Chapter 2

Random Processes

2.1 Preview

In this chapter we illustrate methods for generating random variables and samples of random processes. We begin with the description of a method for generating random variables with a specified probability distribution function. Then we consider Gaussian and Gauss–Markov processes and illustrate a method for generating samples of such processes. The third topic that we consider is the characterization of a stationary random process by its autocorrelation in the time domain and by its power spectrum in the frequency domain. Because linear filters play a very important role in communication systems, we also consider the autocorrelation function and the power spectrum of a linearly filtered random process. The final section of this chapter deals with the characteristics of lowpass and bandpass random processes.

2.2 Generation of Random Variables

Random number generators are often used in practice to simulate the effect of noise-like signals and other random phenomena that are encountered in the physical world. Such noise is present in electronic devices and systems and usually limits our ability to communicate over large distances and to detect relatively weak signals. By generating such noise on a computer, we are able to study its effects through simulation of communication systems and to assess the performance of such systems in the presence of noise.

Most computer software libraries include a uniform random number generator. Such a random number generator generates a number between 0 and 1 with equal probability. We call the output of the random number generator a *random variable*. If A denotes such a random variable, its range is the interval $0 \leq A \leq 1$.

We know that the numerical output of a digital computer has limited precision, and as a consequence it is impossible to represent the continuum of numbers in the interval $0 \leq A \leq 1$. However, we may assume that our computer represents each output by

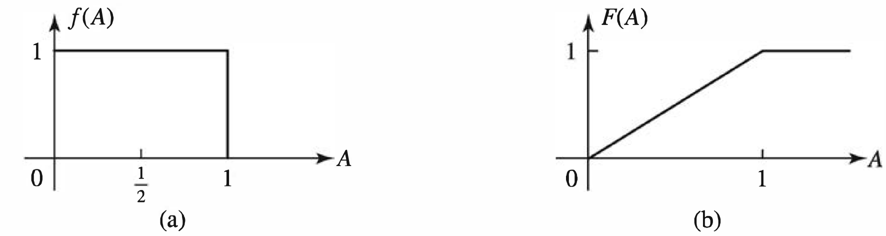

Figure 2.1: Probability density function $f(A)$ and the probability distribution function $F(A)$ of a uniformly distributed random variable A

a large number of bits in either fixed point or floating point. Consequently, for all practical purposes, the number of outputs in the interval $0 \leq A \leq 1$ is sufficiently large so that we are justified in assuming that any value in the interval is a possible output from the generator.

The uniform probability density function for the random variable A, denoted as $f(A)$, is illustrated in Figure 2.1(a). We note that the average value or mean value of A, denoted as m_A, is $m_A = \frac{1}{2}$. The integral of the probability density function, which represents the area under $f(A)$, is called the *probability distribution function* of the random variable A and is defined as

$$F(A) = \int_{-\infty}^{A} f(x)\, dx \tag{2.2.1}$$

For any random variable, this area must always be unity, which is the maximum value that can be achieved by a distribution function. Hence, for the uniform random variable A we have

$$F(1) = \int_{-\infty}^{1} f(x)\, dx = 1 \tag{2.2.2}$$

and the range of $F(A)$ is $0 \leq F(A) \leq 1$ for $0 \leq A \leq 1$. The probability distribution function is shown in Figure 2.1(b).

If we wish to generate uniformly distributed noise in an interval $(b, b+1)$, it can be accomplished simply by using the output A of the random number generator and shifting it by an amount b. Thus a new random variable B can be defined as

$$B = A + b \tag{2.2.3}$$

which now has a mean value $m_B = b + \frac{1}{2}$. For example, if $b = -\frac{1}{2}$, the random variable B is uniformly distributed in the interval $(-\frac{1}{2}, \frac{1}{2})$, as shown in Figure 2.2(a). Its probability distribution function $F(B)$ is shown in Figure 2.2(b).

A uniformly distributed random variable in the range (0,1) can be used to generate random variables with other probability distribution functions. For example, suppose that we wish to generate a random variable C with probability distribution function $F(C)$, as illustrated in Figure 2.3.

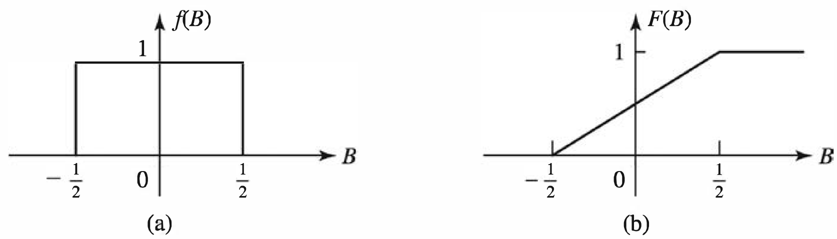

Figure 2.2: Probability density function and the probability distribution function of a zero-mean uniformly distributed random variable

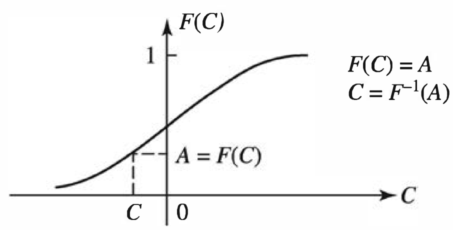

Figure 2.3: Inverse mapping from the uniformly distributed random variable A to the new random variable C

Because the range of $F(C)$ is the interval $(0,1)$, we begin by generating a uniformly distributed random variable A in the range $(0,1)$. If we set

$$F(C) = A \tag{2.2.4}$$

then

$$C = F^{-1}(A) \tag{2.2.5}$$

Thus we solve (2.2.4) for C, and the solution in (2.2.5) provides the value of C for which $F(C) = A$. By this means we obtain a new random variable C with probability distribution function $F(C)$. This inverse mapping from A to C is illustrated in Figure 2.3.

ILLUSTRATIVE PROBLEM

Illustrative Problem 2.1 Generate a random variable C that has the linear probability density function shown in Figure 2.4(a); that is,

$$f(C) = \begin{cases} \frac{1}{2}C, & 0 \le C \le 2 \\ 0, & \text{otherwise} \end{cases}$$

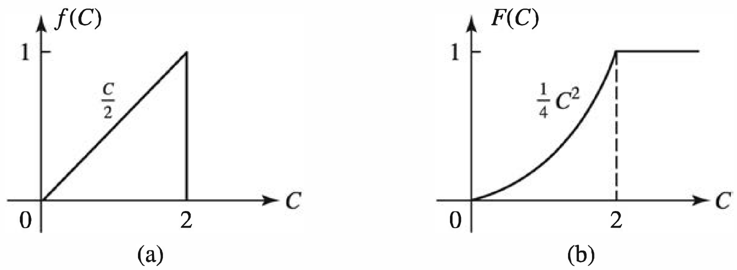

Figure 2.4: Linear probability density function and the corresponding probability distribution function

SOLUTION

This random variable has a probability distribution function

$$F(C) = \begin{cases} 0, & C < 0 \\ \frac{1}{4}C^2, & 0 \le C \le 2 \\ 1, & C > 2 \end{cases}$$

which is illustrated in Figure 2.4(b). We generate a uniformly distributed random variable A and set $F(C) = A$. Hence

$$F(C) = \frac{1}{4}C^2 = A \tag{2.2.6}$$

Upon solving for C, we obtain

$$C = 2\sqrt{A} \tag{2.2.7}$$

Thus we generate a random variable C with probability distribution function $F(C)$, as shown in Figure 2.4(b).

In Illustrative Problem 2.1 the inverse mapping $C = F^{-1}(A)$ was simple. In some cases it is not. This problem arises in trying to generate random numbers that have a normal distribution function.

Noise encountered in physical systems is often characterized by the normal, or Gaussian, probability distribution, which is illustrated in Figure 2.5. The probability density function is given by

$$f(C) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-C^2/2\sigma^2}, \qquad -\infty < C < \infty \tag{2.2.8}$$

where σ^2 is the variance of C, which is a measure of the spread of the probability density function $f(C)$. The probability distribution function $F(C)$ is the area under $f(C)$ over the range $(-\infty, C)$. Thus

$$F(C) = \int_{-\infty}^{C} f(x)\,dx \tag{2.2.9}$$

Unfortunately, the integral in (2.2.9) cannot be expressed in terms of simple functions. Consequently, the inverse mapping is difficult to achieve. A way has been found to circumvent this problem. From probability theory it is known that a Rayleigh distributed random variable R, with probability distribution function

$$F(R) = \begin{cases} 0, & R < 0 \\ 1 - e^{-R^2/2\sigma^2}, & R \geq 0 \end{cases} \tag{2.2.10}$$

is related to a pair of Gaussian random variables C and D through the transformation

$$C = R\cos\Theta \tag{2.2.11}$$

$$D = R\sin\Theta \tag{2.2.12}$$

where Θ is a uniformly distributed variable in the interval $(0, 2\pi)$. The parameter σ^2 is the variance of C and D. Because (2.2.10) is easily inverted, we have

$$F(R) = 1 - e^{-R^2/2\sigma^2} = A \tag{2.2.13}$$

and hence

$$R = \sqrt{2\sigma^2 \ln\left(\frac{1}{1-A}\right)} \tag{2.2.14}$$

where A is a uniformly distributed random variable in the interval (0,1). Now, if we generate a second uniformly distributed random variable B and define

$$\Theta = 2\pi B \tag{2.2.15}$$

then from (2.2.11) and (2.2.12) we obtain two statistically independent Gaussian distributed random variables C and D.

The method described above is often used in practice to generate Gaussian distributed random variables. As shown in Figure 2.5, these random variables have a mean value of zero and a variance σ^2. If a non-zero-mean Gaussian random variable is desired, then C and D can be translated by the addition of the mean value.

The MATLAB script that implements the preceding method for generating Gaussian distributed random variables is given next.

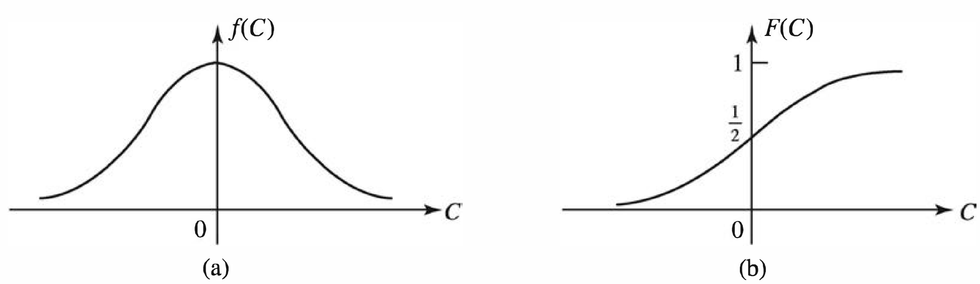

Figure 2.5: Gaussian probability density function and the corresponding probability distribution function

M-FILE

```
function [gsrv1,gsrv2]=gngauss(m,sgma)
%    [gsrv1,gsrv2]=gngauss(m,sgma)
%    [gsrv1,gsrv2]=gngauss(sgma)
%    [gsrv1,gsrv2]=gngauss
%              GNGAUSS generates two independent Gaussian random variables with mean
%              m and standard deviation sgma. If one of the input arguments is missing,
%              it takes the mean as 0.
%              If neither the mean nor the variance is given, it generates two standard
%              Gaussian random variables.
if nargin == 0,
  m=0; sgma=1;
elseif nargin == 1,
  sgma=m; m=0;
end;
u=rand;                                  % a uniform random variable in (0,1)
z=sgma*(sqrt(2*log(1/(1-u))));           % a Rayleigh distributed random variable
u=rand;                                  % another uniform random variable in (0,1)
gsrv1=m+z*cos(2*pi*u);
gsrv2=m+z*sin(2*pi*u);
```

2.2.1 Estimation of the Mean of a Random Variable

Suppose we have N statistically independent observations $x_1, x_2, \ldots, x_n$ of a random variable X. We wish to estimate the mean value of X from the N observations. The estimate of the mean value is

$$\hat{m} = \frac{1}{N} \sum_{k=1}^{N} x_k \tag{2.2.16}$$

Because $\hat{m}$ is a sum of random variables, it is also a random variable. We note that the expected value of the estimate $\hat{m}$ is

$$E[\hat{m}] = \frac{1}{N} \sum_{k=1}^{N} E[x_k] = \frac{1}{N} \cdot mN = m \tag{2.2.17}$$

where m is the actual mean of X. Thus, the estimate $\hat{m}$ is said to be *unbiased*.

The variance of the the estimate $\hat{m}$ is a measure of the spread or dispersion of $\hat{m}$ relative to its mean value. The variance of $\hat{m}$ is defined as

$$\begin{aligned} E\left[(\hat{m}-m)^2\right] &= E\left[\hat{m}^2\right] - 2E[\hat{m}]m + m^2 \\ &= E\left[\hat{m}^2\right] - m^2 \end{aligned}$$

But the $E[\hat{m}^2]$ is

$$\begin{aligned} E\left[\hat{m}^2\right] &= \frac{1}{N^2} \sum_{k=1}^{N} \sum_{n=1}^{N} E[x_k x_n] \\ &= \frac{\sigma^2}{N} + m^2 \end{aligned}$$