Non-Linear Dynamics and Stability

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Class 2

Elements of Stability Theory

Lagrangian formulation (recalling)

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right)-\frac{\partial T}{\partial q_r}+\frac{\partial V}{\partial q_r}=N_r \ , r=1,2,\ldots,n$$

System of second-order differential equations (holonomic constraints)

$$\ddot{\mathbf{q}}=\mathbf{h}\left(\mathbf{q},\dot{\mathbf{q}},t\right) \qquad \ddot{q}_r=h_r\left(q_1,q_2,\ldots,q_n,\dot{q}_1,\dot{q}_2,\ldots,\dot{q}_n,t\right)$$

Example: SDOF linear oscillator

$$\ddot{u}=\gamma\left(t\right)-\omega^2 u-2\xi\omega\dot{u} \quad \text{with} \quad \gamma\left(t\right)=\frac{R\left(t\right)}{m}, \quad \omega=\sqrt{\frac{k}{m}}, \quad \xi=\frac{c}{2m\omega}$$

Example: MDOF linear system

$$\ddot{\mathbf{U}}=\mathbf{M}^{-1}\left[\mathbf{R}\left(t\right)-\mathbf{K}\mathbf{U}-\mathbf{C}\dot{\mathbf{U}}\right]$$

Elements of Stability Theory

Hamiltonian formulation (recalling)

Generalized momenta: $p_r = \dfrac{\partial T}{\partial \dot{q}_r}$

Hamiltonian: $H = \sum_{r=1}^{n} \dot{q}_r p_r - T + V$

System of first-order differential equations (holonomic constraints)

$$\dot{q}_r = \frac{\partial H}{\partial p_r}$$

$$\dot{p}_r = N_r - \frac{\partial H}{\partial q_r}$$

Elements of Stability Theory

Hamiltonian formulation (recalling)

Example: SDOF linear oscillator

$$p = \frac{\partial T}{\partial \dot{q}} = m\dot{q}$$

$$H = p\dot{q} - T + V = \frac{p^2}{2m} + \frac{kq^2}{2}$$

$$N = R(t) - c\dot{q} = R(t) - c\frac{p}{m}$$

$$\Rightarrow \begin{cases} \dot{q} = \dfrac{1}{m}p \\ \dot{p} = R(t) - kq - \dfrac{c}{m}p \end{cases}$$

Elements of Stability Theory

Lagrangian formulation:
from second- to first-order system of differential equations through change of variables

$$\ddot{q}_r = h_r\left(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, t\right)$$

$$\begin{aligned} y_r &= q_r \\ y_{r+n} &= \dot{q}_r \end{aligned} \quad \Rightarrow \quad \begin{cases} \dot{y}_r = y_{r+n} \\ \dot{y}_{r+n} = h_r\left(y_1, y_2, \ldots, y_{2n}, t\right) \end{cases}$$

$$\dot{\mathbf{y}} = \mathbf{g}\left(\mathbf{y}, t\right)$$

Example: SDOF linear oscillator

$$\begin{aligned} y_1 &= q \\ y_2 &= \dot{q} \end{aligned} \quad \Rightarrow \quad \begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = \gamma\left(t\right) - \omega^2 y_1 - 2\xi\omega y_2 \end{cases}$$

Elements of Stability Theory

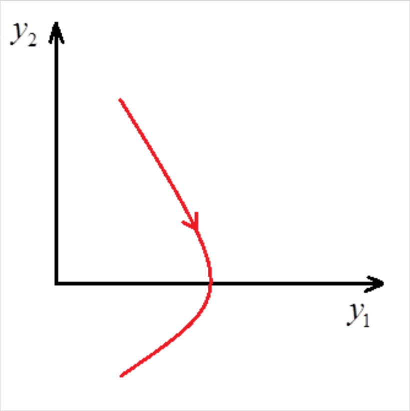
Phase space

Autonomous systems

$$\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y})$$

n-dimensional space

$$y_1 \times y_2 \times \ldots \times y_n$$

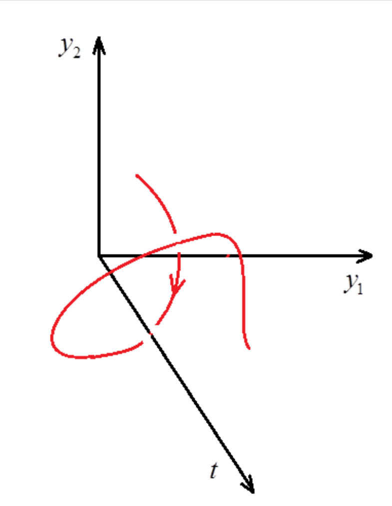


Non-autonomous systems

$$\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y}, t)$$

(n+1)-dimensional space

$$y_1 \times y_2 \times \ldots \times y_n \times t$$



Elements of Stability Theory

Phase space properties for SDOF autonomous systems

Singular phase points (equilibrium points) $\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y}) = 0$

Regular phase points $\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y}) \neq 0$

Phase trajectory tangent

$$\frac{dy_2}{dy_1} = \frac{g_2(y_1, y_2)}{g_1(y_1, y_2)} = \frac{g_2(y_1, y_2)}{y_2}$$

Tangent at singular phase points is indeterminate

$$\frac{dy_2}{dy_1} = \frac{g_2(y_1, y_2)}{g_1(y_1, y_2)} = \frac{0}{0}$$

Tangent at regular phase points with $g_1(y_1, y_2) = y_2 = 0$ and $g_2(y_1, y_2) \neq 0$
is orthogonal to the y_1 axis

Through a regular phase point passes just one phase trajectory
(Theorem of Cauchy-Lipschitz)

Elements of Stability Theory

Non-perturbed solution: $y_r = y_r^0(t), \quad r = 1, 2, \ldots, 2n$

Perturbed solution: $y_r = y_r^0(t) + \delta y_r(t), \quad r = 1, 2, \ldots, 2n$

$$\delta \dot{y}_r = g_r\left(y_1^0 + \delta y_1, y_2^0 + \delta y_2, \ldots, y_{2n}^0 + \delta y_{2n}, t\right) - \dot{y}_r^0$$

Perturbation equations:

$$\delta \dot{y}_r = f_r\left(\delta y_1, \delta y_2, \ldots, \delta y_{2n}, t\right)$$

$$\boldsymbol{\delta}\dot{\mathbf{y}} = \mathbf{f}\left(\boldsymbol{\delta}\mathbf{y}, t\right)$$

$\boldsymbol{\delta}\dot{\mathbf{y}} = \mathbf{A}(t)\boldsymbol{\delta}\mathbf{y} + \mathbf{N}(\boldsymbol{\delta}\mathbf{y}, t)$ with $\mathbf{A}(t) = \left.\dfrac{\partial \mathbf{f}}{\partial \mathbf{y}}\right|_{\mathbf{0}}$ and $\mathbf{N}(\boldsymbol{\delta}\mathbf{y}, t) = \mathbf{f}(\boldsymbol{\delta}\mathbf{y}, t) - \mathbf{A}(t)\boldsymbol{\delta}\mathbf{y}$

Note: the non-perturbed solution corresponds to the trivial solution $\boldsymbol{\delta}\mathbf{y} = \mathbf{0}$ of the perturbation equations

Elements of Stability Theory

Example: SDOF linear oscillator

$$\begin{cases} \delta \dot{y}_1 = \delta y_2 \\ \delta \dot{y}_2 = -\omega^2 \delta y_1 - 2\xi\omega\delta y_2 \end{cases}$$

$$\boldsymbol{\delta}\dot{\mathbf{y}} = \mathbf{f}\left(\boldsymbol{\delta}\mathbf{y}\right)$$

or

$$\boldsymbol{\delta}\dot{\mathbf{y}} = \mathbf{A}\boldsymbol{\delta}\mathbf{y}$$

with

$$\mathbf{A} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{y}}\right|_{\mathbf{0}} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\xi\omega \end{bmatrix}$$

Elements of Stability Theory

Stability concept (Leipholz)

A non-perturbed solution $\mathbf{y}^0(t)$ is stable if the distance $\delta\mathbf{y}(t)$ to the perturbed solutions remains within prescribed bounds for all times and arbitrarily defined perturbations

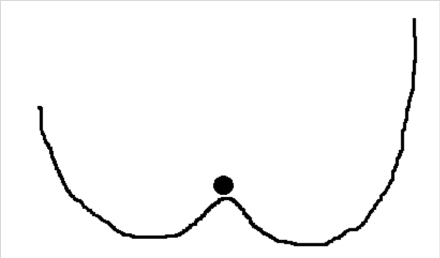
Non-perturbed solution
- Equilibrium $\mathbf{y}^0 = const.$
- Motion $\mathbf{y}^0(t)$

"Type" of perturbation
- Kinematical (initial conditions): $\delta\mathbf{y}(0) \neq \mathbf{0}$
- Topological (perturbation of parameters or perturbation of mathematical model)

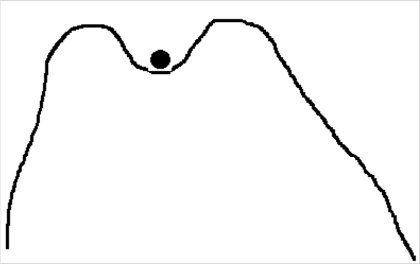
Elements of Stability Theory

Stability concept (Leipholz)

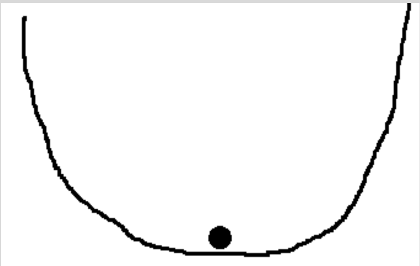
Perturbation "size"
- Local $\left\|\boldsymbol{\delta}\mathbf{y}(0)\right\| < \delta$
- Global



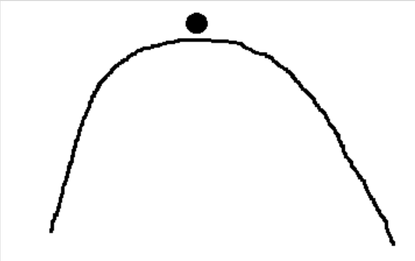
Global S
Local U



Global U
Local S



Global S
Local S



Global U
Local U

Elements of Stability Theory

Stability concept (Leipholz)

"Character" of perturbation
- Deterministic
- Stochastic

Example: <u>definition</u> of stability in the quadratic mean:

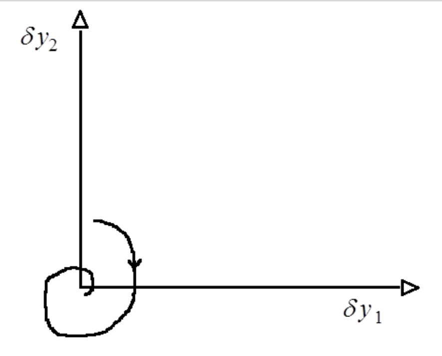
$$\lim_{\tau\to\infty} E_\tau \left\|\boldsymbol{\delta y}(t)\right\|^2 < \varepsilon \qquad \sigma^2_{\boldsymbol{\delta y}} = \int_{-\infty}^{\infty} S_{\boldsymbol{\delta y}}(\omega)\, d\omega < \varepsilon$$

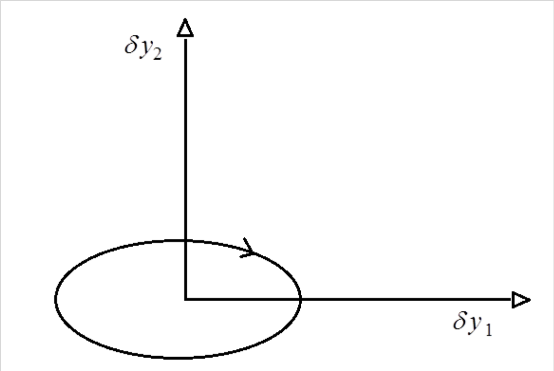
Stability x Reliability x Integrity

Elements of Stability Theory

Stability concept (Leipholz)

Tendency of perturbed solution
- Asymptotic
- Non-asymptotic

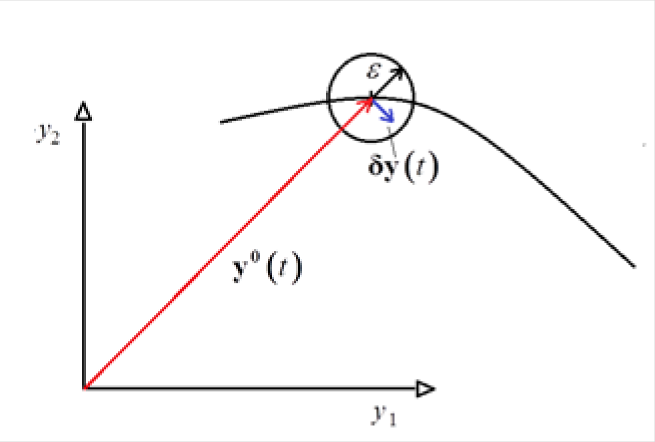


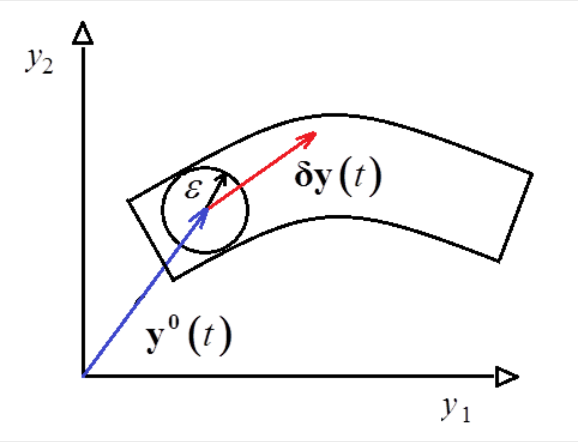


Elements of Stability Theory

Stability concept (Leipholz)

Admissible region for perturbed solution
- Kinetic
- Geometric





Elements of Stability Theory

Stability definitions

Liapunov

Stability of equilibrium of autonomous systems in the sense:
kinematical, local, deterministic, non-asymptotic, kinetic

Poincaré

Stability of motion of autonomous systems in the sense:
kinematical, local, deterministic, non-asymptotic, geometric

Particular case: orbital stability of periodic motions

Structural

Stability of equilibrium or motion in the sense:
topological, local, deterministic, asymptotic

Particular cases: parametric stability; Mathieu stability

Elements of Stability Theory

Liapunov stability

Given $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$, such that,
if $\|\mathbf{\delta y}(0)\| < \delta(\varepsilon)$ then $\|\mathbf{\delta y}(t)\| < \varepsilon$ for $t > 0$

Liapunov's methods

First method (indirect)
Second method (direct)

Elements of Stability Theory

Liapunov's first method

Perturbation equation for the analysis of the stability of equilibrium of the trivial solution $\boldsymbol{\delta y} = \mathbf{0}$

$$\boldsymbol{\delta \dot{y}} = \mathbf{f}\left(\boldsymbol{\delta y}\right) = \mathbf{A}\boldsymbol{\delta y} + \mathbf{N}\left(\boldsymbol{\delta y}\right)$$

with $\mathbf{A} = \left.\dfrac{\partial \mathbf{f}}{\partial \mathbf{y}}\right|_{\mathbf{0}}$ and $\mathbf{N}\left(\boldsymbol{\delta y}\right) = \mathbf{f}\left(\boldsymbol{\delta y}\right) - \mathbf{A}\boldsymbol{\delta y}$

Consider the associated linearized problem

$$\boldsymbol{\delta \dot{y}} = \mathbf{A}\boldsymbol{\delta y}$$

General solution

$$\boldsymbol{\delta y} = \boldsymbol{\delta y}_{\mathbf{0}} e^{\lambda t}$$

Elements of Stability Theory

Liapunov's first method

$$(\mathbf{A} - \lambda \mathbf{I})\boldsymbol{\delta}\mathbf{y}_0 = \mathbf{0}$$

For non-trivial solutions it is required that

$$|\mathbf{A} - \lambda \mathbf{I}| = 0$$

It is the classic eigenvalue problem for matrix $\mathbf{A}$

$$b_0\lambda^{2n} + b_1\lambda^{2n-1} + \ldots + b_{2n-1}\lambda + b_{2n} = 0$$

In the general case, there exists $2n$ complex roots for the characteristic equation

$$\lambda_k = \alpha_k + i\beta_k, \quad \alpha_k \in \mathbb{R} \quad \beta_k \in \mathbb{R}$$

Elements of Stability Theory

Liapunov's first method

Theorem 1 (Liapunov): If $R_k < 0 \quad \forall k = 1, 2 \ldots 2n \Rightarrow \boldsymbol{\delta y} = \mathbf{0}$ is L-stable

Theorem 2 (Liapunov): If $\exists R_k > 0 \quad \Rightarrow \boldsymbol{\delta y} = \mathbf{0}$ is L-unstable

Definition of L-critical case: there exists at least one eigenvalue with zero real part $R_k = 0$, yet none of them with positive real part.

Theorem 3 (Leipholz): In the critical case, if the multiplicity p_k of all the eigenvalues with null real part $(R_k = 0)$ is equal to the rank decrement d_k of the matrix $\mathbf{A} - \lambda_k \mathbf{I}$, then the solution $\boldsymbol{\delta y} = \mathbf{0}$ is L-stable for the linear system. If $p_k > d_k$, then the solution $\boldsymbol{\delta y} = \mathbf{0}$ is L-unstable for the linear system.

Elements of Stability Theory

Liapunov's first method

Theorem 4 (Routh-Hurwitz): If all principal minors of the matrix **B** (below) are positive, then the solution $\boldsymbol{\delta y} = \mathbf{0}$ is L-stable. The reciprocal is also true.

$$\mathbf{B} = \begin{bmatrix} b_1 & b_0 & 0 & 0 & 0 & 0 & \ldots & 0 \\ b_3 & b_2 & b_1 & b_0 & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & 0 & b_{2n} & b_{2n-1} & b_{2n-2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_{2n} \end{bmatrix}$$

$$b_{r>2n} = 0 \quad \text{and} \quad b_{r<0} = 0$$

Elements of Stability Theory

Liapunov's first method

Theorem 5 (Liapunov): Except for the L-critical case, the conclusions drawn from Theorems 1 and 2 for the linearized system $\boldsymbol{\delta\dot{y} = A\delta y}$ can be extended to the non-linear system $\boldsymbol{\delta\dot{y} = A\delta y + N(\delta y)}$

Dynamical systems theory

Theorem 5' (Hartman-Grobman): If a singularity of the linear system $\boldsymbol{\delta\dot{y} = A\delta y}$ is hyperbolic , then the linearized system is topologically equivalent to the non-linear system $\boldsymbol{\delta\dot{y} = A\delta y + N(\delta y)}$ in the singularity neighbourhood, that is, between the phase space flows of the non-linear and the linear systems there exists a diffeomorphism (transformation that is continuous with continuous derivative)

Elements of Stability Theory

Example: stability analysis for the solution $\boldsymbol{\delta y} = \mathbf{0}$ of a SDOF oscillator

$$\begin{cases} \delta \dot{y}_1 = \delta y_2 \\ \delta \dot{y}_2 = -\omega^2 \delta y_1 - 2\xi\omega\,\delta y_2 \end{cases}$$

$$\boldsymbol{\delta \dot{y}} = \mathbf{A}\boldsymbol{\delta y}$$

$$\mathbf{A} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{y}}\right|_{\mathbf{0}} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\xi\omega \end{bmatrix}$$

$$2\xi\omega \rightarrow b \qquad \omega^2 \rightarrow c$$

$$b \in \mathbb{R} \qquad c \in \mathbb{R}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix}$$

characteristic equation $\lambda^2 + b\lambda + c = 0$ ⇨ $\lambda = \dfrac{-b \pm \sqrt{b^2 - 4c}}{2}$

Elements of Stability Theory

Example: stability analysis for the solution $\boldsymbol{\delta y} = \mathbf{0}$ of a SDOF oscillator

Let $\boldsymbol{\delta y} = \mathbf{T}\boldsymbol{\delta x}$ such that $\boldsymbol{\delta \dot{y}} = \mathbf{A}\boldsymbol{\delta y} \Rightarrow \boldsymbol{\delta \dot{x}} = \mathbf{C}\boldsymbol{\delta x}$

with $\mathbf{C}$ being a Jordan canonical form

Remark: $\mathbf{T}$ must be such that $\mathbf{TC} = \mathbf{AT} \Rightarrow \mathbf{C} = \mathbf{T}^{-1}\mathbf{AT}$

Case (a): $\lambda_1 \in \mathbb{R}, \lambda_2 \in \mathbb{R}, \lambda_1 \neq \lambda_2 \rightarrow \mathbf{C} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

$$b^2 - 4c > 0$$

Case (b): $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R} \rightarrow \quad \mathbf{C} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$ ou $\mathbf{C} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

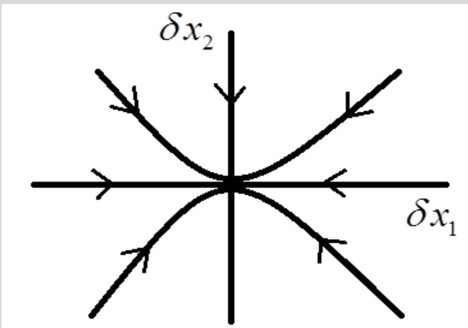
$$b^2 - 4c = 0$$

Case (c): $\lambda_1 = \lambda = \alpha + i\beta \in \mathbb{C}, \lambda_2 = \bar{\lambda} = \alpha - i\beta \in \mathbb{C} \rightarrow \mathbf{C} = \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix}$

$$b^2 - 4c < 0$$

Elements of Stability Theory

Example: stability analysis for the solution $\boldsymbol{\delta x = 0}$ of a SDOF oscillator

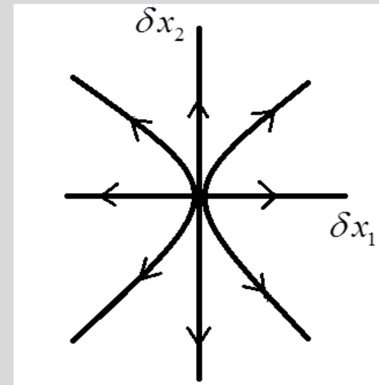


$\lambda_2 < \lambda_1 < 0$

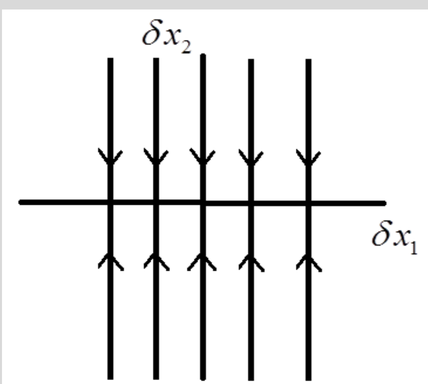
Case (a)

$$\delta x_i = \delta x_i^0 e^{\lambda_i t}$$

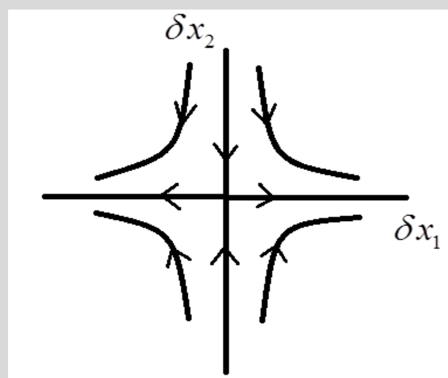
$$\frac{d(\delta x_2)}{d(\delta x_1)} = \left(\frac{\lambda_2}{\lambda_1}\right)\left(\frac{\delta x_2^0}{\delta x_1^0}\right) e^{(\lambda_2 - \lambda_1)t}$$



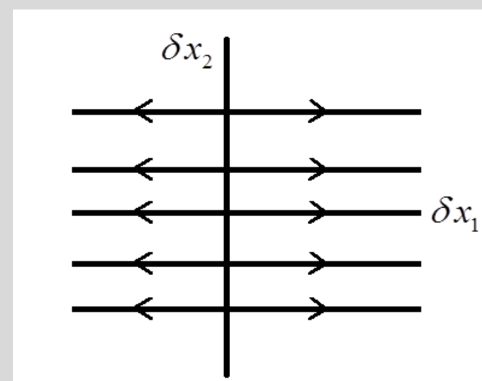
$0 < \lambda_2 < \lambda_1$



$\lambda_2 < \lambda_1 = 0$



$\lambda_2 < 0 < \lambda_1$



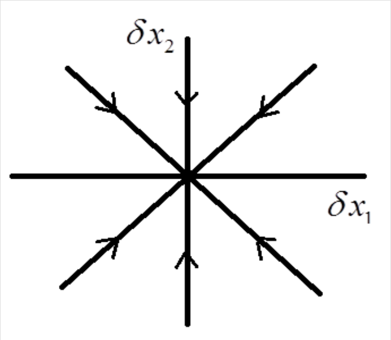
$0 = \lambda_2 < \lambda_1$

Elements of Stability Theory

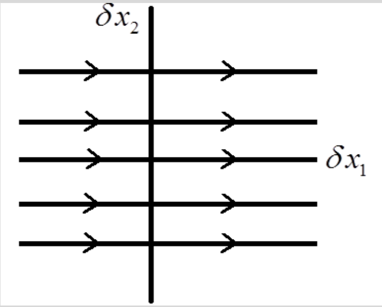
Example: stability analysis for the solution $\boldsymbol{\delta x} = \mathbf{0}$ of a SDOF oscillator

Case (b1)

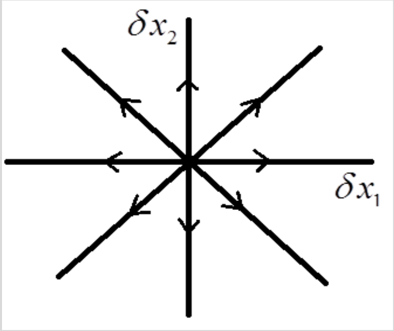
$$\delta x_i = \delta x_i^0 e^{\lambda t} \Rightarrow \frac{d(\delta x_2)}{d(\delta x_1)} = \left(\frac{\delta x_2^0}{\delta x_1^0}\right)$$



$\lambda_2 = \lambda_1 < 0$



$\lambda_2 = \lambda_1 = 0$

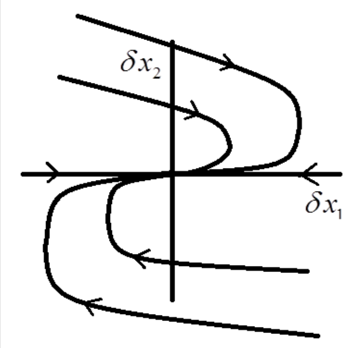


$0 < \lambda_2 = \lambda_1$

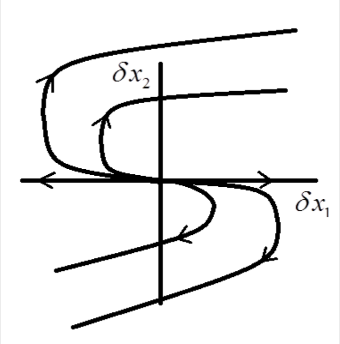
Case (b2)

$$\delta x_1 = \left(\delta x_1^0 + t\delta x_2^0\right)e^{\lambda t} \qquad \delta x_2 = \delta x_2^0 e^{\lambda t}$$

$$\frac{d(\delta x_2)}{d(\delta x_1)} = \frac{\delta x_2^0}{\delta x_1^0 + \left(t + \frac{1}{\lambda}\right)\delta x_2^0} = \frac{1}{\frac{\delta x_1^0}{\delta x_2^0} + \left(t + \frac{1}{\lambda}\right)}$$



$\lambda_2 = \lambda_1 < 0$



$0 < \lambda_2 = \lambda_1$

Elements of Stability Theory

Example: stability analysis for the solution $\boldsymbol{\delta x} = \mathbf{0}$ of a SDOF oscillator

Case (c)

Change variables...

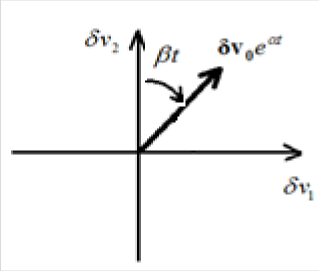
$$\boldsymbol{\delta\dot{x}} = \begin{bmatrix} \alpha + i\beta & 0 \\ 0 & \alpha - i\beta \end{bmatrix} \boldsymbol{\delta x}$$

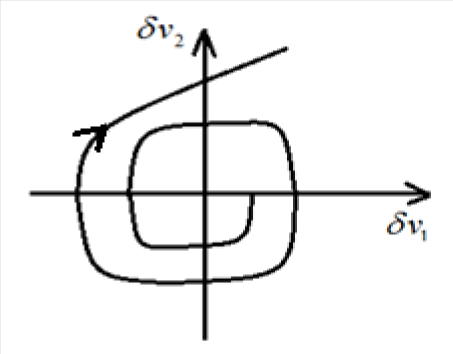
$$\boldsymbol{\delta v} = \begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix} \boldsymbol{\delta x}$$

$$\boldsymbol{\delta\dot{v}} = \begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix} \boldsymbol{\delta\dot{x}} = \begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix} \begin{bmatrix} \alpha + i\beta & 0 \\ 0 & \alpha + i\beta \end{bmatrix} \begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}^{-1} \boldsymbol{\delta v} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \boldsymbol{\delta v}$$

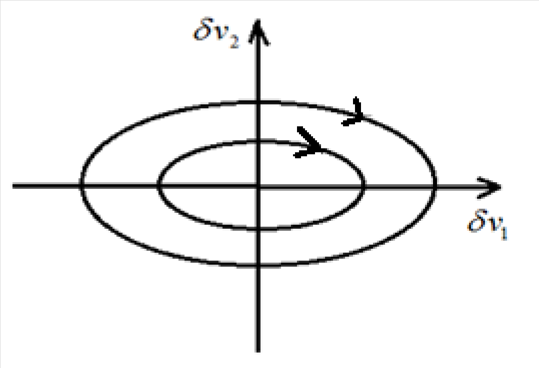
Define vector $\boldsymbol{\delta v} = \delta v_1 + i\delta v_2$ in Argand's plane ...

$$\boldsymbol{\delta\dot{v}} = (\alpha + i\beta)\boldsymbol{\delta v} \Rightarrow \boldsymbol{\delta v} = \boldsymbol{\delta v_0} e^{\alpha t} e^{i\beta t}$$

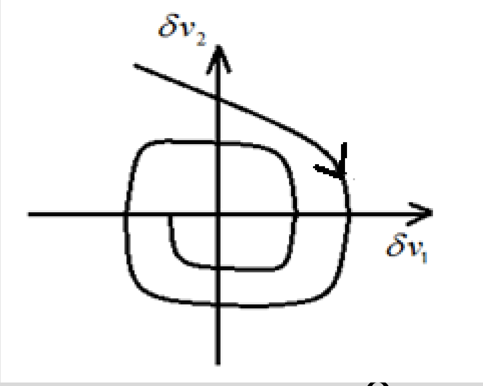




$\alpha > 0$



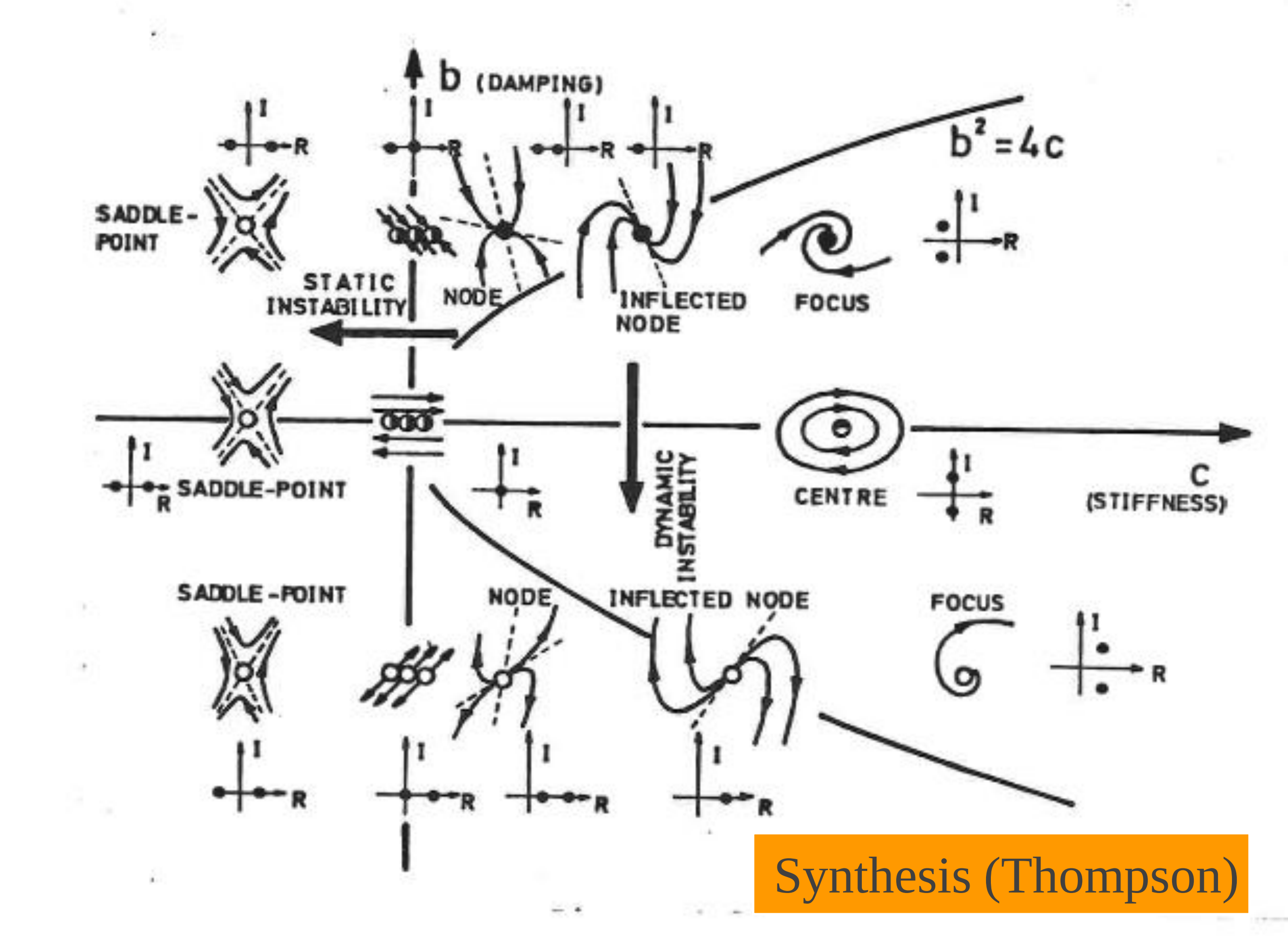
$\alpha = 0$



$\alpha < 0$

Elements of Stability Theory

Example: stability analysis for the solution $\delta x = 0$ of a SDOF oscillator



Synthesis (Thompson)

Elements of Stability Theory

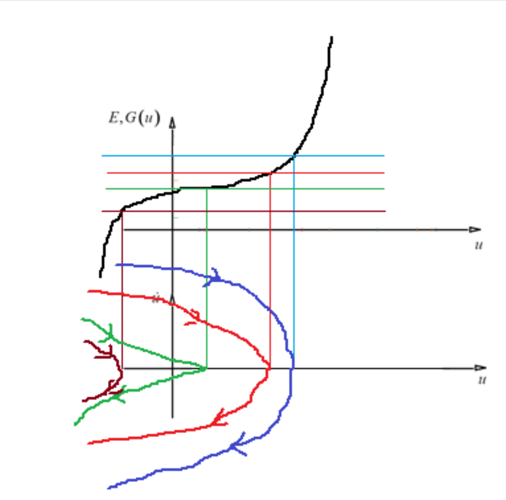
Conservative SDOF oscillator

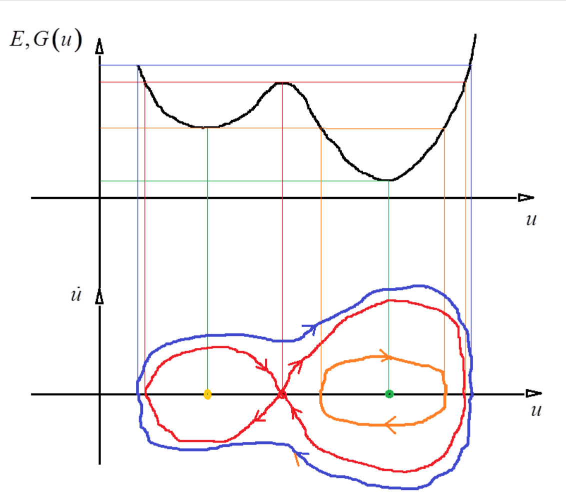
$$\ddot{u} + g(u) = 0 \Rightarrow \ddot{u}\,du + g(u)\,du = 0 \Rightarrow \ddot{u}\dot{u}dt + g(u)\,du = 0$$

Integrating: $$\underbrace{\frac{\dot{u}^2}{2}}_{\text{kinetic energy}} + \underbrace{\int_0^u g(\eta)d\eta}_{\text{potential energy}} = \underbrace{E}_{\text{mechanical energy}} = const.$$

Define: $$G(u) = \int_0^u g(\eta)d\eta \Rightarrow \dot{u} = \pm\sqrt{2[E - G(u)]} \Rightarrow T = \underbrace{2\int_{u(0)}^{u(T/2)} \frac{du}{\sqrt{2[E - G(u)]}}}_{\text{period of motion}}$$

saddle-node





saddle & centres

Elements of Stability Theory

Liapunov's second method

$$\delta\dot{\mathbf{y}} = \mathbf{f}(\delta\mathbf{y}) = \mathbf{A}\delta\mathbf{y} + \mathbf{N}(\delta\mathbf{y})$$

where $\mathbf{A} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{y}}\right|_{\mathbf{0}}$ and $\mathbf{N}(\delta\mathbf{y}) = \mathbf{f}(\delta\mathbf{y}) - \mathbf{A}\delta\mathbf{y}$

Theorem 6 (Liapunov): if there exists a function $F(\delta\mathbf{y}) : E \to \mathbb{R}$ such that:

$$F \geq 0 \quad \forall \delta\mathbf{y}$$

$$F = 0 \Leftrightarrow \delta\mathbf{y} = \mathbf{0}$$

$$\dot{F} = \frac{\partial F}{\partial \delta y_r}\delta\dot{y}_r = \frac{\partial F}{\partial \delta y_r} f_r \leq 0$$

then $\delta\mathbf{y} = \mathbf{0}$ is L-stable

Elements of Stability Theory

Liapunov's second method

Theorem 7 (Liapunov): if there exists a function $F(\boldsymbol{\delta y}): E \to \mathbb{R}$ such that:

$$F \geq 0 \quad \forall \boldsymbol{\delta y}$$

$$F = 0 \Leftrightarrow \boldsymbol{\delta y} = \mathbf{0}$$

$$\dot{F} = \frac{\partial F}{\partial \delta y_r} \delta \dot{y}_r = \frac{\partial F}{\partial \delta y_r} f_r < 0$$

then $\boldsymbol{\delta y} = \mathbf{0}$ is asymptotically stable in Liapunov's sense

Theorem 8 (Chetayev): if there exists a function $F(\boldsymbol{\delta y}): E \to \mathbb{R}$ such that:

$$F \geq 0 \quad \forall \boldsymbol{\delta y}$$

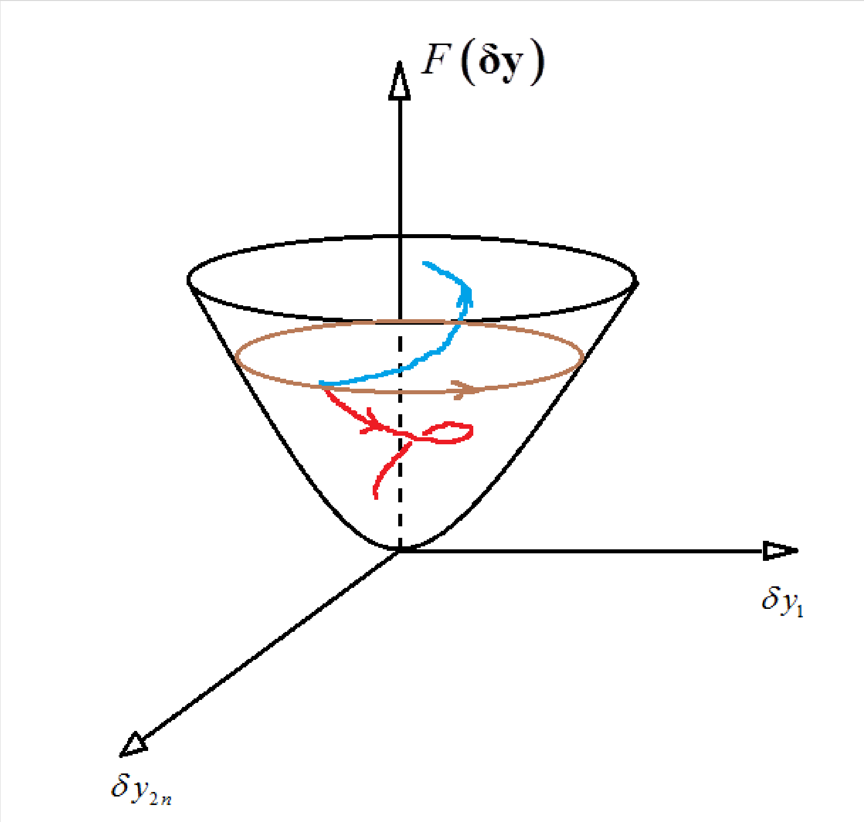
$$F = 0 \Leftrightarrow \boldsymbol{\delta y} = \mathbf{0}$$

$$\dot{F} = \frac{\partial F}{\partial \delta y_r} \delta \dot{y}_r = \frac{\partial F}{\partial \delta y_r} f_r > 0$$

then $\boldsymbol{\delta y} = \mathbf{0}$ is L-unstable

Elements of Stability Theory

Liapunov's second method



$F(\boldsymbol{\delta y})$ is called Liapunov's function

Elements of Stability Theory

Attractor and Basin of Attraction

Attractor is a subset of the phase space to which a solution of the dynamical system tends when $t \to \infty$, for initial conditions in a non-localized subset of the phase space called basin of attraction

- Point attractor: asymptotically stable singularity
- Periodic attractor: asymptotically stable orbit (limit cycle) in the phase space with one dominating frequency or more than one commensurate dominating frequencies
- Limit torus: asymptotically stable manifold in the phase space, with more than one non-commensurate dominating frequency
- Strange attractor (chaotic attractor): coexistence of some of the previous attractors with non-compact (fractal) basins of attraction

Elements of Stability Theory

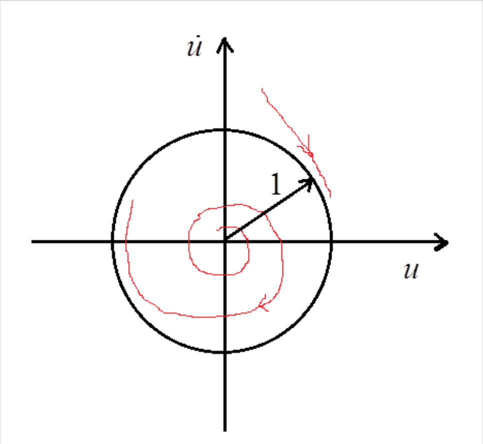
Periodic attractor in autonomous dynamical system $\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y})$

Example: van der Pol equation

$$\ddot{u} - \dot{u} + u + \left(u^2 + \dot{u}^2\right)\dot{u} = 0$$

Trivial solution $u(t) = 0$ is unstable

Periodic attractor $u(t) = \sin t$ is stable



Elements of Stability Theory

Dynamical Systems

Hirsch & Smale: Differential Equations, Dynamical Systems
and Linear Algebra
Guckenheimer & Holmes: Nonlinear Oscillations, Dynamical Systems
And Bifurcation of Vector Fields

Elements of Stability Theory

Orbital stability of autonomous SDOF oscillators

- First Poincaré-Bendixson's Theorem:

If a phase trajectory C remains within a finite region without approaching a singularity, then C is a limit cycle or it tends to one.

- Second Poincaré-Bendixson's Theorem:

Given a region D of the phase space, bounded by two curves C' and C", without a singularity in D, C' e C", if all phase trajectories enter (exit) D through the boundaries C' e C", then there exists at least a stable (unstable) limit cycle in D.

Elements of Stability Theory

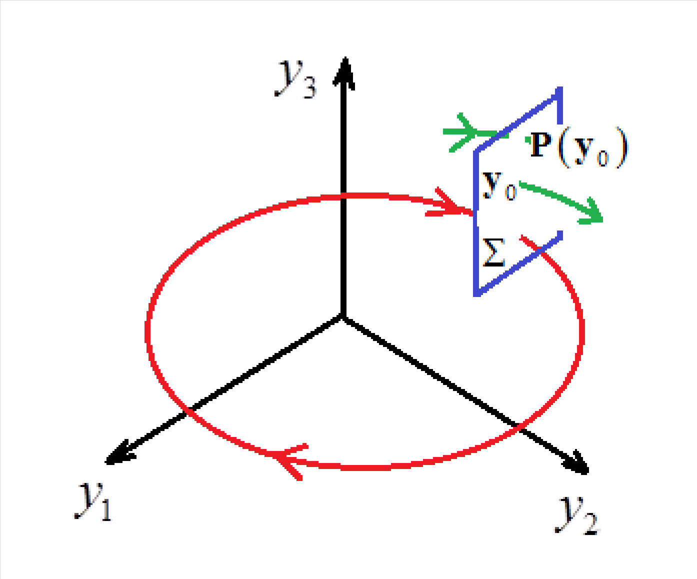
Poincaré's section (map)

- Let $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$ be a flow of an autonomous system in $\mathbb{R}^{2n}$ and Σ a section with normal $\mathbf{N}$ such that $\mathbf{f}(\mathbf{y}) \cdot \mathbf{N} \neq 0$, that is, the section Σ is not parallel to the flow $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$. Consider the mapping $\mathbf{y}_0 \rightarrow \mathbf{P}(\mathbf{y}_0)$ defined by the intersection of the flow $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$ with Σ. $\mathbf{P}(\mathbf{y}_0)$ is called a "Poincaré's section" of the flow $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$ through $\mathbf{y}_0$

- If the system is non-autonomous, defined by the flow $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, t)$, and $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$ is the associated autonomous system $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$ defined in $\mathbb{R}^{2n+1}$ with the addition of $\dot{y}_{2n+1} = 1$, the Poincaré's sections can be defined orthogonally to the axis $y_{2n+1} = t$ at $t = t_0 + iT, \quad i = 1, 2, \ldots$

Elements of Stability Theory

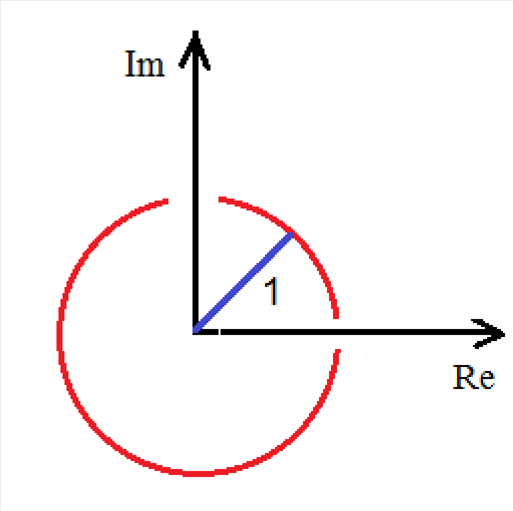
Poincaré's section (map)



Analyse the complex eigenvalues $\lambda_j = \mathrm{Re}_j + i\,\mathrm{Im}_j$ of linearized mapping $\mathbf{DP}(\mathbf{y}_0)$ to test stability.

Stability for $|\lambda_j| < 1$

Instability for $|\lambda_j| > 1$



Elements of Stability Theory

Example of Poincaré’s section (map)

$$\ddot{u}+\left(-1+u^{2}+\dot{u}^{2}\right)\dot{u}+u=0$$

$$\left.\begin{aligned} y_1 &= u \\ y_2 &= \dot{u} \end{aligned}\right\} \Rightarrow \dot{\mathbf{y}} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}\mathbf{y} + \begin{Bmatrix} 0 \\ -\left(y_1^2+y_2^2\right)y_2 \end{Bmatrix}$$

In polar coordinates

$$\left.\begin{aligned} y_1 &= r\sin\theta \\ y_2 &= r\cos\theta \end{aligned}\right\} \Rightarrow r=0 \text{ corresponds to an unstable focus}$$

$$\text{for } r\neq 0 \Rightarrow \begin{cases} \dot{r} = -r\left(r^2-1\right)\cos^2\theta \\ \dot{\theta} = 1+\left(r^2-1\right)\sin\theta\cos\theta \end{cases}$$

It is readily seen that $r=1$ and $\theta=t$ are a limit cycle

Elements of Stability Theory

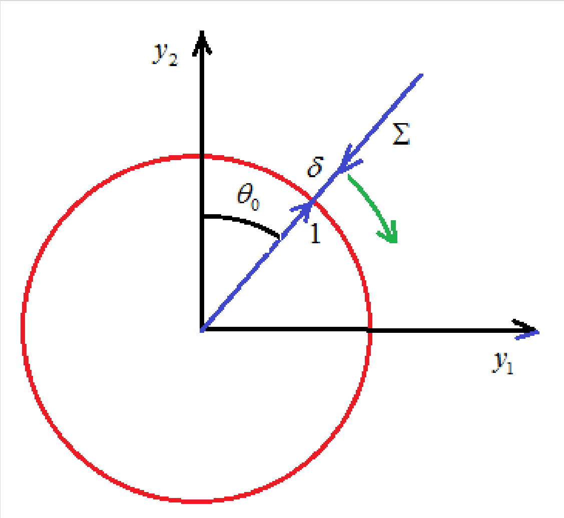
Example of Poincaré's section (map)

$$r_0 = 1 + \varepsilon_0 \rightarrow r_j = 1 + \varepsilon_j \text{ for } \theta = \theta_0 + 2\pi j \qquad j = 1, 2, \ldots$$

Mapping: $$\dot{r}_j = \dot{\varepsilon}_j = -\left(1+\varepsilon_j\right)\left[\left(1+\varepsilon_j\right)^2 - 1\right]\cos^2\theta_0$$

$$\dot{\varepsilon}_j = -\left(2\varepsilon_j + 3\varepsilon_j^2 + \varepsilon_j^3\right)\cos^2\theta_0$$

Linearizing: $\dot{\varepsilon}_j = -\left(2\cos^2\theta_0\right)\varepsilon_j \Rightarrow \varepsilon_j = \varepsilon_0 e^{-4\pi j \cos^2\theta_0}$



Mapping in $\mathbb{R}^1$: $r_j \rightarrow r_{j+1} = P\left(r_j\right) = 1 + \left(r_j - 1\right)e^{-4\pi\cos^2\theta_0}$

$$\mathbf{DP} = \frac{dP\left(r_j\right)}{dr_j} = e^{-4\pi\cos^2\theta_0}$$

asymptotic stability for $\theta_0 \neq \frac{\pi}{2}$ or $\frac{3\pi}{2}$, since $|\lambda| < 1$

Poincaré's section: $\theta = \theta_0$

stability for $\theta_0 = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, since $\dot{\varepsilon}_j = 0 \Rightarrow \varepsilon_j = \varepsilon_0$

Elements of Stability Theory

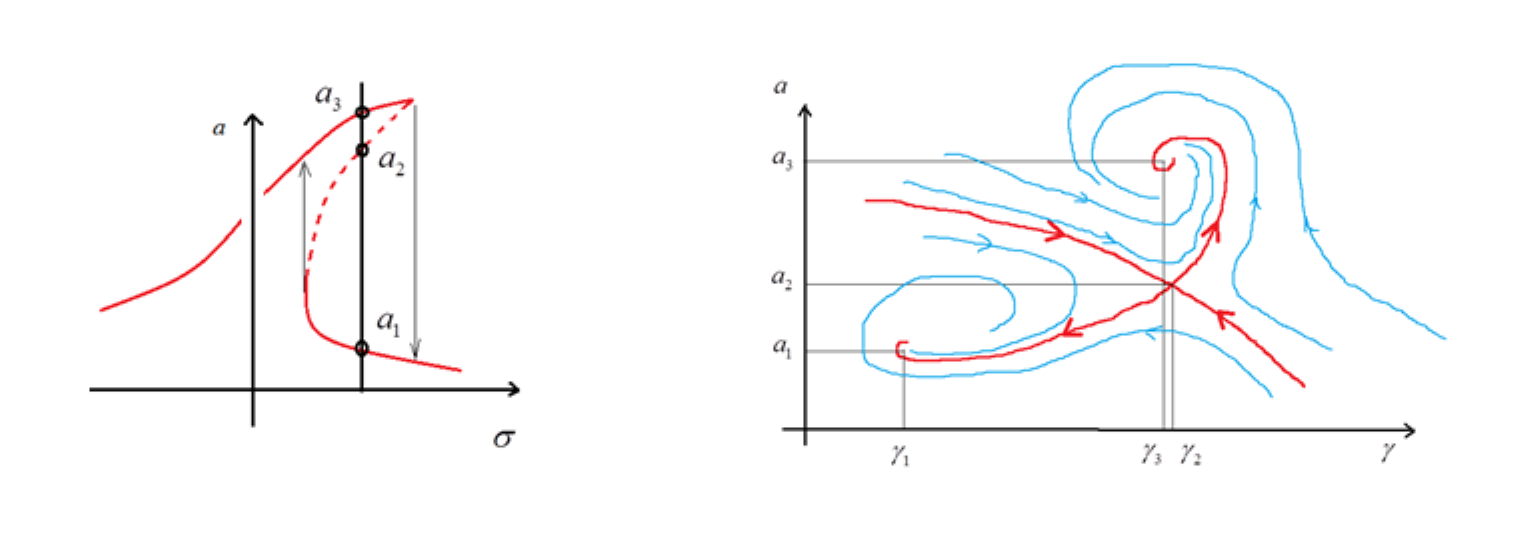
Periodic attractor in non-autonomous dynamical system $\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y}, t)$

Example: forced Duffing's equation

$$\ddot{u} + 2\varepsilon\mu\dot{u} + \omega_0^2 u + \varepsilon\alpha u^3 = \varepsilon k \cos(\omega_0 + \varepsilon\sigma)t \quad \text{with } 0 < \varepsilon << 1$$

There exist periodic attractors

$$u(t) = a\cos\left[(\omega_0 + \varepsilon\sigma)t + \gamma\right] + O(\varepsilon)$$



Estudo recai em estabilidade de singularidades...