7.6 Material constitutive laws

The topic of constitutive laws is a broad one. Basically, a constitutive law is a mathematical expression that aims to describe the material behaviour. In structural analysis, one inputs mainly the strain, strain rate and temperature, to obtain the stress levels.

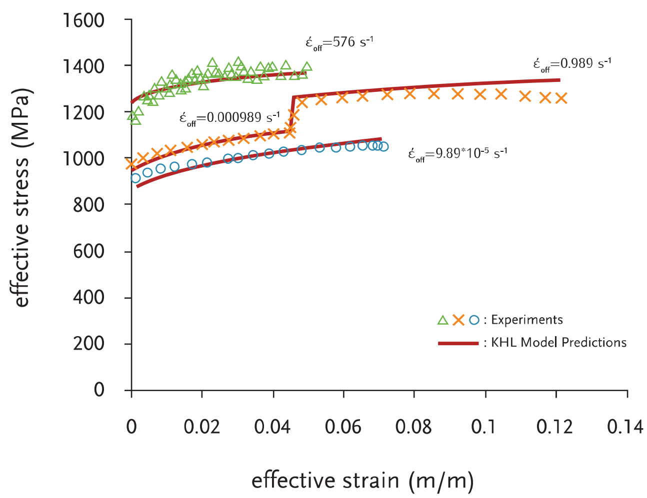
A simple constitutive law is the elastic one and it relates the stress and strain via the single parameter elastic modulus, E,

$$\sigma = E\varepsilon.$$

We have also seen in Chapter 5 the perfectly plastic material constitutive law,

$$\sigma = \sigma_0$$

where σ_0 is the flow stress. Note in this case that there is no unique stress–strain relation, *ie* for any strain value different from zero, the stress level is always the flow stress. Yet, this equation can describe well some materials, classically the mild steel, but also some titanium alloys, as seen in the next figure from the experimental data after the strain rate jump.



Titanium alloy Ti-6Al-4V compressive stress—strain behaviour at various strain rates. Adapted from Experimental and numerical analysis of high strain rate response of Ti-6Al-4V titanium alloy A.Bragov, A. Konstantinov, A. Lomunov, I.Sergeichev and B. Fedulov, DYMAT 2009, p. 1465–1470, 2009.

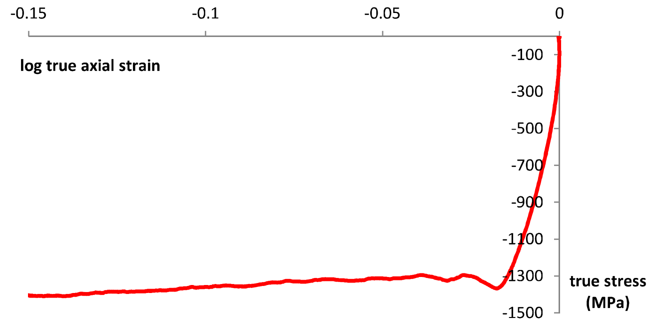
The perfectly plastic material law is a particular case of

$$\sigma = \frac{\sigma_y}{(\sigma_y/E)^n}\varepsilon^n$$

with $n = 0$. The constants σ_y and n are given in the next table for some steel alloys, while the next figure presents a titanium alloy compressed at a strain rate of 1000/s indicating that the rigid—plastic model is a reasonable assumption for this material at such a rate.

Material	σ_y (MPa)	n
16NC6	345	0.17
100C6	430	0.14
35CD4	473	0.09
35NC15	790	0.08
Z38CDV5	390	0.17
Z15CN17-03	804	0.10
Z2CN18-10	200	0.21
42CD4	515	0.10
XC18	350	0.10
XC38	576	0.11
XC48	419	0.16
XC65	345	0.24
XC80	303	0.17

Constitutive parameters for various steel alloys with $E = 210$ GPa. Valid for $\sigma = \frac{\sigma_y}{(\sigma_y/E)^n}\varepsilon^n$ and adapted from A. Nayebi, R. El Abdi, O. Bartier and G. Mauvoisin, New procedure to determine steel mechanical parameters from the spherical indentation technique, Mechanics of Materials, 34, p. 243–254, 2002.



The behaviour of a titanium alloy at a strain rate of 1000/s.

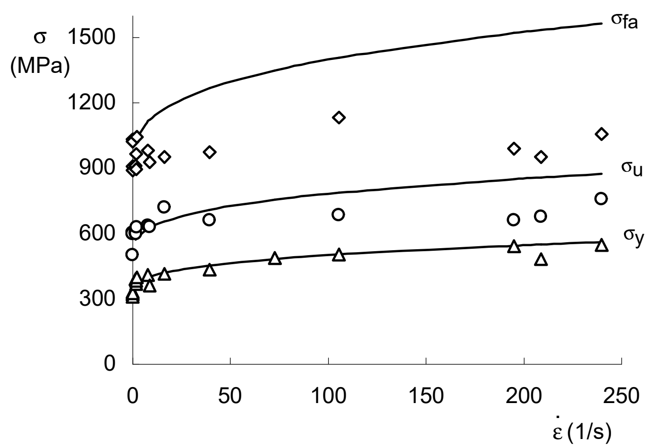
As we have seen in other occasions, some materials increase their flow stress levels when loaded dynamically. If we plot this increase of the flow stress, say $\sigma_d/\sigma_s - 1$, against strain rate, $\dot{\varepsilon}$, we can conjecture, and experimental results for many metals have confirmed, that the resulting relation is linear when using a log scale for the strain rate. It follows then

$$\sigma_d = \sigma_s \left[1 + \left(\frac{\dot{\varepsilon}}{C} \right)^{1/p} \right],$$

with C and p being material constants. This is the well known Cowper—Symonds equation, implemented in most explicit finite element codes. It can also be written as

$$\sigma_d = \sigma_s \left(1 + m\dot{\varepsilon}^n \right), \qquad m = \frac{1}{C^{1/p}} \qquad n = \frac{1}{p}$$

Observe that the equation above can also be adapted to strain hardening materials by writing the static flow stress as a function of the strain, *eg* $\sigma_s = \sigma_y + A\varepsilon^p$. The constants m and n are determined at a given plastic strain, say at the yielding strain ε_y. In this case, only the dynamic yielding stress can be well predicted. Using these constants to predict, for instance, the ultimate stress, σ_u, or the failure stress, σ_{fa}, would lead to an error, as shown in the next figure.



Various stress predictions for the steel alloy BS EN 10025 FE430A according to the Cowper—Symonds equation, with its constants determined from the yielding stress.

A remedy to this situation is to use the equation

$$\sigma_{eq_d} = \sigma_{eq_s} + \bar{m}\dot{\varepsilon}^{\bar{n}},$$

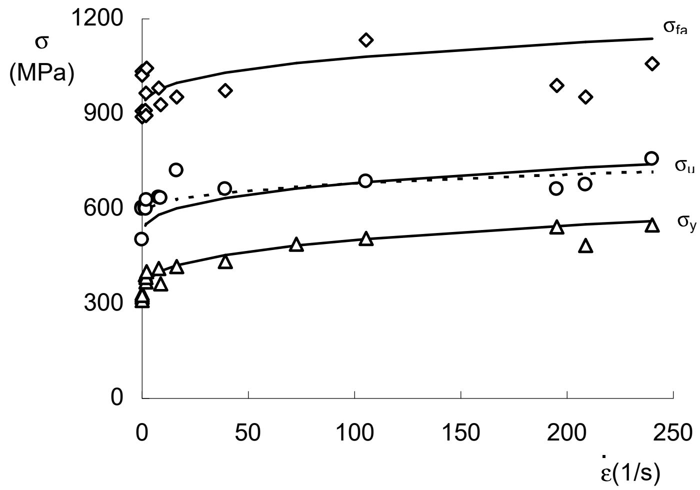
with

$$\bar{m} = \frac{\sigma}{C^{1/q}} \quad \text{and} \quad \bar{n} = \frac{1}{q},$$

valid for materials whose strain hardening is not affected by the strain rate. Here, σ_{eq_d} and σ_{eq_s} are the dynamic and static equivalent stresses, respectively. The coefficients C and q are determined in the same way as in Cowper—Symonds'. The parameters σ and $\dot{\varepsilon}$ can be freely cho-

See M. Alves, Material constitutive law for large strains and strain rates, Journal of Engineering Mechanics, 216–218, 2000.

sen based on the available experimental data or application. Upon this choice, the coefficients $\bar{m}$ and $\bar{n}$ will assume specific values. This equation is just a slight modification of the Cowper—Symonds equation, yet it avoids the problem of dealing with strain dependent coefficients. The next figure shows the predictions of these equations together with experimental data for a steel alloy.



Prediction of the constitutive law $\sigma_{eq_d} = \sigma_{eq_s} + \bar{m}\dot{\varepsilon}^{\bar{n}}$ for steel alloy BS EN 10025 FE430A together with experimental results for △: lower yield, σ_y, ○: ultimate, σ_u, and ◇: failure, σ_{fa}, stresses *versus* strain rate.

Another common constitutive equation, referred to as Norton equation, simply reads

$$\sigma_d = \sigma_0 \left(\frac{\dot{\varepsilon}}{\dot{\varepsilon}_0}\right)^q,$$

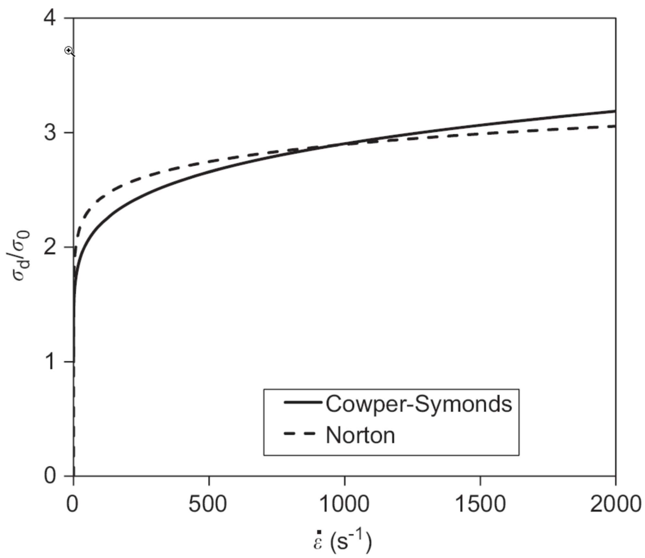
where q is a constant assigned to minimize the error between experimental and predicted data. $\dot{\varepsilon}_0$ is a reference strain rate, usually very small, associated with a quasi–static tensile or compressive flow stress value, σ_0. This equation has the great advantage of facilitating the development of some theoretical models, as we have seen in Chapter 5 and we shall see in Chapter 9. Despite of being simple, this equation compares well with the predictions given by the Cowper—Symonds equation, as illustrated in the next figure for a mild steel.

In impact engineering, a much used constitutive equation is the one due to Johnson and Cook and it reads

$$\sigma_d = (A + B\varepsilon^n)\left(1 + C\ln\frac{\dot{\varepsilon}}{\dot{\varepsilon}_0}\right)\left[1 - \left(\frac{T - T_r}{T_m - T_r}\right)^n\right],$$

with the constants A, B, C, n, m being determined by best fitting a series of stress–strain curve measured at different strain rates and temperatures. Here, T is the temperature and its subscripts refer to melting

Care should be taken with the fact that the term $A + B\varepsilon^n$ may not represent the quasi-static material behaviour but rather its behaviour at $\dot{\varepsilon}_0$.



Predicted dynamic flow stress for a typical mild steel. Continuous line, Cowper—Symonds equation with $p = 5$ and $D = 40$/s. Dashed line, Norton equation with $\dot{\varepsilon}_0 = 0.001$/s and $q = 0.077$. In both predictions $\sigma_0 = 235$ MPa.

and room (or reference) temperature. $\dot{\varepsilon}_0$ is a reference strain rate, usually the unity in the Johnson—Cook work but it can well be associated with the quasi–static tensile test, say 0.001/s. Once the five constants of a material are known, we can enter any temperature, strain rate and plastic strain to obtain the stress level. It is possible to obtain a true equivalent stress if we input the equivalent plastic strain value and in this case, of course, the constants have to be obtained from true equivalent stress–strain pairs. It has been found that this equation performs relatively well for many materials but poor predictions are equally possible.

The next table lists the values of these constants for various materials and as measured in the original work of Johnson and Cook.

material	ρ kg.m^{-3}	E GPa	c m/s	T_{melt} °K	A	B	n	C	m
copper	8960	100	1000	1356	90	292	0.31	0.025	1.09
brass	8520	100	1000	1189	112	505	0.42	0.009	1.68
nickel	8900	100	1000	1726	163	648	0.33	0.006	1.44
1006 Steel	7890	100	1000	1811	350	275	0.36	0.022	1.00
2024 Al.	2770	100	1000	775	337	343	0.41	0.010	1.00
4340 steel	7830	100	1000	1793	792	510	0.26	0.014	1.03
tungsten	17000	100	1000	1723	1506	177	0.12	0.016	1.00

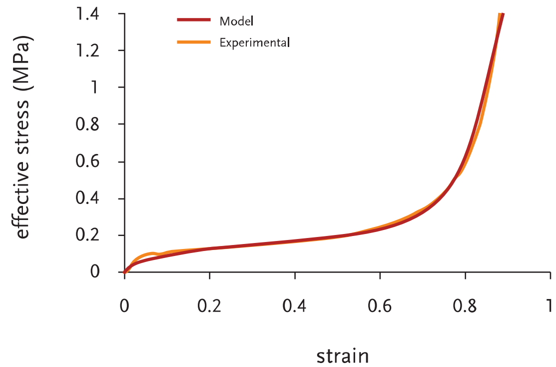
Johnson&Cook constitutive parameters for some metals.

The more complex the behaviour of a material is, so is the constitutive equation. An example of a somewhat complex behaviour is the polymer stress—strain curve depicted in the next figure. The constitutive equation relating engineering stress and strain

$$\sigma = A\left[1 - exp^{(-E\varepsilon/A)(1-\varepsilon)^m}\right] + B\left(\frac{\varepsilon}{1-\varepsilon}\right)^n,$$

with A, B, E, m and n being fitting constants, predicts well this typical polymer behaviour.

See Mechanical models of cellular solids: Parameters identification from experimental tests, M. Avalle, G. Belingardi and A. Ibba, International Journal of Impact Engineering, 2007, 34, p. 3–27.



A polymer stress—strain behaviour and the corresponding model prediction.

In connection with the polymers behaviour, a well known constitutive equation is the Mooney—Rivlin, which can be used for stretching as large as 100%, like in rubber sheets. Because it is obtained from a strain energy function, we refer to it as a hyperelastic model, a subject that will be further explored in the next chapter. For now, we present the relation between true stress and stretching, which reads

$$\sigma = \left(2C_1 + \frac{2C_2}{\lambda}\right)\left(\lambda^2 - \frac{1}{\lambda}\right),$$

with C_1 and C_2 being material constants that are determined from curve fitting to the experimental test. In terms of engineering stress, it follows that

$$s = \left(2C_1 + \frac{2C_2}{\lambda}\right)\left(\lambda - \frac{1}{\lambda^2}\right),$$

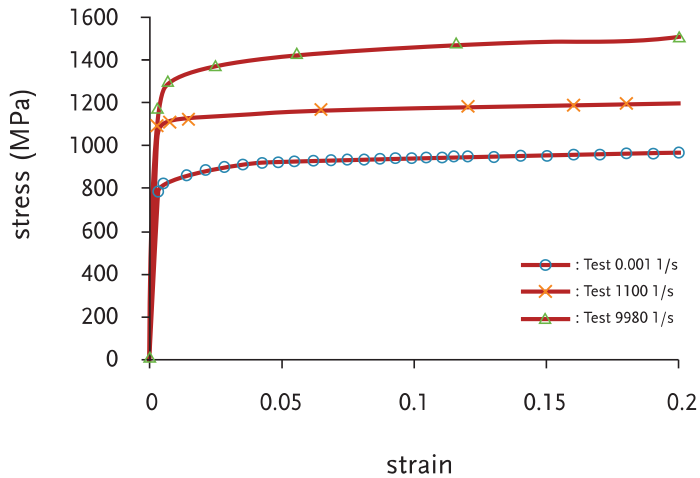
which is useful for obtaining the C_i parameters. We will have the chance to apply this equation in Chapter 10, in connection with the impact of tires.

A material can also be pulled at different speeds during loading. This is exemplified in the next figure, where a titanium alloy is pulled at different strain rates. The various experimental curves are fitted well by the constitutive equation

See A. S. Khan, R. Kazmi and B. Farrokh, Multiaxial and non–proportional loading responses, anisotropy and modelling of Ti-6Al-4V titanium alloy over wide ranges of strain rates and temperatures, International Journal of Plasticity, 2007, 23, p. 931–950

$$\sigma = \left[A + B\left(1 - \frac{\ln\dot{\varepsilon}}{\ln D}\right)^a \varepsilon^b\right]\left(\frac{\dot{\varepsilon}}{\dot{\varepsilon}^*}\right)^c\left(\frac{T_m - T}{T_m - T_{ref}}\right)^d,$$

with A, B, D, a, b, c, d being material constants. Here, T is the current temperature and T_m is the material melting temperature. T_{ref} and $\dot{\varepsilon}^*$ are temperature and strain rate references.



A titanium alloy mechanical behaviour at various strain rates.

Another constitutive equation, based on physics rather than empirical, is the Zerilli—Armstrong one, described by

$$\sigma = C_0 + C_1 e^{-C_3 T + C_4 T \ln\dot{\varepsilon}} + C_5 \varepsilon^n.$$

This equation has been used to predict various material behaviours at a wide range of strains, strain rates and temperatures, as exemplified in the next figure for a tungsten alloy.

7.7 Inverse modelling and image analysis

Another way to obtain the true equivalent stress–strain curve of a material is by trial and error. This can be efficiently done using an optimization algorithm, as available in commercial finite element packages.

The idea is to experimentally load a specimen and record force and displacement. The specimen under the same load and boundary condition configuration is then analysed with the finite element method, a procedure which requires the material stress–strain curve. A set of stress–strain pairs are given, or else some constitutive parameters, and the analysis is run, giving as output a load–displacement curve. This