form and meaning results in difficulties not only in conceiving of a technique for teaching meaning, but also of a *didactical contract* for this purpose. Otherwise stated, one will not be able to ask teachers to use a situation of action, of formulation, of proof, if one cannot find a way of allowing them to negotiate the didactical contract connected with this activity; that is to say, if we cannot negotiate this teaching activity in usable terms.

In geometry, for example, let us imagine that we would like to encourage the student's mastery of spatial relations. It will be difficult to negotiate this objective, except in very small classes, because it doesn't exist as an object of knowledge. It is confused with the teaching of geometry which has nothing to do with it. It is not true that geometry teaches spatial relations.

There are a certain number of mathematical concepts which hold no interest for mathematicians—but which do for *didactique*—and which, because of this, have neither cultural nor social status. For example, the enumerating of a set isn't a mathematically important concept, but it is an important concept for teaching. Has *didactique* the right to introduce concepts which are necessary for itself into the field of mathematics? This is a subject which we shall have to discuss with the mathematical community and with others.

Negotiation, by teachers, of the teaching of understanding and meaning poses a true didactical problem, a technical and theoretical problem for the didactical contract. How should we define and negotiate the object of the activity with the public, the teacher, the student, and with other teachers?

The reader knows very well that there are several types of division, but we have only one word. Division by whole numbers and division by decimals refer, in fact, to different conceptions; this causes a lot of problems. Teachers don't have the possibility of possessing an object which could be called "the meaning of division", which they could say they were working on.

We are attempting to provide a didactical model of meaning negotiable between the teacher and the student, which would allow the student to work on the meaning of division with a vocabulary and concepts that are acceptable and that truly develop her knowing, and which would consist of situations in which she does divisions. This meaning consists of classifications, tools, a terminology. But there is danger in work of this sort; the development of a species of pseudo-knowledge or a ridiculous, useless "mis-knowing".

It must not be thought that *didactique* consists only of presenting as discoveries the things that young children do. Problems must be solved by means of theoretical knowledge and by technical means. It is necessary to propose something in order to have an effect on certain teaching phenomena; but it is necessary, first, to identify them and to explain them. The work of managing the meaning of the didactical contract, relative to meaning by the teacher or between teachers of different levels, is a delicate theoretical problem and one of the principal stakes of *didactique.* Today, teachers of different levels give conclusions which tend to produce a collapse of lower-level activities onto more formal activities because they cannot negotiate anything else.

The reprise by a teacher of all the non-institutionalized old knowing is very difficult. In order to produce new knowledge, she can to some extent use pieces of knowledge that she has tried to introduce herself; this is not very easy. But when it is someone else who has introduced these pieces of knowledge, and if they have seldom been used, the problems become almost insurmountable. The only way to manage is to ask teachers of lower classes to teach, in a rather formal manner, knowledge that the teacher of higher classes identifies, which can serve her at the explicit level in order to construct what she wants to teach.

We know nothing much about interactions between didactical activities; how are they managed over time? We must therefore evolve our conception of the construction of meaning.

4.3. *Epistemology*

Another rôle of the teacher consists of taking on an epistemology. For example, pedagogues extol research of situations which allow the child to be put into contact with real problems. But the more the situation of action realizes this contact with reality, the more complex are problems concerning the status of the knowings. And if the teacher does not have a good control of her epistemological conceptions with respect to this type of situation, the errors are worse in consequence.

In fact, at the same time that she teaches a piece of knowledge, the teacher suggests how to use it. By doing so, she manifests an epistemological position that the student adopts all the more eagerly in that the message remains implicit or even unconscious. This epistemological position is unfortunately difficult to identify, assume and control; on the other hand, it seems to play an important róle in the quality of acquired knowledge.

In order to demonstrate, simultaneously, the importance and the difficulty of the teacher's epistemological rôle, let us take the example of measurement. Whether it is a question of counting a finite set or of calculating the price of a field, most mathematical activities at elementary school pass through the reality or the fiction of a measurement. It is thus an important notion for the school years.

But actual measurement is a complex activity where the manipulation of instruments, the use of numerical structures, and the necessary elementary mathematical knowledge cannot be *really* justified, except for the elucidation of much more complex problems such as approximation, for example, and the calculation of errors.

The classical solution consists of not sparing the didactical relationship from difficulties which are foreign to the knowledge but which must finally be learned at some point. This necessitates teaching successively and above all separately the different necessary pieces of knowledge, beginning preferably with "the simplest". As a result, at the moment of learning, nothing can be justified by the overall problem to be solved. Provisional, partial, or even incompatible, justifications become juxtaposed, they contaminate each other, without becoming truly modified or adapted. Even though mathematicians can keep these explicit pieces of knowledge under the control of their epistemological vigilance, their meaning, in particular the possibilities of their use (by the student), will be profoundly affected by this, as will the róle of knowledge in the activity of the student.

In this hypothesis, the decision of fragmentation of the knowledge made without control leads to its being stripped of its possibilities of functioning.

The notion of measurement is introduced with the sole example of measurement by finite cardinal numbers, illustrated by various discrete measures.

If a student estimates that $3+4 = 6$, the teacher doesn't tell her that she is not far wrong, but that her result can be shown to be wrong. For each measure, there is one true value for an exact, unique measurement. The calculated result coincides perfectly with the "observed" result.

The construction of numerical structures in ($\mathbf{Q}^+$, $\mathbf{D}^+$, $\mathbf{R}^+$) is carried out in such a way that this model is not opened to question. From that moment, actual measuring must become rare. In order not to contradict herself, the teacher must avoid certain conf'rontations between the calculation and reality and she must strongly adjust others.

For example: Does calculation provide a ridiculous precision *vis-à-vis* the possibilities of actual measurement? Then the teacher imposes a standard convention on precision (implicit return to natural numbers) or, perhaps, chooses data that will allow the calculation to come out right.

When confronted by an estimate—calculated and of a real measurement—the calculated value is considered correct, the measurement is more or less "good" according to the amplitude noted of "the error" (!). The latter shows the competcncy of the measurer. The error is therefore a sort of fault or mistake, an insufficiency of the apparatus... even, a breaking of the contract by the teacher, who has imprudently abandoned the comfort of problems where reality is only invoked, and even negotiable.

Real measurements must never be the object of operations since differential calculus applied to the calculation of errors isn't known to the students. In consequence, the data of a problem are very rarely the object of a measurement; also, there is never a real anticipation of an observation, and therefore never a challenge of the theory nor of its deterministic presuppositions.

A student will not be able to start envisaging *real* measurements with a suitable comprehension of the theory which underpins her action and a satisfactory mastery of the necessary techniques, except *after* having seen analysis and integration, differentials and calculation of error, the methodology and calculation of probabilities, and so on, treated seriously.

Before that particular moment:

– either measurements cannot be real (only evoked by a statement, for example),
– or they must be realized in very particular cases (finite sets, discrete measures, etc.),
– or they are within the student's understanding in a suitable reference situation.

In every case, the teacher is obliged to hide questions about relationships between the measuring numbers and the physical sizes which they represent or to treat them in a metaphorical fashion, in particular questions about knowing which operations on the one would allow the prediction of what on the other, and, finally, all questions about relationships between theory and practice.

This results in an epistemological position which is false, but above all purely ideological and accepted as inevitable.

Teachers take this "divorce" between the mathematical concepts taught and the real-world activities of the students badly. They have tried to reduce it and have fought the disappearence of students' activities and contacts with reality. For different reasons, these pedagogical movements have justified themselves by ideological assumptions such as:

"hands-on activities make for better understanding and better learning" (the hand shapes the brain);

"reality avoids errors of understanding" (empiricism/ realism);
"usefulness, the concrete, motivate the student".

I say that the effect of these movements has been the opposite of what was expected; never has the theory-practice conflict been more exacerbated. The chasm between teachers and knowledge has been deepened. Many elementary-school teachers are convinced that theory, "official knowledge", is a discourse, a convention, of relative or doubtful efficiency, to which one can make personal adjustments, or for which one can substitute other "parallel" knowledge. The objections against rationality, science, and even knowledge as means of handling reality have developed at the same time and in the same circles as these pedagogical movements.

In order to support the cause-and-effect relationship between the two phenomena, a little didactical analysis is necessary.

First, "reality" is very much more difficult to "understand" than a theory. It cannot give rise to precise pieces of knowledge, nor can it correct errors, except by way of a specific and very strict organization of the student's activity. Knowledge about didactical situations and epistemology is essential. Without didactical techniques, it naturally "consumes" more motivation that it produces. Immediate utility is only one factor of motivation among several others. Long term utility (such as "Mathematics is useful for Physics") is a very feeble motivator.

Without epistemological and didactical mediation, the fundamental statements are false.

Teachers who increase the number of experiments and real measurement will not be better armed for dealing with their consequences. On the contrary, they will expect more understanding on the students' part but, in fact, in obscurer situations ("Look... don't you see?"). The students do more measurements but if one single value is required, in the end it will be necessary to choose it, as a social (and therefore doubtful) convention or as a truth guaranteed by the teacher.

At every moment, the teacher must underhandedly violate the theory-practice relationships that her pedagogical convictions urge her to profess. She must force theory to spring up, fully armed with reality, and must in fact falsify it or negotiate its use, and manipulate the student's motivations so as to obtain spurious situations and, since this sudden apparition is bound to be fatal, she will tend to assume that the reality is transparent and that the theory is obvious...

The student doesn't come out of this any better; her best manipulations never give her either security or knowledge, which come from elsewhere. Only wheel-spinning, error and deception remain for her, and the conviction that theory does not work very well unless the teacher is working it; or then again ... wasn't this only a convention?

The teacher ends up thinking the same way as the students.

A more thorough study would be necessary to show how a cultural movement of the importance of those to which we allude, among other sources, can feed upon and grow in local didactical relationships.

Let us examine whether an alternative exists to the classical solution and whether the teacher can assume a better epistemological position on the problem of measurement. It is not a question of giving a solution but only a counter-example.

In a fourth grade class (CM1), the teacher is conducting one of the last lessons about measurement. She sets out a large empty container, a glass, a balance, marked weights and a bucket of water. She says, “Watch! I am emptying a glass of water into this container. One of you is going to come up and weigh the whole thing. What weight will we find?”

For the students, it is riddle, a question of estimation. They write their predictions in their exercise books. One student makes a double-weighing, and says “It weighs 225 g”. Everyone compares this with her own estimate. The teacher says, “Who got it right?”. She collects a number of results and writes them on the board. She then says, “Who made the best estimate? Who made the worst?’. With no difficulty, the students use the absolute difference. The teacher says, “Watch, I am now pouring a second glass of water into the container. What weight are we going to find this time?”.

Some students multiply 225 g by 2, but others detect a trap and try to correct their estimates. No discussion; no collection of estimates. This time, weighing gives 282 g. Comparison of the students’ expectations; some see the light; “Ah... I’ve figured something out.” But the teacher doesn’t encourage any comments, saying; “Let us continue. I’ll do a third glass of water”. Already a dozen students are subtracting the first result from the second and adding the difference to the answer; 282–225 = 57, 57+282 = 339. Others play with their numbers; two or three imperturbably multiply the first result by 3. A new student carries out the double weighing; 35 1 grammes. Astonishment, deception and feelings of injustice among those who did the above calculation. The teacher remains neutral. One student has put forward the exact value; the others press her to say how she did it; “I saw that the pointer was rather near there, then I thought about it”; she puffs herself up, she is the best, and furthermore she feels that she is lucky. She is a winner!

The teacher resists her desire to impose the “explanation” on her. The guessing game continues; the students gradually arrive at the understanding that calculation does not necessarily give the value found when the balance is used. The students who used this method of estimation come up and explain it and rebel over its lack of success. The method takes all the essential elements of the problem into account in a way that appears rational; it can be commumicated well.

The students who didn’t invent it use it for comparison; they understand it. The teacher says, “What is the weight of the glass of water; No, no, we are not going to weigh my glass; calculate it”. Depending on the experiments chosen for calculating the differences, the weights are different! The discussion clears the fog. The glass does not hold exactly the same quantity of water each time. One can’t be sure. The teacher has to proceed carefully.

First conclusion: The teacher has to proceed carefully, show that the glass really is full, wait until the water is still.

If differences persist, the students can then be led to think that several weighings of the same object do not give the same value. They will thus make more or less progress in the analysis of errors of measurement.

It is possible to stop this chain of reasoning. It is, for example, sufficient to replace the water with very dry sand and the Roberval balance with a spring scale. The accuracy of reading is of the order of one gramme and the weight of glasses of sand, from one weighing to another, varies much less that one gramme.

The model of one whole, deterministic measurement suits perfectly. In order to obtain the idea that calculation is the best method of estimating results of different weighings despite random measurement errors, it is necessary to arrange a process of activities, of communication of results, of exchanges of guarantees, of reflections and of discussion.

Students are ready to accept the use of intervals in order to lessen the uncertainty of the result, but situations must be organized where the balance between sure estimation and precise estimation takes on its economical meaning.

4.4. *The student's place*

As in the preceding paragraphs, it is a question of showing that problems of teaching are also, and sometimes mainly, didactical problems. The student's place in didactical relationships has been claimed—just like the place of "reality"—in the name of different approaches: psychoanalytical, psychological, pedagogical. In this sense, genetic epistemology has furnished the arguments which are the most serious and the closest to knowledge, but more work is necessary before these contributions can be used. Frequently, the teacher interprets the student's errors as incapacity to reason in general, or at least as errors of logic. In a large didactical contract, the teacher takes charge of representations, the meanings of pieces of knowledge; but, under tougher conditions, she simply has to point out where the student's answer contradicts previous knowledge, neatly avoiding any diagnosis of the causes of the error. Errors, reduced to their most formal aspect, tend to become either an "error of logic" ("You are using false reasoning"; "You are reversing an implication") or ignorance of a theorem or a definition.

In this drastic reduction, the student is identified with an algorithm, the means of production of a proof according to the rules of mathematical logic. The contract allows the teacher the surest defence; she takes charge only of pieces of knowledge agreed upon within her own domain. It is sufficient for her to present them in an axiomatic order and to require axioms as evidence.

But it is clear that children use certain representations or pieces of knowledge different from those which the teacher wants to teach them. Children's logic, and "natural" thought, are already very well known. They cause them to make errors that can be observed and recorded regularly. Some of these pieces of knowledge can generate obstacles (didactical'? ontogenetical? epistemological?) and give rise to cognitive conflicts.

What place, what status, what function should be given to these representations? Should we (can we and, if so, how):

– implicitly reject them on each occasion?
– ignore them?

– accept them without recognizing them?
– manage their evolution without the students knowing it?
– analyze them with the students?
– recognize them, explain them, and explicitly provide a place for them in the teaching project?

We know that the *cognitive subject* uses amalgamated predicates, prelogical connectors, metaphors, metonyms. We know that the development of the student's logical thought consists of an irregular evolution where contradictions between the contextual components go hand-in-hand with the extension of prefunctors and the settling of the predicates, and where syntax and semantics are involved at the same time. Syntax and semantics separate themselves only slowly, at different periods depending on the sectors...

Naïve *didactique* allows only (mathematical) logical exercises on settled components to be given to the *student.* Is knowing the *cognitive subject* sufficient to solve problems related to the *student* ? I do not believe it. The creation and management of teaching situations are not reducible to an art which the teacher can develop spontaneously by means of good attitudes (listen to the child), around simple techniques (the use of games, material, or cognitive conflict, for example). *Didactique* is not reduced to a technology, and its theory is not that of learning but that of the organization of other people's learning, or more generally that of the dissemination and transposition of knowledge.

The debate proposed above has no theoretical framework, no experimental foundation, no solution outside *didactique.*

The student's reasoning is a blind spot for naïve *didactique* because treating it requires a modification of the didactical contract. It is not sufficient to know the cognitive subject, one must have the didactical (and sociocultural) means to understand *it.*

The situation is the same every time the student has responsibility for the implementation of a theory, for example forming an equation that corresponds to a problem or using a physics theory. The initial analysis of the situation and the appeal to theoretical notions happen at first thanks to spontaneous models and a reconnaissance using natural thinking. In case of failure at this stage, the teacher, bound by a contract which mandates the teaching of science but not the way in which science is revealed, can only explain her theory again. This impossibility of treating the material which permits the implementation of the theory leads her to justify herself by means of an erroneous diagnosis ("You don't know your theory"), and finally condemns her to run from failure to failure.

To accept taking charge of the individual means of learning of the student (the cognitive subject) would require:

– a complete modification of the teacher's rôle and training,
– a transformation of knowledge itself,
– other means of individual and social control of teaching,
– a modification of the teacher's epistemology,
– etc.

It is a decision which poses problems that only *didactique* can, perhaps, solve. It is certainly not a decision which comes under teachers' free choices or under teachers' art. Let us stress this contradiction: if the subject does not currently have a place in the teaching relationship (it has one in the pedagogical relationship), it is not because teachers are stubbornly attached to dogmaticism but because they cannot correct the deep didactical causes of this exclusion. We risk paying dearly for errors that entail requiring from voluntarism and ideology what actually depends on knowledge. It is up to research in *didactique* to find explanations and solutions that respect the rules of the game of the teaching profession or to negotiate changes in them on the basis of a scientific knowledge of phenomena! Today, one cannot "teach" students "natural thought", but neither can one continue to let the institution convince failing students that they are idiots, or sick, because we do not want to confront our limits.

Lest my proposals seem too pessimistic, research is advancing as fast as problems are being better formulated. In geometry, the treatment of spatial representation is being studied as a didactical project which is distinct from geometry teaching.

Certain works in recent years show the possibility of handling the child's logical thought within the didactical relationship.

It is a question of situations and contracts allowing the taking charge explicitly of the evolution and rôle of these modes of thought, not only in elaboration of the methods of proof, but also in the shaping of judgement and the regulation of social conduct (interplay of coalitions, aggregation of data, etc.).

We see in these two examples why, on occasion, taking the psychocognitive subject into account goes by way of a definition of the *student* which in fact requires a transformation of the organization of knowledge itself in a didactical transposition *and* a change of contract.

We are studying this phenomenon in connection with, for example, enumeration. This cognitive activity is essential for the student in the learning of number and is useful to her throughout her schooling; but as it does not exist as an object of mathematical knowledge, it has never been possible to teach it correctly and the "practice" could not take students' difficulties with this notion into account.

4.5. *Memory, time*

What the student has in her memory appears to be the final goal of teaching. The characteristics of the subject's memory, its mode of functioning in particular, and its development have therefore possibly appeared as *the* theoretical basis of *didactique.* So one could then reduce teaching to the organization of learning and of the individual student's acquisitions.

A certain number of works show the insufficiencies (the inconveniences) of this conception that notably ignores the relationships between the organization of knowledge (and modifications in the didactical relationship), the organization of the *milieu* and its temporal and institutional requirements in order to produce such and such memorization, and the reorganization and transformation of the knowings that the subject uses. Some phenomena of obsolescence of situations and knowledge, the paradoxical use of the context that is canvassed or rejected according to need, the rapid variations of the status of school knowledge and the related didactical transpositions, and the didactical production of different sorts of memories, attest to the fact that the student's memory is a didactical subject which is very different from the memory of the cognitive subject. Teachers manipulate taught knowledge and students' memories in a complex way. They must organize the forgetting of what was useful at a certain moment and no longer is, as well as the reactivation of what is needed.

This management takes place within a framework of a negotiation which engages the *memory* of the didactical system and not just that of the student.

A teacher who cannot remember what such and such a student has done, or what has been given as common knowledge, or what has been agreed upon, or a teacher who leaves the integration of teaching moments up to the student, is a teacher without memory. She is incapable of exercising the personalized, specific didactical pressures which appear indispensible in the didactical contract. Besides, the "didactical memory" of the teacher and the system administers changes of attitude in the presence (or not) of the resources of the *milieu,* and transformations of language. It is a current observation that students can evoke certain items of knowledge only in the presence of a person who has been part of the history of their relationships with this knowledge, or only in the presence of particular devices which they have used. Transforming memories into knowledge available for further mobilization is a didactical and cognitive operation and not only an individual act of memorization. The organization of didactical memory is a part of the more general management of didactical time.

5. CONCLUSIONS

We have seen that the teacher is a sort of actor. She operates in relation to a text which has been previously written, and to tradition. One can imagine her as an actor in the *Comedia del Arte;* she improvises her play on the spot, within a framework.

This conception is supported by the idea—altogether true—that the teacher needs freedom and creativity in her action. A teacher who recites would not be able to communicate the essential, and if one wanted her to present a situation which she could not cobble up, the teaching would fail. Can she have another, more professional conception of the teacher? Can she use already-made situations in order to recreate learning conditions identical to a known model?

This implies that one could distinguish what the teacher cannot modify from that on which she can make use of her personal talent. In pursuing our compassion, expressed above, the teacher would become an actor whose "text" would be the didactical situation to be managed (obviously not a text in the strict sense).

DIDACTICAL PHENOMENA RELATED TO THE NEGOTIATION OF THE DIDACTICAL CONTRACT

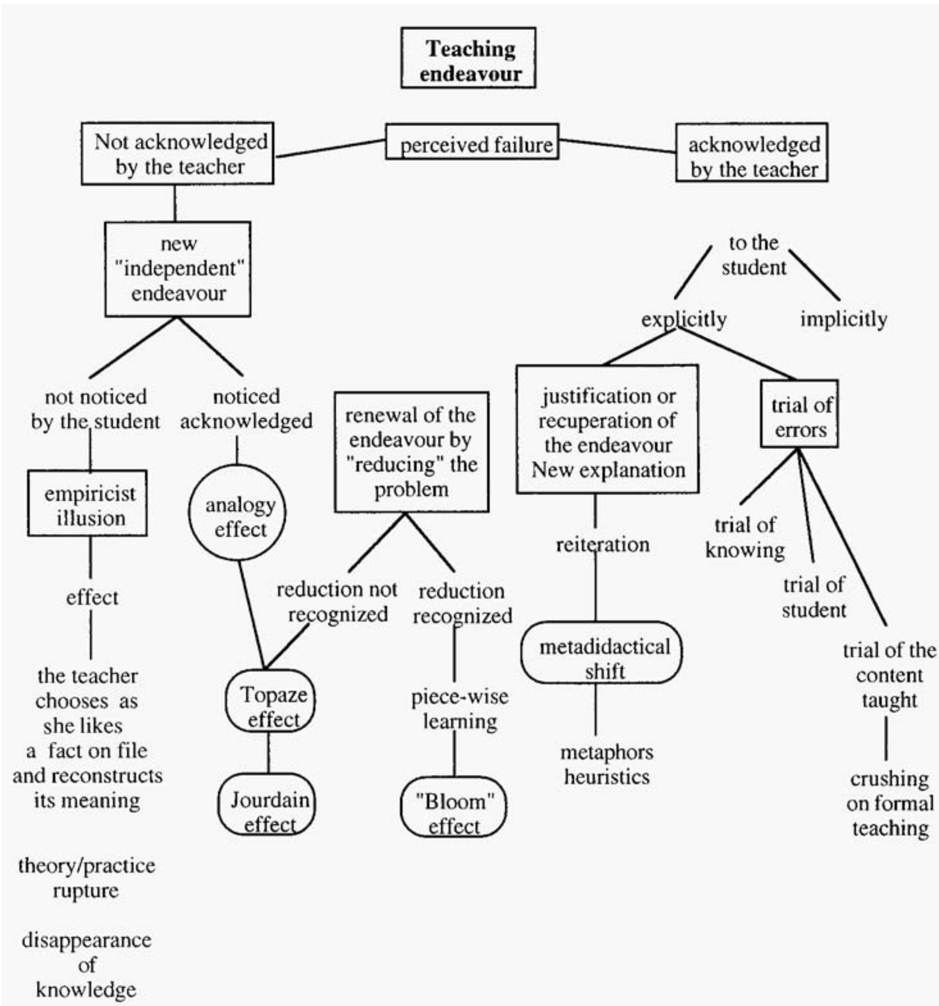
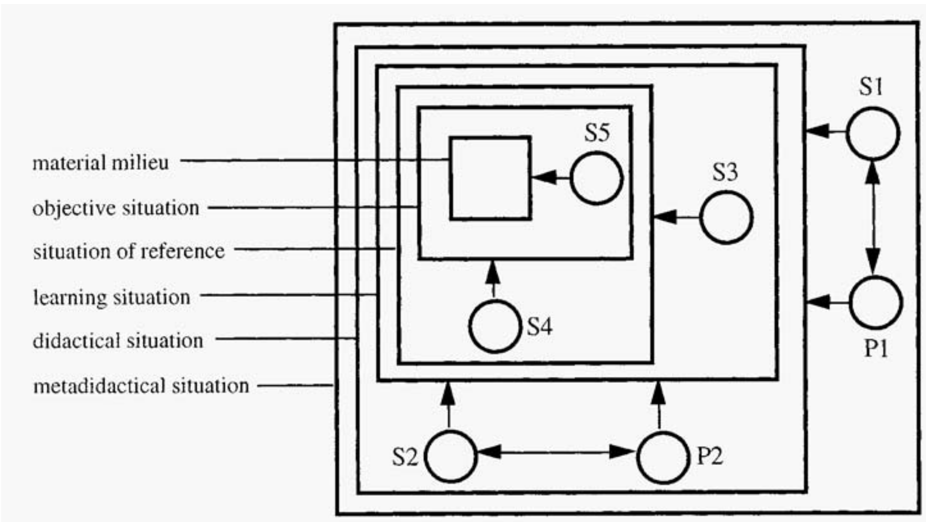


Figure 1

THE DIFFERENT ROLES OF THE TEACHER AND THE STUDENT
OR
THE STRUCTURATION OF THE DIDACTICAL *MILIEU*



S1: the universal subject; S2: the generic student; S3: the subject as a learner; S4: the subject as an actor; *S5:* objective sectors; P1: the teacher preparing her lesson; P2: the teacher teaching acts upon or observes.

Editors' note: *This diagram was proposed tentatively by Brousseau in 1986 under the title "structuration of the didactical milieu ", then used in his lecture in Québec in 1988 in order to suggest the "different rôle of the teacher and of the student". Finally he presented it again in the paper he published in 1990 about the didactical contract and the milieu. It has since been widely used, in particular by Margolinas*[7] *So we have decided to present it here, reconstructing a commentary from the introduction Brousseau gave in the 1990 publication and the more detailed legend of the 1988 publication.*

Figure 2

In order to model the didactical relationships, to predict their different types of functioning, one must undertake:

- to combine interactive systems in order to make the differentiated rôle of the teacher and the student appear by their reciprocal relations and their relations with knowledge.
- to study the compatibility of their respective characteristics.

In this way, one obtains the above diagram fairly easily. One must notice that at each level of relationships, the knowledge and the knowings of the teacher as well as the student are different, even if they refer to the same mathematical notion.

In this diagram, S and P represent the following:

P 1 : The teacher reflects on the sequence that she must construct; she looks upon the teaching situation as an object; she prepares her lesson.

S1 : A student who looks at a teaching situation from outside.

P2: The teacher who is teaching; she is in the didactical situation, she is active and she has in front of her something that is the solution of learning, and she has beside her, independently of the teaching situation, a student to whom she can speak, on whom she can act and who can interact with her.

S2: The student who looks at her own learning situation, with whom P2 is holding a discussion about her learning.

S3: The learning student in a learning situation; she is confronted with a situation which is no longer a didactical situation. She is looking at student S4 who could be herself, in a situation of interacting with the world, someone who takes decisions; this is the situation of reference. S3 is the epistemological subject; S4 is the active subject. S4 looks at the objective situation which makes the subject work. S5 is often hypothetical, those who are in the problem; for example: "Three people who divide up...". The student can identify with this subject but does not intrude at this level.

Considering their functioning, levels S5, S4 and S3 are considered to be non didactical, level S2 is the level of the didactical intervention.

NOTES

1. (Brousseau 1981b).
2. Devolution was an act by which the king, by divine right, gave up power in order to confer it on a Chamber. "Devolution" signifies "It is no longer I who wills, it is you who must will, but I am giving you this right because you cannot take it yourself".
3. R. Skemp sets in opposition these two aspects of mathematical thought: procedural and declarative. In fact, the theory of situations permits an escape from this sort of summary dichotomy by showing the rôles the two aspects play in the "mathematical" relations of the subject with her *milieu* .
4. A long term research project conducted in the COREM of Bordeaux made possible at a first step the production and study of numerous effective didactical situations based on the search for an object satisfying a condition expressed by the student. The best known example is that of fractions introduced by commensuration ("3/4 is the size that, multiplied by 4, is equal to 3 units") rather than the habitual conception of a partition of the unit ("It's the size you get by dividing the unit in four and taking three of those parts.")
5. *Editors' note:* Brousseau (1981b).
6. *Editors' note:* the reader may appreciate being reminded of the meaning of this very rare adjective. Apagogic comes from the Greek and refers to argument by *reductio ad absurdum.*
7. *Editors' note:* Margolinas (1994).

CHAPTER 6 PRELUDE

How does one end a book which presents a theory? The real proof of the validity of the theory is its power to address the problems which provoked its creation. In the case of the theory forged by Brousseau, that proof has been made by the richness and abundance of the research it has produced in the Didactique research community. The Proceedings of the recent conference which celebrated the *20 years of* Didactique *of Mathematics in France* (Artigue *et al.* 1994) bear witness to that richness.

In addition, the presentation of a living theory has no ending. As the editors and Brousseau himself are highly conscious, many interesting developments have occurred since 1990, the year with which this volume concludes, and even more are under way. For example, a lively debate is currently in progress on the theme of the role of the teacher, a theme which was the aim of an important course at the Summer School of Didactique in 1995 under the title: “The teacher in the theory of didactical situations” (Brousseau 1995). Thus this volume has no choice but to be open-ended on the subject of the scientific debate that the theory produces.

On the other hand, not all the debate produced by the theory is theoretical. The reader who is a teacher, or who teaches teachers, is bound to ask about the function of this theoretical construction and about its practical applications. This question was asked to Guy Brousseau by the editors of *Petit x,* published by the IREM of Grenoble, in 1989. We conclude this volume with his response to this question.

The Editors.

CHAPTER 6

DIDACTIQUE: WHAT USE IS IT TO A TEACHER?*

When I was asked to write an article on "what *didactique* of mathematics can bring to *a* teacher" (it is I who have emphasized the indefinite article), I very much wanted to run away, because this task seems to me to be very difficult to achieve.

My reticence results from a number of unfavourable and unfortunate circumstances: *didactique* is difficult to explain, especially to teachers! It is often all the more difficult to explain to teachers in that they expect further results. From this point of view, teaching conditions are so unpleasant that they justify the most urgent expectations. *Didactique* is even more difficult to explain because they think that it must provide them with material assistance, in the form of innovations (we shall see why further on); and even more—the subject that was suggested to me makes the situation even worse: I was asked to present not what *didactique* can do, but how it can change the life of *a* teacher, no matter who he or she is; finally, there is a controversy involving innovators and the defenders of action research in *didactique* about what *didactique* is, what it can do, and what it should do. The misunderstanding is such that it has regularly discouraged IREMs from putting forward a coherent plan for their survival even when the government started wanting to reorganize this type of pedagogical research. *Didactique* has been presented as an alternative, and then as an obstacle, to the aspirations of some teachers who claim a broader conception of "research", and it has, as a result, received none of the resources needed for mere survival.

For a long time, I have refrained from responding in any way other than by my work and through the example of my relationship with teachers in elementary and middle schools. Encouraged by the editors of *Petit x*[1] and the success of their journal, I have now accepted the challenge to change this position in the hope of making a useful contribution. The dedication of teachers to the cause of education makes them capable of hearing and understanding everything. This is the only justification for my optimism.

1. OBJECTS OF *DIDACTIQUE*

Let us take a common teaching problem—an unresolved problem—and see what *didactique* can offer.

---

* Brousseau, G. (1989b) Utilité et intérêt de la didactique pour un professeur de collège. *Petit x* 21, 48–68.

In elementary school, pupils are taught according to the rules of calculation, passing from one term (without a variable) to another—which is equal to it—until they obtain a result in its standard form. "3+4 = 7" is read as "by correctly carrying out the calculation 3+4 you get 7". 3+4 is perhaps equal to 7; it cannot replace it as the answer because it is not equivalent to it.

When these pupils learn algebra, they must be taught to pass from one formula to another which is logically equivalent, until a relationship is found that can he used for whatever purpose they wish. "3+4 = 7 and 7 = 4+x" implies that ‘‘x = 3".

Superficially, we might believe that the teacher and the pupil continue to use the same old knowledge and add on new pieces of knowledge. You just write down what you had previously only thought. All you need is to see "3+4" as a number, "x" as an unknown number, "=" as identity. In fact, it is well known that everything has changed with respect to these familiar symbols: the way they are used, the meaning given to them, the aim of the transformations performed on them. The reader who might be disappointed by this naive, terse introduction should refer to the works of Chevallard on these questions[2].

The pupil must therefore not only learn new knowledge, but also re-learn and organize old knowledge and forget—at least un-learn—part of it.

How far is this observation compatible with the fundamental principles of evaluation as it is practiced today'? Are these principles still valid? Can we improve them directly without involving didactical methods and teachers' conceptions? We do not know enough about distinguishing the different relationships that the teacher and her pupils can have with the same piece of knowledge, nor how to describe the different meanings that this knowledge can take according to the circumstances in which it is used and the person using it.

We do not know enough about how to consider learning in terms of changes in relationship to knowledge or in terms of transformation of students' knowledge. Nor do we know enough about how to describe the rôle of old knowledge in the construction of new knowledge.

What is the place of teaching in this reorganization of old knowledge in terms of the juxtaposition of the outcomes of new learning? Does its place depend on the ideas being learned? Are there pieces of knowledge which become obstacles to later learning? Are there didactical techniques which are more favourable than others with respect to the last question?

At present, the integration of new pieces of knowledge into existing knowledge is left entirely to the pupil, the teacher being happy to communicate, step by step, pieces of "true" knowledge from the present mathematical culture. Can the teacher cut up the child's past, leading her to understand that, apart from algorithms, everything learned earlier is no longer useful? Can students understand what they are taught under these conditions?

Curricula developed today provide nothing but the juxtaposition of blocks of learning activity. What observable consequences can we infer from this "insufficiency", with respect to teaching, with respect to progression from one school level to another, with respect to the didactical and epistemological conceptions of teachers? Are these consequences inevitable? Can they be attributed only to this cause?

Indeed, it would be absurd to conclude that arithmetic should not be taught in primary school, and it does not appear to be easy to teach algebra at this level. Direct teaching of definitive knowledge is impossible and therefore we must stop trying to do it. The use and the destruction of existing knowledge are thus part of the act of learning.

Consequently, we have to accept a certain “didactical reorganization” of knowledge which changes its meaning. We have to accept also, at least in passing, a certain amount of error and misconstruction, not only on the pupils’ part but on the part of the teaching as well.

But how can we transform knowledge so as to make it temporarily intelligible without making it too false, at least with regard to traces which cannot later be erased? And how can we correct these errors at a later stage?

And what right has a teacher to inflict didactical transpositions on common cultural knowledge? How can we control the inevitable distortions? Can this task be entirely the responsibility of a, or even some, teachers’? Can we make them teach false knowledge, even temporarily, without a cultural consensus on this matter? Can this agreement be reached if each of the principal characters is put in a position in which she must ignore all serious analysis? Who will undertake this transaction, which social organization can permit it under conditions that are honest for everyone?

These are some of the “simple” questions, almost naive ones, which are raised about *didactique* of mathematics by one minor phenomenon arising in the field.

As far as this phenomenon is concerned, a first characteristic immediately comes to mind: complexity—complexity which, in a wide range of domains, calls for experimental research and fundamental reflection as well as for ingenuity or engineering research.

This sort of complexity is apparent when necessary research questions are being developed, but it becomes overwhelming when one wishes to take account of some of the results obtained from the research in order to make a decision about practice. This complexity is sufficient to justify suspicion: it would be possible to invest in research on the details of teaching for ever, never obtaining anything except sparse, gratuitous hints.

The second important characteristic is that the questions which arise call for research in very different domains, but they do so in a manner that seems to make it impossible for anyone to give answers independently of others. This remark emphasizes the need for a unitary, systemic approach to research on *didactique.*

2.USEFULNESS OF DIDACTIQUE

It is time to list the different forms in which didactique of mathematics can be expected to manifest itself to a teacher, the results which didactique of mathematics promises, and those which it has already obtained. In addition, I must explain why it is not currently manifesting itself more obviously.

2.1. *Techniques for the teacher*

At the *very least,* the teacher expects *didactique* to provide her with the essentials of *techniques* which are *specifc* to the *content* to be *taught,* compatible with her general conceptions on education and pedagogy:

Local techniques:

– general: preparation of lessons, of teaching material, of ready-made key methods, of methods of management, objectives and evaluations;
– or optional: for students having particular difficulties.

Global techniques: curricula for a whole section of mathematics, long-term programmes covering several years.

This expectation is to be expected, didacticians have started studying several teaching situations, original or not, particularly at the elementary or secondary school level. But these studies are difficult and take a long time to carry out.

In the above example about the treatment of notation when algebra is first introduced, almost everything still needs to be done, even if some possible avenues are beginning to be explored. Can algebra be introduced nowadays as a theoretical rerun of arithmetic and numbers? Or as a system for denoting quantities? Or as an instrument for studying functions? Or as an autonomous formal system? As shorthand for algorithms related to unknown or indeterminate values? Or as a means of generalization or modelling? One can make up a number of settings, but before being able to propose any of them, one must examine their characteristics in terms of a considerable number of requirements. As Dieudonné remarked: “While not much more than a century passed before elementary geometry attained a reasonably definitive form, it was only thirteen centuries after Diophantus that algebra became what we know”. The journey cannot be all that obvious!

Moreover, the amount of knowledge that must be communicated to students and hence the number of specific situations proposed are both very large. In order to communicate in a reasonable way with teachers, *didactique* must therefore also produce unifying concepts and regroup items of knowledge, problems, situations, and students’ behaviours or activities in order to permit generic forms of intervention, depending on the patterns obtained.

The existence of a technique depends at the least on the identification and recognition of practices and their standard results. A piece of engineering relies on a scientific field for support of the techniques that it proposes. Communication, use and reproduction of teaching situations require an appeal to knowledge and specific knowings.

Thus, *didactique* is the only means of pointing out what is an unsolved problem of didactical engineering, of identifying and classifying original work in this domain, of specifying the conditions of its use and reproduction, and therefore of recognizing creations, inventions and processes of research and of scientific production by teachers, and of furthering their recognition.

In defining and bringing respect to the technical part of the teaching profession, *didactique* enables the social negotiation of teachers' work to become a possibility. In this way, it becomes the basis for the professionalization of teachers' activities.

But the development of a repertoire of teaching situations doesn't endow them with any virtue from the point of view of teaching. We need to know what advantages they have for children.

Previous work shows that "good" situations, those that allow the realization of more exacting pedagogical conditions, that is to say lessons which are more reliable for the teacher as well as being more open for the student, are truly communicable only if they have been also properly studied. *Didactique* has introduced situations more varied and better adapted to specific aims, like, for example, making possible the acquisition of a piece of knowledge for action, allowing the acquisition of a language, or a theory, allowing an epistemological obstacle to be dealt with, and so on.

The reader will forgive me for recalling here a personal, but well-known, example, the lesson called "The Puzzle"[3]. Let me briefly describe the context. Following the study of rational numbers, including decimals, as they appear in measures (1.78 m, for example), the aim is to enable students to extend their use of denoting and calculating linear mapping (multiplying by 1.78, for example). Studies have shown that traditional methods of introduction, based mainly on explicit construction by the combination of "natural" mappings already known by the students (multiplying by 178 and dividing by 100), presented difficulties. The idea consisted of leading students to need to make implicit use of such mappings (a rather large family) and to predict their effects and properties (ordering them, estimating their sum, their product, etc.), so that they would be induced to look for a convenient way of referring to them. First, it was necessary for them to appear (a single one to start with and then another, then yet more) as an implicit solution of an "action problem", so as to follow up with communication problems which would explicitly need a designational system.

We chose an action problem in which students have to discover how to enlarge the pieces of a puzzle in such a way that the image of a piece of side 4 units is a piece of side 7 units, and that the image-puzzle works well as a true puzzle. The trick consists of asking small groups (of two students in order to encourage the expression of the cognitive conflicts which will crop up) each to copy one piece of the puzzle (in order to encourage conflicts caused by attempts to adjust the pieces). It seems clear to the students that the image of each length must be calculated by means of elementary arithmetical operations. But which ones? "The measured length plus 3 cm"? "Twice the length less 1"?

Each attempt to match the pieces constitutes real feedback which allows only mappings pretty close to 4/7: "Twice the length less 1" almost works—except, of course, for 0.5 cm.

The students' search for an intellectually satisfying solution will be the source of understanding, and then of making the fundamental property of linearity explicit: the image of the sum of two lengths must be the sum of the images of these lengths. But if it takes some time for the solution to be found, multiplying by 4/7 will take place in an implicit environment of neighbouring functions, in the topological sense, which anticipates the structure to be studied a little later on[4].

The success of this procedure, piloted with fifth-grade pupils in favourable conditions[5], has given me a lot of hope and enthusiasm, in spite of providing evidence that its generalization to all the sixth-grade classes would be difficult[6].

The hopes teachers have for engineering are valid, but they are accompanied by certain assumptions which are not so valid. In order for these techniques to be implemented, it is necessary that no conditions or knowledge be required other than those currently available to teachers, students and their parents. And their usefulness and effectiveness must nevertheless be immediately obvious to everyone, in the form of exceptional, stable advantages over current practice!

Now, these current practices are considered to be well known, but are never described. The number of parameters on which they seem to depend makes them seem hardly homogeneous at all, and so, depending on the moment, they may equally well be accorded every virtue, or every vice.

It is very difficult for a researcher, or for a teacher, to publish for the benefit of her colleagues, and even to read, the detailed description of the conditions needed for a teaching situation (in the present state of *didactique,* this description would cover at least thirty pages). Custom leads her to describe, only and very briefly, the unfolding and effects of the teaching situation; this is entirely insufficient for good replication (it is enough to read reports in journals and books about lessons to see how hazy these relations can be, and to compare them with those that were current fifteen years or so to establish what progress has been accomplished). Do you think it is sufficient to list the moves of chess masters during the game in order to show what strategies they have, and thus make it possible to "reproduce" a winning game? Although description demands considerable effort, all this precision appears superfluous and even insulting to the reader. It comes over as pedanty.

The terms used in such descriptions should be those which teachers use in their work. But if it is indispensable to use concepts that are different from those which teachers already use to designate similar objects, even if it is only in order to control their theoretical definition, then the following dilemma appears: on the one hand, there is a price to be paid for the use of a new, obscure term; on the other hand, the use of the usual term introduces misunderstanding and makes the hypotheses or results sound very much like pompous evidence—putting on airs of being extremely knowledgeable. Finally, the introduction of an evocative but original term quite rightly arouses suspicion and irony when a familiar word could be used instead of jargon.

In brief teachers await *didactique* in a field where they reign as "masters" (that really is the word), in which nothing prepares or motivates them to accept it for what it is.

2.2. *Knowledge about teaching*

The teacher can expect *didactique* to demonstrate its existence through knowledge concerning different aspects of her work:

- concerning students, their behaviour, and their results in specific teaching conditions;
- concerning conditions to be created in teaching and learning situations;
- concerning conditions to be maintained in the management or conduct of teaching;
- concerning didactical phenomena with which they are confronted as well as all other agents involved in the communication of knowledge.

The first point is the one for which results are certainly the most numerous and the most stable. Moreover, teachers are always interested in reports that present students' errors and behaviour; but there remains a lot to be done to ensure that the causes of these errors are understood rather than the errors merely being recorded. So let us look more closely at one of the questions presented in the introduction:

"To what extent does the functioning of 'institutionalized' knowledge depend on non-contextualized knowledge, taught more or less implicitly as a premilinary?" (Institutionalized knowledge is what the teacher teaches explicitly and formally to the student; it is the object of assessments and the student knows that it must be applied.)

For a child, "to understand" is to establish and connect, on her own, phenomena or facts left "independent" by the teacher, the situation, the teacher's language, and previously learned knowledge.

For example, a child can understand the first measurements through counting, understand the properties of the order of numbers using measurement, verify operations using order ("that becomes bigger and therefore we should not be dividing") or some other operation (multiplication as repeated addition), understand counting thanks to operations (thirteen is ten plus three) or to the search for successors—and all possible relationships, true with integers, are good for providing meaning.

This knowledge, linked together entirely by the student on her own, or thanks to the history of the class, is not all institutionalized by the activity of the teacher, but some certainly is in context, and for good reasons. In any case, it appears to be essential for the proper working of the institutionalized knowledge taught by the teacher.

For the student, these properties of natural numbers are those of numbers in general—of all numbers. Now, embedding the set of natural numbers in a super-set such as that of rational or decimal numbers may cause new properties to appear but, at the same time, it causes certain other properties to disappear—either they are no longer true for all numbers, or they are no longer even true for any number; multiplication can now make things smaller; a decimal number no longer has a successor.

The teacher cannot suitably inform the student of this breakdown, for neither culture (and particularly tradition) nor didactical engineering has yet produced the necessary instruments (exercise, warnings, concepts, remarks, paradoxes, etc.). This situation leads the teacher to mix-up and misunderstanding, and causes the student to make mistakes. These erroneous conceptions persist because they are linked to a certain way of understanding the properties of natural numbers, and the effect of the breakdown can be observed over a number of years.

Even more important is the mechanism of this obstacle: it is not taught knowledge which is wrong—in general, teachers cope with this drawback by trying to keep up a discouse that is not misunderstood but correct—it is students' personal instrument of understanding. The student no longer understands because what needs changing is exactly what she would have called "understanding" up to this point.

Didactique cannot produce a solution to such a problem by mere engineering adjustments. It can eventually suggest decontextualization processes, like Régine Douady's *"changements de cadres "*[7] (1985), or situations which allow confrontation with a knowledge obstacle, like socio-cognitive conflicts. It can do nothing when the difficulty arises from the teacher's ignoring students' contextual references, which are necessary so that they do not lose the fruits of their experience when acquiring new knowledge. It is easy to show that the causes of the impossibility of managing the students' memories by a correct memory of the system are often cultural. The problem must be attacked at this level. *Didactique* must interact with culture.

But the facts that we have just mentioned are elements of a much larger phenomenon which contaminates the relationships between parents, teachers and students after a change of class and/or grade. For instance, a teacher wants to remind a student in difficulty about knowledge which she needs. Insofar as the knowledge at stake is well institutionalized, it ought to be available. But its use and comprehension depend on a context; if this context is ignored by the teacher, the situation becomes blocked. Often the student "discovers, after finding the solution, that she knew very well what was required but that she had not understood the question".

The only way for the teacher to understand the circumstances under which the knowledge is created is either to have taught the notion to this student herself or to have at her disposal a set of cultural references, whether they are due to tradition or to professional knowledge (but this sort of "classicism" of *didactique,* if it consists of institutionalization of problems, can prove to be a dangerous solution which kills mathematical thinking). The teacher can then recall learning situations which the she has not actually experienced because they are known by convention.

Faced with many students' difficulties and lack of understanding of how to use simple knowledge, there is a great pressure to take note of the failure (of the student, the teaching, the method, the programme) and to demand that everyone (the student, the teaching, etc.) focus on what can most easily be defined, evaluated and taught: algorithms; in other words, the minimum default—the student knows how to do a division but does not know when it ought to be done.

Result: those responsible are stirred up, teachers consult with each other and, so as to enhance the move from one level to another, the teaching ignores and neglects the contextualized and personal knowledge of the student even more. The process keeps going until the moment when, for both teachers and students, the need arises to stop behaving like talking parrots (repetition stripped of meaning). This is what happened a few years ago: the noosphere[8] (Chevallard, 1985) invited everyone to take part in an endeavour of invention, originality and pedagogical innovation. As a consequence nobody was assumed to know anybody else's personal method. Increasing spontaneous didactical variability always reduces the chance for each teacher to link together and retrieve the old personal knowledge of her students.

The process was made easier by the existence of texts about knowledge which could claim to be universal and definitive. But, also, in return, it caused teaching to be collapsed onto formal knowledge.

Didactique is beginning to bring into view the characteristics of the relatively chaotic evolution of didactical transposition.

The game of the didactical contract is accompanied by a whole family of phenomena where one can perceive, through all the instability and the corrections, the effect of the variables of the system, if not the rules of its evolution.

The study of the didactical contract has led to the discovery of phenomena such as the "Topaze" and "Jourdain" effects, the effect of metadidactical shift, and the analogy effect, which we shall talk about later. They can be inferred from the theoretical model and observed just as well at the level of a classroom lesson as at that of a community group.

The "Topaze effect" consists of this: generally, the student's response is almost determined in advance, and the teacher negotiates the conditions under which it will be produced, and which will give it meaning. She tries to manage the situation so that the meaning is as rich and as accurate as possible by asking the most open questions. If the student fails to answer, the teacher gives her information in order to make the response easier. It can happen that the teacher ends up by accepting conditions which spark the student's response, without requiring it to have the slightest meaning, as in the first scene of Pagnol's "Topaze": "Two... lambz... were safe in a park"[9].

The "Jourdain effect" is a form of the Topaze effect: in order to avoid a discussion about knowledge with the student and possible acknowledgement of failure, the teacher agrees to recognize as the indicator of a piece of knowledge or of an authentic step, an answer or a behaviour generated by the student which in fact has only trivial motives—and is therefore stripped of value and even, from time to time, of meaning. Examples: the scene from Molière's "Bourgeois gentilhomme", where the Professor of Philosophy reveals to Monsieur Jourdain that he has been writing prose, or the scene where the Professor carries on a priggish discourse while Jourdain believes that he is studying spelling by articulating "O"s and "A's.

The metadidactical shift has covered a vast area and has had some important consequences in the recent past. When an attempt at teaching fails, the teacher sometimes has to return to her text in order to explain and complete it. Starting as a means of teaching, this first attempt then turns into an object of study, eventually even into an object of teaching; the form substitutes itself as the substance. Thus, in order to explain the language of set theory, fundamental but a long way from natural thought, the teachers of the seventies wanted to use, in the form of the well known Venn diagrams and other “potatoes”, the diagrams which Euler had invented for Catherine the Great in his “Letters to a German Princess”. But this metaphor is not a good model, and that, aided by the desire for popularization, made it continually necessary to construct new conventions and to teach the means of teaching as if they were its object: language of set theory for logic, diagrams for formal language, the vocabulary of diagrams for drawings, conventions for vocabulary. The metadidactical shift escaped from the control of the community, causing magnificent misunderstandings worldwide for more than ten years, not to mention the resulting after-effects which still influence the public’s and teachers’ epistemology.

But there exist other phenomena of this type which depend on the knowledge which is aimed at and the context. It is not at all certain that cognitive productions for the purposes of research form a ret of concepts that are perfectly adapted to their communication and to the education of students. “Blind points” exist in the culture. Knowledge that is useful for individual development does not feature as a scientific notion in the culture, and teaching obviously cannot fill this gap. For example, the listing of collections—a very minor aspect of combinatorics—or the representation of space for the organization of actions, displacements, measures—a sector swallowed up by the mathematicians’ geometry-debate—or, even more, natural thinking—swallowed up by logic, etc.

More generally, the “natural” mode of‘ production of mathematical activity, which good teachers demand unceasingly, cannot really be received and recognized as such; it is therefore necessary to replace it by the mode of construction and the language in use. If this is not immediately possible, the whole system is faced with the problem of “memorizing” transitory knowledge of uncertain status which is not recognized by Science, and then helping it evolve without being able either to express or to recognize it.

The agreed effort of obtaining knowledge independently of the situations in which it is effective (decontextualization) has as a price the loss of meaning and performance at the time of teaching. The restoration of intelligible situations (recontextualization) has as a price the shift of meaning (didactical transposition). The retransformation of the student’s knowledge or of cultural knowledge takes up the process again and heightens the risk of side-slip. *Didactique* is the means of managing these transformations and, first, of understanding their laws.

Let us return to how students deal with algebraic expressions. Yves Chevallard has demonstrated the phenomenon of displayed conservation of information: an implicit didactical contract gives the student the responsibility of “conserving” information which has been provided. From this point of view, in the expression “3x = 0”, the “3” shows something: it is different from “4”, it must therefore remain present throughout the mathematical transformations. Deriving the result ‘‘x = 0” might run counter to the contract, so the rather inattentive student transforms the result to ‘‘x = 1/3” or ‘‘x = –3”.

The inventory of studies of phenomena linked to the didactical contract is rapidly expending.

Moreover, demonstrating and studying such phenomena is one thing; influencing them is quite another. Work as coherent as that of Michèle Artigue (eg Artigue 1986) is extremely rare; having studied the reproducibility of didactical situations, she observed the obsolescence of those used in the teaching of differential equations and started to put forward alternatives.

2.3. *Conclusions*

Didactique can, in the end, help the teacher change her status, training and relationship with society: by acting directly on the status of the knowledge which she uses, by acting on the knowledge of her professional partners and on that of parents and the general public, in developing better avenues by which the public could use teaching in more satisfying ways, by providing better possibilities for public or private authorities to manage teaching by more appropriate means.

This component of the contribution of *didactique* is certainly not yet ready to be realized, because of the need for a considerable evolution of school structures and attitudes; but everything leads me to believe that its social rôle is there and that we shall progress in that direction.

3. DIFFICULTIES WITH DISSEMINATING *DIDACTIQUE*

Why does knowledge about *didactique* and its techniques spread so slowly to the public and to teachers?

It seems to me that the main reason is that the majority of those who transmit research findings in the area of *didactique* make an inappropriate and invalid use of it. I am not accusing them of doing this on purpose, but only of giving way to a number of converging pressures.

3.1. *How one research finding reached the teaching profession*

1. There is this famous, if not well known, experiment of the equipe élémentaire[10] of the Grenoble IREM, referred to as "the age of the captain". The investigators put problems of the following kind to third-grade and fifth-grade children:

"On a ship there are 26 sheep and 10 goats. What is the age of the captain ?" or "In a class there are 4 rows of 6 students. What is the age of the teacher?"

In third grade, 78% gave answers that you can guess without any difficulty. The investigators then asked the students "what do you think of these problems?". In fifth grade, most of them either refused to reply or expressed reserve, finding these questions either bizarre or even stupid [almost half of them]: "because the age of the captain has nothing to do with the sheep". But the students did their job: they were supposed to reply. Perhaps they thought that the stupidity of questions reflects on those who ask them, as the proverb says.

2. We know today that this experiment shows one of the effects of the didactical contract—this collection of mainly implicit rules which affects both students and teacher, and which influences their work. Was this evident to the designers of the research at that time? If "yes", would they have been allowed publish it? What was their motivation?

I had felt the need to create the concept of the *didactical contract* in the course of observation of children having specific difficulties (see Brousseau and Peres, 198 1) in 1977–78[11]. My first application of this term appeared at the GEDEOP[12] at Pau [November, 19781, where I used it to explain under-comprehension (a finding of Pluvinage) as a contractual phenomenon. In the framework of the theory of situations, it is an important object: the (didactical) interaction game played by the teacher with the student's adidactical situation and the contract appear to the observer only at the time of their inevitable breakdown.

The first results date from the following year. Chevallard and Schneider investigated the breakdown of the contract which appeared during the study of parametric equations. Chevallard and Tonnelle then studied the effect on the contract of the introduction of polynomials. In 1980, Chevallard and Pascal showed how, in solving equations, students think that they have to keep the whole of the explicit information which is provided for them, and hate to see a letter or a coefficient disappear, to the point of preferring a false transformation (Schneider, 1979; Tonnelle, 1979; Pascal, 1980). The term didactical contract did not appear again in any of the above works. It was found in oral debates among didacticians, but was presented for the first time as a research hypothesis only in 1980 (Brousseau, 1980)[13]. We had to wait until the appearance of a text in 1983, in which Chevallard explains "the age of the captain" as an effect of the didactical contract, and this text remained in the "grey" literature[14] until 1988, when it appeared in a confidential publication of IREM de Marseille[15]!

Nevertheless, it is already clear that students' reasoning is formed by a collection of constraints of didactical origin which modify the meanings of their responses and those of the knowledge they are taught; that these constraints are not arbitrary conditions freely imposed by teachers; they exist because they play a certain rôle in the didactical relationship.

This function remains hidden in the classical framework (taken from Psychology) of the study of responses to questionnaires. Furthermore, its study is difficult: theoretical texts are scarce, the word itself is badly understood by some people, who mistake the theoretical character of the concept and think that it functions solely as a means of improving teaching (What contract must be proposed?). It is criticized in the community even by researchers (How can a contract be implicit? Why aren't teachers free to change it?).

The Grenoble researchers conceived an experiment concerning the study of problems and, because they were working in an IREM, had to produce advice for teachers. Their observations were then used to support, in a rather ideological manner, the condemnation of some types of complex or absurd concrete problems. But they obviously had at hand a phenomenon which brings the interpretation of students' performance back into play—and their membership includes confirmed didacticians who were well aware of the didactical contract *problématique,* which must have brought the phenomenon to their attention. But they could not publish this result in the appropriate place with its theoretical environment—the journal *Recherches en didactique des mathématiques* had just been born but was not able to accept such a subject.

3. How can you keep such a fact to yourself when you are in charge of teacher eduction'? And how could teachers avoid seeing this unexplained fact as "disquieting", since they supposed that it was a consequence of their teaching? There were reasons to be alarmed: results were published "anonymously" in the professional magazine *Grand N* and in the *Bulletin de l'APM*[16]. The sensational press seized it, made a big deal of it, interpreted it, and subjected it to of one of those outbursts of indignation about teaching which the public is so fond of "What? 76 out of 97 of our normal children pair sheep and goats together in order to find the age of the captain !" said Stella Baruk indignantly (Baruk, 1985).

This experiment was presented as revealing "the degree of the sinister... even before it is formulated, any mathematical statement whatever, right at the beginning of the game, is totally without meaning" and this state of affairs "should not endure, cannot endure" (ibid, p. 24).

If normal children give abnormal responses it is because "an improper order based on blindness and deafness to the reality of phenomena has sparked their alienation" (ibid, p. 26). It is also the fault of teachers; and proof that teachers are to blame, Stella Baruk believed, could be found in an experiment of Alain Bouvier [this time she named the author but not the text], who showed that teachers themselves, like the students, also answered stupid questions.

Presented in this context, it is not surprising that these "results" provoked reactions from teachers attending conferences. Without the benefit of a more relevant analysis, they attacked the enquiry that the author claimed to defend.

3.2. *What lesson can we draw from this adventure?*

I shall not, in my turn—playing the fourth painter in a Ripolin advertisement, famous in days of yore—write "zero" on Stella Baruk's back, while she writes "zero" on teachers' backs because they write "zero" on students' backs. She is at the end of the chain, but like each one of us, she complies with the conditions of her didactical contract. Educators who simplify in an improper way, researchers who cannabalize their works (or those of their colleagues) in order to write an article "on demand", authors fond of "novelties"—we all yield to pressures which distract us from the truth. And we give way, either to a larger or smaller extent, in order to save time or position, by conformity, by proselytism, by ideology.

In the first place, I shall make three remarks on the substance of this question.

a) The disappearance of meaning, in the didactical relationship as well as in teaching, is a normal phenomenon: teachers, just as much as their students, are right in trying to obtain the expected answer for the minimum cost. The management of meaning must be studied with the same care and with the same respect as students' errors, for the understanding for which Stella Baruk has fought for such a long time; and rightly so. It is the very object of the theory of situations and of didactical transposition.

b) The fact that many students, at the same time that they accept the fiction of the problem, judge it from the exterior must put us on guard against a naïve reading of comparisons with scholastic knowledge. This fact shows simply that it is necessary to distinguish between the knowledge that students bring into play "for private use", learned or taught knowledge, as they function in the practice of didactical relationships, and target knowledge which itself obeys other laws.

It is therefore not surprising that different constraints lead to different acceptable meanings and to contradictory practice. It would, moreover, be absurd to give children credit for their "intelligent" reactions and systematically to debit teachers for all other reactions. The second experiment shows that the didactical contract forces itself peremptorally as such on the student or on the examinee, independently of her mathematical competence and school experience.

c) Finally, we can show that for theoretical reasons the contract forces itself as much on the teacher as on the student and that the teacher is no more at liberty to modify it acording to her fancy than a merchant or even a government is free to modify economic laws.

This kind of declaration or hypothesis is altogether as ill-adapted to the demands of parents who want either peace or solutions as it is to the education of teachers who ask for advice or suggestions, but who are basically convinced of their uselessness. This leads educators to produce novelties or outrageous facts.

Rather than inform the public about the state of scientific reflections as regards the teaching of mathematics, which would not be successful, the teacher educator must prepare as if for a trial[17], and all the more forcefully in that she is not proposing a solution. In addition, the work of Stella Baruk—although fascinating, full of examples, ideas, knowledge, citations, truths—is all directed towards a prosecution and not an understanding of what she describes.

In order to enhance the image of their social importance, teachers would do best to assume full responsibility for everything that happens in the process that they manage, even though doing this has a slight tinge of masochism.

Let us notice that the APMEP[18], Stella Baruk, and her audience behave as if they accept the idea that it must be possible for teachers to perceive all the phenomena that make a communication of knowledge "abnormal" and to prevent them from happening or correct them if they occur.

Demonstrating the existence of laws not already perceived in this domain confirms the idea that imprecations are just as ineffective as decisions.

In order to fight this tendency, researchers would have to produce clearer answers very quickly. But we have had to wait ten years in order to improve our knowledge about the contract by even a small amount; research is too slow. Facts are not exploited, studies are not reported, texts are not published, or not in the most effective way. For example, observations remain in private documents. Accounts of adjustments to teaching situations are systematically eliminated from publications, and even from theses by their authors, even though the design and the adjustment of these situations has taken them dozens of hours of labour in which didactical knowledge is effectively implemented in the most visible manner. Why? Nobody asks for this kind of text which, read by non-specialists, appears to consist of useless trivialities. There are not enough specialists to read this type of work to justify the effort of rewriting needed for publication. But above all, what society naturally demands of the didactician will often kill the object of research: either teacher education or the production of teaching aids, or the alignment of her "scientific" text with the standards of established domains (or the idea that the public holds of them).

Thus the teacher, the teacher-educator, the idealogue and the author, who are all part of the teacher herself, will throw themselves on the findings of the researcher, whose conceptions will be propagated before having being problematized, whose experiments will be known before the data have been analyzed, whose results will be read over her shoulder because of the need for action, and whose conclusions will be selected with respect to their form (have you any statistical results?) or because of their acceptability in all sorts of circles. And when the researcher wants to present her work, the ground will already have been trodden on, making the task more arduous and apparently more useless.

Why, then, does the researcher accept these constraints? I believe that it is so that she can show, as soon and as often as possible, that her research exists and is useful. If the researcher feels obliged to demonstrate endlessly that her research exists, it is because the system being addressed exhibits a certain stubbornness in denying it. Now, we have already started to explain why the public, teachers, and even mathematicians, have an interest in this denial of the organization of research on teaching, and we shall return to this question later on.

We could look at many other more recent examples of how research in *didactique* is nowadays being exploited, rejected, violated and turned back against teachers, depending on the law of the market for ideologies and the sale of books. The path of *didactique* is in opposition to these thundering interventions. Often, the explanation of a failure or a difficulty allows both the teacher and the student to alleviate their guilt and re-orient themselves towards more positive points of view. In medicine, the attribution of tuberculosis to a microbe did not immediately allow the disease to be wiped out, but it allowed guilt to be removed from the afflicted, who had for a long time been suspected of having offended Nature in some way and of now being punished. It suggested, incidentally, that a certain amount of hygiene could very often prevent the disease.

4. *DIDACTIQUE* AND INNOVATION

It is essential for every teacher to begin her daily class as if the knowledge being presented to the students were being discovered for the first time ever, and as if this discovery were decisive for the future of humanity.

If this need to remain alive, as much for the teacher as for the students, has been emphasized many times, it is because the didactical contract tends—legitimately—to solidify the teaching act, to codify the methods, to define scholastic knowledge; it tends to outdate situations that the teacher uses, and outdate the knowledge for the student, too. The way to resist this obsolescence is by regeneration in the different branches of the contract: in relationships with the student, in relationships with knowledge and with the community of mathematicians, in relationships with teaching situations. An attractive, but dangerous, illusion is to maintain that the best way of avoiding obsolescence would be to avoid its chief symptoms, that is to say: all mechanism, all reproduction and, taken to the limit, all teaching. To remain fresh, the teacher would make mathematics with her students, mathematics without reference to the past, entirely justified by circumstances and the students' lives. This empiricist, Rousseauist, radical position leads to the worst: it proclaims, in principle, the very negation of the goal of teaching which is to communicate a dearly acquired cultural knowledge and frames of reference required by a social contract. It bases the didactical relationship on a whole series of burdensome falsehoods. Each of the participants knows very well that the texts dealing with the knowledge have already been written somewhere else and it does not seem legitimate for the teacher to be allowed to change it one iota at the moment of teaching it.

Yes, to remain alert, the teacher must "remake" mathematics in the classical sense of attempting to answer open questions or to ask interesting, novel questions about mathematics, but it is exceptional for research subjects immediately to be good teaching subjects as well.

Yes, to remain alert, the teacher must "remake" known mathematics, looking for the kinds of problem whose solution it facilitates, the sorts of question it causes to be asked, and how its efficiency and presentation can be improved. Recontextualizing mathematics in another way—in particular for students—is the teacher's essential activity. It is simultaneously of a didactical and mathematical nature; it constitutes a stage of the didactical transposition. This renewing of mathematical knowledge in order to prepare for future questioning and future discoveries even constitutes the specific contribution of teaching to the advancement of science. This is no longer recognized today because the community and the knowledge which could allow it to be controlled and taken into account do not exist.

It is therefore true that teaching is, and will remain, in part an act and that, in one way or another, the teacher will have to assume her connection with the goal—the communication of knowledge—and with the means—the didactical situation. In theatre, the reciprocal responsibilities of the actor and the author have been regulated since Shakespeare, Molière and Diderot. Will the teacher become a sort of actor of knowledge? Will *didactique* become a collection of texts with which she must "only" act—which she will not even be able to choose? My answer is "Yes, sometimes, and there is no loss of dignity in honestly clarifying what one is doing".

Can the teacher consider herself an actor without placing her action itself in danger? The scene, which is her lesson, is replayed for the nth time by the same actor, and sometimes for the same audience. She therefore needs a sort of renewal in order to hold the desire and the vigilance of one and all. Today, she prefers to consider herself a *Comedia del Arte* actor and that is why she needs innovations.

A priori, innovation corresponds to what we have defined above as non-teaching didactical activity of the teacher. It is one of the less arbitrary means offered to the teacher to recover her freshness, which is in danger of being lost, because it is supposed to affect the act of teaching itself. It thus appears to be a pressing necessity for every teacher.

But innovation is a didactical mechanism, and therefore social, and an object of lusty investment like research. Its systemic analysis, which is one of the forms of didactical research, shows that the functioning of innovation leads to results which are different from those stated.

Innovation, by definition, cannot remain hidden—it must be communicated. It must therefore merit the greatest dissemination and suggest "things that work" in a format that is communicable to everyone. Thus, its dissemination must be justified by preliminary evidence of the failure of old methods—earlier innovations. It must therefore emphasize the fact that it is new and that it presents at least one essential difference. In order to proceed quickly, it will do best to discredit and reject the past.

In a Ilattering presentation, innovation allows a group of teachers to live like innovators: people who "develop their competencies", who "work for the improvement of teaching conditions", who "state operative conclusions" and who have an impact on their environment[19]. Their goal is generous: to propagate innovation, to extend it and to generalize it by different means of social action.

Innovation needs an audience, it is therefore a phenomenon of an autocatalytic type; its progression, very strong at the beginning (exponential), diminishes rapidly like the concentration of ignorami. Its force and speed of propagation reduce to zero more quickly than foreseen: when more than 20% of teachers share the same point of view, it becomes rather inappropriate for supporting an innovative function. The original innovators, or others, then turn their sights to new horizons—no matter whether the innovations which they are abandoning have been "good" or "bad"—and, the better the success of its dissemination, the more quickly they do so.

The system that we are describing here is the one of fashion. Teachers need fashion, yes fashion! And why not'? I see nothing wrong with this mechanism of incentive for coinsumption. Concretely, the process continues through a more or less profound statement of defeat to a muddled battle in which the first innovators cry "foul play" and demand more ways of training or pressure on their colleagues. Then a new wave of innovation becomes essential for everyone, in order to "pass beyond" this painful period of confusion that everyone is eager to forget. This is why innovation never allows easy lessons to be drawn from the experiences that it incessantly provokes, and thus can contribute nothing to knowledge about *didactique.* In the better cases, it borrows its acquisitions, but for other purposes, as we have shown with "the age of the captain".

The statement of failure is therefore necessary for the self-maintenance of innovation, but is failure itself unavoidable'? No, I think that across these innovations, strongly cyclic in other respects, progress proceeds all the same, but its possibilities of action are very limited. Indeed, in order to be spread very quickly, an innovation needs the rhythm that only processes of fashion permit. In order to allow this rhythm, innovation must touch nothing essential in the profound parts of teachers' practices; clothing fashions can change the collars or lengths of coats but it cannot make holes in them.

The mechanism for disseminating innovation is more complex and depends on the knowledge concerned. The reasons for its success and its failure have been studied in a more precise way in particular cases: that of diagrams, mentioned above, and that of an apparently rather successful innovation, the theories and methods of Diénès.

The phenomena that we have pointed out are interesting: Diénès proposed the Bourbakist organization of mathematics simultaneously as an epistemology, as a cognitive and psychogenetic model, as logic, and as a model of teaching processes. He produced didactical materials which were very much appreciated.

The heart of this marvellous simplification, supported by the structuralist ambience of the period, is the mathematicians' "set-theoretic quotient" which makes the list of classes of equivalent objects correspond to an equivalence relationship and to a set. To generalize means to go from equivalent objects to their common properties. In order to teach, it is necessary to bring "isomorphic" situations to life for the student, that is to say, situations that are equivalent from the point of view of their mathematical structure. After some experiences of this type, the student recognizes the same structure; she can schematize it and then formalize it.

Teachers accepted this theory because they used it implicitly: it corresponded exactly to the current practices of teaching: at the beginning, there were the requirements of the didactical contract: if the student failed, she must be given another chance to work out the solution herself; soon repetition was established as a principle of learning; but, in order to hide the similarity of the questions, it was necessary to vary non-relevant conditions; the student, duly made aware of this, looked for analogies; moreover the teacher invited her to do so.

This abuse of analogy led the student to look for resemblences corresponding to the teacher's intentions and to focus upon irrelevant variables instead of understanding the internal needs of the situation. Thus, she solved her problems more by transfer of algorithms than by understanding the meaning. In fact, without didactical intentions, the empirical process of learning described in this manner did not work.

On the other hand, teaching founded on the hypothesis of this process can really work. It is enough for students to be able to guess the piece of knowledge that the teacher is presenting to them in this way, partly hidden by a didactical fiction. But, if the analogy is rather remote or if they have to make a specific effort of comprehension, the majority of students cannot "read" the teachers' intention and therefore fail to recognize the analogy. On the other hand, they cannot deny the similarity when it is revealed to them; failure is thereby blamed on them and not on their teacher.

In order to succeed, the Diénès method requires the following: an explicit didactical contract: the student must search for similarities; clear didactical signals: it must be possible to bring situations closer in time, number and suitable rhythm; but above all, the teacher's pressure must be sufficient to make the didactical contract work.

One then observes that with innovators who want to show that the method works it is really successful: the students learn what the teacher presents to them. It fails, on the other hand, with teachers who believe in the truth of the didactical theory of Diénès: one after the other, they give out suitable workcards but learning doesn't take place. The explanation is that they are waiting for the process to behave like a physical law and are therefore exerting no pressure on the didactical contract.

I have presented this story in order to draw a few remarks and examples from it.

In the misuse of analogy we see an example of the Topaze effect: the teacher simplifies her task so that the student obtains the correct answer by a trivial reading of the teacher's questions and not by an authentic mathematical activity specific to the proposed structure, and an example of the Jourdain effect: the student obtains the correct answer by a trivial recognition and the teacher attests to the value of this activity by an erudite mathematical and epistemological discourse.

Teachers' acceptance of the theory of Diénès is an example of the false relationships which can become established between research and teaching: the researcher consciously uses or fails to use concepts teachers create to solve problems of teaching management. The researcher rationalizes them to some extent, translates them into erudite terms and returns them, sanctified by a scientific aura, to teachers, who recognize them as "truths" and immediately adopt them with enthusiasm, without changing anything in their work, and ensuring the researcher's social success. This, again, is a Jourdain effect.

Even more subtle, here is what could be called a didactical theorem: if teachers believe in the proper efficiency of a didactical method, to the point of relying on it almost completely, they no longer play their rôle in the negotiation of the didactical contract, and the method fails. The paradox is only an apparent one, but it throws a disquieting light on the future of any didactical technology which could lead us to believe that it is possible to neglect teachers' vigilance and fundamental knowledge of *didactique.*

In summary, any teaching innovation, based on a statement of failure, must cancel out former innovations and references to a progression of knowledge (in contrast to what happens in other domains). It must finally fail and, consequently, no innovation can attack the essential conditions of teaching. It nevertheless has an effect on teaching, but with a necessarily limited efficacy and at a social and epistemological cost which could quickly appear excessive. It will then become necessary for *didactique* to defend and support the innovation which it can generate, recognize and guide (as happens in other scientific fields), whose symbolic importance, at least, it is able to demonstrate. Let us hope it will be strong enough for that!

This re-statement on innovation was necessary, because many people could have understood the question which was put to me as follows: "What could *didactique* bring to a teacher that is more than and different from that produced by innovations put forward by teachers themselves?"

My remarks certainly ought not to make anyone believe that I am an opponent of innovation. First, because I have tried and am still trying to some extent to be "an innovator" myself. I have produced "new" lessons, "new" techniques and "new" ideas. And also because a large part of the means that I have at my disposal for the COREM at the Ecole Jules Michelet[20] at Talence is due to the success of a continuous flow of innovations and suggestions for the sake of the teacher trainers of the region, PEN and IDEN[21]; innovations whose direct dissemination is held back as long as possible so as to allow each person to use it to the best effect in his or her professional activity as a teacher and, consequently, for some time as an innovator. Innovation will allow me to buy and to credit research as long as it is not able to play its rôle plainly. But my right hand, research, has to ignore what my left hand, innovation, is doing.

And again, because innovation produces phenomena without which it would be very difficult to make advances in theoretical reflection, and which we would be surely incapable of (and morally prevented from) producing experimentally.

And finally, because of the extensive meaning that we gave it at the beginning, innovation is the same true principle as the action of teaching; in the same way that no theory of dynamics can exempt a driver from looking at the road and making decisions which he or she alone is able to take, *didactique* cannot replace the teacher in the act of teaching. To refuse the teacher the right of innovation would be to refuse her the right to give meaning to what she does. It is only necessary not to mix up the rôles. Nothing forbids an actor to write plays and act in them; simply, the actor cannot do both at the same time. Teachers can help themselves very much and help *didactique* too by becoming interested in its difficulties, by participating in its progress and in its intellectual challenges. They can take responsibility for using the outcomes of engineering which are its by-products. They can participate in its research and its debates, both as professionals and amateurs (accepting the rules of the game and depending on the time and means available to them); this help is always precious. *Didactique* is their affair, as biology and medicine are for medical practitioners. It has a limited but precise and irreplaceable function; it needs their understanding and support even if it cannot yet lighten their burden very much.

In the choice which we are making to develop one fundamental theory of communication of mathematical knowledge under the name *didactique,* there is no incompatibility with other definitions and other orientations. On the contrary, it is a conception which favours the integration of contributions from other domains and their application to teaching, and which with practice establishes a sound relationship between science and technique and not a prescription for reproduction.

It does not condemn, *a priori,* any action in favour of teaching. But it must be understood that it is a mistake to want to make *didactique* engage in each of these actions at any price and play a rôle which is not its own. In the better cases, ridiculous and impossible challenges are proposed, challenges that one dare not require even of more advanced sciences. In the worst cases, one takes the risk of entrusting experts with responsibilities far beyond their capability and reproducing errors similar to those that can be seen elsewhere (for example: in economics).

Contrary to what some people have suggested, one of the functions of *diductique* could therefore be to contribute by putting a brake on a process which consists of transforming knowledge into algorithms that can be used by robots or by sub-employed humans, and of diminishing the place of noble reflection on all human activities in order to delegate it to a few people.

In order to satisfy the god of so-called efficiency, teaching gives support today to algorithmic reduction and demathematicization. I sincerely hope that *didactique* will be able to fight against this dispossession and dehumanization.

NOTES

1. *Petit x* is a journal focussing mainly on mathematics education at the middle school level.
2. See Y. Chevallard: "Le passage de l'arithmétique à l'algèbre dans I'enseignement au collége"; first part: "L'évolution de la transposition didactique". *Petit x* 5, 51–94; and "second part: "Perspectives curriculaires: la notion de modélisation?" *Petit x* 19, 43–72. *Editors' note:* See also Kieran C. (1992) The learning and teaching of school algebra. In: D. Grouws (Ed) *Handbook of Research on Mathematics Teaching and Learning* (pp. 390–419). New York: Macmillan.
3. I presented and examined this lesson in 1981 in: "Problèmes de didactique des décimaux", *Recherches en didactique des mathématiques* 2(1). This article was referred to in: *Petit x* 9, 1985, 63–65, by Ph. Clapponi and in: *Petit x* 11, 1986, p.20 by F. Pluvinage. In: *Petit x* 17, 1988, 49–56, C. Morin gives a detailed account of students' behaviours observed in the course of replications which she organized. *Editors' note:* see also in this book Chapter 4 section 2.2.
4. The group of procedures is described in detail in: N. and G. Brousseau (1987) *"Rationnels et décimaux dans la scolarité obligatoire",* IREM de Bordeaux.
5. These favourable conditions were provided at the Ecole Michelet in Bordeaux.
6. This lesson is only the introduction of a process which must reach its conclusion to allow the pupils to meet and master all aspects of the use of rational mapping or decimal mapping. Isolated, placed in a confused context, or conducted without rigor, it risks functioning only as an mapping exercise or a metaphorical support for the teaching discourse. But this technique is quite robust: even badly analyzed and badly used, it still holds a real attraction.
7. *Editor' note:* See (Douady 1984) for a developement of Douady's theory of "interplay between settings". A shorter presentation is available in French in (Douady 1986) and in English in (Douady 1985).
8. All people and groups interested in the creation and the communication of knowledge in a certain domain. *Editors' note:* see note 7.
9. *Editors' note:* see Chapter 1, section 2.1.
10. *Editors' note:* Research team working on mathematics teaching in elementary school.
11. C. Amirault and M. Cheret vocational teaching certificate (speech therapy) published by the Bordeaux IREM in 1978. *Editors' note:* see also Brousseau G. and Peres J. (1981).

12. *Editors' note:* GEDEOP stands for Groupe d'Explication, d'Evaluation et d'Expérimentation des Objectifs en Pédagogie, a working group on pedagogical objectives. The text to which Brousseau refers was published by the Bordeaux IREM.
13. *Editors' note:* See also Brousseau (1980a).
14. Such as Alain Mercier (1984) *Eléments pérennes du contract didactique, ruptures locales et ruptures globales.* Marseille IREM.
15. Chevallard Y. (1983) *Remarques sur la notion de contrat didactique.* Aix-Marseille IREM.
16. *"Quel est l'âge du capitaine?"* in: *Grand N* 19, Grenoble IREM, and in Bulletin de l'APMEP 323, 235–243.
17. And with what force! In the debate, Stella Baruk cites J Paulhan: "It is said that the ignorant are the best teachers". (This, in the context, is evidently intended to insinuate the reciprocal.)
18. *Editors' note:* Here, Brousseau uses the usual acronym to stand for the French *Association des Professeurs de Mathématiques de l'Enseignement Public* (APMEP), which is the French association of mathematics teachers.
19. These expressions are used by Alain Bouvier in *"Didactique des mathématiques, le dire et le faire".* In order to characterize action research he contrasted it with research called "traditional", in which the researcher would aim essentially at "the appropriation of the competency of others, the improvement of her status as a researcher and of her career, and the appearence of new questions, new hypotheses" (page 512). I find that my friend Alain sends the Beaujolais cork rather far!
20. *Editors' note:* COREM stands for *COmmission de Réflexion sur l'Enseignement des Mathématiques,* a national working committee on the teaching of mathematics.
21. *Editors' note:* PEN stands for *Professeur d'Ecole Normale,* a mathematics educator of elementary school teachers; IDEN stands for *Inspecteur Départemental de l'Education Nationale,* Inspector of the National Ministry of Education at the level of Département.

APPENDIX

THE CENTER FOR OBSERVATION: THE ECOLE JULES MICHELET AT TALENCE*

Editors introduction: *Together with a theory and a method for the study of didactical phenomena, Brousseau designed an instrument to allow the observation of students and teachers in their original milieu. Not carrying a classroom into a laboratory, but surrounding the classroom with the laboratory, the school, the Center for Observation of the Ecole Jules Michelet in Talence, was built in 1971. It is quite exceptional in its characteristics and its functioning: an ordinary school as a sophisticated research instrument. In the following text, written in 1975, Brousseau presents it, as well as his views on the concrete organisation of research in the field.*

In the present state of things, it is almost impossible for a specialist who is not a teacher to conceive of an experiment which is interesting from a theoretical point of view, and which can be carried out in a school's machinery about which he or she knows nothing. In order to prepare experimental protocols which are satisfactory from the point of view of scientific research and from a pedagogical point of view, it has to be possible to observe children and teachers in favourable conditions.

Hence, the field for experimentation is composed of two distinct parts:

– a school set up for the direct observation of some of the children's activities; within this school, teachers and researchers conceive the protocols of experiments to be carried out in schools responsible for the experimentation;
– a field of experimentation responding to the requirements of the experimental plan, made up of a certain number of schools which have the possibility of signing a contract with the IREM and of undertaking the experiments conceived in the observation school.

But this "observation school" is:

– neither an experimental school, in the sense that outside of the quite short observation phases, there is no imposition of methods, techniques and new programmes that one might wish to evaluate—it is sufficient to have the activities of the children known and compatible with the research;
– nor a pilot or model school (where a supposedly better pedagogy is practiced);

* Taken from the report of the colloquium organized by the *IREM de Bordeaux* at Talence , 13–15 March 1975

- nor an annexe school or an application school (which would serve for the pre-service or in service education of teachers, or for applied research);
- nor a control school (which would serve as a reference for teachers because an adopted, defined, controlled pedagogy is practiced).

Observations must be prepared and realized collectively by researchers of the IREM and the teachers of the school who will have to implement the prepared activities in their classrooms. In this situation, the teachers cannot be acting at the same time as teachers and as objective observers of their own teaching. It is therefore necessary that at other times they be able to observe their colleagues teaching classes. Teachers' participation in research requires that they be relieved of some of their duties, so as to have time for their training and for carrying out the observations. This is why three teachers teach two classes; this gives each of them eighteen hours of teaching and nine hours of observation, training and consultations.

This sort of very particular work can be asked only of volunteers. Like all IREM researchers, these volunteers are seconded to this school for a limited period. It therefore seems impossible to work in a school annexed to a teacher-education college. These two requirements are incompatible with the functioning of Annexed Schools—as they are, too, with the functioning of any school already in existence. What is more, for Annexed Schools, the requirements of the preparation of future teachers (undergoing probation for long periods, visits of FP2[1] classes during practice teaching, meetings with teacher-educators), mean that work in common with the IREM is impossible as soon as the research programme becomes large.

Thus the school chosen was a new school. This excellent working environment, built in Talence in the academic suburb of Bordeaux, has been in operation since the start of the 1972 school year. It is unique in France.

It is a mixed school consisting of ten primary-school classes and four pre-school classes. The curriculum taught is that normally stipulated by the national Ministry of Education; the timetables in place are the official timetables; the children who attend are subject to the same admission conditions as at all other primary and pre-school.

Nevertheless, the parents are informed of the *"experimental"* character of the school, where the children are above all not guinea pigs.

The spirit of analysis which we are developing in them can be surprising in a a classical teaching situation. It is therefore necessary for us to enlarge our contacts with teachers in order to lead them to discover our pedagogical objectives. This fact raises the problem of defining who can be recruited as a volunteer and agree to play the game of observation fully and engage in fundamental research. At the present time, at the Jules Michelet school, for classes having no more that twenty-five children (in primary) and thirty-five enrolled in the pre-school, the staff numbers are as follows: three teachers for the two final-year classes, fifteen teachers for the ten primary classes, six teachers for the pre-school, one pre-school principal and one full-time school psychologist. All the school's teaching staff are under the administrative and pedagogical control of the hierarchical authorities. They are appointed for one year.

Each teacher teaches the topics defined by the official programme. Only mathematics, though it conserves the spirit and the goals, changes the means. That is the aim of this school. The programmes are fixed by the Director of the IREM as a function of the research plans and experiments of its researchers and the requirements of the education of the children.

Within these new, modern buildings, everything is on the ground floor, including some teachers' accomodations, which gives it a very family-like character. Almost at the centre of the school is the mathematics classroom, the laboratory. It is a large hexagonal building about fifteen metres across. The classroom is at the centre. It is equipped with all the didactical material necessary for mathematics. The tables can bc arranged either individually or in groups according to the needs of the lesson or the observation to be carried; a microphone, linked to a control room in which observations can be made and recorded, is suspended from the ceiling above each of the tables; a mobile camera follows the children's activities; a fixed camera is operated by remote control.

All around the classroom, the following facilities are arranged: the control room with its closed-circuit television, its video equipment and its hi-fi receiver; the observation room with a one-way viewing window, a television set and selective microphones which can be aimed at tables to be observed; the interviewing room which contains a computer terminal linked to a computer in the university—delicate, expensive equipment which requires attention.

At the Michelet school, the observation of a lesson is the result of a group project which satisfies a number of requirements: on the one hand, problems raised by teaching as it is practiced nowadays; on the other hand, the objectives of fundamental research. An annual schedule is drawn up in advance. Once the topics to be taught in a particular lesson have been fixed, the different groups which will work on the observation are established: the didactical team, the recording team, the written-record team and the team in charge of evaluation and observation.

The goal of the *didactical team* is to adapt the observation to the year's work and to make it compatible with the official programme. It wishes to create a phenomenon in a precise, reproducible way and to observe it. It will therefore determine very precisely the expected motivations, along with the instructions to be given by the teacher, the children's working time and the material used. This material used by the teachers and the children is subject to a number of requirements; for example, it must be large enough to be picked up clearly by the television camera.

The development of the lesson is provided down to the smallest detail on the didactical proforma. This sheet is given to all observers before the lesson so that they can peruse it.

During the whole lesson, the *recording team* ensures the re-transmission sound, the image, the logging of sequence, and the recording of everything that happens in the classroom.

The *written-record team* records on a simple observation form an account of the lesson, followed from within the room (or on closed-circuit television).

The *evaluation and observation team* is responsible for setting up observation grids and using them during the lesson. After the lesson, this group is responsible for scrutinizing these results and for the immediate treatment of the data. These grids, different for each observation session, must allow the different discoveries made by the children, and their propagation within the groups, to be discerned with ease. The communications of the sender and the receiver, and the content of the communication (verbal or non-verbal), are noted here according to a predetermined code. The children's working papers are numbered in a way that corresponds to a numeral entered on the observation grid.

At the end of the lesson, the children's work is collected; the written records and the observation grids are brought together. These documents are then examined during the working session which takes place after the observation. Everyone offers an opinion. It is necessary to determine in this way whether the predetermined objectives have been attained.

Some visiting personalities have been able to vouch for the research effort and the imagination provided by the children, their particpation and how they are taken by the subject.

Concrete results have been obtained. Since several years ago, the IREM researchers have established an efficient working technique which provides remarkable possibilities for teachers who are willing to leave their routine. Several schools of the *Academie de Bordeaux* are included and are operating with the assistance of IREM staff.

Is it not essential, in this business, to agree to speak the same language? Research in this school is not conducted to the detriment of teaching. The time allocated to observation does not reduce that allocated to study. The prepared observations take from one hour to four periods of one hour. They require one or two months' preparation, and two or three months' analysis. No more than five to ten observations can therefore be undertaken in one year.

The work is already allowing us to gain time compared with classical methods, and it is to the extent that we know that observation does not jeopardize the children's learning activities that it is organized.

An observation therefore develops in three stages:

– a phase in which children are familiarized with the observation system;
– an observation phase under the conditions required by the research without didactical intervention;
– a pedagogical phase in which the teacher turns to the best account what the children have done in order to lead them, from their point of view, to a suitable issue of the experiment.

From the pedagogical point of view, the teaching of mathematics does not take on an undue importance to the detriment of other topics.

For this school, there is not a pedagogy defined in advance, linked to a research project; there is, rather, an effort of harmonization and of adaptation of the options taken up by each teacher in the direction of a renewal of teaching.

The presence of the full-time school psychologist, class size limited to twenty-five students, and the team of three teachers for two classes are elements which permit the following and assisting of each child better than under normal conditions.

The IREM is associated with the general effort towards reforming the teaching of mathematics. The new programmes contain innovations as well as traditional practices. In both cases, there is material for observation, reflection and criticism. Certainly, the mathematics which it uses and which is in command of teaching is current mathematics; it is in no way dogmatic or inflexible. It is simply one of the best ways, with current psychology, current technology and current linguistics, to best fulfill educational intentions.

It would be necessary to explain these intentions, that is to say, those which the IREM believes to be a good training for individuals, in order to understand afterwards, considering the current research, why it selects such-and-such a procedure, no doubt to give an advantage to the children who will benefit from it.

It is by way of a long-term process that fundamental research finds its use. The IREM does not expect immediate benefits for children to result from its observations.

NOTES

1, *Editors' note:* FP2 stands for *Formation Professionnelle, 2° année* (Second Year of Professionnal Training) in the organisation of pre-service teacher training of this time.

The Jules Michelet school, the center for observation

Guy Brousseau in the playground of the Jules Michelet school

A view of the classroom of the center for observation

A snapshop during the observation of the problem – situation of the puzzle (see Chapter 4, section 2.2)

BIBLIOGRAPHY 1970–1990

Brousseau G. (1965) Les mathématiques du cours préparatoire. Paris: Dunod

Brousseau G. (1970) Processus de mathématisation. In: *La mathématique à l'école élémentaire* (pp. 428–457). Paris: APMEP.

Brousseau G. (1971) Vers un enseignement des probabilités à l'école élémentaire. *Enseignement élémentaire des mathématiques,* 11, IREM de Bordeaux

Brousseau G. (1971 a) L'égalité. *Cahier de l'Enseignement Elémentaire* (vol. 1, 25–30). Bordeaux: IREM de Bordeaux.

Brousseau G. (1973) Peut-on améliorer le calcul des produits des nombres naturels? *Cahier de l'Enseignement Elémentaire* (vol. 13, pp. 195–237). Bordeaux: IREM de Bordeaux.

Brousseau G. (1 973a) Recherches sur I'enseignement du calcul numérique. *Séminaire INRDP-SERP de recherche "Mathématique Elémentaire*", Orléans. Published in: *Cahier de I'Enseignement Elémentaire* (vol. 1.5, pp. 68–84). Bordeaux: IREM de Bordeaux.

Brousseau G. (1973b) L'apprentissage des opérations dans les naturels. *Cahier de l'Enseignement Elémentaire* (vol. 13, pages). Bordeaux: IREM de Bordeaux.

Brousseau G. (1 974) Les structures des données dans l'enseignement élémentaire. *Compte-rendu du stage des IDEN.* Published in: *Cahier de l'Enseignement Elémentaire* (vol. 16, pp. 145–1 52). Bordeaux: IREM de Bordeaux.

Brousseau (1 974a) L'enseignement du calcul numérique et les stratégies de l'enseignement. *Cahier de l 'Enseignement Elémentaire* (vol. 14). Bordeaux: IREM de Bordeaux.

Brousseau G. (1974b) Réflexions sur I'enseignement du calcul numérique. *Cahier de I'Enseignement Elémentaire* (vol. 15). Bordeaux: IREM de Bordeaux.

Brousseau G. (1974c) *Etude d'une expérience à probabilités.* Bordeaux: IREM de Bordeaux.

Brousseau G. (1975) *Etude de l'influence des conditions de validation sur l'apprentissage d'un algorithme.* Bordeaux: IREM de Bordeaux

Brousseau G. (1976) Les obstacles épistémologiques et les problèmes en mathématiques. *Comptes-rendus de la XXVIIIe recontre organisée par la Commission Internationale pour l'Etude et l'Amélioration de I'Enseignement des Mathématiques* (pp. 101–1 17), Louvain-la-Neuve.

Brousseau G. (1976a) Peut-on améliorer le calcul des produits des nombres naturels? In: *Actes du Congrès International des Sciences de l'Education* EPI 1, 364–378.

Brousseau G. (1 978) L'observation des activités didactiques. *Revue française de pédagogie* 45, 130–140.

Brousseau G. (1978a) L'étude des processus d'appreutissage en situations scolaires. *Cahier de l'Enseignement Elémentaire* (vol. 18, pp. 2–6). Bordeaux: IREM de Bordeaux.

Brousseau G. (1 979) *Etude de situations,* (*théorie des situations didactique).* IREM de Bordeaux

Brousseau G. (1979a) Evaluation et théories de l'apprentissage en situations scolaires. In: *Educacion matematica en las americas.* UNESCO.

Brousseau *G.* (1980) Problèmes de l'enseignement des décimaux. *Recherches en didactique des mathématiques* 1(1) 11–59.

Brousseau G. (1980a) L'échec et le contrat. *Recherches* 41, 177–182

Brousseau G. (1981) Problèmes de didactique des décimaux. *Recherches en didactique des mathématiques* 2(1), 37–127

Brousseau G. (1981a) Les échecs électifs en mathématiques dans l'enseignement élémentaire. *Revue de laryngologie* 101(3/4) 107–131.

Brousseau G. (1981b) *Le cas de Gaël.* Bordeaux: IREM de Bordeaux.

Brousseau G. (1 982) *Tendances originales des recherches en didactique des mathématiques en France.* IREM de Bordeaux

Brousseau G. (1982a) Les objets de la didactique des mathématiques—Ingénierie didactique. *Actes de la deuxième école d'été de didactique des mathématiques* (pp. 10–60). Orléans: IREM d'Orléans.

Brousseau G. (1983) Les obstacles épistémologiques et les problèmes en mathématiques. *Recherches en didactique des mathématiques* 4(2) 165–198.

Brousseau G. (1983a Etude de questions d'enseignement, un exemple: la géométrie. *Séminaire de didactique des mathématiques et de l'informatique* (pp. 183–226), Grenoble: IMAG.

Brousseau G. (1 984) Le rôle du maître et l'institutionnalisation. *Actes de le IIIe école d'été de didactiques des mathématiques* (pp. 40–44). Grenoble: IMAG.

Brousseau G. (1984a) Le rôle central du contrat didactique dans l'analyse et la construction des situations d'enseignement et d'apprentissage des mathématiques. *Actes de le IIIe école d'été de didactiques des mathématiques* (pp. 99–108) Grenoble: IMAG.

Brousseau G. (1984b) Quelques conduites déterminantes en didactique des mathématiques. In : *Actes du colloque "Réussir à l'école". Perspectives de réussite: Au delà des insuccès scolaires,* Vol *3* (Preprint, 14 pages, IREM de Bordeaux).

Brousseau G. (1984c) The crucial rôle of the didactical contract in the analysis and construction of situations in teaching and learning mathematics. In: Steiner H. G. *et al.* (eds.) *Theory of Mathematics Education* (Occasional Paper 54). Bielefeld: IDM.

Brousseau G. (1984d) The IREM's rôle in helping elementary-school teachers. In: R. Morris (ed.) *Studies in Mathematics Education, the Mathematics Education of Primary-School Teachers*3, 235–251.

Brousseau G. (1985) Quelques concepts fondamentaux en didactique des mathématiques. *EPS, contenus et didactique* (pp. 269–277). Paris: SNEP.

Brousseau G. (1986) *La théorisation des phénomènes d'enseignement des mathématiques.* Thèse d'état. Bordeaux: Université de Bordeaux 1.

Brousseau G. (1986a) Fondements et méthodes de la didactique des mathématiques. *Recherches en didactique des mathématiques* 7(2) 33–115.

Brousseau G. (1986b) La relation didactique: le milieu. In: *Actes de la IVe école d'été de didactiques des mathématiques* (pp. 54–58). Paris: IREM de Paris VII.

Brousseau G. (1987) Didactique des mathématiques et questions d'enseignement: Propositions pour la géometrie. *Les sciences de l'éducation,* 1–2, 69–85

Brousseau G. (1987a) La situation fondamentale pour la géométrie élémentaire en tant que modèle de l'éspace. *Les sciences de l'éducation,* 1–2, 86–100

Brousseau G. (1988) Représentation et didactique du sens de la division. In: Vergnaud, G., Brousseau G., Hulin M. (eds.) *Didactique et acquisition des connaissences scientifiques* (pp. 47–64). Paris: La Pensée Sauvage.

Brousseau G. (1988a) Les différents rôles du maître. *Bulletin de l'Association Mathématique du Québzc* 2/23, 14–24.

Brousseau G. (1988b) L'articulation des apprentissages entre les niveaux scolaires: Problèmes d'apprentissage ou problèmes de didactique? *Communication ICME–6.* Budapest.

Brousseau G. (1988c) *La tour de Babel.* Bordeaux: IREM de Bordeaux.

Brousseau G. (1988d) Basic theory and methods in the didactics of mathematics. In: *Report on the second conference on systematic co-operation between theory and practice in mathematics education* (109–162), ville: Instituut voor Leerplanontwikkeling.

Brousseau G. (1989) Les obstacles épistémologiques et la didactique des mathématiques. In: Bednarz N., Garnier C. (eds.) *Construction des savoirs— Obstacles et conflits* (pp. 41–63). Montréal: Centre interdisciplinaire de recherche sur l'apprentissage et le développement en Education (CIRADE).

Brousseau G. (1989a) Obstacles épistémologiques, conflits socio-cognitifs et ingénierie didactique. In: Bednarz N., Garnier C. (eds.) *Construction des savoirs— Obstacles et conflits* (pp. 277–285). Montréal: Centre interdisciplinaire de recherche sur l'apprentissage et le développement en Education (CIRADE).

Brousseau G. (1989b) Utilité et intérêt de la didactique des mathématiques pour un professeur de collège. *Petit x* 21,48–68.

Brousseau, G. (1990) Le contrat didactique: le milieu. *Recherches en didactique des mathématiques,* 9(3), 309–336

Brousseau, G. (1 990a) *Eléments pour l'étude du sens de la division.* Bordeaux: IREM de Bordeaux.

Brousseau, G. (1 990b) *Une expérience d'épistémologie sur l'enseignement des décimaux.* Bordeaux: IREM de Bordeaux

Brousseau, G. (1990c) *Cours élémentaire de didactique des mathématiques.* Bordeaux: IREM de Bordeaux

Brousseau, G. (1 990d) *L'observation en didactique des mathématiques.* Bordeaux: IREM de Bordeaux

Brousseau, G. (1 990e) *L'ingénierie de la didactique des mathématiques.* Bordeaux: IREM de Bordeaux

Brousseau G. (1995) L'enseignant dans la théorie des situations didactiques. In: Noirfalise R., Perrin-Glorian M.-J. (eds.) *Actes de la VIII^o^ Ecole d'été de didactique des mathématiques* (pp. 3–46). Clermont-Ferrand: IREM de Clermont-Ferrand.

PUBLICATIONS IN COLLABORATION

Brousseau N., Brousseau G. (1987) *Rationnels et décimaux dans la scolarité obligatoire.* Bordeaux: IREM de Bordeaux.

Brousseau G., Centeno J. (1988) *Nécessité de l'analyse de la mémoire du système didactique et de son fonctionnement pour réssoudre les problèmes interniveaux scolaires.* Unpublished Communication at the 6th ICME. Budapest.

Brousseau G., Centeno J. (1991) Rôle de la mémoire didactique de l'enseignant. *Recherches en didactique des mathématiques* 11 (2/3) 167–210.

Brousseau G., Davis R. B., Werner T. (1986) Observing students at work. In: Christiansen B., Howson A. G., Otte M. (eds.) *Perspective on Mathematics Education* (pp. 205–240). Doirdrecht: Reidel Publisher.

Brousseau G., Gabinski P. (1974) *Etude des communications dans un groupe d'enfants. Cahier de l'Enseignement Elémentaire* (vol. 15). Bordeaux: IREM de Bordeaux.

Brousseau G., Gabinski P. (1976) *Etude sur la théorie des automates et son application à la didactique des mathématiques.* Bordeaux: IREM de Bordeaux.

Brousseau, G. and Greslard, D. (1988): La régulation d'un curriculum, problèmes pratiques et théoriques, In: *Compte-rendu des journées de formation des professeurs.* Université de Namur

Brousseau G., Otte M. (1991) The Fragility of Knowledge. In: Bishop A. J., Mellin-Olsen S., van Dormolen J. (eds.) *Mathematical Knowledge: Its Growth Through Teaching* (pp. 13–36). Dordrecht: Kluwer Academic Publishers.

Brousseau G., Peres J. (1981) *Etude du'un enfant en difficulté en mathématiques: "Le cas de Gaël".* Bordeaux: IREM de Bordeaux

Brousseau G., Prouteau C. (1976) *Problèmes d'enseignement des décimaux du cours moyen à la 4e.* Bordeaux: IREM de Bordeaux.

Eyheraguibel J. C., Brousseau G. (1978) Appareillage de mesure automatique des strategies d'apprentissage—Application à un jeu logique: la tour de Hanoi. *Mesure, Régulation, Automatisme* volume, 43–55

Vergnaud G., Brousseau G., Hulin M. (eds.) (1988): *Didactique et acquisition des connaissances scientifiques* (Actes du colloque de Sèvres). Grenoble: La Pensée Sauvage.

TELEVISION FILMS

Brousseau G. (1977) *Quelques réflexions sur une approche des décimaux.* Ateliers de Pédagogie. Film Ofratème

Brousseau G., Briand J. (1975) *Premières découvertes des lois du hasard a l'école élémentaire.* Ateliers de Pédagogie. Film Ofratème

Brousseau G., Briand J. (1975a) *Quelques réflexions sur une approche des décimaux et documents d'accompagnement.* Ateliers de Pédagogie. Film Ofratème.

REFERENCES

Editors' note: *We provide here all the references used by Brousseau in the articles we have selectedfor this book, even when he omitted the related "pointers" in his texts. In several places we indicate the English translations of the texts mentioned. Some references are added either because we introduced them in our notes or, in the case of dissertations or of "grey literature", to offer the reader more accessible publications; these complements are easy to identify.*

Abdeljaouad M. (1978) Vers une épistémologie des décimaux. *Bulletin de l'Association Tunisienne des Sciences Mathématiques* 50, 1–27.

Aebli H. (1959) *Didactique psychologique.* Gcnève: Delachaux et Niestlé.

Alarcos Llorach L. (1968) Acquisition du langage par l'enfant. In: Martinet A. (ed.) *Le langage* (Bibliothèque de la Pléiade). Paris: Editions Gallimard.

Althusser L. (1 967) *Philosophie et philosophie spontanée de savants.* Paris: Librairie François Maspcro.

Arbib M. A. (1976) Memory limitations of stimulus-response models. *Journal of Structural Learning* 5 (1/2) 19–23.

Arbib M. A. (1976a) A reply to Suppes reply. *Journal of Structural Learning* 5 (1/2) 29

Arsac G. (1992) L'évolution d'une théorie en didactique: l'excmple de la transposition didactique. *Recherches en didactique des mathématiques* 12(1) 7–32.

Artigue M. (1984) *Une contribution à l'étude de la reproductibilité des situations didactiques.* Thèse d'état. Paris: Université de Paris VII.

Artigue M. (1986) Etude de la dynamique d'unc situation de classe: une approche de la reproductibilité. *Recherches en didactique des mathématiques* 7(1) 5–62.

Artigue M. (1992) Didactic Engineering. In: Douady R., Mercier A. (eds.) *Recherches en didactique des rnathématiques, selected papers* (pp. 41–65). Grenoble: La Pensée Sauvage.

Artigue M., Gras R., Laborde C., Tavignot P. (eds.) (1994) *Vingt ans de didactique des mathématiques en France. Hommage à Guy Brousseau et Gérard Vergnaud.* Grenoble: La Pensée Sauvage.

Badiou A. (1972) *Le concept de modèle.* Paris: Librairie François Maspero.

Bachclard G. (1938) *La formation de l'esprit scientifique.* Paris: Vrin.

Bachelard G. (1 968) *Essai sur la connaissance approchée* (third edition). Paris: Vrin.

Balacheff N. (1980) *Etude de l'élaboration d'explications par des élèves de 6ème à propos d'un problème de combinatoire* (Rapport de Recherche IMAG n° 224). Grenoble: IMAG.

Balacheff N. (1982) Preuve et démonstration en mathématique. *Recherches en didactique des mathématiques* 3(3) 261–304.

Balacheff N. (1987) Processus de preuve et situations de validation. *Educational Studies in Mathematics* 18(2) 147–176.

Balacheff N. (1991) Treatment of refutations: aspects of the complexity of a constructivist approach of mathematics learning. In: Von Glasersfeld E. (ed.) *Radical Constructivism in Mathematics Education* (pp. 89–110). Dordrecht: Kluwer Academic Publishers.

Balacheff N., Laborde J. M. (1985) Introduction à l'édition française. In: Lakatos I. *Preuves et réfutations* (pp. xii–xx). Paris: Editions Hermann.

Baruk S. (1985) *L'âge du capitaine, de l'erreur en mathématiques.* Paris: Editions du Seuil

Berger and Luckmann (1966) *The social construction of reality; a treatise in the sociology of knowledge.* Garden City, N. Y.: Doubleday.

Berthelot C., Berthelot R. (1 983) *Quelques apports de la théorie des situations à l'introduction de la notion de limite en classe de première.* Mémoire de DEA de Didactique des Mathématiques. Bordeaux: Université de Bordeaux 1.

Bessot A,, Deprez S., Eberhard M., Gomas B. (1993) Une approche didactique de la lecture de graphismes techniques en formation professionnelle de base aux métiers du bâtiment. In: Bessot A., Vérillon P. (eds.) *Espaces graphiques et graphismes d'espaces* (pp. 115–144). Grenoble: La Pensée Sauvage.

Bessot A., Richard F. (1979) *Commande de variable dans une situation didactique pour provoquer I'élargissement des procédures, rôle du schéma.* Thèse. Bordeaux: Université de Bordeaux 1.

Bessot A., Richard F. (1979) Une étude sur le fonctionnement du schéma arbre par la commande de variables d'une situation. *Recherches en didactique des mathématiques* 1(3) 387–422.

Bloom S. (1 975) *Taxonomie des objectifs pédagogiques* (two volumes). Québec: Presses Universitaires du Québec.

Bourdieu P. (1980) *Le sens pratique.* Paris: Editions de Minuit.

Bourdieu P. (1982) *Leçon sur la leçon.* Paris: Editions de Minuit.

Bramaud du Bourcheron G., Champagnol R., Coirier P., Ehrlich S., Ehrlich M.-F. (1970) *Le comportement verbal.* Paris: Dunod.

Briand J. (1985) *Situation didactique et logiciel d'enseignement.* Mémoire de DEA de Didactique des Mathématiques. Bordeaux: Université de Bordeaux I.

Briand J. (1993) *L'énumération dans le mesurage des collections, un dysfonctionnement de la transposition didactique.* Thèse. Bordeaux: Université de Bordeaux I.

Bruner J. S., Goodnow J. J., Austin G. A. (1956) *A study of thinking* (with an appendix on language by Roger W. Brown). New York, Wiley [1956].

Buisson P. (1 976) Evaluation et sondage dans le premier cycle (12–16 ans). *Educational Studies in Mathematics* 6(4) 429–444

Byers V., Herscovics N. (1977) Understanding school mathematics. *Mathematics Teaching* 81, 24–27.

Chevallard Y. (1979) Sur les difficultés "protomathématiques". In: Proceedings of the Conference *"Apport de l' histoire des mathématiques à l'enseignement et à la formation des enseignants"* (Puyricard, 18–19 mai 1979). Marseille: IREM d'Aix-Marseille.

Chevallard Y. (1983) *Remarques sur la notion de contrat didactique.* Marseille: IREM de Marseille.

Chevallard Y. (1 985) *La transposition didactique.* Grenoble: La Pensée Sauvage.

Chevallard Y. (1986) Esquisse d'une théorie formelle du didactique. In: Laborde C. (ed.) *Actes du premier colloque franco-allemand de didactique des mathématiques et de I'informatique* (pp. 97–1 06). Grenoble: La Pensée Sauvage.

Chevallard Y. (1988) *Sur l'analyse didactique. Deux études sur les notions de contrat et de situation* (publication n° 18). Marseille: IREM de Marseille.

Chevallard Y. (1992) Fundamental concepts in didactics: perspectives provided by an anthropological approach. In: Douady R., Mercier A. (eds.) *Research in* Didactique *of Mathematics, Selected Papers* (pp. 131–168). Grenoble: La Pensée Sauvage.

Chevallard Y., Mercier A. (1983) *Sur le temps didactique* (occasional paper). Marseille: IREM d'Aix-Marseille.

Chevallard Y., Mercier A. (1984) La notion de situation didactique. In: *Actes de la troisième Ecole d'été de didactique des mathématiques.* IMAG: Grenoble.

Chomsky N., Miller, G. A. (1968) *L'analyse formelle des langues naturelles.* Paris: Gauthier-Villars. [French translation of Duncan L., Bush R., Galanber E. (eds.) Handbook of Mathematical Psychology, vol. II, Chapters 11 and 12). New York: Wiley (1963–65)].

Choquet G. (1964) *L'enseignement de la géométrie.* Paris: Editions Hermann.

Ciosek M. (1976) On strategies for solving mathematical problems. In: *Comptes-rendus de la XXVIIIe rencontre organisée par la Commission Internationale pour l'Etude et l'Ame'lioration de l'Enseignement des Mathématiques* (pp. 51–56). Louvain-la-Neuve.

Colmez F. (1974) *Documents d'accompagnement du film "Mesure".* Mesure I et II, film RTS. Melun-Bordeaux.

Coquin D. (1 982) *Décomposition d'une notion mathématique en vue de son enseignement et ordre d'acquisition.* Thèse. Bordeaux: Université de Bordeaux I.

Coquin D. (1 985) Complexité mathématique et ordre d'acquisition: une hiérarchie d'acquisitions à propos des relatifs. *Recherches en didactique des mathématiques* 9(3) 133–192.

Cornu B. (1983) *Apprentissage de la notion de limite: conceptions et obstacles.* Thèse. Grenoble: Université Joseph Fourier.

Cronbach J. (1967) *The dependability of Behavioral Measurements.* New York: John Wiley and Sons.

Crozier M., Friedberg E. (1977) *L'acteur et le système.* Paris: Editions du Seuil.

Delattre P. (1 971) *Système, structure, fonction, évolution.* Maloine-Doin Editeurs.

Desjardins M., Hetu J. C. (1974) *Activités mathématiques dans l'enseignement des fractions.* Montréal: Presse de l'Université du Québec.

Devos R. (1976) *Sans dessus dessous* (Le Livre de Poche, n°5102). Paris: Editions Stock.

Diderot D. (1773) *Paradoxe sur le comédien.* (1830). Paris: Editions Flammarion (1981).

Diénès Z. (1970) *Les six étapes du processus d'apprentissage en mathématiques.* Paris: O.C.D.L.

Dieudonné J. (1 964) *Algèbre linéaire et géométrie élémentaire.* Paris: Hermann.

Douady R. (1980) Approche des nombres réels en situation d'apprentissage scolaire. *Recherches en Didactique des Mathématiques.* 1(1) 77–111.

Douady R. (1984) *Rapport enseignement apprentissage: dialectique outil-objet, jeux de cadres.* Thèse d'dtat. Paris: Université de Paris VII.

Douady R (1985) The interplay between different settings. Tools-object dialectic in the extension of mathematical ability. In: Streefland L. (ed.) *Proceedings of the Ninth International Conference for the Psychology of Mathematics Education.* (pp. 33–52). Utrecht: State University of Utrecht.

Douady R. (1986) Jeux de cadres et dialectique outil-objet. *Recherches en Didactique des* Mathématiques 7(2) 5–31.

Duroux A. (1982) *La valeur absolue: difficultés majeures pour une notion mineure.* Mémoire de DEA de Didactique des Mathématiques. Bordeaux: Université de Bordeaux.

Eco U. (1 972) *La structure absente. Paris:* Mercure de France.

El Bouazzaoui H. (1982) *Etude de situations scolaires des enseignements du nombre et de la numération.* Thèse. Bordeaux: Université de Bordeaux 1.

El Bouazzaoui H. (1988) *Conceptions des élèves et des professeurs à propos de la notion de continuité d'une fonction.* Thèse. Québec: Université Laval.

Ermel (1 980) *Apprentissages mathématiques à l'Ecole élémentaire* (Cycle préparatoire, 1 volume; Cycle eIémentaire, 2 volumes). Paris: INRP.

Euler L. (1843) *Lettres à une princesse d'Allemagne.* Paris: Charpentier.

Félix L. (1971) *Dialogues sur la géométrie Dessi-Mati-Logi.* Paris: Librairie Albert Blanchard.

Filloy E., Montes de Oca V., Riestra J. Senderos G. (1979) *Experimentacion en el area de Matematicas del Quinto Grado de educacion primaria.* Mexico. Publicacion de la seccion Matematica Educativa. Mexico.

Foucault M. (1969) *L'archéologie du savoir.* Paris: Gallimard.

Gagné (1980) *Les principes fondamentaux de l'apprentissage.* Montréal: Editions HRW. [French translation of: "The Conditions of Learning", 1970, New York: Holt, Rinehart and Wilson]

Galvez G. (1985) *El aprendizaje de la orientation en el espacio urbano.* Thèse. Mexico.

Glaeser G. (1976) Deux chroniques de processus heuristiques. In: *Comptes-rendus de la XXVIIIe recontre organisée par la Commission Internationale pour l'Etude et l'Amélioration de l'Enseignement des Mathématiques* (pp. 72–80). Louvain-la-Neuve.

Glaeser G. (1981) Epistémologie des nombres relatifs. *Recherches en didactique des mathématiques* 2(3) 303–346.

Glaeser G. (1984) A propos des obstacles épistémologiques. Réponse à Guy Brousseau. *Recherches en didactique des mathématiques.* 5(2) 228–234.

Glaeser G. (1984–1985) *La didactique expérimentale des mathématiques* (Cours de troisième cycle, Chapitre 3). Deuxième rédaction augmentée avec la collaboration de J. B. Romans. Strasbourg: Université Louis Pasteur.

Gobain (1710) *L'arithmétique aisée.*

Gonseth F. (1936) *Les mathématiques et la réalité. Essai sur la méthode axiomatique.* Paris: Librairie A. Blanchard.

Gras R. (1979) *Contribution à l'étude expérimentale et à l'analyse de certaines acquisitions cognitives et de certains objectifs didactiques en mathématiques.* Thèse d'état. Rennes: Université de Rennes 1.

Gras R. (1 980) *Deux méthodes d'analyse de données didactiques: classification implicative et classification hiérarchique. Application a une situation réelle.* Rennes: IREM de Rennes.

Gras R. (1996) *L'implication statistique.* Grenoble: La Pensée Sauvage.

Guittel G. (1975) *Histoire comparée des numérations écrites.* Pans: Flammarion.

Hart K. (1980) From whole numbers to fractions and decimals. *Recherches en didactique des mathématiques* 1(1) 61–75.

Herscovics N. (1980) Constructing meaning for linear equations. A problem of representation. *Recherches en didactique des mathématiques* 1(3) 351–385.

Houzel C., Ovaert J.-L., Raymond P., Sansuc J.-J. (1976) *Philosophie et calcul de l'infini.* Paris: Librairie François Maspero.

Hull L. (1943) *Principles of behaviors.* New York: Appleton Century Crofts.

IREM de Strasbourg (1973) *Le libre du problème.* Paris: Cedic.

Izorche M.L. (1977) *Les réels en classe de seconde.* Mémoire de DEA de Didactique des Mathématiques. Bordeaux: Université de Bordeaux 1.

Janvier C. (1976) L' heuristiques dans l'enseignement des mathématiques. In: *Comptes-rendus de la XXVIIIe rencontre organisée par la Commission Internationale pour l'Etude et l'Ame'lioration de I'Enseignement des Mathématiques* (pp. 48–50). Louvain-la-Neuve.

Katambera I. (1986) *Sur la résolution desproblèmes de soustraction au cours élémentaire.* Thèse. Bordeaux: Université de Bordeaux 1.

Kilpatrick J., Wirszup I. (eds.) (1977) *Soviet studies in the psychology of learning and teaching mathematics* (14 volumes). Reston, USA: N.C.T.M.

Laborde C. (1982) *Langue naturelle et écriture symbolique, deux codes en interaction dans l'enseignement mathématique.* Thèse. Grenoble: Université de Grenoble.

Lakatos I. (1976) *Proofs and Refutations. Cambridge:* Cambridge University Press.

Lalande A. (1972) *Vocabulaire technique et critique de la philosophie.* Paris: PUF.

Léonard F., Sackur-Grivard C. (1981) Sur deux règles implicites utilisées dans la comparaisonde nombres décimaux positifs. *Bulletin de l'APMEP* 327, 47–60.

Léonard F., Sackur-Grivard C. (1991) Connaissances locales et triple approche, une méthodologie de recherche. *Recherches en Didactique des Mathématiques* 10(2/3) 205–240.

Lichnérowicz A. (1969) La mathématique et la réalité. In: *Logique et connaissance scientifique* (pp. 474–485). Bibliothèque de la PIéiade". Paris: Gallimard.

Margolinas C. (1993) *De l'importance du vrai et du faux dans la classe de mathématiques.* Grenoble: La Pensée Sauvage.

Margolinas C. (1994) Double analyse d'un épisode: cercle épistémologique et structuration du milieu. In: Artigue M., Gras R., Laborde C., Tavignot P. (eds.) *Vingt ans de didactique des mathématiques en France* (pp. 250–257). Grenoble: La Pensée Sauvage.

Maudet C. (1979) *Etude critique du processus psycho-dynamique selon Diénès.* Mémoire de DEA de Didactique des Mathématiques. Bordeaux: Université de Bordeaux 1.

Maudet C. (1982) *Les situations et les processus* de *l'apprentissage d'une fonction logique.* Thèse. Bordeaux: Université de Bordeaux 1.

Maysonnave J. (1972) *Expérience d'apprentissage d'une stratégie pour la course à 7 par une série de parties contre un moniteur.* Cahier 11. Bordeaux: IREM de Bordeaux.

Mercier A. (1978) *Etude des notions* "*opérateur*", "*machine*". Mémoire de DEA de Didactique des Mathématiques. Bordeaux: Université de Bordeaux 1.

Mercier A. (1984) *Eléments pérennes du contrat didactique, ruptures locales et ruptures globales.* Paris: IREM de Paris VII.

Mercier A. (1992) *L'élève et les contraintes temporelles de I'enseignement, un cas en calcul algébrique.* Thèse. Bordeaux: Université de Bordeaux 1.

Mercier A. (1995) La biographie didactique d'un élève et les contraintes temporelles de l'enseignement. *Recherches en diductique des mathématiques* 15(1) 97–142.

Moles A. A. (1967) *Sociodynamique de la culture.* Paris: Mouton.

Mopondi B. (1986) *Le sens dans la négociation didactique. Notion de proportionnalité au cours moyen.* Thèse. Bordeaux: Université de Bordeaux 1.

Nelson R. J. (1969) Behaviorism is false. *The Journal of Philosophy* Vol. LXVI (14) 417–452.

Osgood Ch., Suci G. J., Tannenbaum P. H. (1957) *The measurement meaning.* University of Illinois Press.

Ovaert J.-L. (1976) La thèse de Lagrange et la transformation de l'analyse. In: Houzel C., Ovaert J.-L., Raymond P., Sansuc J.-J. (1976) *Philosophie et calcul de l'infini* (pp. 157–198). Paris: Librairie François Maspero.

Pagnol M. (1928) *Topaze.* [English translation: Waldinger R. (1958) Woodbury, N.Y.: Barron's Educational Series, Inc.]

Papy F. (1 970) *L'enfant et les graphes.* Paris: Didier.

Papy G. (1964) *Mathématiques modernes* (vol. 1). Paris: Didier..

Paquette G. (1 976) L'apprentissage heuristique. In: *Comptes-rendus de la XXVIIIe recontre organisée par la Commission Internationale pour l'Etude et l'Amélioration de l'Enseignement des Mathématiques* (pp. 57–62). Louvain-la-Neuve.

Pascal. D (1 980) *Le probléme du zéro: l'économie d'échec dans la classe et la production de l'erreur,* Mémoire de DEA de Didactique des Mathématiques. Marseille : IREM d'Aix-Marseille et de Bordeaux.

Perret-Clermont A. N. (1 979) *La construction de l'intelligence dans l'interaction sociale.* Berne: Peter Lang.

Perrin-Glorian M.-J. (1994) Théorie des situations didactiques: naissance, développement, perspectives. In: Artigue M., Gras R., Laborde C., Tavignot P. (eds.) *Vingt ans de didactique des mathématiques en France* (pp. 96–147). Grenoble: La Pensée Sauvage.

Perez J. (1984) *Construction d'un code de désignation d'objets.* Thèse. Bordeaux: Université de Bordeaux II.

Piaget J. (1 974) *Introduction à l'épistémologie génétique* (2 volumes). Paris: PUF.

Piaget J. (1 975) *L'équilibration des structures cognitives.* Paris: PUF.

Pluvinage F. (1977) *Difficultés des exercices scolaires en mathématiques.* Thèse d'état. Strasbourg: Université de Strasbourg.

Pluvinage F., Dupuis C. (1980) *La proportionalité et son utilisation.* Strasbourg: IRMA.

Polya G. (1957) *How to Solve It.* (Second Edition). Princeton: Princeton University Press.

Quevedo De Villegas B. (1986) *Le rôle de l'énumération dans l'apprentissage du dénombrement.* Thèse. Bordeaux: Université de Bordeaux 1.

Ratsimba-Rajohn H. (1981) *Etudes de deux méthodes de mesures rationnelles: la commensuration et le fractionnement de l'unité en vue de l'élaboration des situations didactiques.* Thèse. Bordeaux: Université de Bordeaux 1.

Ratsimba-Rajohn H. (1982) Eléments d'étude de deux méthodes de mesures rationnelles. *Recherches en didactique des mathématiques* 3(1) 65–113.

Regnier (1976) *La crise du langage scientifique.* Paris: Antropos.

Ricco G. (1978) *Le développement de la notion de fonction linéaire chez l'enfant de 7 à 12 ans.* Travaux du Centre d'Etude des Processus et du Langage, nº11. Paris: Ecole des Hautes Etudes en Sciences Sociales.

Rogalski J. (1 982) Acquisition de notions relatives à la dimensionalité des mesures spatiales (longueur, surface). *Recherches en didactique des mathématiques* 9(3) 343–396.

Rogalski J., Samurçay R. (1994) Modélisation d'un "savoir de référence" et transposition didactique dans la formation de professionnels de haut niveau. In: Arsac G., Chevallard Y., Martinand J.-L., Tiberghien A. (eds.) *La transposition didactique à l'épreuve* (pp. 35–71). Grenoble: La Pensée Sauvage.

Rouchier A. (1980) Situations et processus didactiques dans l'étude des nombres rationnels positifs. *Recherches en Didactique des Mathématiques* 1(2) 225–275.

Rouchier A. (1991) *Etude de la conceptualisation dans le système didactique en mathématiques et informatique élémentaires* : *proportionnalité, structures itéro-récursives, institutionnalisation.* Thèse d'état. Orléans: Université d' Orléans.

Sackur-Grisvard C., Léonard F. (1983) Résurgence de règles implicites dans la comparaison de nombres décimaux. *Bulletin de l'APMEP* 340, 450–459.

Saidan A. S. (1978) *The arithmetic of al-Uqlidisi.* Dordrecht: Reidel Publishing Company.

Salin M. H. (1976) *Le rôle de l'erreur dans l'apprentissage des mathématiques à l'école primaire.* Mémoire de DEA de Didactique des Mathématiques. Bordeaux: Université de Bordeaux.

Schneider O. (1979) *Le passage des équations numériques aux équations paramétriques en classe de seconde.* Mémoire de DEA de Didactique des Mathématiques. Marseille: IREM d'Aix-Marseille et de Bordeaux.

Schubauer-Leoni M. L., Perret-Clermont A.-N. (1980) Interactions sociales et représentations symboliques dans le cadre de problèmes additifs. *Recherches en didactique des mathématiques* 1(3) 297–350.

Seuret (1834) *Traité d'arithmétique.*

Sheffer H. M. (1913) A set of five independent postulates for Boolean algebras. *Transactions of the American Mathematical Society* 14, 481–488.

Sierpinska A. (1 985) Obstacles épistémologiques relatifs à la notion de limite. *Recherches en didactique des mathématiques* 6(1) 5–67.

Sierpinska A. (1987) Humanities students and epistemological obstacles related to limits. *Educational Studies in Mathematics* 18, 371–397.

Skemp R. (1979) *Intelligence, learning and action.* New York: John Wiley and Sons.

Skemp R. (1 976) Relational and instrumental understanding. *Mathematics Teaching* 77, 20–26.

Stevin S. (1585) *La disme* (1634, lère édition).

Suppes P. (1969) Stimulus-response theory of finite automatas. *Journal of Mathematical Psychology* 6, 327–355.

Suppes P. (1 976) Stimulus-response theory of automatas and tote hierarchies: a reply to Arbib. *Journal of Structural Learning* 5(1/2) 11–17.

Schwartzman S. (1994) *The words of Mathematics. The Mathematical Association of America.*

Taton (1966) *Histoire génétrale des sciences.* (see Tome 1: Vercoutter et Itard, la science antique et médiéval; Tome 2: Koyre, la science moderne). Paris: PUF.

Thom R. (1972) *Stabilité culturelle et morphogenèse.* Reading Massachusets.

Tonnelle J. (1979) *Le monde clos de la factorisation au premier cycle.* Mémoire de DEA de Didactiquc des Mathématiques. Marseille: IREM d'Aix-Marseille et de Bordeaux.

Vergnaud G. (1990) La théorie des champs conceptuels. *Recherches en Didactique des Mathématiques* 10(2/3) 133–170.

Vergnaud G., Benhadj J., Dussourt A. (1979) La coordination de l'enseignement des mathématiques entre le cours moyen 2ième année et la classe de 6ième. *Recherches Pédagogiyues* 102.

Vergnaud G., Durand C. (1976) Structures et complexité psychogénétique. *Revue françise de pédagogie* 36, 28–43.

Vergnaud G.. Rouchier A., Ricco G., Marthe P., Metregiste R., Giaccobe J. (1979) *Acquisition des "structures multiplicatives" dans le premier cycle du second degré.* Orléans: IREM d'Orléans.

Verret M. (1975) *Le temps des études* (2 volumes) Paris: Librairie Honoré Champion.

Viennot L. (1 979) *Le raisonnement spontané en dynamique élémentaires.* Paris: Hermann.

Vygotsky L. S. (1983) *Pensée et langage.* Paris: Editions Sociales.

Waliser B. (1977) *Systèmes et modèles.* Paris: Editions du Seuil.

Watzlawick P., Helmick Beavin J., Jackson D. (1978) *Une logique de la communication.* Paris: Editions du Seuil.

Wermuz H. (1978) Esquisse d'un modèle des activités cognitives. *Dialectica* 32(3/4).

Whitney H. (1968) The mathematics of physical quantities. Part 1: Mathematical models for measurement. *The American Mathematical Monthly* 75, 115–138.

Whitney H. (1968a) The mathematics of physical quantities. Part 2: Quantities, structures and dimensional analysis. *The American Mathematical Monthly* 75, 227–256.

Wilson F. (1976) Problem solving project. In: *Comptes-rertdus de la XXVIIIe rencontre organisée par la Commission lnternationale pour l'Etude et l'Amélioration de l'Enseignement des Mathématiques* (pp. 62–72). Louvain-la-Neuve.

Youschkevitch A. P. (1976) *Les mathématiques arabes.* Paris: Vrin.

INDEX OF NAMES

INDEX OF SUBJECTS

Mathematics Education Library

Managing Editor: A.J. Bishop, Melbourne, Australia

1. H. Freudenthal: *Didactical Phenomenology of Mathematical Structures.* 1983 ISBN 90-277-1535-1; Pb 90-277-2261-7
2. B. Christiansen, A. G. Howson and M. Otte (eds.): *Perspectives on Mathematics Education.* Papers submitted by Members of the Bacomet Group. 1986. ISBN 90-277-1929-2; Pb 90-277-2118-1
3. A. Treffers: *Three Dimensions.* A Model of Goal and Theory Description in Mathematics Instruction – The Wiskobas Project. 1987 ISBN 90-277-2165-3
4. *S.* Mellin-Olsen: *The Politics of Mathematics Education.* 1987 ISBN 90-277-2350-8
5. E. Fischbein: *Intuition in Science and Mathematics.* An Educational Approach. 1987 ISBN 90-277-2506-3
6. A.J. Bishop: *Mathematical Enculturation.* A Cultural Perspective on Mathematics Education. 1988 ISBN 90-277-2646-9; Pb (1991) 0-7923-1270-8
7. E. von Glasersfeld (ed.): *Radical Constructivism in Mathematics Education.* 1991 ISBN 0-7923-1257-0
8. L. Streefland *Fractions in Realistic Mathematics Education.* A Paradigm of Developmental Research. 1991 ISBN 0-7923-1282-1
9. H. Freudenthal: *Revisiting Mathematics Education.* China Lectures. 1991 ISBN 0-7923-1299-6
10. A.J. Bishop, S. Mellin-Olsen and J. van Dormolen (eds.): *Mathematical Knowledge: Its Growth Through Teaching.* 1991 ISBN 0-7923-1344-5
11. D. Tall (ed.): *Advanced Mathematical Thinking.* 1991 ISBN 0-7923-1456-5
12. R. Kapadia and M. Borovcnik (eds.): *Chance Encounters: Probability in Education.* 1991 ISBN 0-7923-1474-3
13. R. Biehler, R.W. Scholz, R. Sträßer and B. Winkelmann (eds.): *Didactics of Mathematics as a Scientific Discipline.* 1994 ISBN 0-7923-2613-X
14. S. Lerman (ed.): *Cultural Perspectives on the Mathematics Classroom.* 1994 ISBN 0-7923-2931-7
15. O. Skovsmose: *Towards a Philosophy of Critical Mathematics Education.* 1994 ISBN 0-7923-2932-5
16. H. Mansfield, N.A. Pateman and N. Bednarz (eds.): *Mathematics for Tomorrow's Young Children.* International Perspectives on Curriculum. 1996 ISBN 0-7923-3998-3
17. R. Noss and C. Hoyles: *Windows on Mathematical Meanings.* Learning Cultures and Computers. 1996 ISBN 0-7923-4073-6; Pb 0-7923-4074-4

Mathematics Education Library

Managing Editor: A.J. Bishop, Melbourne, Australia

18. N. Bednarz, C. Kieran and L. Lee (eds.): *Approaches to Algebra.* Perspectives for Research and Teaching. 1996
ISBN 0-7923-4145-7; Pb 0-7923-4168-6
19. G. Brousseau: *Theory of Didactical Situations in Mathematics.* Didactique des Mathématiques 1970–1990. Edited and translated by N. Balacheff, M. Cooper, R. Sutherland and V. Warfield. 1997 ISBN 0-7923-4526-6
20. T. Brown: *Mathematics Education and Language.* Interpreting Hermeneutics and Post-Structuralism. 1997 ISBN 0-7923-4554-1

KLUWER ACADEMIC PUBLISHERS – DORDRECHT / BOSTON / LONDON