On Second Order Sliding Mode Controllers

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1   Introduction

The ability to manage systems in an uncertain context is one of the most intriguing desires of humans; as an example many popular games, such as chess and bridge, are strictly related to such a topic. This expectation can be reduced to a system theory context; indeed the development of the control theory has given rise to a number of sophisticated control techniques devoted to solving the control problem for some classes of uncertain systems. Most of them are based on adaptation methods [24], relying both on identification and observation, and on absolute stability methods [11, 12], which often lead to very complicated control algorithms whose implementation can imply a relevant computational cost and/or the use of very expensive devices.

In the last years an increasing interest for sliding modes led to an almost complete formalization of the mathematical background and of the robustness properties, with respect to system uncertainties and external disturbances, of this control technique [26].

Variable Structure Systems (VSS) in which the control is able to constrain the uncertain system behaviour on an "a priori" specified manifold (the sliding manifold) by "brutal force" seem to be the most obvious and heuristic way to withstand the uncertainty. In such systems the control immediately reacts to any deviation of the system, steering it back to the constraint by means of a sufficiently energetic control effort. Any strictly satisfied equality removes one "uncertainty dimension".

Furthermore, VSS may be considered as a general term for any dynamical system with discontinuous feedback control which can be defined by means of suitable sliding mode techniques.

Classical sliding mode control is based on the possibility of making and keeping identically null an auxiliary output variable (the sliding variable), which represents the deviation from the constraint, by means of a discontinuous control acting on the first time derivative of the sliding variable, and switching between high amplitude opposite values with theoretically infinite frequency. Moreover, due to its regularity properties, any system evolving in a Δ boundary layer of the sliding manifold because of the nonideal realization, both of the system and of the control devices, has the same trajectories as the ideal one apart from some

perturbing terms whose influence grows with the size Δ of the boundary layer [23, 26]. Nevertheless, the implementation of sliding mode control techniques is troublesome because of the large control effort usually needed to assure robustness properties, and the possibility that the so–called chattering phenomenon can arise [26]. In particular the latter may lead to large undesired oscillations which can damage the controlled system.

Recently invented, higher order sliding modes generalize the basic sliding mode idea. They are characterized by a discontinuous control acting on the higher order time derivatives of the sliding variable instead of influencing its first time derivative, as it happens in standard sliding modes. Preserving the main advantages of the original approach with respect to robustness and easiness of implementation, at the same time, they totally remove the chattering effect and guarantee even higher accuracy in presence of plant and/or control devices imperfections [20, 19].

In particular, the sliding order characterizes the smoothness degree of the system dynamics in the vicinity of the sliding mode. Generally speaking, the task in sliding mode control is to keep the system on the sliding manifold defined by the equality of the sliding variable to zero. The sliding order is defined as the number of continuous, and of course null when in sliding mode, total time derivatives of the sliding variable, the zero one included. Furthermore an r–th order real sliding mode provides a sliding precision, i.e., the size of the boundary layer of the sliding manifold, up to the r–th order with respect to plant imperfections which result in delays in the the switchings [20, 19].

Higher order sliding modes can appear naturally when fast dynamic actuators are used in VSS applications [19]. Indeed, when some dynamic actuator is present between a relay and the controlled process, the switching is moved to higher order derivatives of the actual plant input. As a result some new modes appear providing for exact satisfaction of the constraint, and actually being higher order sliding modes. This phenomenon reveals itself by the spontaneous disappearance of chattering in VSS.

Some controllers which are able to induce asymptotically stable sliding behaviours of any order have been presented in the literature [13, 10, 25, 1]. In this contribution we deal with the first generation of specifically designed controllers which give rise to finite time second order sliding behaviours in VSS [13, 14, 20, 3, 4]. In particular, the most effective second order sliding algorithms presented by the authors are described.

2   Second Order Sliding Modes

VSS dynamics is characterized by differential equations with a discontinuous right-hand side. According to the definition by Filippov, any discontinuous differential equation $\dot{\mathbf{x}} = v(\mathbf{x})$, where $\mathbf{x} \in I\!R^n$ and v is a locally bounded measurable vector function, is replaced by an equivalent differential inclusion $\dot{\mathbf{x}} \in V(\mathbf{x})$ [15]. In the simplest case, when v is continuous almost everywhere, $V(\mathbf{x})$ is the convex closure of the set of all possible limits of $v(\mathbf{z})$ as $\mathbf{z} \to \mathbf{x}$, while $\{\mathbf{z}\}$ are continuity

points of $v(\mathbf{z})$. Any solution of the differential equation is defined as an absolute continuous function $\mathbf{x}(t)$ satisfying the differential inclusion almost everywhere. The extension to the non–autonomous case is straightforward by considering time t as an element of vector $\mathbf{x}$.

Consider an uncertain single–input nonlinear system whose dynamics is defined by the differential system

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), t, u(t)) \tag{1}$$

where $\mathbf{x} \in X \subset \mathbb{R}^n$ is the state vector, $u \in U \subset \mathbb{R}$ is the bounded input, t is the independent variable time, and $f : \mathbb{R}^{n+2} \to \mathbb{R}^n$ is a sufficiently smooth uncertain vector function. Assume that the control task is fulfilled by constraining the state trajectory on a proper sliding manifold in the state space defined by the vanishing of a corresponding sliding variable $s(t)$, i.e.,

$$s(t) = s(\mathbf{x}(t), t) = 0 \tag{2}$$

where $s : \mathbb{R}^{n+1} \to \mathbb{R}$ is a known single valued function such that its total time derivatives $s^{(k)}$, $k = 0, 1, \ldots, r - 1$, along the system trajectories exist and are single valued functions of the system state $\mathbf{x}$. The latter assumption means that discontinuity does not appear in the first $r-1$ total time derivatives of the sliding variable s.

Definition 1. Given the constraint function (2), its r–th order sliding set is defined by the r equalities

$$s = \dot{s} = \ddot{s} = \ldots = s^{(r-1)} = 0 \tag{3}$$

which constitute an r–dimensional condition on the system dynamics [19]. □

Definition 2. Let the r–th order sliding set (3) be not empty, and assume that it is locally an integral set in the Filippov sense, i.e., it consists of Filippov's trajectories of the discontinuous dynamic system. The corresponding motion of system (1) satisfying (3) is called r–th order sliding mode with respect to the constraint function s. □

For shortening purposes, the words "r-th order sliding" will be abridged below to "r–sliding".

On the basis of Definition 2 system (1) evolves featuring a 2–sliding mode on the sliding manifold (2) iff its state trajectories lie on the intersection of the two manifolds $s = 0$ and $\dot{s} = 0$ in the state space. It is easy to see that at 2–sliding points Filippov's set of admissible velocities lies in the tangential space to $s = 0$ (Fig.1).
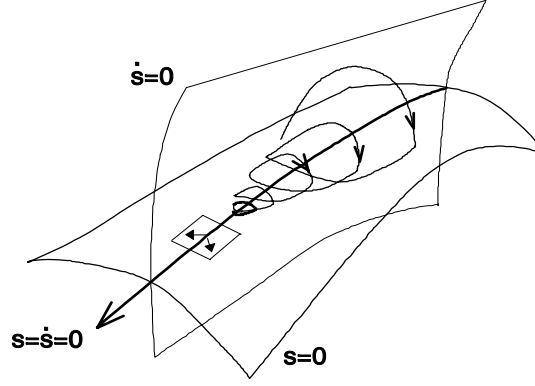
Fig. 1. Second order sliding mode trajectory

2.1 The Sliding Variable Dynamics

Consider system (1) and assume that the control task is fulfilled by its zero dynamics [16] with respect to a properly defined output variable $s(\mathbf{x}, t)$ as in (2). By differentiating the sliding variable s twice, the following relationships are derived

$$\dot{s}(t) = \dot{s}(\mathbf{x}(t), t, u(t)) = \frac{\partial}{\partial t} s(\mathbf{x}, t) + \frac{\partial}{\partial \mathbf{x}} s(\mathbf{x}, t) f(\mathbf{x}, t, u) \qquad (4)$$

$$\begin{aligned}
\ddot{s}(t) = \ddot{s}(\mathbf{x}(t), t, u(t), \dot{u}(t)) = \\
= \frac{\partial}{\partial t}\dot{s}(\mathbf{x}, t, u) + \frac{\partial}{\partial \mathbf{x}}\dot{s}(\mathbf{x}, t, u) f(\mathbf{x}, t, u) + \frac{\partial}{\partial u}\dot{s}(\mathbf{x}, t, u)\dot{u}(t)
\end{aligned} \qquad (5)$$

Depending on the relative degree [16] of the nonlinear SISO system (1), (2), different cases should be considered

a) relative degree $p = 1$, i.e., $\frac{\partial}{\partial u}\dot{s} \neq 0$;
b) relative degree $p \geq 2$, i.e., $\frac{\partial}{\partial u}s^{(i)} = 0$ $(i = 1, 2, \dots, p-1)$, $\frac{\partial}{\partial u}s^{(p)} \neq 0$.

In case a) the classical approach to VSS by means of 1–sliding mode control solves the control problem, nevertheless 2–sliding mode control can be used in order to avoid chattering as well. In fact, if the time derivative of the plant control, $\dot{u}(t)$, is considered as the actual control variable, the 2–sliding mode control approach allows the definition of a discontinuous control $\dot{u}$ steering both the sliding variable s and its time derivative $\dot{s}$ to zero, so that the plant control u is continuous and chattering is avoided [20, 6].

In case b), because of the system uncertainties and the possible not complete availability of the system state, the r–sliding mode approach, with $r \geq p$, is the most appropriate control technique. In this contribution we limit our interest to 2–sliding mode control problems, and, therefore, to systems with $p = 2$. These class of control problems can arise when the output control problem of systems with relative degree two is faced [4], or when the differentiation of a smooth signal is considered [21, 7].

Chattering avoidance; the generalized constraint fulfillment problem

When considering classical VSS the control variable $u(t)$ is a feedback-designed relay output. The most direct application of 2–sliding mode control is that of attaining the sliding motion on the sliding manifold (2) by means of a continuous bounded input $u(t)$. This means that $u(t)$ can be considered as a continuous output of a suitable first–order dynamical system which can be driven by a proper discontinuous signal. Such first–order dynamics can be either inherent to the control device [19] or specially introduced for chattering elimination purposes [6], and the feedback control signal generated by the 2–sliding control algorithm is mostly the time derivative of the plant input $u(t)$.

Consider system (1) and the constraint function (2); assume that f and s are respectively $\mathcal{C}^1$ and $\mathcal{C}^2$ functions, and that the only available information consists of the current values of t, $u(t)$, $s(\mathbf{x}, t)$ and, possibly, of the sign of the time derivative of the latter. The control goal for a 2–sliding mode controller is that of steering s to zero in a finite time by means of a control $u(t)$ continuously dependent on time.

In order to define the control problem the following conditions must be assumed:

1) $U = \{u : |u| \leq U_\mathrm{M}\}$, where $U_\mathrm{M} > 1$ is a real constant; furthermore the solution of (1) is well defined for all t, provided that $u(t)$ is continuous and $u(t) \in U \; \forall t$.

2) There exists $u_1 \in (0, 1)$ such that for any continuous function $u(t)$ with $|u(t)| > u_1$, there is t_1 such that $s(t)u(t) > 0$ for each $t > t_1$. Hence, the control $u(t) = -\mathrm{sign}[s(t_0)]$, where t_0 is the initial value of time, provides hitting the manifold (2) in finite time.

3) Let $\dot{s}(\mathbf{x}, t, u)$ be the total time derivative of the sliding variable $s(\mathbf{x}, t)$ as defined in (4). There are positive constants s_0, $u_0 < 1$, Γ_m, Γ_M such that if $|s(\mathbf{x}, t)| < s_0$ then

$$\Gamma_\mathrm{m} \leq \frac{\partial}{\partial u}\dot{s}(\mathbf{x}, t, u) \leq \Gamma_\mathrm{M} \quad , \forall u \in U, \mathbf{x} \in X \tag{6}$$

and the inequality $|u| > u_0$ entails $\dot{s}u > 0$.

4) There is a positive constant Φ such that within the region $|s| < s_0$ the following inequality holds $\forall t, \mathbf{x} \in X, u \in U$

$$\left| \frac{\partial}{\partial t}\dot{s}(\mathbf{x}, t, u) + \frac{\partial}{\partial \mathbf{x}}\dot{s}(\mathbf{x}, t, u)f(\mathbf{x}, t, u) \right| \leq \Phi \tag{7}$$

Condition 2 means that starting from any point of the state space it is possible to define a proper control $u(t)$ steering the sliding variable within a set such that the boundedness conditions on the sliding dynamics defined by conditions 3 and 4 are satisfied. In particular such conditions state that the second time derivative of the sliding variable s, evaluated with fixed values of the control u, is uniformly bounded in a bounded domain.

It follows from the theorem on implicit function that there is a function $u_\mathrm{eq}(t, \mathbf{x})$, which can be viewed as Utkin's equivalent control [26], satisfying the

equation $\dot{s} = 0$. Once $s = 0$ is achieved, the control $u = u_{\text{eq}}(t, \mathbf{x})$ would provide for the exact constraint fulfillment. Conditions 3 and 4 mean that $|s| < s_0$ implies $|u_{\text{eq}}| < u_0 < 1$, and that the velocity of the u_{eq} changes is bounded. This opens the possibility to approximate u_{eq} by a Lipschitzian control.

Note that the unit upper bound for u_0 and u_1 can be considered as a scaling factor, and somewhere in the following it is not explicitly considered. Note also that linear dependence on control u is not required and that the usual form of the uncertain systems dealt with by the VSS theory, i.e., systems affine in the control, is a special case of the considered systems (1), (2).

Systems with relative degree two The conditions to define the control problem for system (1), (2) in case of relative degree two could be derived from those above by considering the variable u as a state variable and $\dot{u}$ as the actual control. Nevertheless, they will be re–stated in case the system dynamics is affine in the control law, i.e.,

$$f(\mathbf{x}, t, u) = a(\mathbf{x}, t) + b(\mathbf{x}, t)u(t) \tag{8}$$

where $a : I\!R^{n+1} \to I\!R^n$ and $b : I\!R^{n+1} \to I\!R^n$ are sufficiently smooth uncertain vector functions.

By substituting (8) in (4) and (5) the following relationships are derived

$$\dot{s}(t) = \frac{\partial}{\partial t}s(\mathbf{x}, t) + \frac{\partial}{\partial \mathbf{x}}s(\mathbf{x}, t)a(\mathbf{x}, t) + \frac{\partial}{\partial \mathbf{x}}s(\mathbf{x}, t)b(\mathbf{x}, t)u(t) \tag{9}$$

$$\ddot{s}(t) = \frac{\partial^2}{\partial t^2}s(\mathbf{x}, t) + \left[\frac{\partial^2}{\partial t \partial \mathbf{x}}s(\mathbf{x}, t) + a^T(\mathbf{x}, t)\frac{\partial^2}{\partial \mathbf{x}^2}s(\mathbf{x}, t) \right. \\ \left. + \frac{\partial}{\partial \mathbf{x}}s(\mathbf{x}, t)\frac{\partial}{\partial \mathbf{x}}a(\mathbf{x}, t)\right] [a(\mathbf{x}, t,) + b(\mathbf{x}, t)u(t)] \tag{10}$$

In order to define the control problem the following conditions must be assumed:

I) Let (9) and (10) be, respectively, the first and second total time derivative of the sliding variable $s(\mathbf{x}, t)$, such that $\dot{s} = \dot{s}(\mathbf{x}, t)$, $\ddot{s} = \ddot{s}(\mathbf{x}, t, u)$, i.e.,

$$\frac{\partial}{\partial \mathbf{x}}s(\mathbf{x}, t)b(\mathbf{x}, t) \equiv 0$$
$$\left[\frac{\partial^2}{\partial t \partial \mathbf{x}}s(\mathbf{x}, t) + a^T(\mathbf{x}, t)\frac{\partial^2}{\partial \mathbf{x}^2}s(\mathbf{x}, t) + \frac{\partial}{\partial \mathbf{x}}s(\mathbf{x}, t)\frac{\partial}{\partial \mathbf{x}}a(\mathbf{x}, t)\right] b(\mathbf{x}, t) \neq 0 \tag{11}$$
$$\forall t, u \in U, \mathbf{x} \in X$$

$U = \{u : |u| \leq U_{\text{M}}\}$, where U_{M} is a real constant, so that $u(t)$ is a bounded discontinuous function of time; furthermore, the differential equation with discontinuous right–hand side (1) , (8) admits solutions in the Filippov sense on the 2–sliding manifold $s = \dot{s} = 0$ for all t.

II) There exists $u_1 \in (0, U_{\text{M}})$ such that for any continuous function $u(t) \in U$ with $|u(t)| > u_1$, there is t_1 such that $\dot{s}u(t) > 0$ for each $t > t_1$. Hence, the control $u(t) = -U_{\text{M}}\text{sign}[\dot{s}(t_0)]$, where t_0 is the time initial value, provides hitting the manifold $\dot{s} = 0$ in finite time.

III) There are positive constants s_0, Γ_{m}, Γ_{M} such that if $|s(\mathbf{x}, t)| < s_0$ then

$$\Gamma_{\mathrm{m}} \leq \left[\frac{\partial^2}{\partial t \partial \mathbf{x}} s(\mathbf{x}, t) + a^T(\mathbf{x}, t) \frac{\partial^2}{\partial \mathbf{x}^2} s(\mathbf{x}, t) \right. \\ \left. + \frac{\partial}{\partial \mathbf{x}} s(\mathbf{x}, t) \frac{\partial}{\partial \mathbf{x}} a(\mathbf{x}, t) \right] b(\mathbf{x}, t) \qquad \leq \Gamma_{\mathrm{M}} \tag{12}$$
$$\forall t > t_1, \mathbf{x} \in X$$

IV) There is a positive constant Φ such that within the region $|s| < s_0$ the following inequality holds $\forall t > t_1, \mathbf{x} \in X$

$$\left| \frac{\partial^2}{\partial t^2} s(\mathbf{x}, t) + \left[\frac{\partial^2}{\partial t \partial \mathbf{x}} s(\mathbf{x}, t) + a^T(\mathbf{x}, t) \frac{\partial^2}{\partial \mathbf{x}^2} s(\mathbf{x}, t) \right. \right. \\ \left. \left. + \frac{\partial}{\partial \mathbf{x}} s(\mathbf{x}, t) \frac{\partial}{\partial \mathbf{x}} a(\mathbf{x}, t) \right] a(\mathbf{x}, t,) \right| \leq \Phi \tag{13}$$

Condition I states that the differential equation with discontinuous right–hand side (1), (8) admits solution in the Filippov sense on a 2–sliding manifold, while condition II means that starting from any point of the state space it is possible to define a proper control $u(t)$ steering the sliding variable within a set such that the boundedness conditions on the sliding dynamics defined by III and IV are satisfied. In particular, they state that the second time derivative of the sliding variable s is uniformly bounded in a bounded domain, for all $u \in U$.

The auxiliary problem The sliding variable s can be considered as a suitable output variable of the uncertain system (1), and the control aim is that of steering this output to zero in a finite time. The 2–sliding mode approach allows for the finite time stabilization of both the output variable s and its time derivative $\dot{s}$ by defining a suitable discontinuous control function which can be either the actual control plant or its time derivative, depending on the system relative degree.

Consider the local coordinates (y_1, y_2), where $y_1 \equiv s$, $y_2 \equiv \dot{s}$, on the base of the previous definitions and conditions, apart from a proper initialization phase, the 2–sliding mode control problem is equivalent to the finite time stabilization problem for the following uncertain second order system

$$\begin{cases} \dot{y}_1(t) = y_2(t) \\ \dot{y}_2(t) = \varphi(\mathbf{y}(t), t) + \gamma(\mathbf{y}(t), t) v(t) \end{cases} \tag{14}$$

with $y_2(t)$ unmeasurable but with a possibly known sign, and $\varphi(\mathbf{y}(t), t)$ and $\gamma(\mathbf{y}(t), t)$ uncertain functions such that

$$\begin{matrix} |\varphi(\mathbf{y}(t), t)| \leq \Phi \\ 0 < \Gamma_{\mathrm{m}} \leq \gamma(\mathbf{y}(t), t) \leq \Gamma_{\mathrm{M}} \end{matrix} \quad \mathbf{y} \in Y \subset I\!\!R^2 \tag{15}$$

in which, referring to the previous notation, v is the actual control plant if system (1) has relative degree $p = 2$, with respect to y_1, or its time derivative if $p = 1$, and Y is a bounded region within which the boundedness of the uncertain sliding dynamics is assured, i.e., $|y_1| \leq s_0$.

Since y_2 is not available and $\varphi(\mathbf{y}(t), t)$, $\gamma(\mathbf{y}(t), t)$ are uncertain, this problem is not easily solvable by consolidated theory. It has been solved, recently, in

previous papers by the authors [20, 3, 4]. Hereafter, a synthetic and qualitative presentation of the solution procedure is provided for the readers' convenience.

Consider a double integrator $\dot{y}_1 = y_2$, $\dot{y}_2 = v$, that is $\varphi(\mathbf{y}(t), t)$, $\gamma(\mathbf{y}(t), t)$ are perfectly known and evaluable functions. In this case, there are algorithms capable of causing the finite time reaching of the origin. One, in particular, can be derived by the well–known time optimal bang–bang control approach according to the following propositions.

Proposition 3. *The time optimal switching line [17, 18]*

$$y_1(t) + \frac{y_2(t)|y_2(t)|}{2V_{\mathrm{M}}} = 0$$

when $y_1(0)y_2(0) \geq 0$, *can be replaced by*

$$y_1(t) - \frac{1}{2}y_1(t_{\mathrm{M}_1}) = 0$$

with t_{M_1} *such that* $y_2(t_{\mathrm{M}_1}) = 0$. $\qquad\qquad\square$

Proposition 4. *When* $y_1(t)y_2(t) < 0$, *the initialization control*

$$v(t) = -V_{\mathrm{M}}\mathrm{sign}\left(y_1(t) - \frac{1}{2}y_1(0)\right) = 0$$

guarantees that the condition $y_1(t)y_2(t) > 0$ *is achieved in finite time.* $\qquad\square$

The combination of the initialization phase with the use of the modified optimal switching line steers the state trajectory to the origin of the state plane with at most two commutations, instead of the single commutation characterizing the optimal bang–bang control (Fig.2). In this sense, such a combination gives rise to a suboptimal strategy.

Fig. 2. Modified time optimal switchings

Another way to steer the state of a double integrator to zero in finite time is that produced by the so–called twisting algorithm by Levant [20], i.e.

$$v(t) = -\alpha(t)V_{\mathrm{M}}\mathrm{sign}(y_1(t))$$

with

$$\alpha(t) = \begin{cases} 1 & \text{if } y_1(t)y_2(t) > 0 \\ \alpha^* \in (0,1) & \text{if } y_1(t)y_2(t) < 0 \end{cases}$$

With reference to this algorithm the convergence to zero in finite time can be proved following a procedure which can be used also to deal with the case of perturbed couple of integrators. More specifically, it can be verified that the sequence of values $y_1(t_{\mathrm{M}_i})$ is contractive, that is

$$\frac{y_1(t_{\mathrm{M}_{i+1}})}{y_1(t_{\mathrm{M}_i})} \leq q < 1$$

and that $\lim_{i \to \infty} y_1(t_{\mathrm{M}_i}) = 0$. Moreover, the reaching time is a series of positive elements upperbounded by a geometric series with ratio strictly less than one. Therefore, $\lim_{i \to \infty} t_{\mathrm{M}_i} = T < \infty$.

Now consider system (14): it can be viewed as a double integrator with perturbation uncertain terms. The solution to the perturbed case proposed in [4] is based on the suboptimal algorithm indicated through Proposition 1 and 2. The point is to prove that a contractive behaviour between two successive singular points is achieved despite the uncertainties φ and γ. The analysis has been carried out by considering, as the worst case, that in which uncertainties act always against the attainment of the contraction.
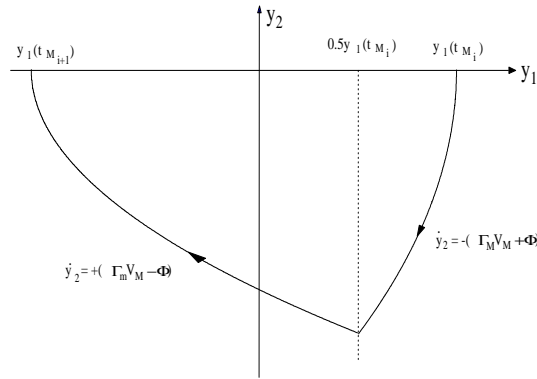

Fig. 3. Perturbed sub time optimal trajectory (worst case)

Assume that, for a certain choice of V_{M}, the worst case behaviour, sketched in Fig.3, is not characterized by the contraction effect. There are three ways to achieve the prefixed control objective:

a) to increase the control amplitude V_M, that is to reduce $\frac{\Gamma_M V_M + \Phi}{\Gamma_m V_M - \Phi}$ (Fig.4), but this is effective only if $3\Gamma_m > \Gamma_M$ [3];

b) to use an asymmetric commutation logic as in the twisting algorithm, (Fig.5);

c) to anticipate the commutation at the moment when $y_1 = \beta y_1(t_{M_i})$, with $\beta \in [\frac{1}{2}, 1)$ (Fig.6).

With each of the above actions, the worst case turns out to be characterized by a finite time damped oscillatory time response. A particular possible behaviour which has to be further considered is that which corresponds to uncertainties producing series of two successive singular points with the same sign, (Fig.7). In this case, $y_1(t_{M_i})y_1(t_{M_{i+1}}) > 0$, and the strategy is to change the sign of the control not only when $y_1(t) = \beta y_1(t_{M_i})$, but also at $t_{M_{i+1}}$.
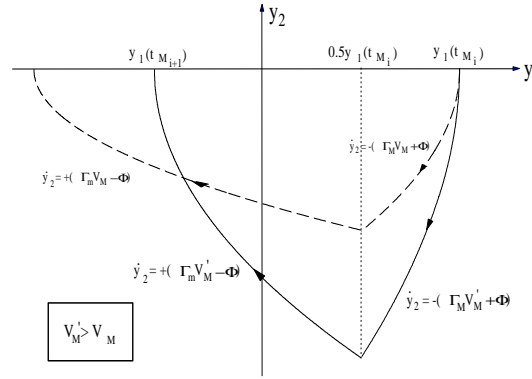
Fig. 4. Perturbed sub time optimal trajectory with increased control authority (worst case)

2.2 The Initialization Phase

In most cases the bounds (15) are not global even if the uncertain functions φ and γ are bounded in any bounded domain, i.e., $Y \subset I\!\!R^2$. Furthermore it can be assumed, without loss of generality, that function s is such that

$$\|\mathbf{x}\| \leq N \iff \|[s, \dot{s}]\| \leq M$$

where N and M are real nonnegative constants, so that bounded domains of the state variables correspond to bounded domains of the sliding variable and its time derivative, and vice–versa.

Usually 2–sliding control algorithms are defined with respect to constant bounds of the uncertain dynamics (14) and are such that y_1 and y_2 converge to the origin of the phase state plane, and, therefore, once set Y is reached, the y_1 and y_2 trajectories never leave it. The above considerations imply that, apart
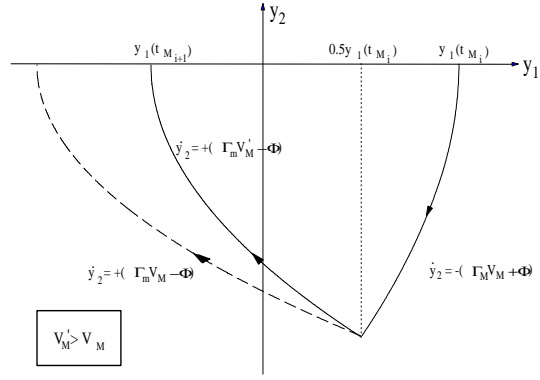
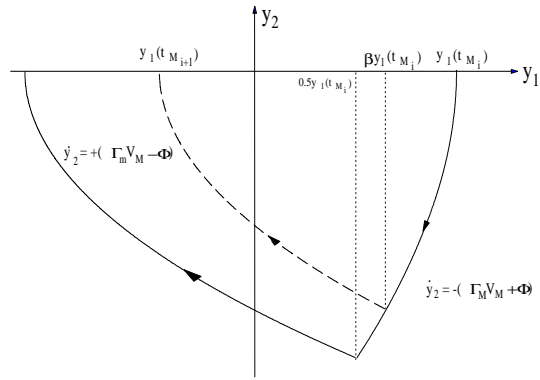**Fig. 5.** Perturbed sub time optimal trajectory with control magnitude modulation (worst case)

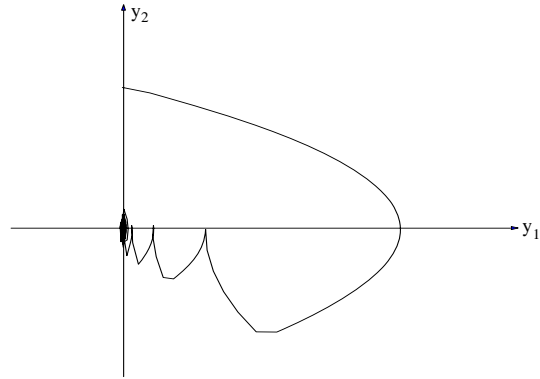**Fig. 6.** Perturbed sub time optimal trajectory with switching antecipation (worst case)

Fig. 7. Perturbed sub time optimal trajectory with constant signs of y_1 and y_2

from the case in wich bounds (15) are gobal, an initialization phase in which the domain Y is reached in finite time must be implemented.

The control law during the initialization phase depends on the class of system to which a 2-sliding control strategy is applied:

a) systems with relative degree one;
b) systems with relative degree two.

Indeed, in case a) the possibility of using the plant control u directly allows to assign the sign of the y_2 variable, and not only that of its time derivative $\dot{y}_2$ as in case b).

In case of relative degree one systems, 2–sliding mode control can be used in order to avoid chattering in VSS; this means that the plant control u is computed by integration of the discontinuous control generated by a 2–sliding controller. Due to the availability of both the control and its time derivative two different strategies are possible in order to reach the domain Y.

If condition 2 is verified then control u is able to assign the sign of the y_2 variable. By choosing u such that $y_1 y_2 \leq -k^2$, a point of the ordinate axis of the $y_1 O y_2$ plane, i.e., $y_1 = 0$, is reached in finite time, at least by using

$$u(t) = -U_{\mathrm{M}}\mathrm{sign}(y_1(t_0)) \ t \geq t_0$$

During the initialization phase a time varying control magnitude can be used if some knowledge about a function of the available signals which upperbounds the uncertain dynamics is available. As an example, assume that the sliding surface is a linear manifold in the state space

$$s(\mathbf{x}) = C\mathbf{x} \tag{16}$$

where C is a n–dimensional row vector whose elements define a Hurwitz polinomial. Furthermore the uncertain system dynamics (1) (8) is such that

$$\begin{aligned} \|a(\mathbf{x},t)\| &\leq A + K_{\mathrm{a}}\|\mathbf{x}\| \\ 0 < B_{\mathrm{m}} &\leq b_i(\mathbf{x},t) \leq B + K_{\mathrm{b}}\|\mathbf{x}\| \ i = 1, 2, \ldots, n \end{aligned} \tag{17}$$

where A, B, B_{m}, K_{a} and K_{b} are known positive constants. In this case an initialization phase with the following control law guarantees the finite time reaching of the ordinate axis of the $y_1 O y_2$ plane

$$u(\mathbf{x}, y_1) = -\frac{A + K_{\mathrm{a}}\|\mathbf{x}\| + h^2}{B_{\mathrm{m}}}\mathrm{sign}(y_1)$$

When systems with relative degree two are considered, the plant control v acts on the y_2 time derivative so that the previous approach cannot be applied. If condition II is verified then control v is able to assign the sign of the time derivative of the y_2 variable. By choosing v such that $y_2 \dot{y}_2 \leq -k^2$, a point of the abscissa axis of the $y_1 O y_2$ plane, i.e., $y_2 = 0$, is reached in finite time, at least by using

$$v(t) = -V_{\mathrm{M}}\mathrm{sign}(y_2(t_0)) \ t \geq t_0$$

Also in this case the knowledge of bound functions both for the uncertain dynamics and its partial derivatives can allow to design time varying control magnitudes during the initialization phase. Assume that the uncertain dynamics (14) is such that the following bounds are satisfied

$$|\varphi(\mathbf{y}(t),t)| \leq P + \mathcal{P}(y_1) + (Q + \mathcal{Q}(y_1)) \, \|\mathbf{y}\|$$
$$0 < \Gamma_{\mathrm{m}} \leq \gamma(\mathbf{y}(t),t)v(t) \leq R + \mathcal{R}(y_1)\|\mathbf{y}\|$$

(18)

where P, Q, and R are nonnegative constants, and $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ are positive semi–definite functions. In this case an initialization phase with the following control law guarantees the finite time reaching of the abscissa axis of the y_1Oy_2 plane [2]

$$v(y_1) = -V_{in}(y_1)\mathrm{sign}(y_2(t_0))$$
$$V_{in}(y_1) = \tfrac{1+h^2}{\Gamma_{\mathrm{m}}} \left(P + \mathcal{P}(y_1) + (Q + \mathcal{Q}(y_1)) \left\| [y_1 ; y_2(t_0)]^T \right\| \right)$$

Often $\mathrm{sign}(y_2(t_0))$ is unknown but it can be evaluated by the first difference of the available quantity y_1, that is $\mathrm{sign}(y_2(t_0))$ is estimated by $\mathrm{sign}(y_1(t_0 + \delta) - y_1(t_0))$, where δ is an arbitray small time interval.

The last initialization procedure is effective if the chattering avoidance is dealt with as well. Furthermore, in this case, it could be possible to assign the initial sign of the y_2 variable by a proper choice of the initial value of the available plant control u [6].

Once the abscissa or the ordinate axis of the y_1Oy_2 plane is reached, the constant bounds (15) exist and can be evaluated if the global uncertainty bounds have a first order dependence on the not available signal y_2 [2, 3, 6]. If $\|\mathbf{y}\|$ in (18) is an usual m–order norm defined in an Euclidean space, i.e., $\|\mathbf{y}\| = [|y_1|^m + |y_2|^m]^{\frac{1}{m}}$ $(m \geq 1)$, the previous condition is satisfied, so that after a finite time transient a subspace, in which any of the controllers described in the sequel can be applied, is reached.

3 Second Order Sliding Controllers

Each of the following controllers is characterized by few constant parameters. These parameters have to be tuned in order to achieve the control goal for the considered class of processes and sliding functions which will be defined in terms of the constants Φ, Γ_{m}, Γ_{M} and s_0. By increasing the constants Φ, Γ_{m}, Γ_{M} and reducing s_0 at the same time, it is possible to enlarge the class of controlled systems too. Such algorithms are obviously insensitive to any model perturbations and external disturbances which do not move the dynamic system from the given class.

It must be pointed out that, given the system to be controlled and the desired sliding manifold, it is possible to define the above constants by uncertainty maximization. Nevertheless this evaluation procedure usually defines very large parameters, and, as a consequence, very large control signals as well, which are not really needed for controlling the real system. In practice, the convergence conditions for the control algorithms are only sufficient but not necessary and the tuning of the controller is often better made heuristically.

3.1 Twisting Algorithm

This algorithm is characterized by a twisting around the origin of the $y_1 O y_2$ 2–sliding plane (Fig.8). The finite time convergence to the origin of the plane is due to the switching of the control amplitude between two different values such that the abscissas and ordinates axes are crossed nearer and nearer to the origin. The control amplitude commutes at each axis crossing, and the sign of the sliding variable time derivative y_2 is needed.
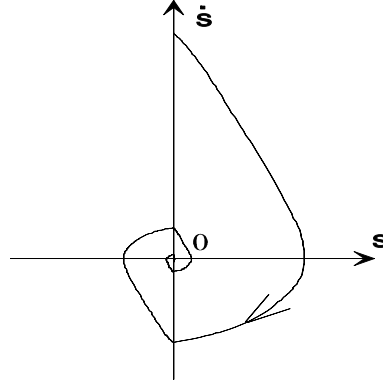

Fig. 8. Twisting algorithm phase trajectory

The control algorithm is defined by the following control law [20], in which the condition on $|u|$ must be taken into account when considering the chattering avoidance problem,

$$
v(t) = \begin{cases} -u & if \ \ |u| > 1 \\ -V_{\mathrm{m}}\mathrm{sign}(y_1) \ if \ \ y_1 y_2 \leq 0; \ |u| \leq 1 \\ -V_{\mathrm{M}}\mathrm{sign}(y_1) \ if \ \ y_1 y_2 > 0; \ |u| \leq 1 \end{cases} \tag{19}
$$

and the corresponding sufficient conditions for the finite time convergence to the sliding manifold are [20]

$$
\begin{cases} V_{\mathrm{M}} > V_{\mathrm{m}} \\ V_{\mathrm{m}} > \frac{4\Gamma_{\mathrm{M}}}{s^0} \\ V_{\mathrm{m}} > \frac{\Phi}{\Gamma_{\mathrm{m}}} \\ \Gamma_{\mathrm{m}} V_{\mathrm{M}} - \Phi > \Gamma_{\mathrm{M}} V_{\mathrm{m}} + \Phi \end{cases} \tag{20}
$$

By taking into account the different limit trajectories arising from the uncertain dynamics (14) and evaluating the time intervals between subsequent crossings of the abscissa axis, it is possible to define the following upper bound for the convergence time

$$
t_{\mathrm{tw}\infty} \leq t_{M_1} + \Theta_{\mathrm{tw}} \frac{1}{1 - \theta_{\mathrm{tw}}} \sqrt{|y_{1_{M_1}}|} \tag{21}
$$

where $y_{1_{M_1}}$ is the value of the y_1 variable at the first abscissa crossing in the $y_1 O y_2$ plane, t_{M_1} the corresponding time instant and

$$\Theta_{tw} = \sqrt{2} \frac{\Gamma_m V_M + \Gamma_M V_m}{(\Gamma_m V_M - \Phi)\sqrt{\Gamma_M V_m + \Phi}}$$
$$\theta_{tw} = \sqrt{\frac{\Gamma_M V_m + \Phi}{\Gamma_m V_M - \Phi}}$$

In many practical cases the y_2 variable is completely unmeasurable, then its sign can be estimated by the sign of the first difference of the available sliding variable y_1 in a time interval δ, i.e., $\mathrm{sign}[y_2(t)]$ is estimated by $\mathrm{sign}[y_1(t) - y_1(t - \delta)]$. In this case only 2–sliding precision with respect to the measurement time interval is provided, and the size of the boundary layer of the sliding manifold is $\Delta \sim \mathcal{O}(\delta^2)$ [20].

3.2 Sub–Optimal Algorithm

This 2–sliding control algorithm derives from a sub–optimal feedback implementation of the classical time optimal control for a double integrator. The trajectories on the $y_1 O y_2$ plane are confined within limit parabolic arcs which include the origin, so that both twisting and jumping (in which y_1 and y_2 do not change sign) behaviours are allowed (Fig.9)
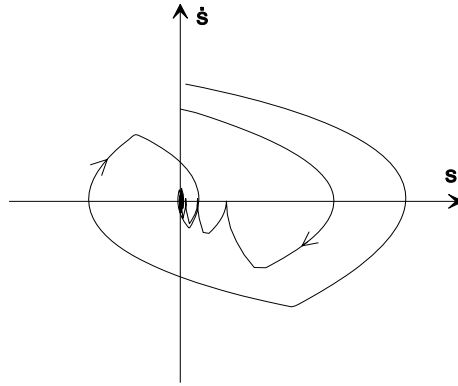

Fig. 9. Sub–Optimal algorithm phase trajectories

Apart from a possible initialization phase [2, 3, 6], the control algorithm is defined by the following control law

$$v(t) = -\alpha(t) V_M \mathrm{sign}(y_1(t) - \tfrac{1}{2} y_{1_M})$$
$$\alpha(t) = \begin{cases} \alpha^* \ if \ [y_1(t) - \tfrac{1}{2} y_{1_M}][y_{1_M} - y_1(t)] > 0 \\ 1 \quad if \ [y_1(t) - \tfrac{1}{2} y_{1_M}][y_{1_M} - y_1(t)] \le 0 \end{cases} \qquad (22)$$

where y_{1_M} represents the last extremal value of the $y_1(t)$ function, i.e., the last local maximum, local minimum or horizontal flex point of $y_1(t)$. The corresponding sufficient conditions for the finite time convergence to the sliding manifold are [3]

$$
\begin{cases}
\alpha^* \in (0, 1] \cap (0, \frac{3\Gamma_m}{\Gamma_M}) \\
V_M > \max \left(\frac{\Phi}{\alpha^* \Gamma_m}, \frac{4\Phi}{3\Gamma_m - \alpha^* \Gamma_M} \right)
\end{cases}
\tag{23}
$$

Also in this case an upper bound for the convergence time can be defined [3]

$$
t_{opt_\infty} \leq t_{M_1} + \Theta_{opt} \frac{1}{1 - \theta_{opt}} \sqrt{|y_{1_{M_1}}|}
\tag{24}
$$

where $y_{1_{M_1}}$ and t_{M_1} are those defined previously, and

$$
\begin{aligned}
\Theta_{opt} &= \frac{(\Gamma_m + \alpha^* \Gamma_M) V_M}{(\Gamma_m V_M - \Phi)\sqrt{\alpha^* \Gamma_M V_M + \Phi}} \\
\theta_{opt} &= \sqrt{\frac{(\alpha^* \Gamma_M - \Gamma_m) V_M + 2\Phi}{2(\Gamma_m V_M - \Phi)}}
\end{aligned}
$$

In [2, 3, 8] the effectiveness of the above algorithm was extended to larger classes of uncertain systems, while in [4] it was proved that in case of unit gain function the control law (22) can be simplified by setting $\alpha = 1$ and choosing $V_M > 2\Phi$.

The sub–optimal algorithm needs a device capable of detecting local maxima, local minima and horizontal flexes of the available sliding variable. In practical cases y_{1_M} can be estimated by checking the sign of the quantity $D(t) = [y_1(t - \delta) - y_1(t)] y_1(t)$, in which $\frac{\delta}{2}$ is the estimation delay. In this case the control amplitude V_M needs to belong to a finite set instead of a semi–infinite one, so that the second of (23) is modified into the following [5]

$$
V_M \in \left(\max \left(\frac{\Phi}{\alpha^* \Gamma_m}, V_{M_1}(\delta, y_{1_M}) \right), V_{M_2}(\delta; y_{1_M}) \right)
\tag{25}
$$

where $V_{M_1} < V_{M_2}$ are the solutions of the second order algebraic equation

$$
\left[(3\Gamma_m - \alpha^* \Gamma_M) \frac{V_{M_i}}{\Phi} - 4 \right] \frac{y_{1_M}}{\Phi \delta^2} - \frac{V_{M_i}}{8\Phi} [\Gamma_m + \Gamma_M (2 - \alpha^*)] \left(\Gamma_M \frac{V_{M_i}}{\Phi} + 1 \right) = 0
$$

In accordance with the definition of real higher order sliding mode, in the case of approximated evaluation of the y_{1_M} values the size of the boundary layer of the sliding manifold is $\Delta \sim \mathcal{O}(\delta^2)$, and it can be minimized by choosing V_M as follows [5]

$$
V_M = \frac{4\Phi}{3\Gamma_m - \alpha^* \Gamma_M} \left[1 + \sqrt{1 + \frac{3\Gamma_m - \alpha^* \Gamma_M}{4\Gamma_M}} \right]
$$

An extension of the real sub–optimal 2–sliding control algorithm to a class of sampled data systems characterized by a constant gain function in (14), i.e., $\gamma = 1$, was recently presented in [9]. A development of the cited digital controller is presented in the next chapter.

3.3 Super–Twisting Algorithm

This algorithm has been developed for the case of systems with relative degree one in order to avoid chattering in VSS. Also in this case the trajectories on the 2–sliding plane are characterized by twistings around the origin (Fig.10), but the continuous control law $u(t)$ is constituted by two terms. The first is defined by means of its discontinuous time derivative, while the other, which is present during the reaching phase only, is a continuous function of the available sliding variable.
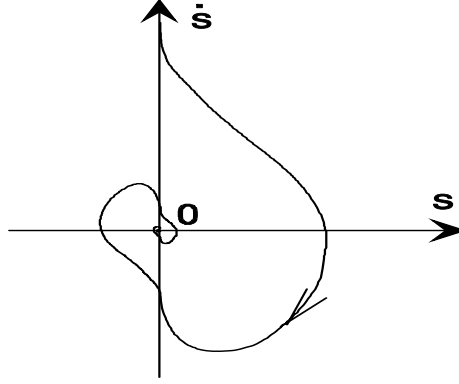

Fig. 10. Super–Twisting algorithm phase trajectory

The control algorithm is defined by the following control law [20]

$$u(t) = u_1(t) + u_2(t)$$
$$\dot{u}_1(t) = \begin{cases} -u & if \ |u| > 1 \\ -W \operatorname{sign}(y_1) & if \ |u| \le 1 \end{cases}$$
$$u_2(t) = \begin{cases} -\lambda |s_0|^\rho \operatorname{sign}(y_1) & if \ |y_1| > s_0 \\ -\lambda |y_1|^\rho \operatorname{sign}(y_1) & if \ |y_1| \le s_0 \end{cases} \tag{26}$$

and the corresponding sufficient conditions for the finite time convergence to the sliding manifold are [20]

$$\begin{cases} W > \frac{\Phi}{\Gamma_m} \\ \lambda^2 \ge \frac{4\Phi}{\Gamma_m^2} \frac{\Gamma_M(W+\Phi)}{\Gamma_m(W-\Phi)} \\ 0 < \rho \le 0.5 \end{cases} \tag{27}$$

The above algorithm does not need the evaluation of the sign of the time derivative of the sliding variable. An exponentially stable 2–sliding mode appears if the control law (26) with $\rho = 1$ is used. The choice $\rho = 0.5$ assures that the maximum real sliding order for 2–sliding realization is achieved.

3.4 Drift Algorithm

When using the drift algorithm the phase trajectories on the 2-sliding plane are characterized by loops with constant sign of the sliding variable y_1 (Fig.11), furthermore it is characterized by the use of sampled values of the available signal y_1 with sampling period δ.
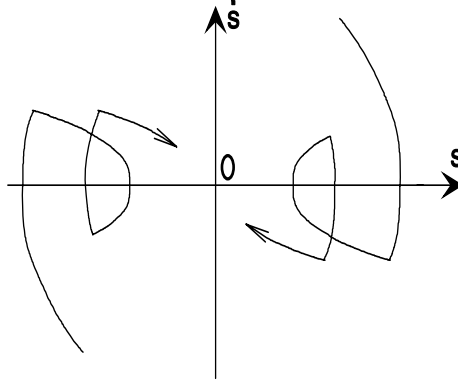

Fig. 11. Drift algorithm phase trajectories

The control algorithm is defined by the following control law, in which the condition on $|u|$ must be considered when dealing with the chattering avoidance problem. [14, 20]

$$v(t) = \begin{cases} -u & if \quad |u| > 1 \\ -V_{\mathrm{m}}\mathrm{sign}(\Delta y_{1_i}) & if \quad y_1 \Delta y_{1_i} \le 0; \ |u| \le 1 \\ -V_{\mathrm{M}}\mathrm{sign}(\Delta y_{1_i}) & if \quad y_1 \Delta y_{1_i} > 0; \ |u| \le 1 \end{cases} \qquad (28)$$

where V_{m} and V_{M} are suitable positive constants such that $V_{\mathrm{m}} < V_{\mathrm{M}}$ and $\frac{V_{\mathrm{M}}}{V_{\mathrm{m}}}$ sufficiently large, and $\Delta y_{1_i} = y_1(t_i) - y_1(t_i - \delta)$, $t \in [t_i, t_{i+1})$. The corresponding sufficient conditions for the convergence to the sliding manifold are rather cumbersome [14] and are omitted here for the sake of simplicity.

After substituting y_2 for Δy_{1_i} a first order sliding mode on $y_2 = 0$ would be achieved. This implies $y_1 = const.$, but, since an artificial switching time delay appears, we ensure a real sliding on y_2 with most of time spent in the set $y_1 y_2 < 0$, and therefore, $y_1 \to 0$. The accuracy of the real sliding on $y_2 = 0$ is proportional to the sampling time interval δ; hence the duration of the transient process is proportional to δ^{-1}.

Such an algorithm does not satisfy the definition of a real sliding algorithm [20] requiring the convergence time to be uniformly bounded with respect to δ. Let us consider a variable sampling time $\delta_{i+1}[y_1(t_i)] = t_{i+1} - t_i$, $i = 0, 1, 2, \ldots$ with $\delta = \max(\delta_{\mathrm{M}}, \min(\delta_{\mathrm{m}}, \eta|y_1(t_i)|^\rho))$ himself where $0.5 \le \rho \le 1$, $\delta_{\mathrm{M}} > \delta_{\mathrm{m}} > 0$, $\eta > 0$. Then with η, $\frac{V_{\mathrm{m}}}{V_{\mathrm{M}}}$ sufficiently small and V_{m} sufficiently large the drift

algorithm constitutes a second order real sliding algorithm with respect to $\delta \to 0$. This algorithm has no overshoot if parameters are chosen properly [14].

3.5 Algorithm with a Prescribed Law of Variation of

This class of sliding control algorithms is characterized by the fact that the switchings of the control time derivative depend on suitable functions of the sliding variable. Therefore the convergence properties are strictly related to the considered function (Fig.12).
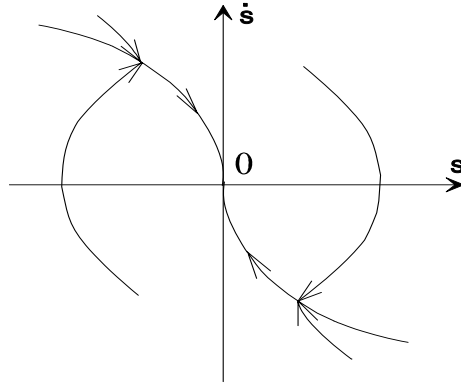

Fig. 12. Phase trajectories for the algorithm with prescribed law of variation of s

The general formulation of such a class of 2–sliding control algorithms is the following

$$v(t) = \begin{cases} -u & if \ |u| > 1 \\ -V_{\mathrm{M}}\mathrm{sign}(y_2 - g(y_1)) \ if \ |u| \leq 1 \end{cases} \tag{29}$$

where V_{M} is a positive constant and the continuous function $g(y_1)$ is smooth everywhere but in $y_1 = 0$.

Function g must be chosen so that all the solutions of the equation $\dot{y}_1 = g(y_1)$ vanish in a finite time, and function $g' \cdot g$ is bounded. For example, the following function can be used

$$g(y_1) = -\lambda |y_1|^\rho \mathrm{sign}(y_1)$$
$$\lambda > 0 \ , 0.5 \leq \rho < 1$$

The sufficient condition for the finite time convergence to the sliding manifold is defined by the following inequality

$$V_{\mathrm{M}} > \frac{\Phi + \sup(g'(y_1)g(y_1))}{\Gamma_{\mathrm{m}}} \tag{30}$$

and the convergence time depends on the function g [13, 20, 27].

This algorithm needs the variable y_2 to be known and that is not always the case. The substitution of y_2 with the first difference of the available y_1, i.e.,

$\text{sign}[y_2 - g(y_1)] \rightarrow \text{sign}[\Delta y_{1_i} - \delta_i g(y_1)]$ $(t \in [t_i\,; t_{i+1}), \quad \delta_i = t_i - t_{i-1})$, turns this algorithm into a real sliding algorithm, and its order equals two if g is chosen as in the above example with $\rho = 0.5$ [20].

An extension of the control algorithm with prescribed law of variation of s to arbitrary order sliding mode control of uncertain system was recently presented in [22].

4   Conclusions

In this paper a collection of control algorithms which are able to give rise to 2–sliding modes have been presented, and for each of them the sufficient convergence conditions are given. Furthermore, the real sliding behaviour is briefly considered, and, in some cases, the upper bound of the convergence time is given.

2–sliding mode control seems to be an effective tool for the control of uncertain nonlinear systems since it overcomes the main drawbacks of the classical sliding mode control, i.e., the chattering phenomenon and the large control effort. Its real implementation implies very simple control laws and assures an improvement of the sliding accuracy with respect to real 1–sliding mode control.

The main difficulty in using 2–sliding mode controllers is the tuning of the parameters which characterize the various algorithms. Their values depend on the bounds of the uncertain dynamics and on the chosen sliding manifold, and only sufficient conditions for the convergence to the sliding behaviour are known. These conditions are very conservative, and, in practice, the parameters are heuristically tuned. It depends on the engineer's experience to define which of the presented algorithms is more suitable for the specific control problem, even if, in the authors' opinion, the super–twisting and the sub–optimal ones seem to be able to cover a large class of control problems with a remarkable implementation easiness.

Acknowledgments

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