Aanvulling syllabus "Lie theorie voor natuurkundigen" van G.J. Heckman (Mathematisch Instituut KU Nijmegen) door P.E.S. Wormer (Instituut voor Theoretische Chemie KU Nijmegen).

September 12, 2003.

Introduction

Prof. Heckman gives in his lecture notes (in Dutch) a quantum mechanical/group theoretical treatment of the Kepler problem. He mentions without proof a few properties of the Laplace-Runge-Lenz-Pauli vector, which are used in his subsequent development. In these notes these properties are formulated in Theorem 2 and proved. The proofs are made somewhat more transparent by introduction of the vector operator and its properties. Considering the length of the present notes, it is evident why prof. Heckman skipped the proof.

Vector operators

Consider two arbitrary linear operators A and B acting on the same linear space. Then we define the operator $\mathrm{Ad}\,A$ and its n^{th} power ($n = 1, 2, \ldots$) acting on the operator B, by

$$(\mathrm{Ad}\,A)^0 B \equiv B \tag{1}$$

$$(\mathrm{Ad}\,A)^n B \equiv \underbrace{[A, [A, [A, \cdots, B]] \ldots]}_{n \text{ times}} \tag{2}$$

The following result is a well-known lemma needed in the proof of the Baker-Campbell-Hausdorff theorem.

$$e^A\, B\, e^{-A} = \exp[\mathrm{Ad}\,(A)]B \equiv \sum_{k=0}^{\infty} \frac{(\mathrm{Ad}\,A)^k}{k!} B$$

The proof of this result, although not difficult, is skipped.

Remember that rotation of a vector $\boldsymbol{a} \in \mathbb{R}^3$ around a unit vector $\boldsymbol{n}$ over an angle ψ, which moves $\boldsymbol{a}$ to $\boldsymbol{a}'$, is given by the operator

$$\boldsymbol{a}' = \mathcal{R}(\boldsymbol{n}, \psi)\, \boldsymbol{a} \equiv \boldsymbol{a} \cos\psi + \boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{a})(1 - \cos\psi) + (\boldsymbol{n} \times \boldsymbol{a}) \sin\psi.$$

Definitions

1. Angular momentum operator: $\boldsymbol{L} \equiv \boldsymbol{x} \times \boldsymbol{p}$, where $\boldsymbol{p} = -i\boldsymbol{\nabla}$.

2. For the definition of the vector operator $\boldsymbol{A} \equiv (A_1, A_2, A_3)$ we consider a rotation around a unit vector $\boldsymbol{n}$ over an angle ψ. If $\boldsymbol{A}$ satisfies

$$\begin{aligned} e^{-i\psi \boldsymbol{n}\cdot\boldsymbol{L}}\,\boldsymbol{A}\,e^{i\psi\boldsymbol{n}\cdot\boldsymbol{L}} &= \mathcal{R}(\boldsymbol{n},-\psi)\boldsymbol{A} \\ &\equiv \boldsymbol{A}\cos\psi + \boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{A})(1-\cos\psi) - (\boldsymbol{n}\times\boldsymbol{A})\sin\psi, \end{aligned} \tag{3}$$

then it is a vector operator.

Sum over repeated indices is implied everywhere and recall that ϵ_{ijk}, the antisymmetric Levi-Civita tensor, obeys the following contraction rule,

$$\epsilon_{kia}\epsilon_{bja} = \delta_{kb}\delta_{ij} - \delta_{kj}\delta_{ib}. \tag{4}$$

Theorem 1. *An operator $\boldsymbol{A} = (A_1, A_2, A_3)$ is a vector operator if and only if it satisfies the commutation relations*

$$[L_i, A_j] = i\epsilon_{ijk}A_k. \tag{5}$$

In order to prove this theorem we will need the following lemma.

Lemma 1. *Let $\boldsymbol{n}$ be a unit vector and let the components of $\boldsymbol{A}$ satisfy the commutation relations of Eq. (5), then*

$$(\mathrm{Ad}\,\boldsymbol{n}\cdot\boldsymbol{L})^{2k+1}\boldsymbol{A} = (\mathrm{Ad}\,\boldsymbol{n}\cdot\boldsymbol{L})\boldsymbol{A} \qquad \text{for } k \geq 0, \tag{6}$$

$$(\mathrm{Ad}\,\boldsymbol{n}\cdot\boldsymbol{L})^{2k}\boldsymbol{A} = \boldsymbol{A} - (\boldsymbol{n}\cdot\boldsymbol{A})\boldsymbol{n} \quad \text{for } k \geq 1. \tag{7}$$

Proof. The proof is by mathematical induction. Using Eq. (4) we find

$$\begin{aligned} (\mathrm{Ad}\,\boldsymbol{n}\cdot\boldsymbol{L})^2 A_i &= [\boldsymbol{n}\cdot\boldsymbol{L}, [\boldsymbol{n}\cdot\boldsymbol{L}, A_i]] \\ &= n_j n_k [L_j, [L_k, A_i]] = n_j n_k i\epsilon_{kia}[L_j, A_a] \\ &= -n_j n_k \epsilon_{kia}\epsilon_{jab}A_b = n_j n_k(\delta_{kj}\delta_{ib} - \delta_{kb}\delta_{ij})A_b \\ &= A_i - \boldsymbol{n}\cdot\boldsymbol{A}n_i, \end{aligned}$$

which proves the second statement for $k = 1$. It is easily shown that $[\boldsymbol{n}\cdot\boldsymbol{L}, \boldsymbol{n}\cdot\boldsymbol{A}]$= 0. Indeed, the expression

$$[\boldsymbol{n}\cdot\boldsymbol{L}, \boldsymbol{n}\cdot\boldsymbol{A}] = in_i n_j \epsilon_{ijk}A_k$$

is a product of $n_i n_j$, which is symmetric in i and j and ϵ_{ijk}, which is antisymmetric in i and j. Consider now

$$\begin{aligned} (\mathrm{Ad}\,\boldsymbol{n}\cdot\boldsymbol{L})^3\boldsymbol{A} &= (\mathrm{Ad}\,\boldsymbol{n}\cdot\boldsymbol{L})(\mathrm{Ad}\,\boldsymbol{n}\cdot\boldsymbol{L})^2\boldsymbol{A} \\ &= [\boldsymbol{n}\cdot\boldsymbol{L}, \boldsymbol{A} - (\boldsymbol{n}\cdot\boldsymbol{A})\boldsymbol{n}] = (\mathrm{Ad}\,\boldsymbol{n}\cdot\boldsymbol{L})\boldsymbol{A}, \end{aligned}$$

which proves the first statement for $k=1$. For $k=0$ this statement is an identity. The general induction step is easy

$$\begin{aligned}(\mathrm{Ad}\,\boldsymbol{n}\cdot\boldsymbol{L})^{2k}\boldsymbol{A} &= (\mathrm{Ad}\,\boldsymbol{n}\cdot\boldsymbol{L})(\mathrm{Ad}\,\boldsymbol{n}\cdot\boldsymbol{L})^{2k-1}\boldsymbol{A}\\ &= [\boldsymbol{n}\cdot\boldsymbol{L},[\boldsymbol{n}\cdot\boldsymbol{L},\boldsymbol{A}]] = \boldsymbol{A}-\boldsymbol{n}\cdot\boldsymbol{A}\boldsymbol{n}.\end{aligned}$$

And

$$\begin{aligned}(\mathrm{Ad}\,\boldsymbol{n}\cdot\boldsymbol{L})^{2k+1}\boldsymbol{A} &= (\mathrm{Ad}\,\boldsymbol{n}\cdot\boldsymbol{L})(\mathrm{Ad}\,\boldsymbol{n}\cdot\boldsymbol{L})^{2k}\boldsymbol{A}\\ &= [\boldsymbol{n}\cdot\boldsymbol{L},\boldsymbol{A}-(\boldsymbol{n}\cdot\boldsymbol{A})\boldsymbol{n}] = (\mathrm{Ad}\,\boldsymbol{n}\cdot\boldsymbol{L})\boldsymbol{A}.\end{aligned}$$

□

Proof of Theorem 1. If $\boldsymbol{A}$ satisfies the commutation relations, then

$$\begin{aligned}\exp[\mathrm{Ad}\,-i\psi(\boldsymbol{n}\cdot\boldsymbol{L})]\boldsymbol{A} &= \boldsymbol{A}+\sum_{k=1}^{\infty}\frac{(-i\psi)^{2k}}{(2k)!}(\mathrm{Ad}\,\boldsymbol{n}\cdot\boldsymbol{L})^{2k}\boldsymbol{A}\\ &\quad+\sum_{k=0}^{\infty}\frac{(-i\psi)^{2k+1}}{(2k+1)!}(\mathrm{Ad}\,\boldsymbol{n}\cdot\boldsymbol{L})^{2k+1}\boldsymbol{A}\\ &= \boldsymbol{A}+\sum_{k=1}^{\infty}(-1)^k\frac{\psi^{2k}}{(2k)!}\big(\boldsymbol{A}-(\boldsymbol{n}\cdot\boldsymbol{A})\boldsymbol{n}\big)\\ &\quad+\sum_{k=0}^{\infty}(-1)^k\frac{\psi^{2k+1}}{(2k+1)!}[-i\boldsymbol{n}\cdot\boldsymbol{L},\boldsymbol{A}]\\ &= \boldsymbol{A}+(\cos\psi-1)\big(\boldsymbol{A}-(\boldsymbol{n}\cdot\boldsymbol{A})\boldsymbol{n}\big)-\sin\psi\,\boldsymbol{n}\times\boldsymbol{A}\\ &= \mathcal{R}(\boldsymbol{n},-\psi)\,\boldsymbol{A}.\end{aligned}$$

If $\boldsymbol{A}$ is a vector operator it satisfies Eq. (3). Differentiate this equation with respect to ψ at $\psi=0$

$$-i(\boldsymbol{n}\cdot\boldsymbol{L})\boldsymbol{A}+i\boldsymbol{A}(\boldsymbol{n}\cdot\boldsymbol{L}) = i[\boldsymbol{A},\boldsymbol{n}\cdot\boldsymbol{L}] = -\boldsymbol{n}\times\boldsymbol{A}.$$

Then, taking for $\boldsymbol{n}$ the coordinate axis $\boldsymbol{e}_i$, i.e., $(\boldsymbol{e}_i)_k=\delta_{ik}$,

$$i[A_j,L_i] = -\left(\boldsymbol{e}_i\times\boldsymbol{A}\right)_j = -\epsilon_{jkl}\delta_{ik}A_l = \epsilon_{ijl}A_l.$$

□

Corollary: $[L_i,A_i]=0$ and $\boldsymbol{L}\cdot\boldsymbol{A}=\boldsymbol{A}\cdot\boldsymbol{L}$.

Lemma 2.

1. *$\boldsymbol{x}$ and $\boldsymbol{p}$ are vector operators.*

2. *The outer product of any two vector operators is a vector operator.*

Proof. Use $[x_i, x_j] = [p_i, p_j] = 0$ for all $i, j = 1, 2, 3$. Use also $[p_i, x_j] = -i\delta_{ij}$.

$$[L_i, x_j] = \epsilon_{abi}[x_a p_b, x_j] = \epsilon_{abi} x_a [p_b, x_j] = \epsilon_{abi}(-i)\delta_{bj} = i\epsilon_{ija} x_a.$$

The proof that $\boldsymbol{p}$ is a vector operator is analogous. Consider the vector operators $\boldsymbol{A}$ and $\boldsymbol{B}$. It is easily proved by the use of Eq. (4) that

$$[L_i, (\boldsymbol{A} \times \boldsymbol{B})_j] \equiv \epsilon_{abj}[L_i, A_a B_b] = iA_i B_j - iA_j B_i = i\epsilon_{ijk}(\boldsymbol{A} \times \boldsymbol{B})_k.$$

□

Corollary: The angular momentum operator $\boldsymbol{L}$ is a vector operator.

Lemma 3. *Any vector operator $\boldsymbol{A}$ satisfies*

$$\boldsymbol{A} \times \boldsymbol{L} = -\boldsymbol{L} \times \boldsymbol{A} + 2i\boldsymbol{A}. \tag{8}$$

Proof.

$$(\boldsymbol{A} \times \boldsymbol{L})_i = \epsilon_{jki} A_j L_k = \epsilon_{jki}(L_k A_j - i\epsilon_{kjl} A_l).$$

By inspection it is shown that $\epsilon_{jki}\epsilon_{kjl} = -2\delta_{il}$, then

$$\boldsymbol{A} \times \boldsymbol{L} = -\boldsymbol{L} \times \boldsymbol{A} + 2i\boldsymbol{A}.$$

□

Corollary: $\boldsymbol{L} \times \boldsymbol{L} = i\boldsymbol{L}$.

Because $[x_i, x_j] = 0$ and $[p_i, p_j] = 0$ it follows that

$$\boldsymbol{x} \times \boldsymbol{x} = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{p} \times \boldsymbol{p} = \boldsymbol{0}. \tag{9}$$

Lemma 4. *Any three vector operators satisfy:*

$$\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{A} \times \boldsymbol{B}) \cdot \boldsymbol{C}. \tag{10}$$

Proof.

$$\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = A_i(\epsilon_{ijk} B_j C_k) = (\epsilon_{kij} A_i B_j) C_k = (\boldsymbol{A} \times \boldsymbol{B}) \cdot \boldsymbol{C}.$$

□

Lemma 5.

$$\boldsymbol{x} \cdot \boldsymbol{L} = \boldsymbol{L} \cdot \boldsymbol{x} = 0 \tag{11}$$

$$\boldsymbol{p} \cdot \boldsymbol{L} = \boldsymbol{L} \cdot \boldsymbol{p} = 0 \tag{12}$$

Proof. $\boldsymbol{x} \cdot \boldsymbol{L} = \boldsymbol{x} \cdot (\boldsymbol{x} \times \boldsymbol{p}) = (\boldsymbol{x} \times \boldsymbol{x}) \cdot \boldsymbol{p} = 0 = \boldsymbol{L} \cdot \boldsymbol{x}$,

$\boldsymbol{L} \cdot \boldsymbol{p} = (\boldsymbol{x} \times \boldsymbol{p}) \cdot \boldsymbol{p} = \boldsymbol{x} \cdot (\boldsymbol{p} \times \boldsymbol{p}) = 0 = \boldsymbol{p} \cdot \boldsymbol{L}$, where we used Eqs. (10) and (9) and the fact that both $\boldsymbol{x}$ and $\boldsymbol{p}$ are vector operators. □

The Runge-Lenz-Pauli operator

Definitions

1. Runge-Lenz-Pauli operator

$$\boldsymbol{K} \equiv \frac{1}{2}\,\boldsymbol{L} \times \boldsymbol{p} - \frac{1}{2}\,\boldsymbol{p} \times \boldsymbol{L} + \frac{\boldsymbol{x}}{r}.$$

   is a vector operator (by second statement of lemma 2).

2. Hydrogen atom Hamiltonian:

$$H \equiv \frac{1}{2}(p^2 - \frac{2}{r}).$$

Theorem 2.

$$\begin{aligned}
\left[L_i, K_j\right] &= i\epsilon_{ijk}K_k \\
\boldsymbol{K} \cdot \boldsymbol{L} &= \boldsymbol{L} \cdot \boldsymbol{K} = 0 \\
K^2 &= 2H(L^2 + 1) + 1 \\
\left[H, \boldsymbol{K}\right] &= \boldsymbol{0} \\
\left[K_i, K_j\right] &= -2i\epsilon_{ijk}L_k H
\end{aligned}$$

Proof. The first assertion follows directly from the fact that $\boldsymbol{K}$ is a vector operator. Consider $\boldsymbol{L} \cdot \boldsymbol{K}$:

$$\begin{aligned}
\boldsymbol{L} \cdot (\boldsymbol{L} \times \boldsymbol{p}) &= (\boldsymbol{L} \times \boldsymbol{L}) \cdot \boldsymbol{p} = i\boldsymbol{L} \cdot \boldsymbol{p} = 0 \\
\boldsymbol{L} \cdot (\boldsymbol{p} \times \boldsymbol{L}) &= \boldsymbol{L} \cdot (-\boldsymbol{L} \times \boldsymbol{p} + 2i\boldsymbol{p}) = 0 \\
\boldsymbol{L} \cdot \frac{\boldsymbol{x}}{r} &= \frac{1}{r}(\boldsymbol{L} \cdot \boldsymbol{x}) + \boldsymbol{x} \cdot (\boldsymbol{L}\frac{1}{r}) + \frac{1}{r}(\boldsymbol{x} \cdot \boldsymbol{L}) = 0
\end{aligned}$$

where the middle term vanishes because $\boldsymbol{L}(V(r)) = \boldsymbol{0}$ for any central symmetric function $V(r)$. Hence, $\boldsymbol{L}\cdot\boldsymbol{K} = 0$. The operator $\boldsymbol{K}$ is a vector operator, so that $\boldsymbol{K} \cdot \boldsymbol{L} = \boldsymbol{L} \cdot \boldsymbol{K} = 0$. □

Turning to K^2 we need two more lemmas.

Lemma 6.

$$\boldsymbol{p} \cdot (\boldsymbol{p} \times \boldsymbol{L}) = 0 \qquad (13)$$

$$\boldsymbol{p} \cdot (\boldsymbol{L} \times \boldsymbol{p}) = 2ip^2 \qquad (14)$$

Proof. $\boldsymbol{p}\cdot(\boldsymbol{p}\times\boldsymbol{L}) = (\boldsymbol{p}\times\boldsymbol{p})\cdot\boldsymbol{L} = 0$.
$\boldsymbol{p}\cdot(\boldsymbol{L}\times\boldsymbol{p}) = \boldsymbol{p}\cdot(-\boldsymbol{p}\times\boldsymbol{L}+2i\boldsymbol{p}) = 2ip^2$ [by Eq. (8)]. □

Lemma 7.

$$\begin{aligned}
(\boldsymbol{p}\times\boldsymbol{L})\cdot(\boldsymbol{p}\times\boldsymbol{L}) &= p^2\,L^2 &(15)\\
(\boldsymbol{L}\times\boldsymbol{p})\cdot(\boldsymbol{p}\times\boldsymbol{L}) &= -p^2\,L^2 &(16)\\
(\boldsymbol{L}\times\boldsymbol{p})\cdot(\boldsymbol{L}\times\boldsymbol{p}) &= p^2\,L^2 &(17)\\
(\boldsymbol{p}\times\boldsymbol{L})\cdot(\boldsymbol{L}\times\boldsymbol{p}) &= -p^2\,L^2-4p^2 &(18)
\end{aligned}$$

Proof. Use Eq. (4), then

$$\begin{aligned}
(\boldsymbol{p}\times\boldsymbol{L})\cdot(\boldsymbol{p}\times\boldsymbol{L}) &= \epsilon_{ijk}\epsilon_{abk}p_iL_jp_aL_b = p_iL_jp_iL_j - p_iL_jp_jL_i\\
&= p_i(p_iL_j - i\epsilon_{ijk}p_k)L_j - p_i(\boldsymbol{L}\cdot\boldsymbol{p})L_i\\
&= p^2L^2 + i(\boldsymbol{p}\times\boldsymbol{p})\cdot\boldsymbol{L} = p^2\,L^2.
\end{aligned}$$

It is easily shown with the use of Eqs. (8), (14) and (15) that

$$\begin{aligned}
(\boldsymbol{p}\times\boldsymbol{L})\cdot(\boldsymbol{L}\times\boldsymbol{p}) &= (\boldsymbol{p}\times\boldsymbol{L})\cdot(-\boldsymbol{p}\times\boldsymbol{L}+2i\boldsymbol{p}) = -p^2\,L^2 + 2i\boldsymbol{p}\cdot(\boldsymbol{L}\times\boldsymbol{p})\\
&= -p^2\,L^2 - 4p^2.
\end{aligned}$$

Equations (16) and (17) are proved likewise. □

The operator K^2 consists of 9 terms, 4 of which are given by the previous lemma. Consider

$$(\boldsymbol{L}\times\boldsymbol{p})\cdot\boldsymbol{x} = \boldsymbol{L}\cdot(\boldsymbol{p}\times\boldsymbol{x}) = -L^2,$$

so that (recalling that L^2 commutes with $1/r$) a further term of K^2 is:

$$(\boldsymbol{L}\times\boldsymbol{p})\cdot\boldsymbol{x}\,\frac{1}{r} = -\frac{L^2}{r}. \tag{19}$$

Using Eq. (8) we find

$$(\boldsymbol{p}\times\boldsymbol{L})\cdot\boldsymbol{x}\,\frac{1}{r} = \frac{L^2}{r} + 2i\boldsymbol{p}\cdot\boldsymbol{x}\frac{1}{r}.$$

where

$$\begin{aligned}
i\boldsymbol{p}\cdot\boldsymbol{x}\frac{1}{r} &= \boldsymbol{\nabla}\cdot\boldsymbol{x}\frac{1}{r} = (3+\boldsymbol{x}\cdot\boldsymbol{\nabla})\,\frac{1}{r}\\
&= \frac{3}{r} - \frac{\boldsymbol{x}\cdot\boldsymbol{x}}{r^3} + \frac{1}{r}\boldsymbol{x}\cdot\boldsymbol{\nabla} = \frac{2}{r} + \frac{i}{r}\boldsymbol{x}\cdot\boldsymbol{p}.
\end{aligned}$$

Hence,

$$(\boldsymbol{p}\times\boldsymbol{L})\cdot\boldsymbol{x}\,\frac{1}{r} = \frac{L^2}{r} + \frac{4}{r} + \frac{2i}{r}\boldsymbol{x}\cdot\boldsymbol{p}. \tag{20}$$

Use that

$$\frac{1}{r}\boldsymbol{x}\cdot(\boldsymbol{p}\times\boldsymbol{L}) = \frac{L^2}{r} \tag{21}$$

$$\frac{1}{r}\boldsymbol{x}\cdot(\boldsymbol{L}\times\boldsymbol{p}) = -\frac{L^2}{r} + \frac{2i}{r}\boldsymbol{x}\cdot\boldsymbol{p} \tag{22}$$

and we get finally from Eqs. (15), (18), (16), (17), (19), (20), (21), (22) and from the definition of r: $\boldsymbol{x}\cdot\boldsymbol{x}/r^2 = 1$,

$$K^2 = (p^2 - \frac{2}{r})L^2 + p^2 - \frac{2}{r} + 1 \equiv 2H(L^2+1)+1,$$

which proves the third statement of Theorem 2.

We will now show that $[H,\boldsymbol{K}] = \boldsymbol{0}$. Because p^2 is rotationally invariant, $[L_i, p^2] = 0$ and since $[p_i, p^2] = 0$ it follows directly that $[(\boldsymbol{L}\times\boldsymbol{p})_i, p^2] = \epsilon_{ijk}[L_j p_k, p^2] = 0$. Similarly $[\boldsymbol{p}\times\boldsymbol{L}, p^2] = 0$. The operator $\boldsymbol{x}$ commutes with any operator depending on x_i only, hence

$$\left[\frac{\boldsymbol{x}}{r}, \frac{-1}{r}\right] = \boldsymbol{0}.$$

It is fairly tedious to show the mutual cancellation of the two remaining terms. We will need

$$[p_i, \frac{1}{r}] = \frac{i x_i}{r^3},$$

and the equations in the following lemma.

Lemma 8.

$$\boldsymbol{x}\times\boldsymbol{L} = \boldsymbol{x}(\boldsymbol{x}\cdot\boldsymbol{p}) - r^2\boldsymbol{p} \tag{23}$$

$$\boldsymbol{L}\times\boldsymbol{x} = -(\boldsymbol{p}\cdot\boldsymbol{x})\boldsymbol{x} + \boldsymbol{p}r^2 \tag{24}$$

Proof.

$$(\boldsymbol{x}\times\boldsymbol{L})_i = \epsilon_{ijk}\epsilon_{abk}x_j x_a p_b = x_j x_i p_k - x_j x_j p_i = x_i(\boldsymbol{x}\cdot\boldsymbol{p}) - r^2 p_i$$

$$(\boldsymbol{L}\times\boldsymbol{x})_i = -\epsilon_{ikj}\epsilon_{abj}p_a x_b x_k = p_i x_k x_k - p_k x_i x_k = p_i r^2 - (\boldsymbol{p}\cdot\boldsymbol{x})x_i.$$

□

The first non-vanishing term in $[H,\boldsymbol{K}]$ is given by the following lemma.

Lemma 9.

$$[\frac{1}{2}p^2, \frac{\boldsymbol{x}}{r}] = \frac{i}{2r^3}(\boldsymbol{x} \times \boldsymbol{L} - \boldsymbol{L} \times \boldsymbol{x})$$

Proof. Consider

$$[p_i, \frac{x_j}{r}] = x_j[p_i, \frac{1}{r}] + \frac{1}{r}[p_i, x_j] = i(\frac{x_i x_j}{r^3} - \frac{\delta_{ij}}{r}) \tag{25}$$

and

$$\begin{aligned} [p^2, \frac{x_j}{r}] &= p_i[p_i, \frac{x_j}{r}] + [p_i, \frac{x_j}{r}]p_i \\ &= ip_i(\frac{x_i x_j}{r^3} - \frac{\delta_{ij}}{r}) + i(\frac{x_i x_j}{r^3} - \frac{\delta_{ij}}{r})p_i. \end{aligned}$$

Using Eqs. (23) and (24)

$$\begin{aligned} [p^2, \frac{\boldsymbol{x}}{r}] &= i\big[\boldsymbol{p} \cdot \boldsymbol{x}\frac{\boldsymbol{x}}{r^3} + \frac{\boldsymbol{x}}{r^3}\boldsymbol{x} \cdot \boldsymbol{p} - \boldsymbol{p}\frac{1}{r} - \frac{1}{r}\boldsymbol{p}\big] \\ &= \frac{i}{r^3}\boldsymbol{x} \times \boldsymbol{L} - \boldsymbol{L} \times \boldsymbol{x}\frac{i}{r^3}. \end{aligned}$$

Observing that $\boldsymbol{x}$ and $\boldsymbol{L}$ commute with $1/r^3$, the result follows. □

The last non-vanishing term of $[H, \boldsymbol{K}]$ is given by the following lemma.

Lemma 10.

$$[-\frac{1}{r}, \frac{1}{2}(\boldsymbol{L} \times \boldsymbol{p} - \boldsymbol{p} \times \boldsymbol{L})] = \frac{i}{2r^3}\big[(\boldsymbol{L} \times \boldsymbol{x}) - (\boldsymbol{x} \times \boldsymbol{L})\big].$$

Proof. Use $[L_i, 1/r] = [L_i, 1/r^3] = 0$, then

$$\begin{aligned} \epsilon_{ijk}[\frac{1}{r}, L_i p_j + p_j L_i] &= \epsilon_{ijk}\Big\{L_i[\frac{1}{r}, p_j] + [\frac{1}{r}, p_j]L_i\Big\} \\ &= \epsilon_{ijk}\big(L_i\frac{-ix_j}{r^3} + \frac{-ix_j}{r^3}L_i\big) \\ &= \frac{-i}{r^3}\big[(\boldsymbol{L} \times \boldsymbol{x})_k - (\boldsymbol{x} \times \boldsymbol{L})_k\big]. \end{aligned}$$

□

Finally we see that the results of the last two lemmas cancel each other, so that indeed $[H, \boldsymbol{K}] = \boldsymbol{0}$ (the fourth assertion of Theorem 2).

To evaluate $[K_i, K_j]$ we need a few more lemmas.

Lemma 11.

$$[(\boldsymbol{L} \times \boldsymbol{p})_i, p_j] = i(p_i p_j - \delta_{ij} p^2)$$

Proof.

$$\begin{aligned} [(\boldsymbol{L} \times \boldsymbol{p})_i, p_j] &= \epsilon_{abi}[L_a p_b, p_j] = \epsilon_{abi}[L_a, p_j] p_b \\ &= \epsilon_{abi} i \epsilon_{ajk} p_k p_b = i(p_i p_j - \delta_{ij} p_k p_k). \end{aligned}$$

□

Note that this lemma implies that $[(\boldsymbol{L} \times \boldsymbol{p})_i, p_j]$ is symmetric in i and j. Using Eq. (8) and this symmetry, one shows easily that

$$\frac{1}{4}[(\boldsymbol{L} \times \boldsymbol{p} - \boldsymbol{p} \times \boldsymbol{L})_i, (\boldsymbol{L} \times \boldsymbol{p} - \boldsymbol{p} \times \boldsymbol{L})_j] = [(\boldsymbol{L} \times \boldsymbol{p})_i, (\boldsymbol{L} \times \boldsymbol{p})_j]. \qquad (26)$$

Lemma 12.

$$[(\boldsymbol{L} \times \boldsymbol{p})_i, (\boldsymbol{L} \times \boldsymbol{p})_j] = -i\epsilon_{ijk} p^2 L_k.$$

Proof. We write $A_i \equiv (\boldsymbol{L} \times \boldsymbol{p})_i$ and use that it is a vector operator. Remember that $[p^2, L_i] = 0$.

$$\begin{aligned} [A_i, (\boldsymbol{L} \times \boldsymbol{p})_j] &= \epsilon_{abj}[A_i, L_a p_b] = \epsilon_{abj} L_a [A_i, p_b] + \epsilon_{abj}[A_i, L_a] p_b \\ &= i\epsilon_{abj} L_a (p_i p_b - \delta_{ib} p^2) - i\epsilon_{abj}\epsilon_{aik} A_k p_b \\ &= i(\boldsymbol{L} \times \boldsymbol{p})_j p_i - i\epsilon_{aij} p^2 L_a - i A_j p_i = -i\epsilon_{aij} p^2 L_a. \end{aligned}$$

□

Use of Eq. (26) and this lemma shows that the first terms of $[K_i, K_j]$ satisfy

$$\frac{1}{4}[(\boldsymbol{L} \times \boldsymbol{p} - \boldsymbol{p} \times \boldsymbol{L})_i, (\boldsymbol{L} \times \boldsymbol{p} - \boldsymbol{p} \times \boldsymbol{L})_j] = -2i\epsilon_{ijk} \frac{1}{2} p^2 L_k.$$

In order to reduce the last two terms of $[K_i, K_j]$ we introduce the following short hand notation for them:

$$Q_{ij} \equiv \frac{1}{2}[(\boldsymbol{L} \times \boldsymbol{p} - \boldsymbol{p} \times \boldsymbol{L})_i, \frac{x_j}{r}] + \frac{1}{2}[\frac{x_i}{r}, (\boldsymbol{L} \times \boldsymbol{p} - \boldsymbol{p} \times \boldsymbol{L})_j],$$

we use again Eq. (8) and the fact that $[p_i, x_j/r]$ is symmetric in i and j [cf. Eq. (25)] so that

$$Q_{ij} = [(\boldsymbol{L} \times \boldsymbol{p})_i, \frac{x_j}{r}] + [\frac{x_i}{r}, (\boldsymbol{L} \times \boldsymbol{p})_j] \qquad (27)$$

Note that $Q_{ij} = -Q_{ji}$ so that only the case $i \neq j$ must be considered. We need the following result:

Lemma 13.

$$(\boldsymbol{L} \times \boldsymbol{x})_i x_j - (\boldsymbol{L} \times \boldsymbol{x})_j x_i = r^2(p_i x_j - p_j x_i).$$

Proof. By Eq. (24):

$$(\boldsymbol{L} \times \boldsymbol{x}) \times \boldsymbol{x} = \boldsymbol{p} \times \boldsymbol{x} r^2.$$

Or,

$$\epsilon_{ijk}(\boldsymbol{L} \times \boldsymbol{x})_i x_j = r^2 \epsilon_{ijk} p_i x_j$$

Multiply by $\epsilon_{i'j'k}$ sum over k, use Eq. (4) and remember that a sum over i and j is implied,

$$(\delta_{i'i}\delta_{j'j} - \delta_{i'j}\delta_{j'i})(\boldsymbol{L} \times \boldsymbol{x})_i x_j = r^2(\delta_{i'i}\delta_{j'j} - \delta_{i'j}\delta_{j'i}) p_i x_j,$$

from which the lemma follows. □

Lemma 14.

$$Q_{ij} = 2i\epsilon_{ijk}\frac{L_k}{r}$$

Proof. Use Eqs. (25) and (27), the fact that $\boldsymbol{x}/r$ is a vector operator and dropping a term in δ_{ij}, we get

$$\begin{aligned}
[(\boldsymbol{L} \times \boldsymbol{p})_i, \frac{x_j}{r}] &= \epsilon_{abi}(L_a[p_b, \frac{x_j}{r}] + [L_a, \frac{x_j}{r}]p_b) \\
&= \epsilon_{abi} L_a \frac{i(x_b x_j - r^2 \delta_{bj})}{r^3} + i\epsilon_{abi}\epsilon_{ajk}\frac{x_k}{r} p_b \\
&= \frac{i}{r^3}(\boldsymbol{L} \times \boldsymbol{x})_i x_j - i\epsilon_{aji}\frac{L_a}{r} + i\delta_{bj}\delta_{ik}\frac{x_k}{r} p_b \\
&= \frac{i}{r^3}(\boldsymbol{L} \times \boldsymbol{x})_i x_j + i\epsilon_{ijk}\frac{L_k}{r} + i\frac{x_i}{r} p_j
\end{aligned}$$

Likewise

$$-[(\boldsymbol{L} \times \boldsymbol{p})_j, \frac{x_i}{r}] = -\frac{i}{r^3}(\boldsymbol{L} \times \boldsymbol{x})_j x_i + i\epsilon_{ijk}\frac{L_k}{r} - i\frac{x_j}{r} p_i$$

Use of the previous lemma and the observation that $[x_i, p_j] = 0$, because $i \neq j$, proves the lemma. □

The last assertion of Theorem 2 follows now by recalling that Q_{ij} is a short hand notation for the two remaining terms in $[K_i, K_j]$.