UNIVERSIDADE DE SÃO PAULO

ESCOLA POLITÉCNICA

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GISE Grupo de Inovação em Sistemas Estruturais

PEF-5750
Estruturas Leves

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Discrete Systems:
ARGYRIS' NATURAL MEMBRANE ELEMENT

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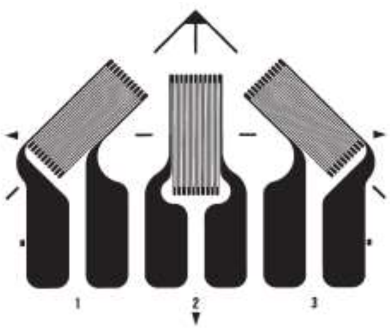
Argyris' Natural Membrane Element

Argyris ~1974
A membrane finite element based on natural deformations
(mesured along the sides of the element),
able to cope with large displacements and large deformations.

- Akin to a "strain rosette" plane stress finite element:

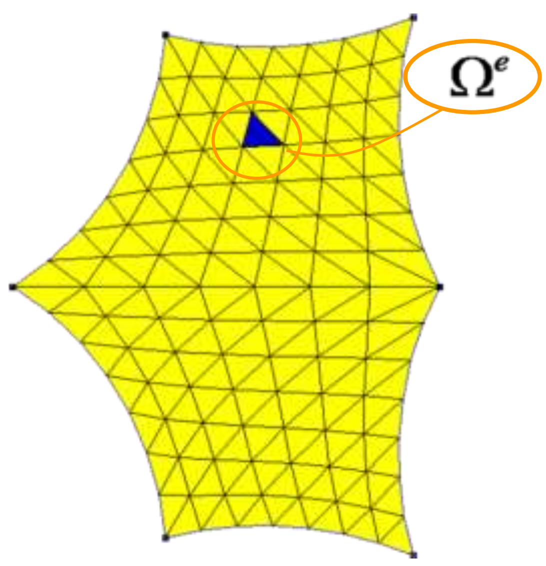


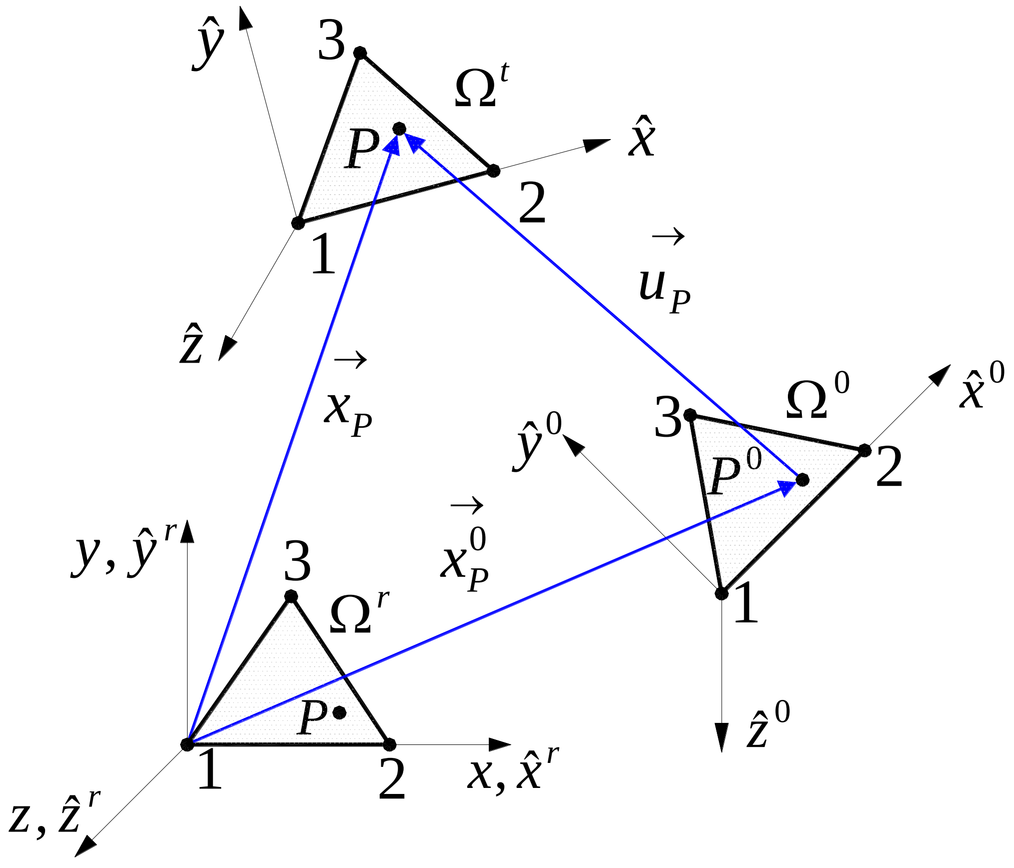
Meek ~1991
•A corrotational description;
•Small strains.

Pauletti ~2003
•a more concise notation;
•distinction between the constitutive and geometric
parts of the element tangent stiffness;
•the "simplest possible membrane finite element":
•large displacements / small strains (a few percent...)

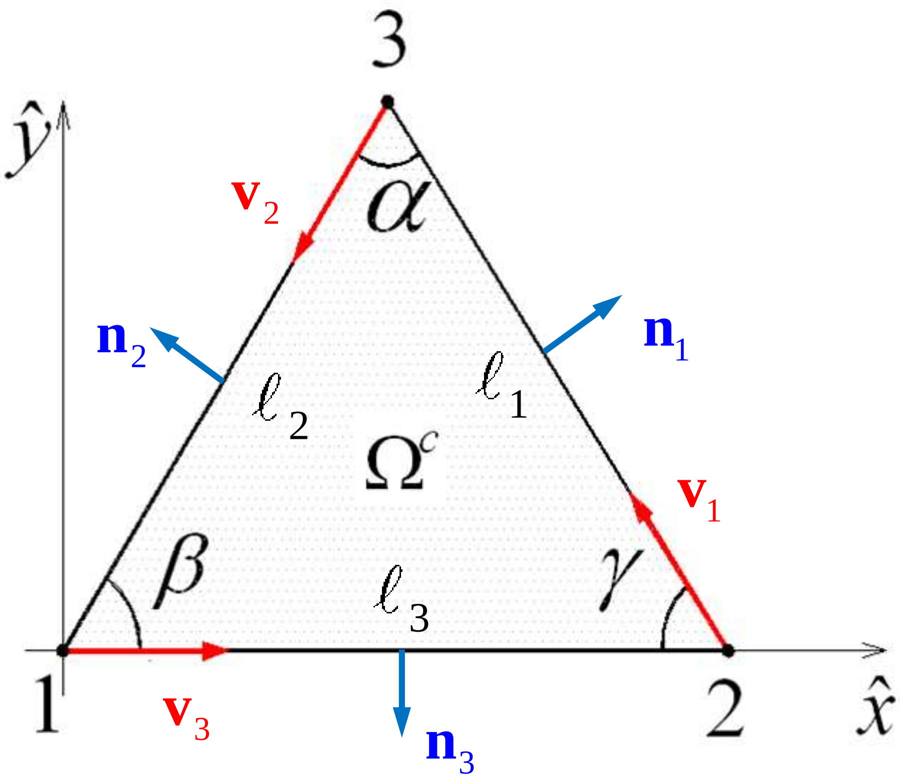
Pauletti (2006)
• first publication on the natural force density concept
R.M.O. Pauletti, "An extension of the force density procedure to membrane structures"
IASS Symposium / APCS Conference – New Olympics, New Shell and Spacial Structures, Beijing, 2006

Reference, Initial and Current Configurations For Argyris Element





Element Description



$$\mathbf{x}_{0,c} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix}_{0,c} \qquad \mathbf{u} = \mathbf{x}_c - \mathbf{x}_0 = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \end{bmatrix}$$

Side lenghts:

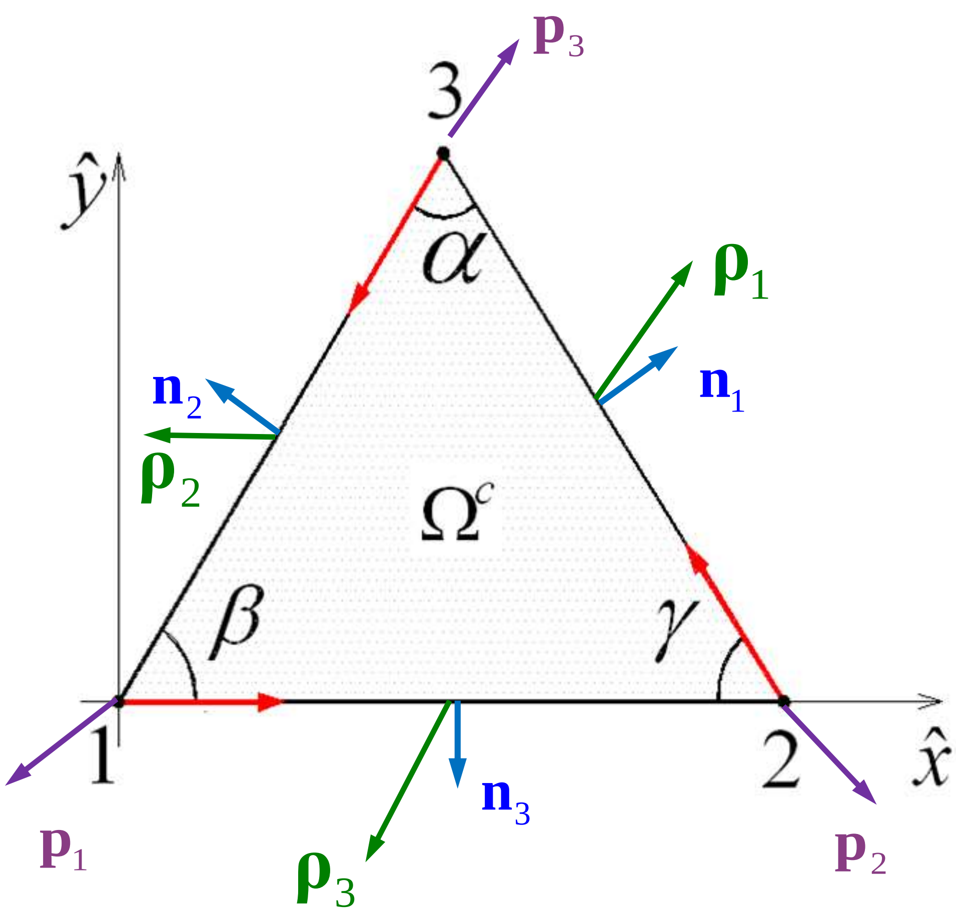
$$\ell_i = \left\| \mathbf{x}_k - \mathbf{x}_j \right\| \qquad \{i, j, k\} \equiv \{1, 2, 3\}$$

(in cyclic permutation)

Unit side vectors: $\mathbf{v}_i = \dfrac{\mathbf{x}_k^e - \mathbf{x}_j^e}{\ell_i}$

Unit normal vectors: $\mathbf{n}_i = -\hat{\mathbf{k}} \times \mathbf{v}_i$

Element Stress Field and Vector of Internal Nodal Forces



Cauchy Plane Stress Tensor:

$$\hat{\boldsymbol{\sigma}} = \begin{bmatrix} \sigma_{\hat{x}} & \tau_{\hat{x}\hat{y}} & 0 \\ \tau_{\hat{x}\hat{y}} & \sigma_{\hat{y}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \equiv \begin{bmatrix} \sigma_{\hat{x}} \\ \sigma_{\hat{y}} \\ \tau_{\hat{x}\hat{y}} \end{bmatrix}$$

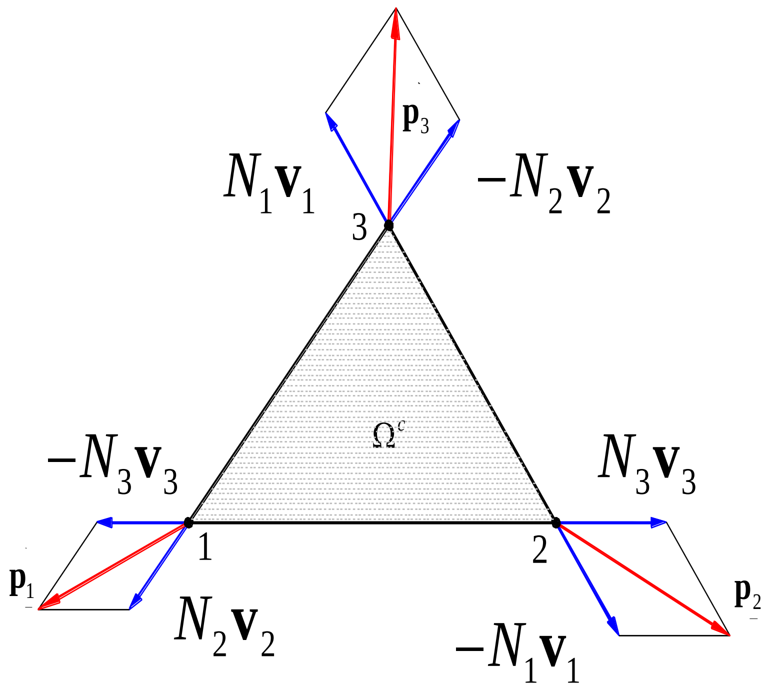
Side stress vectors:

$$\boldsymbol{\rho}_i = \hat{\boldsymbol{\sigma}} \mathbf{n}_i$$

Vector of internal nodal forces:

$$\mathbf{p}^e = \begin{bmatrix} \mathbf{p}_1^e \\ \mathbf{p}_2^e \\ \mathbf{p}_3^e \end{bmatrix} = \frac{t}{2} \begin{bmatrix} \ell_2 \boldsymbol{\rho}_2 + \ell_3 \boldsymbol{\rho}_3 \\ \ell_1 \boldsymbol{\rho}_1 + \ell_3 \boldsymbol{\rho}_3 \\ \ell_1 \boldsymbol{\rho}_1 + \ell_2 \boldsymbol{\rho}_2 \end{bmatrix}$$

Vector of Natural Forces



The vector of internal forces can be decomposed into components parallel to the element sides:

$$\mathbf{p}^e = \begin{bmatrix} N_2\mathbf{v}_2^e - N_3\mathbf{v}_3^e \\ N_3\mathbf{v}_3^e - N_1\mathbf{v}_1^e \\ N_1\mathbf{v}_1^e - N_2\mathbf{v}_2^e \end{bmatrix}$$

Vector of Natural Forces

$$\mathbf{N} = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix}$$

Natural Stresses

Comparing both expression available for p^e:

$$\begin{bmatrix} N_2\mathbf{v}_2^e - N_3\mathbf{v}_3^e \\ N_3\mathbf{v}_3^e - N_1\mathbf{v}_1^e \\ N_1\mathbf{v}_1^e - N_2\mathbf{v}_2^e \end{bmatrix} = \frac{t}{2}\begin{bmatrix} \ell_2\boldsymbol{\rho}_2 + \ell_3\boldsymbol{\rho}_3 \\ \ell_1\boldsymbol{\rho}_1 + \ell_3\boldsymbol{\rho}_3 \\ \ell_1\boldsymbol{\rho}_1 + \ell_2\boldsymbol{\rho}_2 \end{bmatrix}$$

We obtain the Vector of Natural Forces, as function of Cauchy Stresses, and we identity some "Natural Stresses" ($\sigma_1, \sigma_2, \sigma_3$):

$$\mathbf{N} = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} = \frac{t}{2}\begin{bmatrix} h_1\left(\frac{\cos\beta}{\sin\gamma\sin\alpha}\sigma_{\hat{y}} - \frac{\sin\beta}{\sin\gamma\sin\alpha}\tau_{\hat{x}\hat{y}}\right) \\ h_2\left(\frac{\cos\gamma}{\sin\beta\sin\alpha}\sigma_{\hat{y}} + \frac{\sin\gamma}{\sin\beta\sin\alpha}\tau_{\hat{x}\hat{y}}\right) \\ h_3\left(\sigma_{\hat{x}} - \frac{\cos\beta\cos\gamma}{\sin\beta\sin\gamma}\sigma_{\hat{y}} + \frac{\cos(\beta-\gamma)}{\sin\beta\sin\gamma}\tau_{\hat{x}\hat{y}}\right) \end{bmatrix}$$

$$\mathbf{N} = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} = \frac{t}{2}\begin{bmatrix} h_1\sigma_1 \\ h_2\sigma_2 \\ h_3\sigma_3 \end{bmatrix}$$

Vector of Natural Stresses

We group the Natural Stresses" ($\sigma_1, \sigma_2, \sigma_3$) in a *Vector of Natural Stresses*:

$$\boldsymbol{\sigma}_n = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{\cos\beta}{\sin\gamma\sin\alpha} & -\dfrac{\sin\beta}{\sin\gamma\sin\alpha} \\ 0 & \dfrac{\cos\gamma}{\sin\beta\sin\alpha} & \dfrac{\sin\gamma}{\sin\beta\sin\alpha} \\ 1 & -\dfrac{\cos\beta\cos\gamma}{\sin\beta\sin\gamma} & \dfrac{\sin(\beta-\gamma)}{\sin\beta\sin\gamma} \end{bmatrix} \begin{bmatrix} \sigma_{\hat{x}} \\ \sigma_{\hat{y}} \\ \tau_{\hat{x}\hat{y}} \end{bmatrix}$$

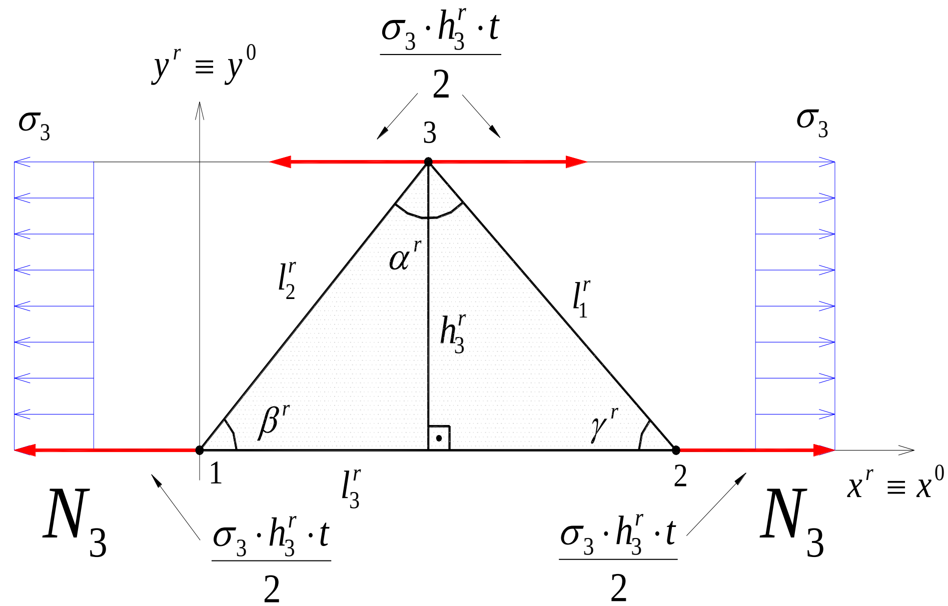
$$\boldsymbol{\sigma}_n = \mathbf{T}^{-T}\hat{\boldsymbol{\sigma}}$$

Exercise 11. Verify the above expression!

Vector of Natural Stresses

Each natural force N_i can be understood as the nodal resultant of each natural normal stress field σ_i

$$N_i = \frac{t}{2} h_i \sigma_i$$



And since $h_i = \dfrac{2A}{\ell_i}$ ∴

$$N_i = V \frac{\sigma_i}{\ell_i}$$

Relationship between the Vectors of Natural Forces and Stresses

In matrix form:

$$\mathbf{N} = V\mathcal{L}^{-1}\boldsymbol{\sigma}_n$$

Length matrix:

$$\mathcal{L} = \begin{bmatrix} \ell_1 & 0 & 0 \\ 0 & \ell_2 & 0 \\ 0 & 0 & \ell_3 \end{bmatrix}$$

$$\boldsymbol{\sigma}_n = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix}$$

Vector of Natural Stresses

Vector of Natural Deformations

The deformations along the sides of the element are collected in a 'Vector of Natural Deformations':

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} = \begin{bmatrix} \cos^2\gamma & \sin^2\gamma & -\sin\gamma\cos\gamma \\ \cos^2\beta & \sin^2\beta & -\sin\beta\cos\beta \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{\hat{x}} \\ \varepsilon_{\hat{y}} \\ \gamma_{\hat{x}\hat{y}} \end{bmatrix}$$

Linearized Green Strains

$$\boldsymbol{\varepsilon}_n = \mathbf{T}\hat{\boldsymbol{\varepsilon}}$$

Exercise 12. Verify the above expression!

We remark that $\boldsymbol{\sigma}_n$ and $\boldsymbol{\varepsilon}_n$ are energetically conjugate.

Indeed, by the Principle of Virtual Work:

$$\delta\hat{\boldsymbol{\varepsilon}}^T\hat{\boldsymbol{\sigma}} = \delta\boldsymbol{\varepsilon}_n{}^T\boldsymbol{\sigma}_n \quad , \forall\,\delta\hat{\boldsymbol{\varepsilon}}$$

$$\delta\hat{\boldsymbol{\varepsilon}}^T\hat{\boldsymbol{\sigma}} = \left(\mathbf{T}\,\delta\hat{\boldsymbol{\varepsilon}}\right)^T\boldsymbol{\sigma}_n = \delta\hat{\boldsymbol{\varepsilon}}^T\mathbf{T}^T\boldsymbol{\sigma}_n \quad , \forall\,\delta\hat{\boldsymbol{\varepsilon}}$$

Thus: $\boldsymbol{\sigma}_n = \mathbf{T}^{-T}\hat{\boldsymbol{\sigma}}$, as deduced before.

Tangent Stiffness Matrix for Argyris' Element

$$\mathbf{k}_t = \frac{\partial \mathbf{g}}{\partial \mathbf{u}} = \mathbf{N}^T \frac{\partial \mathbf{C}}{\partial \mathbf{u}} + \mathbf{C}\frac{\partial \mathbf{N}}{\partial \mathbf{u}} - \frac{\partial \mathbf{f}}{\partial \mathbf{u}}$$

Geometric Stiffness Matrix

$$\mathbf{k}_g = \mathbf{N}^T \frac{\partial \mathbf{C}}{\partial \mathbf{u}} = \begin{bmatrix} N_2 \dfrac{\partial \mathbf{v}_2}{\partial \mathbf{u}} - N_3 \dfrac{\partial \mathbf{v}_3}{\partial \mathbf{u}} \\ -N_1 \dfrac{\partial \mathbf{v}_1}{\partial \mathbf{u}} + N_3 \dfrac{\partial \mathbf{v}_3}{\partial \mathbf{u}} \\ N_1 \dfrac{\partial \mathbf{v}_1}{\partial \mathbf{u}} - N_2 \dfrac{\partial \mathbf{v}_2}{\partial \mathbf{u}} \end{bmatrix}$$

$$\mathbf{k}_g = \begin{bmatrix} \frac{N_2}{\ell_2}\left(\mathbf{I}_3 - \mathbf{v}_2\mathbf{v}_2^T\right) + \frac{N_3}{\ell_3}\left(\mathbf{I}_3 - \mathbf{v}_3\mathbf{v}_3^T\right) & -\frac{N_3}{\ell_3}\left(\mathbf{I}_3 - \mathbf{v}_3\mathbf{v}_3^T\right) & -\frac{N_2}{\ell_2}\left(\mathbf{I}_3 - \mathbf{v}_2\mathbf{v}_2^T\right) \\ -\frac{N_3}{\ell_3}\left(\mathbf{I}_3 - \mathbf{v}_3\mathbf{v}_3^T\right) & \frac{N_1}{\ell_1}\left(\mathbf{I}_3 - \mathbf{v}_1\mathbf{v}_1^T\right) + \frac{N_3}{\ell 3}\left(\mathbf{I}_3 - \mathbf{v}_3\mathbf{v}_3^T\right) & -\frac{N_1}{\ell_1}\left(\mathbf{I}_3 - \mathbf{v}_1\mathbf{v}_1^T\right) \\ -\frac{N_2}{\ell_2}\left(\mathbf{I}_3 - \mathbf{v}_2\mathbf{v}_2^T\right) & -\frac{N_1}{\ell_1}\left(\mathbf{I}_3 - \mathbf{v}_1\mathbf{v}_1^T\right) & \frac{N_1}{\ell_1}\left(\mathbf{I}_3 - \mathbf{v}_1\mathbf{v}_1^T\right) + \frac{N_2}{\ell_2}\left(\mathbf{I}_3 - \mathbf{v}_2\mathbf{v}_2^T\right) \end{bmatrix}$$

Exact!

Tangent Stiffness Matrix for Argyris' Element

$$\mathbf{k}_t = \frac{\partial \mathbf{g}}{\partial \mathbf{u}} = \mathbf{N}^T \frac{\partial \mathbf{C}}{\partial \mathbf{u}} + \mathbf{C}\frac{\partial \mathbf{N}}{\partial \mathbf{u}} - \frac{\partial \mathbf{f}}{\partial \mathbf{u}}$$

<u>*External Stiffness Matrix*</u>

$$\mathbf{k}_{\text{ext}} = \frac{\partial \mathbf{f}}{\partial \mathbf{u}}$$

External force vector:

$$\mathbf{f} = \mathbf{f}_{\text{weight}} - \mathbf{f}_{\text{wind}} = \frac{V\rho}{3}\begin{bmatrix}\mathbf{I}_3 \\ \mathbf{I}_3 \\ \mathbf{I}_3\end{bmatrix}\mathbf{g} - \frac{pA}{3}\begin{bmatrix}\mathbf{I}_3 \\ \mathbf{I}_3 \\ \mathbf{I}_3\end{bmatrix}\mathbf{n}$$

$$\mathbf{k}_{\text{ext}} = \frac{\partial \mathbf{f}_{\text{wind}}}{\partial \mathbf{u}} = \ldots = \frac{p}{6}\begin{bmatrix}\mathbf{\Lambda}_1 & \mathbf{\Lambda}_2 & \mathbf{\Lambda}_3 \\ \mathbf{\Lambda}_1 & \mathbf{\Lambda}_2 & \mathbf{\Lambda}_3 \\ \mathbf{\Lambda}_1 & \mathbf{\Lambda}_2 & \mathbf{\Lambda}_3\end{bmatrix}$$

$$\mathbf{\Lambda}_i = \text{Skew}(\mathbf{l}_i) = \ell_i \begin{bmatrix} 0 & -\text{v}_i^z & \text{v}_i^y \\ \text{v}_i^z & 0 & -\text{v}_i^x \\ -\text{v}_i^y & \text{v}_i^x & 0 \end{bmatrix}, \; i = 1, 2, 3$$

Exact!

Exercise 13. Verify the above expression for k_{ext}!

Tangent Stiffness Matrix for Argyris' Element

$$\mathbf{k}_t = \frac{\partial \mathbf{g}}{\partial \mathbf{u}} = \mathbf{N}^T \frac{\partial \mathbf{C}}{\partial \mathbf{u}} + \mathbf{C}\frac{\partial \mathbf{N}}{\partial \mathbf{u}} - \frac{\partial \mathbf{f}}{\partial \mathbf{u}}$$

<u>Constitutive Stiffness Matrix</u> $\mathbf{k}_c = \mathbf{C}\frac{\partial \mathbf{N}}{\partial \mathbf{u}}$ *Exact!*

Defining the vector of Natural Displacements $\mathbf{a} = \begin{bmatrix} \ell_1 - \ell_1^0 \\ \ell_2 - \ell_2^0 \\ \ell_3 - \ell_3^0 \end{bmatrix}$

There exist some kind of relationship $\mathbf{N} = \mathbf{N}(\mathbf{a})$ *so that*

$$\mathbf{k}_c = \mathbf{C}\frac{\partial \mathbf{N}}{\partial \mathbf{a}}\frac{\partial \mathbf{a}}{\partial \mathbf{u}} = \mathbf{C}\mathbf{k}_n\mathbf{C}^T$$

$$\mathbf{k}_n = \frac{\partial \mathbf{N}}{\partial \mathbf{a}}$$ *'is the <u>Natural Stiffness Matrix</u>'*

Tangent Stiffness Matrix for Argyris' Element

A simplification: *Linear elastic material behavior*

Thus, a linear relationship $\mathbf{N} = \mathbf{k}_n^r \mathbf{a}$ *exists*

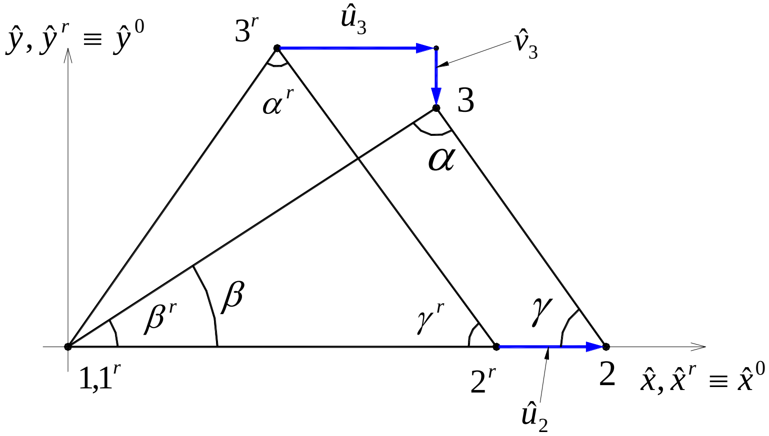
Where $\mathbf{k}_n^r = \dfrac{\partial \mathbf{N}}{\partial \mathbf{a}}$ *is a 3x3 constant natural stiffness matrix*

And therefore

$$\mathbf{k}_c = \mathbf{C}\mathbf{k}_n^r\mathbf{C}^T$$

?!

A linear elastic simplification for K_c



$$\hat{\boldsymbol{\varepsilon}} = \begin{bmatrix} \varepsilon_{\hat{x}} \\ \varepsilon_{\hat{y}} \\ \gamma_{\hat{x}\hat{y}} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \hat{u}}{\partial \hat{x}} \\ \dfrac{\partial \hat{v}}{\partial \hat{y}} \\ \dfrac{\partial \hat{v}}{\partial \hat{x}} + \dfrac{\partial \hat{u}}{\partial \hat{y}} \end{bmatrix} = \begin{bmatrix} \dfrac{u_2}{x_2^r} \\ \dfrac{v_3}{y_3^r} \\ \dfrac{x_2^r u_3 - x_3^r u_2}{x_2^r y_3^r} \end{bmatrix}$$

$$\boldsymbol{\varepsilon}_n = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} = \begin{bmatrix} \dfrac{\Delta\ell_1}{\ell_1^r} \\ \dfrac{\Delta\ell_2}{\ell_2^r} \\ \dfrac{\Delta\ell_3}{\ell_3^r} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\ell_1^r} & 0 & 0 \\ 0 & \dfrac{1}{\ell_2^r} & 0 \\ 0 & 0 & \dfrac{1}{\ell_3^r} \end{bmatrix} \begin{bmatrix} \Delta\ell_1 \\ \Delta\ell_2 \\ \Delta\ell_3 \end{bmatrix} = \mathcal{L}_r^{-1} \mathbf{a}$$

$$\boldsymbol{\varepsilon}_n = \mathbf{T}_r \hat{\boldsymbol{\varepsilon}}$$

$$\mathbf{T}_r = \begin{bmatrix} \cos^2\gamma_r & \sin^2\gamma_r & -\sin\gamma_r\cos\gamma_r \\ \cos^2\beta_r & \sin^2\beta_r & -\sin\beta_r\cos\beta_r \\ 1 & 0 & 0 \end{bmatrix}$$

A linear elastic simplification for K_c

Hooke's Law:

$$\hat{\boldsymbol{\sigma}} = \begin{bmatrix} \sigma_{\hat{x}} \\ \sigma_{\hat{y}} \\ \tau_{\hat{x}\hat{y}} \end{bmatrix} = \hat{\mathbf{D}}\hat{\boldsymbol{\varepsilon}} + \hat{\boldsymbol{\sigma}}_0$$

$$\hat{\mathbf{D}} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

But, now:

$$\boldsymbol{\sigma}_n = \mathbf{T}_r^{-T}\hat{\boldsymbol{\sigma}} = \mathbf{T}_r^{-T}\left(\hat{\mathbf{D}}\hat{\boldsymbol{\varepsilon}} + \hat{\boldsymbol{\sigma}}_0\right) = \mathbf{T}_r^{-T}\hat{\mathbf{D}}\hat{\boldsymbol{\varepsilon}} + \mathbf{T}_r^{-T}\hat{\boldsymbol{\sigma}}_0$$

$$\boldsymbol{\sigma}_n = \mathbf{T}_r^{-T}\hat{\mathbf{D}}\mathbf{T}_r^{-1}\boldsymbol{\varepsilon}_n + \mathbf{T}_r^{-T}\hat{\boldsymbol{\sigma}}_0$$

That is $\boldsymbol{\sigma}_n = \mathbf{D}_n\boldsymbol{\varepsilon}_n + \boldsymbol{\sigma}_{n0}$

Where $\mathbf{D}_n = \mathbf{T}_r^{-T}\hat{\mathbf{D}}\mathbf{T}_r^{-1}$

A linear elastic simplification for K_c

Recaling the Natural Forces: $\mathbf{N} = V^r \mathcal{L}_r^{-1} \boldsymbol{\sigma}_n$

$$\mathbf{N} = V^r \mathcal{L}_r^{-1} \mathbf{D}_n \boldsymbol{\varepsilon}_n = V^r \mathcal{L}_r^{-1} \left(\mathbf{T}_r^{-T} \hat{\mathbf{D}} \mathbf{T}_r^{-1} \right) \mathcal{L}_r^{-1} \mathbf{a}$$

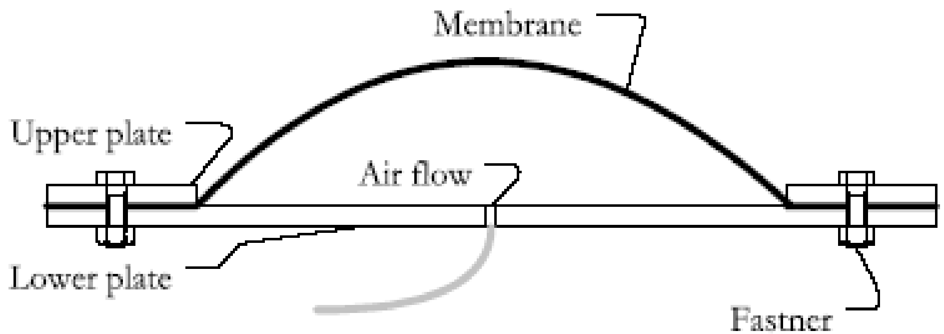
And we arrive to the Natural Stiffness Matrix, (considering small deformations):

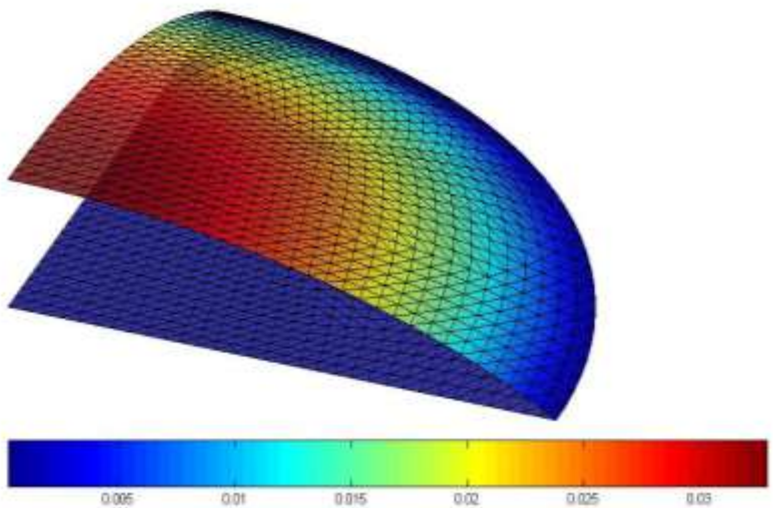
$$\mathbf{k}_n^r = \frac{\partial \mathbf{N}}{\partial \mathbf{a}} = V^r \mathcal{L}_r^{-1} \left(\mathbf{T}_r^{-T} \hat{\mathbf{D}} \mathbf{T}_r^{-1} \right) \mathcal{L}_r^{-1}$$

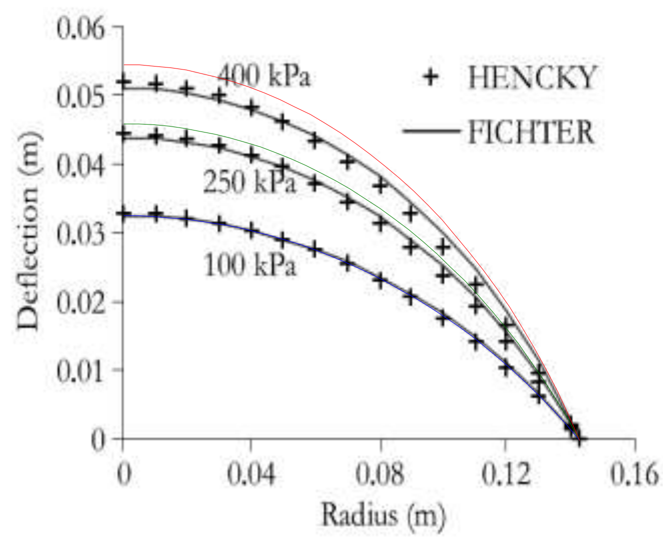
An order 3, symmetric matrix, that can be calculated and stored at the start, and rotated at each Newton's iteration, according to the co-rotational element coordinate system:

$$\mathbf{k}_c = \mathbf{C} \mathbf{k}_n^r \mathbf{C}^T$$

A benchmark:
an axisymmetric pressurized membrane







deformed shape,
as calculated by SATS

comparison between SATS
and theoretical results