Prova final (21/11/2018)

GABARITO.

Questão 1. Aceleração v. desaceleração.

(a) Equação de Friedmann: $\dot{a}^2 = \frac{8\pi G}{3} a^2 \rho - K$.

Derivando: $\frac{d}{dt}\dot{a}^2 = 2\dot{a}\ddot{a} = \frac{8\pi G}{3}\left(2\dot{a}a\rho + a^2\dot{\rho}\right)$.

Mas, da equação da continuidade: $\dot{\rho} = -3H(\rho + P)$.

Portanto: $2\ddot{a} = \frac{8\pi G}{3}\left[2a\rho + \frac{a^2}{\dot{a}}(-3H)(\rho+P)\right]$

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3}\left[2\rho - 3\rho - 3P\right]$$

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)}$$

(b) $w = -\frac{1}{3} \Rightarrow$ Eq. da continuidade: $\rho \sim a^{-2}$.

Eq. de Friedmann: $\dot{a}^2 = \frac{8\pi G}{3}\rho_0 \Rightarrow \dot{a} =$ constante.

$\Rightarrow a \sim t$, $\boxed{H = \frac{1}{t}}$, $\boxed{a(t) = \frac{t}{t_0}}$

(c) Não: $\chi = \int dt/a(t) = \ln t \xrightarrow[t\to 0]{t\to\infty} \infty$.

(d) O universo expande de $a=0$ a $a=1$.

$da = \frac{da}{dt}dt = \frac{dt}{t_0}$ $\qquad \frac{\dot{a}}{a} = \frac{1}{t_0}\frac{t_0}{t} = \frac{1}{t}$

$\int_0^1 da = \int_0^{t_0} dt/t_0$ $\qquad H_0 = t_0^{-1} \Rightarrow t_0 = H_0^{-1} \approx 10$ Gyrs.

(e) Por medidas de distância: $\chi(z)$ é idêntica, mas ...

$$\begin{cases} d_L = \dfrac{(1+z)}{\sqrt{K}} \sin(\sqrt{K}\,\chi), & K > 0. \\ d_L = (1+z)\,\chi, & K = 0 \Leftarrow \\ d_L = \dfrac{(1+z)}{\sqrt{-K}} \sinh(\sqrt{-K}\,\chi), & K < 0. \end{cases}$$

Questão 2. Crescimento de Estruturas.

(a) $$\dot{\delta} + \frac{1}{a}\vec{\nabla}\cdot\vec{v} = 0 \Rightarrow a^2\dot{\delta} + a\,\vec{\nabla}\cdot\vec{v} = 0 \qquad (1)$$

$$\dot{\vec{v}} + H\vec{v} = -\frac{1}{a}\vec{\nabla}(c_s^2\delta + \phi) \Rightarrow \cancel{\frac{1}{a}}\frac{d}{dt}(a\vec{v}) = -\cancel{\frac{1}{a}}\vec{\nabla}(c_s^2\delta + \phi) \qquad (2)$$

$$\frac{d}{dt}(1) \Rightarrow \frac{d}{dt}(a^2\dot{\delta}) + \frac{d}{dt}(a\vec{\nabla}\cdot\vec{v}) = 0$$

$$\vec{\nabla}\cdot(2) \Rightarrow \frac{d}{dt}(a\nabla\cdot\vec{v}) = -\nabla^2(c_s^2\delta + \phi)$$

$$\therefore\ 2a\dot{a}\dot{\delta} + a^2\ddot{\delta} - \nabla^2(c_s^2\delta + \phi) = 0$$

$$\Rightarrow \ddot{\delta} + 2H\dot{\delta} - \frac{1}{a^2}c_s^2\nabla^2\delta - \frac{1}{a^2}\nabla^2\phi = 0 \qquad \frac{1}{a^2}\nabla^2\phi = 4\pi G\bar{\rho}_m\delta \quad (*)$$

$$\overset{(*)}{\Rightarrow} \ddot{\tilde{\delta}}_K + 2H\dot{\tilde{\delta}}_K = \left(\frac{c_s^2K^2}{a^2} - 4\pi G\bar{\rho}_m\right)\delta_K$$

onde usamos $\tilde{\delta}$ (transformada de Fourier do δ: $\nabla \to -K^2$).

(b) Escala de Jeans:

$$\frac{c_s^2 k^2}{a^2} - 4\pi G\bar{\rho}_m = 0 \Rightarrow k_J^2 = \frac{4\pi G}{c_s^2}\bar{\rho}_m a^2 \qquad (\lambda = 2\pi/k)$$

$$\Rightarrow \lambda_J = \frac{c_s}{a}\sqrt{\frac{\pi}{G\bar{\rho}_m}}$$

(c) Estático: $H=0$, $a=1$, $c_s^2 = 0$ (matéria).

$$\ddot{\delta}_m + \underbrace{4\pi G\bar{\rho}_m}_{\alpha^2}\delta_m = 0 \Rightarrow \delta_m \sim e^{\pm\alpha t}$$

$e^{-\alpha t} \to 0$, $\therefore\ \delta_m(t) \propto e^{\alpha t}$ → crescimento de estruturas é muito eficiente quando $H=0$.

(d) Universo em expansão só com matéria: $\bar{\rho}_m = \rho_0 a^{-3}$, $a \sim t^{2/3}$, $H = 2/3t$.

$$\ddot{\delta}_m + 2\left(\frac{2}{3t}\right)\dot{\delta} = -4\pi G\bar{\rho}_m \qquad 3H^2 = 3\left(\frac{2}{3t}\right)^2 = 8\pi G\bar{\rho}_m$$

$$= -\frac{3}{2}\left(\frac{2}{3t}\right)^2\delta_m$$

$$\Rightarrow \ddot{\delta}_m + \frac{4}{3t}\dot{\delta} + \frac{2}{3t^2}\delta_m = 0$$

$$\delta_m \propto t^p; \quad \dot{\delta}_m = p\,t^{p-1}; \quad \ddot{\delta}_m = p(p-1)t^{p-2}.$$

$$\Rightarrow p(p-1) + \frac{4}{3}p + \frac{2}{3} = 0$$

$$p^2 + \frac{1}{3}p + \frac{2}{3} = 0$$

$p_+ = 2/3$, $p_- = -1$
↳ cresce ↳ decai. $\Rightarrow \delta_m(t) \propto t^{2/3}$
↳ crescimento menos eficiente c/ $H \neq 0$!

(e) Universo dominado por constante cosmológica Λ:

$$a \sim e^{H_\Lambda t}, \quad H_\Lambda \text{ constante}$$

$$3H_\Lambda = 8\pi G \, (\rho_\Lambda + \rho_m)$$

↳ muito pequeno

$$\therefore \quad \ddot{\delta}_m + 2H\dot{\delta}_m \approx 0 \quad \Rightarrow \quad \delta_m \propto \begin{cases} \text{Constante} \\ e^{-2H_\Lambda t} \end{cases}$$

↳ decai exponencialmente

$\therefore$ δ_m é constante $\Rightarrow$ perturbações "congelam" no tempo.

Questão 3. Matriz de Fisher.

(a) $\theta^\alpha = \{\Omega_m, w\}$

$$F^A_{\alpha\beta} = \begin{pmatrix} 245/2 & 21 \\ 21 & 98/15 \end{pmatrix}, \quad \det F^A = \frac{245.98}{10} - (21)^2 = 1960.$$

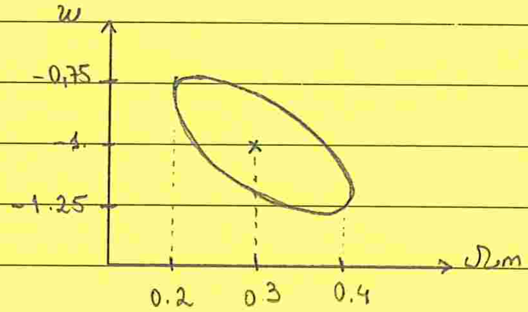
$$C^A = (F^A)^{-1} = \frac{1}{1960} \begin{pmatrix} 98/15 & -21 \\ -21 & 245/2 \end{pmatrix} = \begin{pmatrix} 0.01 & -0.0107 \\ -0.0107 & 0.0625 \end{pmatrix}$$

$$\boxed{\sigma^2_{\Omega_m} = 0.01 \Rightarrow \sigma_{\Omega_m} = 0.1}$$

$$\boxed{\sigma^2_w = 1/16 \Rightarrow \sigma_w = 0.25}$$

(b) A correlação entre Ω_m e w é negativa.

Portanto:



(c)

$$F^{B} = \begin{pmatrix} X & 0 \\ 0 & 27/5 \end{pmatrix}$$

$$F^{Tot} = F^{A} + F^{B} = \begin{pmatrix} \frac{245}{2} + X & 21 \\ 21 & \frac{125}{5} \end{pmatrix} = \begin{pmatrix} \tilde{X} & 21 \\ 21 & 25 \end{pmatrix}$$

$$C^{tot} = \left(F^{tot}\right)^{-1} \cong \begin{pmatrix} 1/\tilde{X} & \approx 0 \\ \approx 0 & 1/25 \end{pmatrix}$$

$$\sigma_{w}^{tot} = \frac{1}{\sqrt{25}} = \frac{1}{5} = 0.2$$

(d)