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ECE 350 Lecture Notes

## 3. Wave Equation from Maxwell's Equations

## Lossless Medium

In a source free region, Maxwell's equations are

$$
\begin{align*}
\nabla \times \mathbf{H} & =\frac{\partial \mathbf{D}}{\partial t}  \tag{1}\\
\nabla \times \mathbf{E} & =-\frac{\partial \mathbf{B}}{\partial t}  \tag{2}\\
\nabla \cdot \mathbf{B} & =0  \tag{3}\\
\nabla \cdot \mathbf{D} & =0 \tag{4}
\end{align*}
$$

where $\mathbf{B}=\mu \mathbf{H}$ and $\mathbf{D}=\epsilon \mathbf{E}$. Taking the curl of (2), we have

$$
\begin{equation*}
\nabla \times(\nabla \times \mathbf{E})=-\mu \frac{\partial}{\partial t} \nabla \times \mathbf{H} \tag{5}
\end{equation*}
$$

Substituting (1) into (5), we obtain

$$
\begin{equation*}
\nabla \times \nabla \times \mathbf{E}=-\mu \epsilon \frac{\partial^{2}}{\partial t^{2}} \mathbf{E} \tag{6}
\end{equation*}
$$

Making use of the vector identity that

$$
\begin{equation*}
\nabla \times \nabla \times \mathbf{E}=\nabla(\nabla \cdot \mathbf{E})-\nabla^{2} \mathbf{E} \tag{7}
\end{equation*}
$$

we have

$$
\begin{equation*}
\nabla(\nabla \cdot \mathbf{E})-\nabla^{2} \mathbf{E}=-\mu \epsilon \frac{\partial^{2}}{\partial t^{2}} \mathbf{E} \tag{8}
\end{equation*}
$$

Since the region is source free, and $\nabla \cdot \mathbf{E}=0$, we have

$$
\begin{equation*}
\nabla^{2} \mathbf{E}=\mu \epsilon \frac{\partial^{2}}{\partial t^{2}} \mathbf{E} \tag{9}
\end{equation*}
$$

which is the vector wave equation in freespace where $\nabla \cdot \mathbf{E}=0$. Similarly, we can show that

$$
\begin{equation*}
\nabla^{2} \mathbf{H}=\mu \epsilon \frac{\partial^{2}}{\partial t^{2}} \mathbf{H} \tag{10}
\end{equation*}
$$

if $\nabla \cdot \mathbf{H}=0$, which is, of course, true in free space.

## Plane Wave Solutions to the Vector Wave Equations

The condition for arriving at Equation (9) is that $\nabla \cdot \mathbf{E}=0$. We can have solutions of the form

$$
\begin{align*}
& \mathbf{E}=\hat{x} E_{x}(z, t),  \tag{11}\\
& \mathbf{E}=\hat{y} E_{y}(z, t), \tag{12}
\end{align*}
$$

but not

$$
\begin{equation*}
\mathbf{E}=\hat{z} E_{z}(z, t) \tag{13}
\end{equation*}
$$

because (13) violates $\nabla \cdot \mathbf{E}=0$ unless $E_{z}$ is independent of $z$. If $\mathbf{E}$ is of the form (11), then

$$
\begin{equation*}
\nabla^{2} \mathbf{E}=\hat{x}\left(\frac{\partial^{2}}{\partial x^{2}}+\frac{\partial^{2}}{\partial y^{2}}+\frac{\partial^{2}}{\partial z^{2}}\right) E_{x}(z, t)=\hat{x} \frac{\partial^{2}}{\partial z^{2}} E_{x} \tag{14}
\end{equation*}
$$

with both $\frac{\partial^{2}}{\partial x^{2}}$ and $\frac{\partial^{2}}{\partial y^{2}}$ equal to zero. Then (9) becomes

$$
\begin{equation*}
\frac{\partial^{2}}{\partial z^{2}} E_{x}(z, t)-\mu \epsilon \frac{\partial^{2}}{\partial t^{2}} E_{x}(z, t)=0 \tag{15}
\end{equation*}
$$

Similarly, if $\mathbf{H}=\hat{y} H_{y}(z, t),(10)$ becomes

$$
\begin{equation*}
\frac{\partial^{2}}{\partial z^{2}} H_{y}(z, t)-\mu \epsilon \frac{\partial^{2}}{\partial t^{2}} H_{y}(z, t)=0 \tag{16}
\end{equation*}
$$

Equations (15) and (16) are scalar, one dimensional wave equations of the form

$$
\begin{equation*}
\frac{\partial^{2}}{\partial z^{2}} y(z, t)-\frac{1}{v^{2}} \frac{\partial^{2}}{\partial t^{2}} y(z, t)=0 \tag{17}
\end{equation*}
$$

where $v=1 / \sqrt{\mu \epsilon}$. The solution to (17) is of the form $y=f(z+a t)$. We can show that

$$
\begin{align*}
\frac{\partial}{\partial z} f & =f^{\prime}(z+a t), & \frac{\partial f}{\partial t} & =a f^{\prime}(z+a t)  \tag{18}\\
\frac{\partial^{2}}{\partial z^{2}} f & =f^{\prime \prime}(z+a t), & \frac{\partial^{2} f}{\partial t^{2}} & =a^{2} f^{\prime \prime}(z+a t) \tag{19}
\end{align*}
$$

Substituting (19) into (17), we have

$$
\begin{equation*}
f^{\prime \prime}(z+a t)-\frac{a^{2}}{v^{2}} f^{\prime \prime}(z+a t)=0 \tag{20}
\end{equation*}
$$

which is possible only if $a= \pm v$. Hence, the general solution to the wave equation is

$$
\begin{equation*}
y=f(z-v t)+g(z+v t) \tag{21}
\end{equation*}
$$

where $f$ and $g$ are arbitrary functions.

The solution $f(z-v t)$ moves in the positive $z$-direction for increasing $t$.





The solution $g(z+v t)$ moves in the negative $z$-direction for increasing $t$.

The shapes of the functions $f$ and $g$ are undistorted as they move along. We can observe wavelike behavior in a pond when we drop a pebble into it. Solutions to (9) and (10) that correspond to a plane wave is of the form

$$
\begin{equation*}
\mathbf{E}=\hat{x} f_{1}(z-v t), \quad \mathbf{H}=\hat{y} f_{2}(z-v t) . \tag{22}
\end{equation*}
$$

The wave is propagating in the $z$-direction, but the electric and magnetic fields are transverse to the direction of propagation. Such a wave is known as the Transverse Electro Magnetic wave or TEM wave.

If one substitutes (22) into Equation (2), one has

$$
\begin{equation*}
\nabla \times \mathbf{E}=\hat{y} \frac{\partial}{\partial z} E_{x}=-\mu \frac{\partial}{\partial t} \mathbf{H} \tag{23}
\end{equation*}
$$

or

$$
\begin{equation*}
\frac{\partial}{\partial z} f_{1}(z-v t)=-\mu \frac{\partial}{\partial t} f_{2}(z-v t) \tag{24}
\end{equation*}
$$

or

$$
\begin{equation*}
f_{1}^{\prime}(z-v t)=\mu v f_{2}^{\prime}(z-v t) \tag{25}
\end{equation*}
$$

or

$$
\begin{equation*}
f_{2}(z-v t)=\sqrt{\frac{\epsilon}{\mu}} f_{1}(z-v t) \tag{26}
\end{equation*}
$$

Hence, for a plane TEM wave,

$$
\begin{equation*}
\frac{E_{x}}{H_{y}}=\sqrt{\frac{\mu}{\epsilon}}=377 \Omega, \quad \text { for free space. } \tag{27}
\end{equation*}
$$

The quantity

$$
\begin{equation*}
Z=\sqrt{\frac{\mu}{\epsilon}} \tag{28}
\end{equation*}
$$

is also known as the intrinsic impedance of free-space.

