parameter. This can be done if $M(j\omega)$ is a rational function of ω since the operation of taking the determinant of $(I - M\Delta)$ will preserve this property. This removes the frequency scanning of the test at the price that the order of the system that must be analyzed is increased by one real perturbation.

Appendix: Some Definitions

Sylverster's Resultant matrix S is the $(m+n) \times (m+n)$ matrix associated with the two polynomials $A(x) = \sum_{i=0}^{n} a_i x^i$ and $B(x) = \sum_{i=0}^{m} b_i x^i$

$$S = \begin{bmatrix} a_0 & a_1 & \cdots & a_n & & 0 & \cdots & 0 \\ 0 & a_0 & a_1 & \cdots & a_n & 0 & \cdots & 0 \\ \vdots & & & & & & & \\ 0 & \cdots & & 0 & a_0 & a_1 & \cdots & a_n \\ b_0 & b_1 & \cdots & b_m & & 0 & \cdots & 0 \\ 0 & b_0 & b_1 & \cdots & b_m & 0 & \cdots & 0 \\ \vdots & & & & & & & \\ 0 & \cdots & & 0 & b_0 & b_1 & \cdots & b_m \end{bmatrix} \begin{matrix} \left.\right\} m \text{ rows} \\ \left.\right\} n \text{ rows} \end{matrix}.$$

The resultant of two polynomials is defined as the determinant of the Sylvester's Resultant matrix associated with these polynomials. The resultant depends only on polynomials' coefficients and can be used to test if polynomials have a common factor without determining their roots explicitly. If at least one of the leading coefficients of two polynomials is nonzero, the resultant of these polynomials is zero if and only if they have a common factor.

A number of real roots of a real polynomial on a given interval can be found by Sturm's theorem. The Sturm sequence for polynomials $f(x)$ and $f'(x)$ begins with these two polynomials, and each new term is the remainder of a division of the two previous terms (Euclid's algorithm), but with the signs of the remainders reversed

$$f_0(x) = f(x), f_1(x) = f_0'(x)$$
$$f_i(x) = f_{i-1}(x)g_{i-1}(x) - f_{i-2}(x) \qquad \text{for } i = 2, \cdots, k$$

where $g_{i-1}(x)$ is the quotient obtained by dividing $f_{i-2}(x)$ by $f_{i-1}(x)$ and $-f_i(x)$ is the remainder, with $f_k(x)$ being the last nonzero remainder in the sequence (the greatest common divisor of $f(x)$ and $f'(x)$).

Sturm's theorem states that N, the number of distinct real roots of $f(x)$ between α and β can be computed as

$$N = V(\alpha) - V(\beta)$$

where $V(\theta)$ denotes the number of variations in sign in the Sturm sequence for a fixed value $x = \theta$. Instead of a computation of remainders by polynomial division, the standard Routh algorithm to polynomials $f(x)$ and $f'(x)$ can be applied (proofs and details appear in standard texts, e.g., Barnett [2, Ch. 3]).

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Chattering Avoidance by Second-Order Sliding Mode Control

G. Bartolini, A. Ferrara, and E. Usai

***Abstract*— Relying on the possibility of generating a second-order sliding motion by using, as control, the first derivative of the control signal instead of the actual control, a new solution to the problem of chattering elimination in variable structure control (VSC) is presented. Such a solution, inspired by the classical bang–bang optimal control strategy, is first depicted and expressed in terms of a control algorithm by introducing a suitable auxiliary problem involving a second-order uncertain system with unavailable velocity. Then, the applicability of the algorithm is extended, via suitable modifications, to the case of nonlinear systems with uncertainties of more general types. The proposed algorithm does not require the use of observers and differential inequalities and can be applied in practice by exploiting such commercial components as peak detectors or other approximated methods to evaluate the change of the sign of the derivative of the quantity accounting for the distance to the sliding manifold.**

I. Introduction

When considering the chattering phenomenon, that is, the high-frequency finite amplitude control signal generated by the sliding mode method, some authors (see, for instance, [1] and [2]) appear to relate this behavior to the discontinuity of the sign function on the sliding manifold. In other terms, they simply propose to replace this function with a smooth approximation in order to counteract the chattering effect at the price of a small deterioration in performances.

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G. Bartolini and E. Usai are with the Department of Electrical and Electronic Engineering, University of Cagliari, 09123 Cagliari, Italy.

A. Ferrara is with the Department of Communication, Computer and System Sciences, University of Genova, Genova, Italy (e-mail: ferrara@dist.unige.it).

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Their assumption is motivated by the fact that some important systems, like complex mechanical systems with backlash in the gear boxes, do not tolerate an abrupt commutation in the applied forces and torques.

Nevertheless, the use of smoothing devices, which are characterized by a high gain to have a small approximation error, does not guarantee that oscillations will disappear. Indeed, the approximate sliding motion so originated is guaranteed to lay in a small vicinity of the sliding manifold, but nothing can be told about the behavior inside this vicinity. The variable which accounts for the distance from the sliding manifold can oscillate at an unpredictable frequency, and this fact, through the high gain of the approximate switching devices, reflects on the control signal which could be characterized by a finite amplitude and an unpredictable frequency behavior. Since the above-mentioned mechanical systems do not tolerate changes in the sign of the control law at any frequency higher than some critical value, the promised counteraction of the chattering phenomenon could not be ensured. Further, it must be taken into account that often the rigid body assumption, which makes one neglect the presence of distributed or concentrated elasticity, is not motivated if the control input frequencies belong to a range such that neglected resonant modes could be excited. Indeed, this assumption turns out to be motivated only for frequencies sufficiently far from such a range, both lower and greater than the extreme critical frequencies, and it is not guaranteed that the approximated sliding motion resulting from the use of smoothing devices presents this feature.

Another approach, which is effective in the presence of unmodeled dynamics, is based on the introduction of observers for the modeled part of the system with a sliding manifold defined in terms of the observer states [5]. The almost ideal high-frequency observer control signal is filtered by the high-gain fast dynamical part of the system so that a smooth control is actually applied.

Recently, this problem has been addressed within a general framework with reference to known nonlinear systems [10]. A procedure inspired by this approach, but effective also for uncertain systems is that described in [8] and [9]. This procedure solves the following problem. Given the system

$$\begin{cases} \dot{x}_i(t) = x_{i+1}(t) & i = 1, \cdots, n-1 \\ \dot{x}_n(t) = f[\mathbf{x}(t)] + g[\mathbf{x}(t)]u(t) \end{cases} \tag{1}$$

with $\mathbf{x}(t) = [x_1, x_2, \cdots, x_n]^T$ representing the completely available state and $f[\mathbf{x}(t)]$ and $g[\mathbf{x}(t)]$ uncertain smooth functions satisfying the classical condition for the existence and uniqueness of the solution as well as the following inequalities:

$$0 < G_1 \leq g[\mathbf{x}(t)] \leq G_2 \tag{2}$$

$$|f[\mathbf{x}(t)]| \leq P_f + Q_f\|\mathbf{x}(t)\| \tag{3}$$

$$\left\|\frac{\partial f[\mathbf{x}(t)]}{\partial \mathbf{x}}\right\| \leq P_{df} + Q_{df}\|\mathbf{x}(t)\| \tag{4}$$

$$\left\|\frac{\partial g[\mathbf{x}(t)]}{\partial \mathbf{x}}\right\| \leq P_{dg} + Q_{dg}\|\mathbf{x}(t)\| \tag{5}$$

with G_1, G_2, P_f, Q_f, P_{df}, Q_{df}, P_{dg}, and Q_{dg} real positive known constants, find a continuous control $u(t)$ such that in spite of the uncertainties (2)–(5) the states of (1) are steered to zero exponentially. Note that conditions (4) and (5) mean that, in each time instant, $\|\partial f[\mathbf{x}(t)]/\partial \mathbf{x}\|$ and $\|\partial g[\mathbf{x}(t)]/\partial \mathbf{x}\|$ are bounded by known (measurable) quantities.

To determine the desired continuous control, the following steps need to be taken [9].

1) Differentiate the second equation of (1) and set

$$x_{n+1}(t) = f[\mathbf{x}(t)] + g[\mathbf{x}(t)]u(t)$$

and consider the system

$$\begin{cases} \dot{x}_i(t) = x_{i+1}(t) & (i = 1, 2, \cdots, n-1). \\ \dot{x}_n(t) = x_{n+1}(t) \\ \dot{x}_{n+1} = \dfrac{d}{dt} f[\mathbf{x}(t)] + \dfrac{d}{dt} g[\mathbf{x}(t)]u(t) \\ \qquad + g[\mathbf{x}(t)] \dfrac{d}{dt} u(t). \end{cases} \tag{6}$$

2) Choose a sliding manifold in the augmented state space, i.e.,

$$s[\mathbf{x}(t)] = x_{n+1}(t) + \sum_{i=1}^{n} c_i x_i(t) = 0. \tag{7}$$

On this manifold, (6) behaves like a linear autonomous system with eigenvalues coinciding with the roots of $P(z) = z^n + c_n z^{n-1} + \cdots + c_1$.

3) Since x_{n+1} is not measurable, standard variable structure control (VSC) cannot be applied. Then, as suggested in [9], introduce an observer of such a quantity, i.e.,

$$\dot{z}(t) = -p(t)\hat{s}(t) - \sum_{i=1}^{n} c_i x_i(t) + w(t) \tag{8}$$

where

$$\begin{aligned} \hat{s}(t) &= z(t) + \sum_{i=1}^{n} c_i x_i(t) \\ s(t) &= x_{n+1}(t) + \sum_{i=1}^{n} c_i x_i(t) \\ E(t) &= s(t) - \hat{s}(t) \end{aligned} \tag{9}$$

and $p(t)$, the parameter of the observer, is any piecewise continuous function and constitutes a design degree of freedom. The signal $w(t)$ is chosen in such a way that the dynamics of the whole system is characterized by the following second-order equation:

$$\begin{cases} \dot{\hat{s}}(t) = -p(t)\hat{s}(t) + c_n E(t) \\ \dot{E}(t) = -p(t)\hat{s}(t) + c_n E(t) + \mu(x, u) \end{cases} \tag{10}$$

where $\mu(t)$ is a nonlinear signal depending on the uncertain dynamics and affinely on the derivative of the control signal $u(t)$.

In [8] it has been proved that if the derivative of the control signal $u(t)$ is chosen so that $\mu(t)$ is discontinuous on $\hat{s}(t) = 0$ and, at the same time, inside the region of continuity of $\mu(t)$ the following differential inequality holds $\dot{\mu}(t) < [p(t) + c_n]\mu(t)$, then, the states $\hat{s}(t)$ and $E(t)$ tend simultaneously (asymptotically) to zero; as a consequence, $\mathbf{x}(t)$ tends to zero asymptotically with a control $u(t)$, which turns out to be a continuous function, being the integral of a bounded signal.

The purpose of this paper is to provide a solution to the problem of steering to zero the states of (1) by forcing them to lie on a suitable sliding manifold in the original (nonaugmented) state space by using a continuous control (thereby, avoiding the chattering effect as in [8], [9]) but also presenting the following advantages over the previous proposals.

1) It does not require the introduction of any observer.
2) The differential inequalities are replaced by algebraic inequalities.
3) The convergence to the sliding manifold of the state trajectories takes place in a finite time.

The proposed methodology relies on the possibility of generating a second-order sliding motion. The latter naturally arises when the control $v(t) = \dot{u}(t)$ is used instead of the actual control $u(t)$. Indeed, if we choose $s[\mathbf{x}(t)] = 0$, $\mathbf{x}(t) \in \mathbb{R}^n$ as sliding manifold, it turns out that $v(t)$ affects $\ddot{s}[\mathbf{x}(t)]$ but not $\dot{s}[\mathbf{x}(t)]$, and the problem becomes that of steering $s[\mathbf{x}(t)]$ to zero by acting on its second derivative. The proposed solution procedure is outlined in this paper first dealing with a simple case, namely a second-order system with inaccessible states, to motivate and clarify the basic control algorithm, then highlighting the modifications to make to the basic algorithm extend its applicability to the actual case in question.

The paper is organized as follows. In Section II the control problem is formulated. An auxiliary problem is stated and solved in Section III. Finally, in Section IV it is shown how the original chattering elimination problem can be solved by analogy to the solution procedure envisaged, in the previous section, with reference to the auxiliary problem.

II. Problem Formulation

Consider (1) with uncertainties (2)–(5). Choose an nth-order sliding manifold

$$s[\mathbf{x}(t)] = x_n(t) + \sum_{i=1}^{n-1} c_i x_i(t) = 0 \tag{11}$$

with c_i, $i = 1, \cdots, n-1$, real positive constants such that the characteristic equation $z^{n-1} + \sum_{i=1}^{n-1} c_i z^{i-1} = 0$ has all roots with negative real parts. Consider the first and second time derivatives of $s[\mathbf{x}(t)]$, namely

$$\dot{s}[\mathbf{x}(t)] = f[\mathbf{x}(t)] + g[\mathbf{x}(t)]u(t) + \sum_{i=1}^{n-1} c_i x_{i+1}(t) \tag{12}$$

$$\ddot{s}[\mathbf{x}(t)] = \frac{d}{dt} f[\mathbf{x}(t)] + \frac{d}{dt} g[\mathbf{x}(t)]u(t) + c_{n-1}[f[\mathbf{x}(t)] + g[\mathbf{x}(t)]u(t)] + \sum_{i=1}^{n-2} c_i x_{i+2}(t) + g[\mathbf{x}(t)]\dot{u}(t). \tag{13}$$

If it is possible to steer $s[\mathbf{x}(t)]$ to zero in a finite time by using a discontinuous control signal $\dot{u}(t)$, then the corresponding $u(t)$ is continuous, thereby eliminating the undesired high-frequency oscillations of $u(t)$ typical of the standard VSC design. Once on $s[\mathbf{x}(t)] = 0$, the system performs like a reduced-order linear system with a stable transfer function. Assume $y_1(t) = s[\mathbf{x}(t)]$ and $y_2(t) = \dot{s}[\mathbf{x}(t)]$, then, relying on (11), the system dynamics (1) and the relevant uncertain dynamics (12), (13) can be rewritten as

$$\begin{cases} \dot{\hat{\mathbf{x}}}(t) = A\hat{\mathbf{x}}(t) + By_1(t) \\ x_n(t) = -\mathbf{C}\hat{\mathbf{x}} + y_1(t) \\ \dot{y}_1(t) = y_2(t) \\ \dot{y}_2(t) = F[\mathbf{x}(t), u(t)] + g[\mathbf{x}(t)]v(t) \end{cases} \tag{14}$$

where $\hat{\mathbf{x}} = [x_1, x_2, \cdots, x_{n-1}]^T$, $\mathbf{C} = [c_1, c_2, \cdots, c_{n-1}]$, A is a $(n-1)\times(n-1)$-matrix in companion form whose last row coincides with vector $-\mathbf{C}$, $B = [0, \cdots, 0, 1]^T \in \mathbb{R}^{n-1}$, $v(t) = \dot{u}(t)$ and $F[\cdot, \cdot]$ collects all the uncertainties not involving $v(t)$. The first two lines of (14) correspond to a linear system controlled by $y_1(t)$, and this system is stable by assumption. The second two equations of (14) correspond to a nonlinear uncertain second-order system ($y_2(t)$ is not available for measurement) with control $v(t)$. If the control $v(t)$ steers to zero both $y_1(t)$ and $y_2(t)$, then the linear system becomes an autonomous system evolving on the manifold defined by (11). Note that the last two equations of (14) are coupled with the previous ones through the uncertainties $F[\mathbf{x}(t), u(t)]$, $g[\mathbf{x}(t)]$.

III. The Auxiliary Problem

As a preliminary step of our treatment, we assume that instead of bounds (2)–(5), the following particular bounds:

$$0 < G_1 \le g[\mathbf{x}(t)] \le G_2 \tag{15}$$

$$|F[\mathbf{x}(t), u(t)]| < F \tag{16}$$

are considered. This assumption will be dispensed with in the next section. On this basis, one can deal with the last two equations of (14) as if they were independent of the others. That is, the following *auxiliary* problem can be faced.

Auxiliary Problem: Given a second-order system

$$\begin{cases} \dot{z}_1(t) = z_2(t) \\ \dot{z}_2(t) = H[z_1(t), z_2(t)] + d[z_1(t), z_2(t)]w(t) \end{cases} \tag{17}$$

with $z_2(t)$ unmeasurable and bounds

$$|H[z_1(t), z_2(t)]| < H \tag{18}$$

$$0 < D_1 \le d[z_1(t), z_2(t)] \le D_2 \tag{19}$$

find a control law $w(t)$ such that $z_1(t)$, $z_2(t)$ are steered to zero in a finite time in spite of the uncertainties.

By analogy to the well-known solution to the time optimal control problem, the control $w(t)$ can be chosen as a bang–bang control switching between two extreme values $-U_{\mathrm{Max}}, +U_{\mathrm{Max}}$. The classical switching logic for a double integrator ($H[z_1(t), z_2(t)] = 0$, $D_1 = D_2 = 1$) is

$$w(t) = \begin{cases} -U_{\mathrm{Max}} \text{ if } \left\{ z_1(t) > -\frac{1}{2}\frac{z_2(t)|z_2(t)|}{U_{\mathrm{Max}}} \right\} \\ \quad \bigcup \left\{ z_1(t) = -\frac{1}{2}\frac{z_2(t)|z_2(t)|}{U_{\mathrm{Max}}} \bigcap z_1(t) < 0 \right\} \\ +U_{\mathrm{Max}} \text{ if } \left\{ z_1(t) < -\frac{1}{2}\frac{z_2(t)|z_2(t)|}{U_{\mathrm{Max}}} \right\} \\ \quad \bigcup \left\{ z_1(t) = -\frac{1}{2}\frac{z_2(t)|z_2(t)|}{U_{\mathrm{Max}}} \bigcap z_1(t) > 0 \right\}. \end{cases} \tag{20}$$

Such a switching logic, instead of being based on the sign of $z_1(t) + (z_2(t)|z_2(t)|)/(2U_{\mathrm{Max}})$ and $z_1(t)$, and therefore depending on both $z_1(t)$ and $z_2(t)$, could be expressed only in terms of $z_1(t)$ which, by assumption, is available for measurement. Indeed, it is easy to verify the following.

The optimal trajectory is a sequence of two parabolic arcs. The second arc of the trajectory lies on the switching line $z_1(t) + (z_2(t)|z_2(t)|)/(2U_{\mathrm{Max}}) = 0$. The modulus of the $z_1(t)$ component of the initial point of this second arc is equal to one-half of the maximum modulus of the $z_1(t)$ component of the points of the previous part of the trajectory.

Assume that the extremal value of $z_1(t)$ along each parabolic arc of the trajectory can be evaluated, and denote it by z_{Max}. Then, the foregoing considerations can be summarized by the following algorithm, which is equivalent to the optimal one in the case $H[\cdot, \cdot] = 0$, $D_1 = D_2 = 1$, $z_1(0)z_2(0) > 0$, $\alpha^* = 1$, where α^* is a parameter appearing in the algorithm.

Algorithm 1:

1) Set $\alpha^* \in (0, 1] \cap (0, 3D_1/D_2)$.
2) Set $z_{\mathrm{Max}} = z_1(0)$.
   Repeat, for any $t > 0$, the following steps.
3) If $[z_1(t) - \frac{1}{2} z_{\mathrm{Max}}][z_{\mathrm{Max}} - z_1(t)] > 0$, then set $\alpha = \alpha^*$, else set $\alpha = 1$.
4) If $z_1(t)$ is extremal, then set $z_{\mathrm{Max}} = z_1(t)$.

5) Apply the control law

$$w(t) = -\alpha U_{\rm Max}\ \text{sign}\{z_1(t) - \tfrac{1}{2} z_{\rm Max}\} \tag{21}$$

until the end of the control time interval. □

The aim of this section is to prove that this algorithm is valid also with $H[\cdot, \cdot] \neq 0$, $D_1 \neq D_2 \neq 1$, $z_1(0)z_2(0)$ not necessarily positive, in the sense that it allows the origin of the $z_1(t)$, $z_2(t)$ state space to be reached in a finite time. To this end, the following result can be proved.

Theorem 1: Given the state equation (17) with bounds as in (18) and (19) and $z_2(t)$ not available for measurement, then if the extremal value of $z_1(t)$ is evaluated with ideal precision, for any $z_1(0)$, $z_2(0)$, the suboptimal control strategy defined by Algorithm 1 with the additional constraint

$$U_{\rm Max} > \max\left(\frac{H}{\alpha^* D_1};\ \frac{4H}{3D_1 - \alpha^* D_2}\right) \tag{22}$$

causes the generation of a sequence of states with coordinates $(z_{{\rm Max}_i}, 0)$ featuring the following contraction property:

$$|z_{{\rm Max}_{i+1}}| < |z_{{\rm Max}_i}|, \qquad i = 1, 2, \cdots. \tag{23}$$

Moreover, the convergence of the system trajectory to the origin of the state plane takes place in a finite time.

Proof—Part 1 (Proof of the Contraction Property): Consider the state equation (17) with $|w(t)| \leq U_{\rm Max}$ and bounds (18) and (19). Moreover

$$U_{\rm Max} > \frac{H}{\alpha^* D_1} \tag{24}$$

and α^* as in 1) of Algorithm 1, since, by assumption, (22) is satisfied. Depending on the initial conditions $z_1(0)$, $z_2(0)$, one can distinguish among the following cases.

Case 1: ($z_1(0) = z_{\rm Max} > 0$, $z_2(0) = 0$, i.e., the initial point lies on the right side of the abscissa axis.) In this case, $w(t) = -\alpha^* U_{\rm Max}$, and by integration of (17) it is trivial to show that when the commutation occurs at time instant t_c such that $z_1(t_c) = \frac{1}{2} z_{\rm Max}$, according to Algorithm 1, the corresponding value of $z_2(t_c)$ belongs to the interval

$$\left[-\sqrt{z_{\rm Max}(\alpha^* D_2 U_{\rm Max} + H)},\ -\sqrt{z_{\rm Max}(\alpha^* D_1 U_{\rm Max} - H)}\right].$$

Starting from any point in this interval, for $t > t_c$, integrating system (17) with $w(t) = U_{\rm Max}$ ($\alpha = 1$), one can easily show that the state trajectory crosses the abscissa axis within the interval

$$\left[-\frac{1}{2}\,\frac{(\alpha^* D_2 - D_1)U_{\rm Max} + 2H}{D_1 U_{\rm Max} - H}\, z_{\rm Max},\ \frac{1}{2}\,\frac{(D_2 - \alpha^* D_1)U_{\rm Max} + 2H}{D_2 U_{\rm Max} + H}\, z_{\rm Max}\right].$$

The right extreme of this interval is nearer to the origin than the considered starting point. Then, to assess the contraction it is sufficient that the modulus of the left bound of the previous interval is less than $z_{\rm Max}$. This sufficient condition, considering also (24) and 1) of Algorithm 1, can be expressed by the following system of inequalities:

$$\begin{cases} \alpha^* \leq 1 \\ \alpha^* D_1 U_{\rm Max} > H \\ \dfrac{(\alpha^* D_2 - D_1)U_{\rm Max} + 2H}{D_1 U_{\rm Max} - H} < 2. \end{cases} \tag{25}$$

By means of simple computations, one can find the interval solutions of (25) with respect to α^* and $U_{\rm Max}$, that is

$$U_{\rm Max} > \begin{cases} \dfrac{H}{\alpha^* D_1}, & \text{if } \alpha^* \in \left(0,\ \dfrac{3D_1}{4D_1 + D_2}\right] \\ \dfrac{4H}{3D_1 - \alpha^* D_2}, & \text{if } \alpha^* \in \left(\dfrac{3D_1}{4D_1 + D_2},\ 1\right] \\ & \quad \cap \left(\dfrac{3D_1}{4D_1 + D_2},\ \dfrac{3D_1}{D_2}\right). \end{cases} \tag{26}$$

It is also easy to verify that if $\alpha^* > 3D_1/(4D_1 + D_2)$, then $H/\alpha^* D_1 < 4H/(3D_1 - \alpha^* D_2)$, if $\alpha^* < 3D_1/(4D_1 + D_2)$, then $H/\alpha^* D_1 > 4H/(3D_1 - \alpha^* D_2)$, and that the intersection between the two pieces of the limiting curve for $U_{\rm Max}$ occurs at $\alpha^* = 3D_1/(4D_1 + D_2)$. According to (22) that is true by assumption.

Case 2: ($z_1(0) = z_{\rm Max} < 0$, $z_2(0) = 0$, i.e., the initial point lies on the left side of the abscissa axis.) The proof is the same as in Case 1 but with reversed extremes of the relevant intervals.

Case 3: ($z_1(0)z_2(0) > 0$, $z_1(0)z_2(0) < 0$, $z_1(0) = 0$, $z_2(0) \neq 0$, i.e., all the other possible initial conditions.) It is trivial to see that after at most a finite-time interval, the system trajectory reaches a point of the types considered in Cases 1 and 2.

Part 2: Proof of the fact that the convergence to the origin takes place in a finite time.

Algorithm 1 defines a sequence $\{t_{{\rm Max}_k}\}$ of the time instants when an extremal value of $z_1(t)$ occurs. It can be proved that each term of this sequence is upperbounded by the corresponding term of the sequence

$$\hat{t}_{{\rm Max}_{k+1}} = \hat{t}_{{\rm Max}_k} + \frac{(D_1 + \alpha^* D_2)U_{\rm Max}}{(D_1 U_{\rm Max} - H)\sqrt{\alpha^* D_2 U_{\rm Max} + H}} \cdot \sqrt{|z_{{\rm Max}_k}|}. \tag{27}$$

From (27), recursively

$$\begin{aligned} \hat{t}_{{\rm Max}_{k+1}} &= \frac{(D_1 + \alpha^* D_2)U_{\rm Max}}{(D_1 U_{\rm Max} - H)\sqrt{\alpha^* D_2 U_{\rm Max} + H}} \cdot \sum_{j=1}^{k} \sqrt{|z_{{\rm Max}_j}|} + t_{{\rm Max}_1} \\ &= \beta \sum_{j=1}^{k} \sqrt{|z_{{\rm Max}_j}|} + t_{{\rm Max}_1} \end{aligned} \tag{28}$$

with $t_{{\rm Max}_1}$ being the time interval from $t = 0$ to the time instant when the first extremal value occurs. From previous relationships

$$|z_{{\rm Max}_j}| < \left|\frac{1}{2}\,\frac{(\alpha^* D_2 - D_1)U_{\rm Max} + 2H}{D_1 U_{\rm Max} - H}\right|^{j-1} |z_{{\rm Max}_1}| \tag{29}$$

so that, in compact form, with implicit definition of the symbols

$$\begin{aligned} t_{{\rm Max}_{k+1}} &< \beta \sum_{j=1}^{k} \gamma^{j-1} \sqrt{|z_{{\rm Max}_1}|} + t_{{\rm Max}_1} \\ &= \beta' \sum_{j=1}^{k} \gamma^{j-1} + t_{{\rm Max}_1}. \end{aligned} \tag{30}$$

By assumption (22) is true and, obviously, $\gamma < 1$. Therefore, from (29)

$$\lim_{k \to \infty} z_{{\rm Max}_k} = 0 \tag{31}$$

and from (30)

$$\lim_{k \to \infty} t_{{\rm Max}_k} < \frac{\beta'}{1 - \gamma} + t_{{\rm Max}_1} \tag{32}$$

which concludes the proof. □

The convergence of the sequence $\{z_{{\rm Max}_j}\}$ in a finite time implies the convergence to zero of the phase trajectories, since in any time interval $[t_{\text{Max}_j}, t_{\text{Max}_{j+1}}]$ the maximum value of $|z_2(t)|$ is bounded by a function of $\sqrt{|z_{\text{Max}_j}|}$, and this becomes zero in a finite time.

IV. The Solution to the Chattering Elimination Problem

In this section, it will be proved that it is possible to solve the chattering elimination problem by relying on the results obtained with reference to the auxiliary problem in the previous section. To this end, consider (14), which can be viewed as the connection of two systems coupled through the signal $y_1(t)$ and the nonlinear term $F[\mathbf{x}(t), u(t)] + g[\mathbf{x}(t)]v(t)$. Now, we simply assume that (15) holds, which is reasonable in many practical situations, while $F[\mathbf{x}(t), u(t)]$, though bounded in any bounded domain, cannot be assumed *a priori* to be bounded since proving the boundedness of its arguments is an objective of this treatment. Thus, the aim of the following analysis is to prove that after an initialization phase, the state trajectories reach regions of the state space including the origin. Once such regions are reached, the application of Algorithm 1, with minor modifications, leads to a contractive process steering $y_1(t)$, $y_2(t)$ to zero in a finite time. After that time, the further evolution of the system states is that of an autonomous linear exponentially stable system. In order to formally describe this procedure, the following theorem needs to be proved.

Theorem 2: Given (14), the norm of the state vector $\mathbf{x}(t)$, in any finite interval, can be upperbounded by a function of the norm of the initial value $\mathbf{x}(t_i)$ and of the maximum value assumed by $y_1(t)$.

Proof: Consider (14). One has

$$\hat{\mathbf{x}}(t) = \exp\{A(t-t_i)\}\hat{\mathbf{x}}(t_i) + \int_{t_i}^{t} \exp\{A(t-\tau)\}By_1(t)\,d\tau. \tag{33}$$

Thus, $\|\hat{\mathbf{x}}(t)\|$ can be bounded from above as

$$\|\hat{\mathbf{x}}(t)\| \le \|\hat{\mathbf{x}}(t_i)\| + \max_{t_i\le\tau\le t} |y_1(\tau)| \int_{t_i}^{\infty} \cdot \|\exp\{A(t-\tau)\}B\|\,d\tau. \tag{34}$$

Note that the integral can be evaluated since the first two equations of (14) describe a perfectly known stable system. Then

$$|x_n(t)| \le c_{\text{Max}}\left\{\|\hat{\mathbf{x}}(t_i)\| + \max_{t_i\le\tau\le t}|y_1(\tau)| \int_{t_i}^{\infty} \cdot\|\exp\{A(t-\tau)\}B\|\,d\tau\right\} + \max_{t_i\le\tau\le t}|y_1(\tau)| \tag{35}$$

where

$$c_{\text{Max}} = \max_{i=1,\cdots,n-1} c_i$$

with c_i being the component of vector $\mathbf{C}$. As a result

$$\|\mathbf{x}(t)\| \le P_{\mathbf{x}}\|\mathbf{x}(t_i)\| + Q_{\mathbf{x}}Y_1(t_i,t) \tag{36}$$

with

$$Y_1(t_i,t) = \max_{t_i\le\tau\le t}|y_1(\tau)|.$$

□

Now, one has to exploit the known bounds relevant to the uncertain dynamics to obtain an expression of the bound F in terms of $y_1(t)$ and Y_1. From (12)–(14) one has

$$y_2(t) = f[\mathbf{x}(t)] + g[\mathbf{x}(t)]u(t) + \sum_{i=1}^{n-1} c_i x_{i+1}(t) \tag{37}$$

$$F[\mathbf{x}(t), u(t)] = \left\{\frac{d}{dt}f[\mathbf{x}(t)] + \frac{d}{dt}g[\mathbf{x}(t)]u(t) + c_{n-1}[f[\mathbf{x}(t)] + g[\mathbf{x}(t)]u(t)] + \sum_{i=1}^{n-2} c_i x_{i+2}(t)\right\}. \tag{38}$$

On the basis of the previous relationship $F[\mathbf{x}(t), u(t)]$ can be written as

$$F[\mathbf{x}(t), u(t)] = \Theta_1[\mathbf{x}(t)] + \Theta_2[\mathbf{x}(t)]u(t) + \Theta_3[\mathbf{x}(t)]u^2(t). \tag{39}$$

Using (2)–(5) and (36), it is possible to express the upperbounds of $|\Theta_i[\cdot]|$, $i = 1, 2, 3$, in any finite interval (t_i, t_f), as

$$|\Theta_i[\mathbf{x}(t)]| < F_i[Y_{1_{(t_i,t_f)}}], \qquad i = 1, \cdots, 3 \tag{40}$$

with $F_i[\cdot]$ being an increasing positive function of its argument, i.e., of the maximum value of $y_1(t)$ in the interval (t_i, t_f). Hence, in any finite interval one can define an upperbound of $|F[\mathbf{x}(t), u(t)]|$ as

$$F^*[Y_{1_{(t_i,t_f)}}] = F_1[Y_{1_{(t_i,t_f)}}] + F_2[Y_{1_{(t_i,t_f)}}]|u(t)| + F_3[Y_{1_{(t_i,t_f)}}]u^2(t). \tag{41}$$

Note that the term depending on $u^2(t)$ would not appear in case $g[\mathbf{x}(t)]$ should not be dependent on $x_n(t)$.

The foregoing considerations will be used in the following way. First, it will be proved that it is possible to reach the axis $y_2(t) = 0$ in a finite time, starting from any initial condition. Then, it will be clarified how from the time instant t_{Max_1} when the axis $y_2(t) = 0$ is crossed, the control algorithm will force $y_1(t)$ either to reach directly a new extremal value with the same sign of $y_1(t_{\text{Max}_1})$, but contracted in modulus, or a value equal to $y_1(t_{\text{Max}_1})/2$, for which a commutation of the control signal will occur. As a consequence, the control algorithm will cause the reaching of a new extremal value of $y_1(t)$ at the time instant t_{Max_2}. Finally, it will be observed that $|y_1(t_{\text{Max}_2})| < |y_1(t_{\text{Max}_1})|$, that is, a contraction occurs and repeats itself during the subsequent time intervals $[t_{\text{Max}_i}, t_{\text{Max}_{i+1}}]$, until the convergence of both $y_1(t)$ and $y_2(t)$ to zero takes place. Thus, the following theorem can be proved.

Theorem 3: Given (14), starting from any initial condition and relying on the knowledge of the bounds (2)–(5) on the uncertain system dynamics, it is possible to design a control signal such that an extremal value of $y_1(t)$ is reached in a finite time.

Proof: In the initialization phase, the fact that the control signal $u(t)$ can be expressed as

$$u(t) = u(0) + \int_0^t \dot{u}(\tau)\,d\tau \tag{42}$$

needs to be exploited. Note that it is possible, at isolated time instants (e.g., $t = 0$), to modify the sign and the amplitude of $\dot{s}[\mathbf{x}(t)]$ through $u(t)$. In other terms, one can initialize the algorithm by measuring $y_1(0)$ and choosing $u(0)$ so as to obtain $y_1(0)y_2(0) > 0$. This can be accomplished, taking into account (3) and (12), by selecting

$$u(0) = \frac{1}{G_1}\left\{P_f + Q_f\|\mathbf{x}(0)\| + \left|\sum_{i=1}^{n-1} c_i x_{i+1}(0)\right| + h^2\right\} \cdot \text{sign}[y_1(0)] \tag{43}$$

where h is any nonzero real constant. Starting from this initial choice, it is possible to identify a signal $\dot{u}(t)$ which forces the trajectories in the plane $[y_1(t), y_2(t)]$ to reach the axis $y_2(t) = 0$ in a finite time. The time instant when such an axis is reached constitutes the actual initial instant for the subsequent contraction procedure. One can choose $\dot{u}(t)$ as

$$\dot{u}(t) = -M[\mathbf{x}(t), u(t)]\text{sign}[y_1(t)] \tag{44}$$

where

$$M[\mathbf{x}(t), u(t)] = \frac{1}{G_1}\left(\Theta_1^*[\mathbf{x}(t)] + \Theta_2^*[\mathbf{x}(t)]|u(t)| + \Theta_3^*[\mathbf{x}(t)]u^2(t)\right)$$

with $\Theta_i^*[\cdot]$ being positive functions upperbounding the modulus of $\Theta_i[\cdot]$ appearing in (39). Then, $|y_2(t)|$ is decreasing: indeed, the sign of $y_1(t)$ does not change during this phase, and, consequently, from (14) and (44) $y_2(t)\dot{y}_2(t) < 0$. □

Let t_{Max_1} be the time instant when the axis is reached, and consider t_{Max_1} as the new initial time to which the state $\mathbf{x}(t_{\text{Max}_1}) = \mathbf{x}_1$ corresponds. As the first extremal value of $y_1(t)$ is reached, the procedure previously outlined can be activated, provided that the control $v(t) = \dot{u}(t)$ is suitably chosen in order to force the state trajectories to cross again the $y_2(t) = 0$ axis in a finite time, say at the time instant t_{Max_2}, with a new extremal value $y_1(t_{\text{Max}_2})$ such that a contraction is registered, that is, $|y_1(t_{\text{Max}_2})| < |y_1(t_{\text{Max}_1})|$. To this end, the following theorem can be proved.

Theorem 4: Given (14), provided that for $t \in [t_{\text{Max}_1}, t_{\text{Max}_2}]$ the control signal $v(t)$ is chosen as

$$\begin{aligned} v(t) = &-\alpha V_{\text{Max}}\ \text{sign}\{y_1(t) - \tfrac{1}{2}\, y_1(t_{\text{Max}_1})\} \\ &- [F_2[y_1(t_{\text{Max}_1})]|u(t) - u(t_{\text{Max}_1})| \\ &+ F_3[y_1(t_{\text{Max}_1})]|u^2(t) - u^2(t_{\text{Max}_1})|]\text{sign}\{y(t_{\text{Max}_1}\} \end{aligned} \tag{45}$$

where α is defined according to Algorithm 1 and V_{Max} is chosen equal to U_{Max} specified in (22) with H given by

$$\begin{aligned} H &= F^*[y_1(t_{\text{Max}_1})] \\ &= F_1[y_1(t_{\text{Max}_1})] + F_2[y_1(t_{\text{Max}_1})]|u(t_{\text{Max}_1})| \\ &\quad + F_3[y_1(t_{\text{Max}_1})]u^2(t_{\text{Max}_1}) \end{aligned} \tag{46}$$

then, the trajectories of $y_1(t)$ and $y_2(t)$ are internal to the limiting curve defined by

$$\begin{aligned} \dot{y}_{2_l}(t) &= -(H + \alpha G_2 V_{\text{Max}})\text{sign}\{y_1(t_{\text{Max}_1})\} \\ &\qquad t \in [t_{\text{Max}_1}, t_{c_1}] \\ \dot{y}_{2_l}(t) &= (H - G_1 V_{\text{Max}})\text{sign}\{y_1(t_{\text{Max}_1})\} \\ &\qquad t \in (t_{c_1}, t_{\text{Max}_2}] \end{aligned} \tag{47}$$

and the axis $y_2(t) = 0$, t_{c_1} being the time instant when the first commutation occurs.

Proof: The limiting curve (47) corresponds to the case in which a constant upperbound H of $F[\mathbf{x}(t), u(t)]$ can be identified. But, as previously observed, such an upperbound cannot be found in practice because of the linear and quadratic dependence of $F[\mathbf{x}(t), u(t)]$ on $u(t)$. By choosing the control signal $v(t)$ as indicated in (45), one obtains an equation for $\dot{y}_2(t)$ which is similar to (17) and (18) plus a term

$$\begin{aligned} &-[F_2[y_1(t_{\text{Max}_1})]|u(t) - u(t_{\text{Max}_1})| \\ &\quad + F_3[y_1(t_{\text{Max}_1})]|u^2(t) - u^2(t_{\text{Max}_1})|]\text{sign}\{y(t_{\text{Max}_1}\}. \end{aligned} \tag{48}$$

If this term were neglected, the results of the previous section could also be applied in the present case. This means that the system trajectories would lie inside the region delimited by the two pieces of the curve (47). Since any feasible trajectory included in such a region enjoys the convergence properties of the limiting one, to prove the theorem it is sufficient to note that the effect of the term (48) is such that the corresponding trajectories reach the axis $y_2(t) = 0$ at a point which is internal to the segment $[y_1(t_{\text{Max}_1}), y_{1_l}(t_{\text{Max}_2})]$, which ensures a contractive behavior for any real trajectory. Indeed, because of the presence of the additional term (48), during the interval $[t_{\text{Max}_1}, t_{c_1}]$ the time derivative $\dot{y}_2(t)$ along the trajectories obtained by applying the control signal (45) is in modulus less than the modulus of $\dot{y}_{2_l}(t)$ corresponding to the first part of the limiting curve (i.e., in the interval $[t_{\text{Max}_1}, t_{c_1}]$), while, in the interval $(t_{c_1}, t_{\text{Max}_2}]$, $|\dot{y}_2(t)| > |\dot{y}_{2_l}(t)|$. Then, in the first interval $|y_2(t)|$ increases less than $|y_{2_l}(t)|$, while the opposite happens in the second interval. Then, it is proved that the trajectories are included within the region delimited by (47) and the axis $y_2(t) = 0$. □

Note that (46) is compatible with (41) in the sense that it implies that $Y_{1_{(t_{\text{Max}_1}, t_{\text{Max}_2})}} = y_1(t_{\text{Max}_1})$. Indeed, it can be easily verified that this fact is true for the system controlled via (45).

To conclude, it should be noted that Theorem 4 also holds for the subsequent control intervals $[t_{\text{Max}_i}, t_{\text{Max}_{i+1}}]$, $i = 2, 3, \cdots$. Relying on Theorem 4, it turns out that the results proved before (in particular Theorem 1) can be extended to solve the chattering elimination problem. Indeed, as a consequence of Theorems 3 and 4, the trajectories generated by applying, during the initialization phase, the control signal (43), and, once the first extremal value of $y_1(t)$ is reached, Algorithm 1 with the control signal (21) replaced by (45) feature the same behavior as that of (17) controlled as indicated in Theorem 1. As a result, both $y_1(t) = s[\mathbf{x}(t)]$ and $y_2(t) = \dot{s}[\mathbf{x}(t)]$ tend to zero in a finite time.

V. Conclusions

In this paper, a new contribution to the solution to the chattering elimination problem in VSC of single-input nonlinear systems is presented. The proposed algorithm results in being simpler and more feasible than the one previously presented by one of the authors, in the sense that it does not require the use of observers and differential inequalities. In the paper, the chattering elimination problem has been first formulated, then a suitable auxiliary problem involving a second-order uncertain system with unavailable velocity has been introduced and solved by using an algorithm inspired by the classical bang–bang optimal control strategy. The problem in question has been solved finally by using an algorithm which coincides with that used to cope with the auxiliary problem, apart from an additional term in the control law. Thanks to the introduction of such a term, the original system has been proved to have the same convergence properties as the auxiliary one. As far as the practical feasibility of the proposed control strategy is concerned, it is strictly related to the availability of commercial components, like peak detectors or other approximated methods, to evaluate the change of the sign of the derivative of the constraint $s[\mathbf{x}(t)]$. Experiences in the field of vibration damping of elastic robotic structures are presently in progress to confirm the attractive features of the chattering elimination approach herein proposed.

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