Proper Orthogonal Decomposition - POD

PEF5737 - Non-linear dynamics and stability

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Outline

1. Objetivo
2. Introdução
3. Definitions
4. Geometric interpretation
5. Applications to linear systems
   - Free vibrations
   - Forced vibrations
6. Non-linear modes
7. Final remarks

Outline

Objective

To present an introduction to the Proper Orthogonal Decomposition (POD) method. Some examples are also presented. This class is based on the paper [1].

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Introduction

- POD is an empirical method for dynamic analysis and allows conclusions *a posteriori* about the investigated system;

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- Besides its use in dynamics of structures, POD is also used in flow analyses (turbulence), image processing among other applications. This class focuses on its use for vibration analyses.

Outline

Investigated problem

- Consider the vibrations of a structure. Let M be the number of degrees of freedom (properly sorted) samples at N time instants. The displacement on the m-th degree of freedom at t_n is $x_m(t_n)$;

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- Consider the vibrations of a structure. Let M be the number of degrees of freedom (properly sorted) samples at N time instants. The displacement on the m-th degree of freedom at t_n is $x_m(t_n)$;
- Let $\mathbf{X}$ the response matrix given by:

$$\mathbf{X} = \begin{pmatrix} x_1(t_0) & x_2(t_0) & x_3(t_0) & \dots & x_M(t_0) \\ x_1(t_1) & x_2(t_1) & x_3(t_1) & \dots & x_M(t_1) \\ x_1(t_2) & x_2(t_2) & x_3(t_2) & \dots & x_M(t_2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_1(t_N) & x_2(t_N) & x_3(t_N) & \dots & x_M(t_N) \end{pmatrix}$$

Correlation matrix

- The correlation matrix $\mathbf{R} = \frac{1}{N}\mathbf{X}^{\mathsf{T}}\mathbf{X}$.

Correlation matrix

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- The correlation matrix $\mathbf{R} = \frac{1}{N}\mathbf{X}^{\mathsf{T}}\mathbf{X}$.
- As $\mathbf{X}$ is symmetric and real, it can be made a diagonal matrix. The eigenvalues of $\mathbf{R}$ are the proper orthogonal values (POVs) and the corresponding eigenvectors are the proper orthogonal modes (POMs).

Outline

Geometric interpretation

Let $\mathbf{v}$ be a normalized POM. Consequently, $\mathbf{Rv} = \lambda \mathbf{v}$. Como $\mathbf{R} = \frac{1}{N}\mathbf{X^T X}$, we obtain $(\mathbf{Xv})^T(\mathbf{Xv}) = \lambda N \leftrightarrow \frac{1}{N}(\mathbf{Xv})^T(\mathbf{Xv}) = \lambda$

Each row of $\mathbf{X}$ can be interpreted as an snapshot (a “photograph”). Defining $\mathbf{p}_j$ as the snapshot at a particular instant t_j, $\mathbf{Xv}$ is given by:

$$\begin{bmatrix} \mathbf{v}^T\mathbf{p_1} \\ \mathbf{v}^T\mathbf{p_2} \\ \vdots \\ \mathbf{v}^T\mathbf{p_N} \end{bmatrix}$$

Geometric interpretation

Xv can be interpreter as the projection of the experimental data onto **v**. Consequently, λ plays the role of a mean squared distance from the origin. In mechanical systems, such as distance is associated with the energy.

Outline

Formulação

Undamped system under free vibrations $\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = 0$. Its solution can be written as functions of the natural modes v_i by means of:

$$\mathbf{x}(t) = \sum_{i=1}^{M} A_i \sin(\omega_i t - \phi_i)\mathbf{v_i}$$

- As showed, an eigenvector of $\mathbf{R}$ (POM) converges to a modal vector. This is valid for low-damped systems.

- As showed, an eigenvector of **R** (POM) converges to a modal vector. This is valid for low-damped systems.
- POD can be used as an empiric scheme for determining the modal shape (a possible alternative do other methods such as, for example, the Circle Adjust Method)

A brief note

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For a harmonically forced system, the POM do not tend to the modal vectors. However, close to the resonance, in which one mode dominates the response and the corresponding POV is much large than the others, the associated POM is a good approximation for the excited mode.

Outline

Definition

- A non-linear mode can be interpreted as an invariant manifold in the state-space;

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- Fenny & Kappagantu (1998) deal with synchronous non-linear modes.

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- POMs are an optimal linear representation for the non-linear normal modes;
- The POD technique can be used obtaining reduced-order models for non-linear systems

Outline

Final remarks

- For several applications, POMs can be assumed to match the natural modes of a linear system. This allows the definition of modal shapes in a quick way from experiments;

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- Even for forced vibrations, the modal shapes can be obtained from POD;

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- For synchronous non-linear normal modes, POMs consist of the best linear fit (based on energy criterion);

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- POD can be easily programmed and is a powerful toll for both quick or intricate analyses;
- For synchronous non-linear normal modes, POMs consist of the best linear fit (based on energy criterion);
- Feeny & Kappagantu suggest, as further works, investigations on the POD use for non-synchronous non-linear modes.

Bibliography

B. F. Feeny and R. Kappagantu.
On the physical interpretation of proper orthogonal modes in vibrations.
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