CHAPTER SEVEN

Uncertainty

In this chapter we explore some of the basic elements of the theory of individual behavior in uncertain situations. We discuss why individuals do not like risk and the various methods (buying insurance, acquiring more information, and preserving options) they may adopt to reduce it. More generally, the chapter is intended to provide a brief introduction to issues raised by the possibility that information may be imperfect when individuals make utility-maximizing decisions. The Extensions section provides a detailed application of the concepts in this chapter to the portfolio problem, a central problem in financial economics. Whether a well-informed person can take advantage of a poorly informed person in a market transaction (asymmetric information) is a question put off until Chapter 18.

MATHEMATICAL STATISTICS

Many of the formal tools for modeling uncertainty in economic situations were originally developed in the field of mathematical statistics. Some of these tools were reviewed in Chapter 2, and in this chapter we will make a great deal of use of the concepts introduced there. Specifically, four statistical ideas will recur throughout this chapter.

- ***Random variable:*** A random variable is a variable that records, in numerical form, the possible outcomes from some random event.[1]
- ***Probability density function (PDF):*** A function $f(x)$ that shows the probabilities associated with the possible outcomes from a random variable.
- ***Expected value of a random variable:*** The outcome of a random variable that will occur "on average." The expected value is denoted by $E(x)$. If x is a discrete random variable with n outcomes, then $E(x) = \sum_{i=1}^{n} x_i f(x_i)$. If x is a continuous random variable, then $E(x) = \int_{-\infty}^{+\infty} x f(x)\,dx$.
- ***Variance and standard deviation of a random variable:*** These concepts measure the dispersion of a random variable about its expected value. In the discrete case, $\text{Var}(x) = \sigma_x^2 = \sum_{i=1}^{n} [x_i - E(x)]^2 f(x_i)$; in the continuous case, $\text{Var}(x) = \sigma_x^2 = \int_{-\infty}^{+\infty} [x - E(x)]^2 f(x)\,dx$. The standard deviation is the square root of the variance.

As we shall see, all these concepts will come into play when we begin looking at the decision-making process of a person faced with a number of uncertain outcomes that can be conceptually represented by a random variable.

[1]When it is necessary to distinguish between random variables and nonrandom variables, we will use the notation $\tilde{x}$ to denote the fact that the variable x is random in that it takes on a number of potential randomly determined outcomes. Often, however, it will not be necessary to make the distinction because randomness will be clear from the context of the problem.

FAIR GAMBLES AND THE EXPECTED UTILITY HYPOTHESIS

A "fair" gamble is a specified set of prizes and associated probabilities that has an expected value of zero. For example, if you flip a coin with a friend for a dollar, the expected value of this gamble is zero because

$$E(x) = 0.5(+\$1) + 0.5(-\$1) = 0, \tag{7.1}$$

where wins are recorded with a plus sign and losses with a minus sign. Similarly, a game that promised to pay you \$10 if a coin came up heads but would cost you only \$1 if it came up tails would be "unfair" because

$$E(x) = 0.5(+\$10) + 0.5(-\$1) = \$4.50. \tag{7.2}$$

This game can easily be converted into a fair game, however, simply by charging you an entry fee of \$4.50 for the right to play.

It has long been recognized that most people would prefer not to take fair gambles.[2] Although people may wager a few dollars on a coin flip for entertainment purposes, they would generally balk at playing a similar game whose outcome was +\$1 million or −\$1 million. One of the first mathematicians to study the reasons for this unwillingness to engage in fair bets was Daniel Bernoulli in the eighteenth century.[3] His examination of the famous St. Petersburg paradox provided the starting point for virtually all studies of the behavior of individuals in uncertain situations.

St. Petersburg paradox

In the St. Petersburg paradox, the following gamble is proposed: A coin is flipped until a head appears. If a head first appears on the nth flip, the player is paid $\$2^n$. This gamble has an infinite number of outcomes (a coin might be flipped from now until doomsday and never come up a head, although the likelihood of this is small), but the first few can easily be written down. If x_i represents the prize awarded when the first head appears on the ith trial, then

$$x_1 = \$2,\ x_2 = \$4,\ x_3 = \$8, \ldots,\ x_n = \$2^n. \tag{7.3}$$

The probability of getting a head for the first time on the ith trial is $\left(\frac{1}{2}\right)^i$; it is the probability of getting $(i - 1)$ tails and then a head. Hence the probabilities of the prizes given in Equation 7.3 are

$$\pi_1 = \frac{1}{2},\ \pi_2 = \frac{1}{4},\ \pi_3 = \frac{1}{8}, \ldots,\ \pi_n = \frac{1}{2^n}. \tag{7.4}$$

Therefore, the expected value of the gamble is infinite:

$$\begin{aligned} E(x) &= \sum_{i=1}^{\infty} \pi_i x_i = \sum_{i=1}^{\infty} 2^i (1/2^i) \\ &= 1 + 1 + 1 + \cdots + 1 + \cdots = \infty. \end{aligned} \tag{7.5}$$

---

[2]The gambles discussed here are assumed to yield no utility in their play other than the prizes; hence the observation that many individuals gamble at "unfair" odds is not necessarily a refutation of this statement. Rather, such individuals can reasonably be assumed to be deriving some utility from the circumstances associated with the play of the game. Therefore, it is possible to differentiate the consumption aspect of gambling from the pure risk aspect.

[3]The paradox is named after the city where Bernoulli's original manuscript was published. The article has been reprinted as D. Bernoulli, "Exposition of a New Theory on the Measurement of Risk," *Econometrica* 22 (January 1954): 23–36.

Some introspection, however, should convince anyone that no player would pay very much (much less than infinity) to take this bet. If we charged \$1 billion to play the game, we would surely have no takers, despite the fact that \$1 billion is still considerably less than the expected value of the game. This then is the paradox: Bernoulli's gamble is in some sense not worth its (infinite) expected dollar value.

EXPECTED UTILITY

Bernoulli's solution to this paradox was to argue that individuals do not care directly about the dollar prizes of a gamble; rather, they respond to the utility these dollars provide. If we assume that the marginal utility of wealth decreases as wealth increases, the St. Petersburg gamble may converge to a finite *expected utility* value even though its expected monetary value is infinite. Because the gamble only provides a finite expected utility, individuals would only be willing to pay a finite amount to play it. Example 7.1 looks at some issues related to Bernoulli's solution.

EXAMPLE 7.1 Bernoulli's Solution to the Paradox and Its Shortcomings

Suppose, as did Bernoulli, that the utility of each prize in the St. Petersburg paradox is given by

$$U(x_i) = \ln(x_i). \tag{7.6}$$

This logarithmic utility function exhibits diminishing marginal utility (i.e., $U' > 0$ but $U'' < 0$), and the expected utility value of this game converges to a finite number:

$$\begin{aligned} \text{expected utility} &= \sum_{i=1}^{\infty} \pi_i U(x_i) \\ &= \sum_{i=1}^{\infty} \frac{1}{2^i} \ln(2^i). \end{aligned} \tag{7.7}$$

Some manipulation of this expression yields[4] the result that the expected utility from this gamble is 1.39. Therefore, an individual with this type of utility function might be willing to invest resources that otherwise yield up to 1.39 units of utility (a certain wealth of approximately \$4 provides this utility) in purchasing the right to play this game. Thus, assuming that the large prizes promised by the St. Petersburg paradox encounter diminishing marginal utility permitted Bernoulli to offer a solution to the paradox.

Unbounded utility. Unfortunately, Bernoulli's solution to the St. Petersburg paradox does not completely solve the problem. As long as there is no upper bound to the utility function, the paradox can be regenerated by redefining the gamble's prizes. For example, with the logarithmic utility function, prizes can be set as $x_i = e^{2^i}$, in which case

$$U(x_i) = \ln[e^{2^i}] = 2^i \tag{7.8}$$

and the expected utility from the gamble would again be infinite. Of course, the prizes in this redefined gamble are large. For example, if a head first appears on the fifth flip, a person would

[4] *Proof:*

$$\text{expected utility} = \sum_{i=1}^{\infty} \frac{i}{2^i} \cdot \ln 2 = \ln 2 \sum_{i=1}^{\infty} \frac{i}{2^i}.$$

But the value of this final infinite series can be shown to be 2.0. Hence expected utility $= 2 \ln 2 = 1.39$.

win $e^{2^5} = \$79$ trillion, although the probability of winning this would be only $1/2^5 = 0.031$. The idea that people would pay a great deal (say, trillions of dollars) to play games with small probabilities of such large prizes seems, to many observers, to be unlikely. Hence in many respects the St. Petersburg game remains a paradox.

QUERY: Here are two alternative solutions to the St. Petersburg paradox. For each, calculate the expected value of the original game.

1. Suppose individuals assume that any probability less than 0.01 is in fact zero.
2. Suppose that the utility from the St. Petersburg prizes is given by

$$U(x_i) = \begin{cases} x_i & \text{if } x_i \le 1{,}000{,}000, \\ 1{,}000{,}000 & \text{if } x_i > 1{,}000{,}000. \end{cases}$$

THE VON NEUMANN–MORGENSTERN THEOREM

Among many contributions relevant to Part 3 of our text, in their book *The Theory of Games and Economic Behavior*, John von Neumann and Oscar Morgenstern developed a mathematical foundation for Bernoulli's solution to the St. Petersburg paradox.[5] In particular, they laid out basic axioms of rationality and showed that any person who is rational in this way would make choices under uncertainty as though he or she had a utility function over money $U(x)$ and maximized the expected value of $U(x)$ (rather than the expected value of the monetary payoff x itself). Although most of these axioms seem eminently reasonable at first glance, many important questions about their tenability have been raised.[6] We will not pursue these questions here, however.

The von Neumann–Morgenstern utility index

To begin, suppose that there are n possible prizes that an individual might win by participating in a lottery. Let these prizes be denoted by $x_1, x_2, \ldots, x_n$, and assume that these have been arranged in order of ascending desirability. Therefore, x_1 is the least preferred prize for the individual and x_n is the most preferred prize. Now assign arbitrary utility numbers to these two extreme prizes. For example, it is convenient to assign

$$\begin{aligned} U(x_1) &= 0, \\ U(x_n) &= 1, \end{aligned} \qquad \textbf{(7.9)}$$

but any other pair of numbers would do equally well.[7] Using these two values of utility, the point of the von Neumann–Morgenstern theorem is to show that a reasonable way exists to assign specific utility numbers to the other prizes available. Suppose that we choose any other prize, say, x_i. Consider the following experiment. Ask the individual to state the probability, say, π_i, at which he or she would be indifferent between x_i with

[5] J. von Neumann and O. Morgenstern, *The Theory of Games and Economic Behavior* (Princeton, NJ: Princeton University Press, 1944). The axioms of rationality in uncertain situations are discussed in the book's appendix.

[6] For a discussion of some of the issues raised in the debate over the von Neumann–Morgenstern axioms, especially the assumption of independence, see C. Gollier, *The Economics of Risk and Time* (Cambridge, MA: MIT Press, 2001), chap. 1.

[7] Technically, a von Neumann–Morgenstern utility index is unique only up to a choice of scale and origin—that is, only up to a "linear transformation." This requirement is more stringent than the requirement that a utility function be unique up to a monotonic transformation.

certainty, and a *gamble* offering prizes of x_n with probability π_i and x_1 with probability $(1 - \pi_i)$. It seems reasonable (although this is the most problematic assumption in the von Neumann–Morgenstern approach) that such a probability will exist: The individual will always be indifferent between a gamble and a sure thing, provided that a high enough probability of winning the best prize is offered. It also seems likely that π_i will be higher the more desirable x_i is; the better x_i is, the better the chance of winning x_n must be to get the individual to gamble. Therefore, the probability π_i measures how desirable the prize x_i is. In fact, the von Neumann–Morgenstern technique defines the utility of x_i as the expected utility of the gamble that the individual considers equally desirable to x_i:

$$U(x_i) = \pi_i U(x_n) + (1 - \pi_i)U(x_1). \tag{7.10}$$

Because of our choice of scale in Equation 7.9, we have

$$U(x_i) = \pi_i \cdot 1 + (1 - \pi_i) \cdot 0 = \pi_i. \tag{7.11}$$

By judiciously choosing the utility numbers to be assigned to the best and worst prizes, we have been able to devise a scale under which the utility index attached to any other prize is simply the probability of winning the top prize in a gamble the individual regards as equivalent to the prize in question. This choice of utility indices is arbitrary. Any other two numbers could have been used to construct this utility scale, but our initial choice (Equation 7.9) is a particularly convenient one.

Expected utility maximization

In line with the choice of scale and origin represented by Equation 7.9, suppose that a utility index π_i has been assigned to every prize x_i. Notice in particular that $\pi_1 = 0$, $\pi_n = 1$, and that the other utility indices range between these extremes. Using these utility indices, we can show that a "rational" individual will choose among gambles based on their expected "utilities" (i.e., based on the expected value of these von Neumann–Morgenstern utility index numbers).

As an example, consider two gambles. Gamble A offers x_2 with probability a and x_3 with probability $(1 - a)$. Gamble B offers x_4 with probability b and x_5 with probability $(1 - b)$. We want to show that this person will choose gamble A if and only if the expected utility of gamble A exceeds that of gamble B. Now for the gambles:

$$\begin{aligned}\text{expected utility of } A &= aU(x_2) + (1 - a)U(x_3),\\ \text{expected utility of } B &= bU(x_4) + (1 - b)U(x_5).\end{aligned} \tag{7.12}$$

Substituting the utility index numbers (i.e., π_2 is the "utility" of x_2, and so forth) gives

$$\begin{aligned}\text{expected utility of } A &= a\pi_2 + (1 - a)\pi_3,\\ \text{expected utility of } B &= b\pi_4 + (1 - b)\pi_5.\end{aligned} \tag{7.13}$$

We wish to show that the individual will prefer gamble A to gamble B if and only if

$$a\pi_2 + (1 - a)\pi_3 > b\pi_4 + (1 - b)\pi_5. \tag{7.14}$$

To show this, recall the definitions of the utility index. The individual is indifferent between x_2 and a gamble promising x_1 with probability $(1 - \pi_2)$ and x_n with probability π_2. We can use this fact to substitute gambles involving only x_1 and x_n for all utilities in Equation 7.13 (even though the individual is indifferent between these, the assumption that this substitution can be made implicitly assumes that people can see through complex lottery combinations). After a bit of messy algebra, we can conclude that gamble A is equivalent to a gamble promising x_n with probability $a\pi_2 + (1 - a)\pi_3$, and gamble B is equivalent to a gamble promising x_n with probability $b\pi_4 + (1 - b)\pi_5$. The individual will presumably prefer the gamble with the higher probability of winning the best prize. Consequently, he or she will choose gamble A if and only if

$$a\pi_2 + (1 - a)\pi_3 > b\pi_4 + (1 - b)\pi_5. \tag{7.15}$$

But this is precisely what we wanted to show. Consequently, we have proved that an individual will choose the gamble that provides the highest level of expected (von Neumann–Morgenstern) utility. We now make considerable use of this result, which can be summarized as follows.

OPTIMIZATION PRINCIPLE

Expected utility maximization. If individuals obey the von Neumann–Morgenstern axioms of behavior in uncertain situations, they will act as though they choose the option that maximizes the expected value of their von Neumann–Morgenstern utility.

RISK AVERSION

Economists have found that people tend to avoid risky situations, even if the situation amounts to a fair gamble. For example, few people would choose to take a $10,000 bet on the outcome of a coin flip, even though the average payoff is 0. The reason is that the gamble's money prizes do not completely reflect the utility provided by the prizes. The utility that people obtain from an increase in prize money may increase less rapidly than the dollar value of these prizes. A gamble that is fair in money terms may be unfair in utility terms and thus would be rejected.

In more technical terms, extra money may provide people with decreasing marginal utility. A simple example can help explain why. An increase in income from, say, $40,000 to $50,000 may substantially increase a person's well-being, ensuring he or she does not have to go without essentials such as food and housing. A further increase from $50,000 to $60,000 allows for an even more comfortable lifestyle, perhaps providing tastier food and a bigger house, but the improvement will likely not be as great as the initial one.

Starting from a wealth of $50,000, the individual would be reluctant to take a $10,000 bet on a coin flip. The 50 percent chance of the increased comforts that he or she could have with $60,000 does not compensate for the 50 percent chance that he or she will end up with $40,000 and perhaps have to forgo some essentials.

These effects are only magnified with riskier gambles, that is, gambles having even more variable outcomes.[8] The person with initial wealth of $50,000 would likely be even more reluctant to take a $20,000 bet on a coin flip because he or she would face the prospect of ending up with only $30,000 if the flip turned out badly, severely cutting into life's essentials. The equal chance of ending up with $70,000 is not adequate compensation. On the other hand, a bet of only $1 on a coin flip is relatively inconsequential. Although the person may still decline the bet, he or she would not try hard to do so because his or her ultimate wealth hardly varies with the outcome of the coin toss.

Risk aversion and fair bets

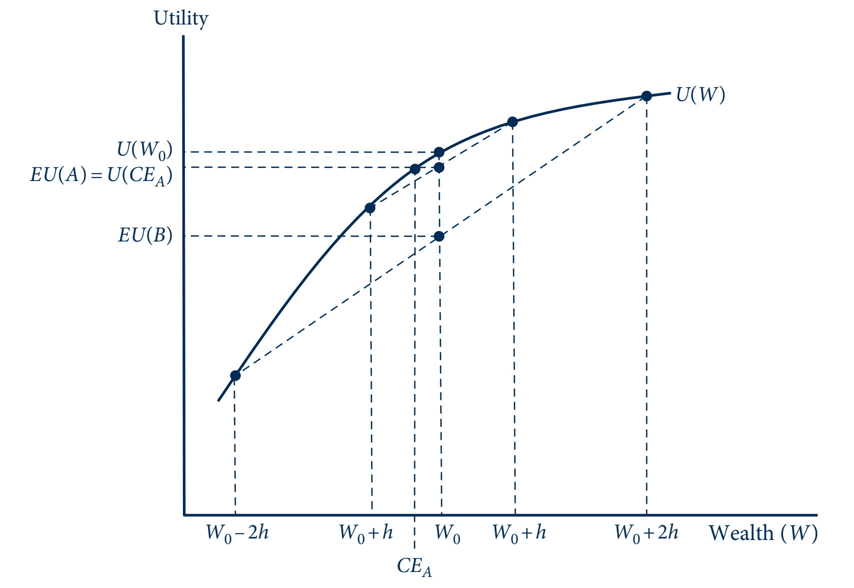
This argument is illustrated in Figure 7.1. Here W_0 represents an individual's current wealth and $U(W)$ is a von Neumann–Morgenstern utility index (we will call this a utility

[8]Often the statistical concepts of variance and standard deviation are used to measure. We will do so at several places later in this chapter.

FIGURE 7.1

Utility of Wealth from Two Fair Bets of Differing Variability

If the utility-of-wealth function is concave (i.e., exhibits a diminishing marginal utility of wealth), then this person will refuse fair bets. A 50–50 chance of winning or losing h dollars, for example, yields less expected utility [$EU(A)$] than does refusing the bet. The reason for this is that winning h dollars means less to this individual than does losing h dollars.



function from now on) that reflects how he or she feels about various levels of wealth.[9] In the figure, $U(W)$ is drawn as a concave function of W to reflect the assumption of a diminishing marginal utility. Now suppose this person is offered two fair gambles: gamble A, which is a 50–50 chance of winning or losing \$$h$, and gamble B, which is a 50–50 chance of winning or losing \$$2h$. The utility of current wealth is $U(W_0)$, which is also the expected value of current wealth because it is certain. The expected utility if he or she participates in gamble A is given by $EU(A)$:

$$EU(A) = \frac{1}{2}U(W_0 + h) + \frac{1}{2}U(W_0 - h), \tag{7.16}$$

and the expected utility of gamble B is given by $EU(B)$:

$$EU(B) = \frac{1}{2}U(W_0 + 2h) + \frac{1}{2}U(W_0 - 2h). \tag{7.17}$$

Equation 7.16 shows that the expected utility from gamble A is halfway between the utility from the unfavorable outcome $W_0 - h$ and the utility from favorable outcome $W_0 + h$. Likewise, the expected utility from gamble B is halfway between the utilities from unfavorable and favorable outcomes, but payoffs in these outcomes vary more than with gamble A.

[9]Technically, $U(W)$ is an indirect utility function because it is the consumption allowed by wealth that provides direct utility. In Chapter 17 we will take up the relationship between consumption-based utility functions and their implied indirect utility of wealth functions.

It is geometrically clear from the figure that[10]

$$U(W_0) > EU(A) > EU(B). \tag{7.18}$$

Therefore, this person will prefer to keep his or her current wealth rather than taking either fair gamble. If forced to choose a gamble, the person would prefer the smaller one (A) to the large one (B). The reason for this is that winning a fair bet adds to enjoyment less than losing hurts.

Risk aversion and insurance

As a matter of fact, this person might be willing to pay some amount to avoid participating in any gamble at all. Notice that a certain wealth of CE_A provides the same expected utility as does participating in gamble A. CE_A is referred to as the *certainty equivalent* of gamble A.

The individual would be willing to pay up to $W_0 - CE_A$ to avoid participating in the gamble. This explains why people buy insurance. They are giving up a small, certain amount (the insurance premium) to avoid the risky outcome they are being insured against. The premium a person pays for automobile collision insurance, for example, provides a policy that agrees to repair his or her car should an accident occur. The widespread use of insurance would seem to imply that aversion to risk is prevalent.

In fact, the person in Figure 7.1 would pay even more to avoid taking the larger gamble, B. As an exercise, try to identify the certainty equivalent CE_B of gamble B and the amount the person would pay to avoid gamble B on the figure. The analysis in this section can be summarized by the following definition.

DEFINITION

Risk aversion. An individual who always refuses fair bets is said to be *risk averse*. If individuals exhibit a diminishing marginal utility of wealth, they will be risk averse. As a consequence, they will be willing to pay something to avoid taking fair bets.

EXAMPLE 7.2 Willingness to Pay for Insurance

To illustrate the connection between risk aversion and insurance, consider a person with a current wealth of \$100,000 who faces the prospect of a 25 percent chance of losing his or her \$20,000 automobile through theft during the next year. Suppose also that this person's von Neumann–Morgenstern utility function is logarithmic; that is, $U(W) = \ln(W)$.

If this person faces next year without insurance, expected utility will be

$$\begin{aligned} EU(\text{no insurance}) &= 0.75U(100{,}000) + 0.25U(80{,}000) \\ &= 0.75 \ln 100{,}000 + 0.25 \ln 80{,}000 \\ &= 11.45714. \end{aligned} \tag{7.19}$$

In this situation, a fair insurance premium would be \$5,000 (25 percent of \$20,000, assuming that the insurance company has only claim costs and that administrative costs are \$0).

[10]Technically this result is a direct consequence of Jensen's inequality in mathematical statistics. The inequality states that if x is a random variable and $f(x)$ is a strictly concave function of that variable, then $E[f(x)] < f[E(x)]$. In the utility context, this means that if utility is concave in a random variable measuring wealth (i.e., if $U'(W) > 0$ and $U''(W) < 0$), then the expected utility of wealth will be less than the utility associated with the expected value of W. With gamble A, for example, $EU(A) < U(W_0)$ because, as a fair gamble, A provides expected wealth W_0.

Consequently, if this person completely insures the car, his or her wealth will be \$95,000 regardless of whether the car is stolen. In this case then,

$$\begin{aligned} EU(\text{fair insurance}) &= U(95{,}000) \\ &= \ln(95{,}000) \\ &= 11.46163. \end{aligned} \tag{7.20}$$

This person is made better off by purchasing fair insurance. Indeed, he or she would be willing to pay more than the fair premium for insurance. We can determine the maximum insurance premium (x) by setting

$$\begin{aligned} EU(\text{maximum-premium insurance}) &= U(100{,}000 - x) \\ &= \ln(100{,}000 - x) \\ &= 11.45714. \end{aligned} \tag{7.21}$$

Solving this equation for x yields

$$100{,}000 - x = e^{11.45714}, \tag{7.22}$$

or

$$x = 5{,}426. \tag{7.23}$$

This person would be willing to pay up to \$426 in administrative costs to an insurance company (in addition to the \$5,000 premium to cover the expected value of the loss). Even when these costs are paid, this person is as well off as he or she would be when facing the world uninsured.

QUERY: Suppose utility had been linear in wealth. Would this person be willing to pay anything more than the actuarially fair amount for insurance? How about the case where utility is a convex function of wealth?

MEASURING RISK AVERSION

In the study of economic choices in risky situations, it is sometimes convenient to have a quantitative measure of how averse to risk a person is. The most commonly used measure of risk aversion was initially developed by J. W. Pratt in the 1960s.[11] This risk aversion measure, $r(W)$, is defined as

$$r(W) = -\frac{U''(W)}{U'(W)}. \tag{7.24}$$

Because the distinguishing feature of risk-averse individuals is a diminishing marginal utility of wealth $[U''(W) < 0]$, Pratt's measure is positive in such cases. The measure is invariant with respect to linear transformations of the utility function, and therefore not affected by which particular von Neumann–Morgenstern ordering is used.

Risk aversion and insurance premiums

A useful feature of the Pratt measure of risk aversion is that it is proportional to the amount an individual will pay for insurance against taking a fair bet. Suppose the winnings from such a fair bet are denoted by the random variable h (which takes on both

[11] J . W. Pratt, "Risk Aversion in the Small and in the Large," *Econometrica* (January/April 1964): 122–36.

positive and negative values). Because the bet is fair, $E(h) = 0$. Now let p be the size of the insurance premium that would make the individual exactly indifferent between taking the fair bet h and paying p with certainty to avoid the gamble:

$$E[U(W+h)] = U(W-p), \tag{7.25}$$

where W is the individual's current wealth. We now expand both sides of Equation 7.25 using Taylor's series.[12] Because p is a fixed amount, a linear approximation to the right side of the equation will suffice:

$$U(W-p) = U(W) - pU'(W) + \text{higher-order terms.} \tag{7.26}$$

For the left side, we need a quadratic approximation to allow for the variability in the gamble, h:

$$E[U(W+h)] = E\left[U(W) + hU'(W) + \frac{h^2}{2}U''(W) + \text{ higher-order terms}\right] \tag{7.27}$$

$$= U(W) + E(h)U'(W) + \frac{E(h^2)}{2}U''(W) + \text{ higher-order terms.} \tag{7.28}$$

If we recall that $E(h) = 0$ and then drop the higher-order terms and use the constant k to represent $E(h^2)/2$, we can equate Equations 7.26 and 7.28 as

$$U(W) - pU'(W) \cong U(W) - kU''(W) \tag{7.29}$$

or

$$p \cong -\frac{kU''(W)}{U'(W)} = kr(W). \tag{7.30}$$

That is, the amount that a risk-averse individual is willing to pay to avoid a fair bet is approximately proportional to Pratt's risk aversion measure.[13] Because insurance premiums paid are observable in the real world, these are often used to estimate individuals' risk aversion coefficients or to compare such coefficients among groups of individuals. Therefore, it is possible to use market information to learn a bit about attitudes toward risky situations.

Risk aversion and wealth

An important question is whether risk aversion increases or decreases with wealth. Intuitively, one might think that the willingness to pay to avoid a given fair bet would decrease as wealth increases because decreasing marginal utility would make potential losses less serious for high-wealth individuals. This intuitive answer is not necessarily correct, however, because decreasing marginal utility also makes the gains from winning gambles less attractive. Thus, the net result is indeterminate; it all depends on the precise shape of the utility function. Indeed, if utility is quadratic in wealth,

$$U(W) = a + bW + cW^2, \tag{7.31}$$

[12]Taylor's series provides a way of approximating any differentiable function around some point. If $f(x)$ has derivatives of all orders, it can be shown that $f(x + h) = f(x) + hf'(x) + (h^2/2)f''(x)$ + higher-order terms. The point-slope formula in algebra is a simple example of Taylor's series.

[13]In this case, the factor of proportionality is also proportional to the variance of h because $\text{Var}(h) = E[h - E(h)]^2 = E(h^2)$. For an illustration where this equation fits exactly, see Example 7.3.

where $b > 0$ and $c < 0$, then Pratt's risk aversion measure is

$$r(W) = -\frac{U''(W)}{U'(W)} = \frac{-2c}{b + 2cW}, \tag{7.32}$$

which, contrary to intuition, increases as wealth increases.

On the other hand, if utility is logarithmic in wealth,

$$U(W) = \ln(W), \tag{7.33}$$

then we have

$$r(W) = -\frac{U''(W)}{U'(W)} = \frac{1}{W}, \tag{7.34}$$

which does indeed decrease as wealth increases.

The exponential utility function

$$U(W) = -e^{-AW} = -\exp(-AW) \tag{7.35}$$

(where A is a positive constant) exhibits constant absolute risk aversion over all ranges of wealth because now

$$r(W) = -\frac{U''(W)}{U'(W)} = \frac{A^2 e^{-AW}}{Ae^{-AW}} = A. \tag{7.36}$$

This feature of the exponential utility function[14] can be used to provide some numerical estimates of the willingness to pay to avoid gambles, as the next example shows.

EXAMPLE 7.3 Constant Risk Aversion

Suppose an individual whose initial wealth is W_0 and whose utility function exhibits constant absolute risk aversion is facing a 50–50 chance of winning or losing \$1,000. How much (f) would he or she pay to avoid the risk? To find this value, we set the utility of $W_0 - f$ equal to the expected utility from the gamble:

$$\begin{aligned} -\exp[-A(W_0 - f)] = &-0.5\exp[-A(W_0 + 1{,}000)] \\ &-0.5\exp[-A(W_0 - 1{,}000)]. \end{aligned} \tag{7.37}$$

Because the factor $-\exp(-AW_0)$ is contained in all the terms in Equation 7.37, this may be divided out, thereby showing that (for the exponential utility function) the willingness to pay to avoid a given gamble is independent of initial wealth. The remaining terms

$$\exp(Af) = 0.5\exp(-1{,}000A) + 0.5\exp(1{,}000A) \tag{7.38}$$

can now be used to solve for f for various values of A. If $A = 0.0001$, then $f = 49.9$; a person with this degree of risk aversion would pay approximately \$50 to avoid a fair bet of \$1,000. Alternatively, if $A = 0.0003$, this more risk-averse person would pay $f = 147.8$ to avoid the gamble. Because intuition suggests that these values are not unreasonable, values of the risk aversion parameter A in these ranges are sometimes used for empirical investigations.

Normally distributed risk. The constant risk aversion utility function can be combined with the assumption that a person faces a random shock to his or her wealth that follows a Normal distribution (see Chapter 2) to arrive at a particularly simple result. Specifically, if a person's

[14]Because the exponential utility function exhibits constant (absolute) risk aversion, it is sometimes abbreviated by the term *CARA utility*.

risky wealth follows a Normal distribution with mean μ and variance σ^2, then the probability density function for wealth is given by $f(W) = (1/\sqrt{2\pi})e^{-z^2/2}$, where $z = [(W - \mu)/\sigma]$. If this person has a utility function for wealth given by $U(\mathrm{W}) = -e^{-AW}$, then expected utility from his or her risky wealth is

$$E[U(W)] = \int_{-\infty}^{\infty} U(W)f(W)\,dW = \frac{1}{\sqrt{2\pi}}\int -e^{-AW}e^{-[(W-\mu)/\sigma]^2/2}\,dW. \tag{7.39}$$

Perhaps surprisingly, this integration is not too difficult to accomplish, although it does take patience. Performing this integration and taking a variety of monotonic transformations of the resulting expression yields the final result that

$$E[U(W)] \cong \mu - \frac{A\sigma^2}{2}. \tag{7.40}$$

Hence expected utility is a linear function of the two parameters of the wealth probability density function, and the individual's risk aversion parameter (A) determines the size of the negative effect of variability on expected utility. For example, suppose a person has invested his or her funds so that wealth has an expected value of \$100,000 but a standard deviation (σ) of \$10,000. Therefore, with the Normal distribution, he or she might expect wealth to decrease below \$83,500 about 5 percent of the time and increase above \$116,500 a similar fraction of the time. With these parameters, expected utility is given by $E[U(W)] = 100{,}000 - (A/2)(10{,}000)^2$. If $A = 0.0001 = 10^{-4}$, expected utility is given by $100{,}000 - 0.5 \cdot 10^{-4} \cdot (10^4)^2 = 95{,}000$. Hence this person receives the same utility from his or her risky wealth as would be obtained from a certain wealth of \$95,000. A more risk-averse person might have $A = 0.0003$, and in this case the certainty equivalent of his or her wealth would be \$85,000.

QUERY: Suppose this person had two ways to invest his or her wealth: Allocation 1, $\mu_1 = 107{,}000$ and $\sigma_1 = 10{,}000$; Allocation 2, $\mu_2 = 102{,}000$ and $\sigma_2 = 2{,}000$. How would this person's attitude toward risk affect his or her choice between these allocations?[15]

Relative risk aversion

It seems unlikely that the willingness to pay to avoid a given gamble is independent of a person's wealth. A more appealing assumption may be that such willingness to pay is inversely proportional to wealth and that the expression

$$rr(W) = Wr(W) = -W\frac{U''(W)}{U'(W)} \tag{7.41}$$

might be approximately constant. Following the terminology proposed by J. W. Pratt,[16] the $rr(W)$ function defined in Equation 7.41 is a measure of *relative risk aversion*. The power utility function

$$U(W, R) = \begin{cases} W^R/R & \text{if } R < 1,\ R \neq 0 \\ \ln W & \text{if } R = 0 \end{cases} \tag{7.42}$$

[15]This numerical example (roughly) approximates historical data on real returns of stocks and bonds, respectively, although the calculations are illustrative only.

[16]Pratt, "Risk Aversion."

exhibits diminishing absolute risk aversion,

$$r(W) = -\frac{U''(W)}{U'(W)} = -\frac{(R-1)W^{R-2}}{W^{R-1}} = \frac{1-R}{W}, \tag{7.43}$$

but constant relative risk aversion:[17]

$$rr(W) = Wr(W) = 1 - R. \tag{7.44}$$

Empirical evidence is generally consistent with values of R in the range of -3 to -1. Hence individuals seem to be somewhat more risk averse than is implied by the logarithmic utility function, although in many applications that function provides a reasonable approximation. It is useful to note that the constant relative risk aversion utility function in Equation 7.42 has the same form as the general CES utility function we first described in Chapter 3. This provides some geometric intuition about the nature of risk aversion that we will explore later in this chapter.

EXAMPLE 7.4 Constant Relative Risk Aversion

An individual whose behavior is characterized by a constant relative risk aversion utility function will be concerned about proportional gains or loss of wealth. Therefore, we can ask what fraction of initial wealth (f) such a person would be willing to give up to avoid a fair gamble of, say, 10 percent of initial wealth. First, we assume $R = 0$, so the logarithmic utility function is appropriate. Setting the utility of this individual's certain remaining wealth equal to the expected utility of the 10 percent gamble yields

$$\ln[(1-f)W_0] = 0.5\ln(1.1W_0) + 0.5\ln(0.9W_0). \tag{7.45}$$

Because each term contains $\ln W_0$, initial wealth can be eliminated from this expression:

$$\ln(1-f) = 0.5[\ln(1.1) + \ln(0.9)] = \ln(0.99)^{0.5};$$

hence

$$(1-f) = (0.99)^{0.5} = 0.995$$

and

$$f = 0.005. \tag{7.46}$$

Thus, this person will sacrifice up to 0.5 percent of wealth to avoid the 10 percent gamble. A similar calculation can be used for the case $R = -2$ to yield

$$f = 0.015. \tag{7.47}$$

Hence this more risk-averse person would be willing to give up 1.5 percent of his or her initial wealth to avoid a 10 percent gamble.

QUERY: With the constant relative risk aversion function, how does this person's willingness to pay to avoid a given absolute gamble (say, of 1,000) depend on his or her initial wealth?

[17]Some authors write the utility function in Equation 7.42 as $U(W) = W^{1-a}/(1-a)$ and seek to measure $a = 1 - R$. In this case, a is the relative risk aversion measure. The constant relative risk aversion function is sometimes abbreviated as *CRRA utility*.

METHODS FOR REDUCING UNCERTAINTY AND RISK

We have seen that risk-averse people will avoid gambles and other risky situations if possible. Often it is impossible to avoid risk entirely. Walking across the street involves some risk of harm. Burying one's wealth in the backyard is not a perfectly safe investment strategy because there is still some risk of theft (to say nothing of inflation). Our analysis thus far implies that people would be willing to pay something to at least reduce these risks if they cannot be avoided entirely. In the next four sections, we will study each of four different methods that individuals can take to mitigate the problem of risk and uncertainty: insurance, diversification, flexibility, and information.

INSURANCE

We have already discussed one such strategy: buying insurance. Risk-averse people would pay a premium to have the insurance company cover the risk of loss. Each year, people in the United States spend more than half a trillion dollars on insurance of all types. Most commonly, they buy coverage for their own life, for their home and cars, and for their health care costs. But insurance can be bought (perhaps at a high price) for practically any risk imaginable, ranging from earthquake insurance for a house along a fault line to special coverage for a surgeon against a hand injury.

A risk-averse person would always want to buy fair insurance to cover any risk he or she faces. No insurance company could afford to stay in business if it offered fair insurance (in the sense that the premium exactly equals the expected payout for claims). Besides covering claims, insurance companies must also maintain records, collect premiums, investigate fraud, and perhaps return a profit to shareholders. Hence an insurance customer can always expect to pay more than an actuarially fair premium. If people are sufficiently risk averse, they will even buy unfair insurance, as shown in Example 7.2; the more risk averse they are, the higher the premium they would be willing to pay.

Several factors make insurance difficult or impossible to provide. Large-scale disasters such as hurricanes and wars may result in such large losses that the insurance company would go bankrupt before it could pay all the claims. Rare and unpredictable events (e.g., war, nuclear power plant accidents) offer reliable track record for insurance companies to establish premiums. Two other reasons for absence of insurance coverage relate to the informational disadvantage the company may have relative to the customer. In some cases, the individual may know more about the likelihood that they will suffer a loss than the insurance company. Only the "worst" customers (those who expect larger or more likely losses) may end up buying an insurance policy. This *adverse selection problem* may unravel the whole insurance market unless the company can find a way to control who buys (through some sort of screening or compulsion). Another problem is that having insurance may make customers less willing to take steps to avoid losses, for example, driving more recklessly with auto insurance or eating fatty foods and smoking with health insurance. This so-called *moral hazard problem* again may impair the insurance market unless the insurance company can find a way to cheaply monitor customer behavior. We will discuss the adverse selection and moral hazard problems in more detail in Chapter 18, and discuss ways the insurance company can combat these problems, which besides the above strategies include offering only partial insurance and requiring the payment of deductibles and copayments.

DIVERSIFICATION

A second way for risk-averse individuals to reduce risk is by diversifying. This is the economic principle behind the adage, "Don't put all your eggs in one basket." By suitably spreading risk around, it may be possible to reduce the variability of an outcome without lowering the expected payoff.

The most familiar setting in which diversification comes up is in investing. Investors are routinely advised to "diversify your portfolio." To understand the wisdom behind this advice, consider an example in which a person has wealth W to invest. This money can be invested in two independent risky assets, 1 and 2, which have equal expected values (the mean returns are $\mu_1 = \mu_2$) and equal variances (the variances are $\sigma_1^2 = \sigma_2^2$). A person whose undiversified portfolio, UP, includes just one of the assets (putting all his or her "eggs" in that "basket") would earn an expected return of $\mu_{UP} = \mu_1 = \mu_2$ and would face a variance of $\sigma_{UP}^2 = \sigma_1^2 = \sigma_2^2$.

Suppose instead the individual chooses a diversified portfolio, DP. Let α_1 be the fraction invested in the first asset and $\alpha_2 = 1 - \alpha_1$ in the second. We will see that the person can do better than the undiversified portfolio in the sense of getting a lower variance without changing the expected return. The expected return on the diversified portfolio does not depend on the allocation across assets and is the same as for either asset alone:

$$\mu_{DP} = \alpha_1\mu_1 + (1 - \alpha_1)\mu_2 = \mu_1 = \mu_2. \tag{7.48}$$

To see this, refer back to the rules for computed expected values from Chapter 2. The variance will depend on the allocation between the two assets:

$$\sigma_{DP}^2 = \alpha_1^2\sigma_1^2 + (1 - \alpha_1)^2\sigma_2^2 = (1 - 2\alpha_1 + 2\alpha_1^2)\sigma_1^2. \tag{7.49}$$

This calculation again can be understood by reviewing the section on variances in Chapter 2. There you will be able to review the two "facts" used in this calculation: First, the variance of a constant times a random variable is that constant squared times the variance of a random variable; second, the variance of independent random variables, because their covariance is 0, equals the sum of the variances.

Choosing α_1 to minimize Equation 7.49 yields $\alpha_1 = \frac{1}{2}$ and $\sigma_{DP}^2 = \frac{\sigma_1^2}{2}$. Therefore, the optimal portfolio spreads wealth equally between the two assets, maintaining the same expected return as an undiversified portfolio but reducing variance by half. Diversification works here because the assets' returns are independent. When one return is low, there is a chance the other will be high, and vice versa. Thus, the extreme returns are balanced out at least some of the time, reducing the overall variance. Diversification will work in this way as long as there is not perfect correlation in the asset returns so that they are not effectively the same asset. The less correlated the assets are, the better diversification will work to reduce the variance of the overall portfolio.

The example, constructed to highlight the benefits of diversification as simply as possible, has the artificial element that asset returns were assumed to be equal. Diversification was a "free lunch" in that the variance of the portfolio could be reduced without reducing the expected return compared with an undiversified portfolio. If the expected return from one of the assets (say, asset 1) is higher than the other, then diversification into the other asset would no longer be a "free lunch" but would result in a lower expected return. Still, the benefits from risk reduction can be great enough that a risk-averse investor would be willing to put some share of wealth into the asset with the lower expected return. A practical example of this idea is related to advice one would give to an employee of a firm with a stock purchase plan. Even if the plan allows employees to buy shares of the company's stock at a generous discount compared with the market, the employee may still be advised not to invest all savings in that stock because otherwise the employee's entire savings, to say nothing of his or her salary and perhaps even house value (to the extent house values depend on the strength of businesses in the local economy), is tied to the fortunes of a single company, generating a tremendous amount of risk.

The Extensions provide a much more general analysis of the problem of choosing the optimal portfolio. However, the principle of diversification applies to a much broader range of situations than financial markets. For example, students who are uncertain about where their interests lie or about what skills will be useful on the job market are well advised to register for a diverse set of classes rather than exclusively technical or artistic ones.

FLEXIBILITY

Diversification is a useful method to reduce risk for a person who can divide up a decision by allocating small amounts of a larger sum among a number of different choices. In some situations, a decision cannot be divided; it is all or nothing. For example, in shopping for a car, a consumer cannot combine the attributes that he or she likes from one model (say, fuel efficiency) with those of another (say, horsepower or power windows) by buying half of each; cars are sold as a unit. With all-or-nothing decisions, the decision-maker can obtain some of the benefit of diversification by making flexible decisions. Flexibility allows the person to adjust the initial decision, depending on how the future unfolds. The more uncertain the future, the more valuable this flexibility. Flexibility keeps the decision-maker from being tied to one course of action and instead provides a number of options. The decision-maker can choose the best option to suit later circumstances.

A good example of the value of flexibility comes from considering the fuels on which cars are designed to run. Until now, most cars were limited in how much biofuel (such as ethanol made from crops) could be combined with petroleum products (such as gasoline or diesel) in the fuel mix. A purchaser of such a car would have difficulties if governments passed new regulations increasing the ratio of ethanol in car fuels or banning petroleum products entirely. New cars have been designed that can burn ethanol exclusively, but such cars are not useful if current conditions continue to prevail because most filling stations do not sell fuel with high concentrations of ethanol. A third type of car has internal components that can handle a variety of types of fuel, both petroleum-based and ethanol, and any proportions of the two. Such cars are expensive to build because of the specialized components involved, but a consumer might pay the additional expense anyway because the car would be useful whether or not biofuels become more important over the life of the car.[18]

Types of options

The ability of "flexible-fuel" cars to be able to burn any mix of petroleum-based fuels and biofuels is valuable because it provides the owner with more options relative to a car that can run on only one type of fuel. Readers are probably familiar with the notion that options are valuable from another context where the term is frequently used—financial markets—where one hears about stock options and other forms of options contracts. There is a close connection between the option implicit in the flexible-fuel cars and these option contracts that we will investigate in more detail. Before discussing the similarities between the options arising in different contexts, we introduce some terms to distinguish them.

[18]While the current generation of flexible-fuel cars involve state-of-the-art technology, the first such car, produced back in 1908, was Henry Ford's Model-T, one of the top-selling cars of all time. The availability of cheap gasoline may have swung the market toward competitors' single-fuel cars, spelling the demise of the Model-T. For more on the history of this model, see L. Brooke, *Ford Model T: The Car That Put the World on Wheels* (Minneapolis: Motorbooks, 2008).

DEFINITION

Financial option contract. A *financial option contract* offers the right, but not the obligation, to buy or sell an asset (such as a share of stock) during some future period at a certain price.

DEFINITION

Real option. A *real option* is an option arising in a setting outside of financial markets.

The flexible-fuel car can be viewed as an ordinary car combined with an additional real option to burn biofuels if those become more important in the future.

Financial option contracts come in a variety of forms, some of which can be complex. There are also many different types of real options, and they arise in many different settings, sometimes making it difficult to determine exactly what sort of option is embedded in the situation. Still, all options share three fundamental attributes. First, they specify the underlying transaction, whether it is a stock to be traded or a car or fuel to be purchased. Second, they specify a period over which the option may be exercised. A stock option may specify a period of 1 year, for example. The option embedded in a flexible-fuel car preserves the owner's option during the operating life of the car. The longer the period over which the option extends, the more valuable it is because the more uncertainty that can be resolved during this period. Third, the option contract specifies a price. A stock option might sell for a price of \$70. If this option is later traded on an exchange, its price might vary from moment to moment as the markets move. Real options do not tend to have explicit prices, but sometimes implicit prices can be calculated. For example, if a flexible-fuel car costs \$5,000 more than an otherwise equivalent car that burns one type of fuel, then this \$5,000 could be viewed as the option price.

Model of real options

Let x embody all the uncertainty in the economic environment. In the case of the flexible-fuel car, x might reflect the price of fossil fuels relative to biofuels or the stringency of government regulation of fossil fuels. In terms of the section on statistics in Chapter 2, x is a random variable (sometimes referred to as the "state of the world") that can take on possibly many different values. The individual has some number, $I = 1, \ldots, n$, of choices currently available. Let $A_i(x)$ be the payoffs provided by choice i, where the argument (x) allows each choice to provide a different pattern of returns depending on how the future turns out.

Figure 7.2a illustrates the case of two choices. The first choice provides a decreasing payoff as x increases, indicated by the downward slope of A_1. This might correspond to ownership of a car that runs only on fossil fuels; as biofuels become more important than fossil fuels, the value of a car burning only fossil fuels decreases. The second choice provides an increasing payoff, perhaps corresponding to ownership of a car that runs only on biofuels. Figure 7.2b translates the payoffs into (von Neumann–Morgenstern) utilities that the person obtains from the payoffs by graphing $U(A_i)$ rather than A_i. The bend introduced in moving from payoffs to utilities reflects the diminishing marginal utility from higher payoffs for a risk-averse person.

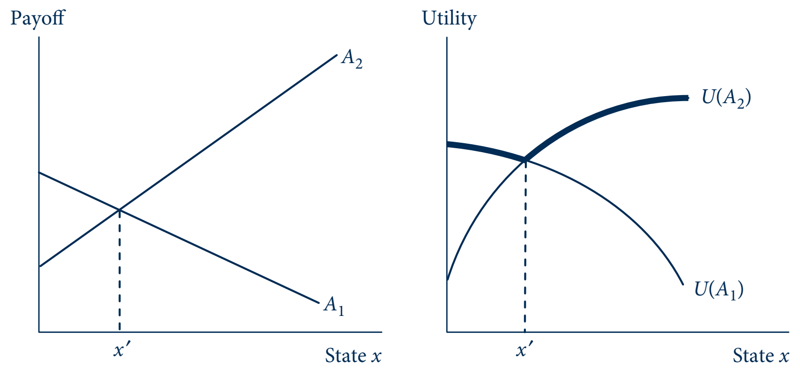
If the person does not have the flexibility provided by a real option, he or she must make the choice before observing how the state x turns out. The individual should choose the single alternative that is best on average. His or her expected utility from this choice is

$$\max\{E[U(A_1)], \ldots, E[U(A_n)]\}. \tag{7.50}$$

FIGURE 7.2

The Nature of a Real Option

Panel (a) shows the payoffs and panel (b) shows the utilities provided by two alternatives across states of the world (x). If the decision has to be made upfront, the individual chooses the single curve having the highest expected utility. If the real option to make either decision can be preserved until later, the individual can obtain the expected utility of the upper envelope of the curves, shown in bold.



(a) Payoffs from alternatives

(b) Utilities from alternatives

Figure 7.2 does not provide enough information to judge which expected utility is higher because we do not know the likelihoods of the different x's, but if the x's are about equally likely, then it looks as though the individual would choose the second alternative, providing higher utility over a larger range. The individual's expected utility from this choice is $E[U(A_2)]$.

On the other hand, if the real option can be preserved to make a choice that responds to which state of the world x has occurred, the person will be better off. In the car application, the real option could correspond to buying a flexible-fuel car, which does not lock the buyer into one fuel but allows the choice of whatever fuel turns out to be most common or inexpensive over the life of the car. In Figure 7.2, rather than choosing a single alternative, the person would choose the first option if $x < x'$ and the second option if $x > x'$. The utility provided by this strategy is given by the bold curve, which is the "upper envelope" of the curves for the individual options. With a general number (n) of choices, expected utility from this upper envelope of individual options is

$$E\{\max[U(A_1), \ldots, U(A_1)]\}. \quad \textbf{(7.51)}$$

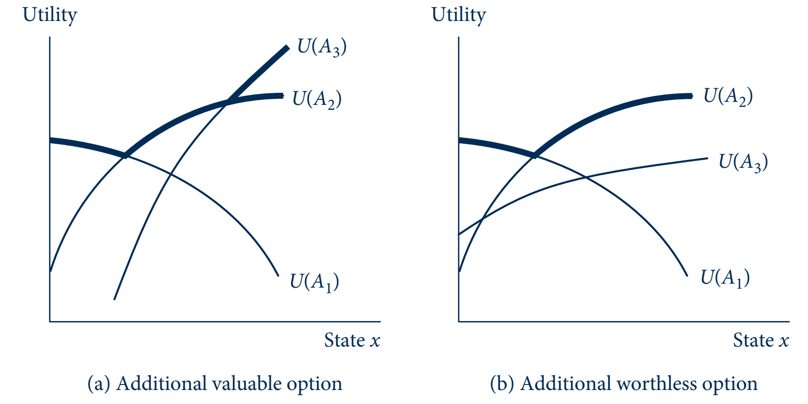
The expected utility in Equation 7.51 is higher than in 7.50. This may not be obvious at first glance because it seems that simply swapping the order of the expectations and "max" operators should not make a difference. But indeed it does. Whereas Equation 7.50 is the expected utility associated with the best single utility curve, Equation 7.51 is the expected utility associated with the upper envelope of all the utility curves.[19]

[19]The result can be proved formally using Jensen's inequality, introduced in footnote 10. The footnote discusses the implications of Jensen's inequality for concave functions: $E[f(x)] \leq f[E(x)]$. Jensen's inequality has the reverse implication for convex functions: $E[f(x)] \geq f[E(x)]$. In other words, for convex functions, the result is greater if the expectations operator is applied outside of the function than if the order of the two is reversed. In the options context, the "max" operator has the properties of a convex function. This can be seen from Figure 7.2b, where taking the upper envelope "convexifies" the individual curves, turning them into more of a V-shape.

FIGURE 7.3

More Options Cannot Make the Individual Decision-Maker Worse Off

The addition of a third alternative to the two drawn in Figure 7.2 is valuable in (a) because it shifts the upper envelope (shown in bold) of utilities up. The new alternative is worthless in (b) because it does not shift the upper envelope, but the individual is not worse off for having it.



More options are better (generally)

Adding more options can never harm an individual decision-maker (as long as he or she is not charged for them) because the extra options can always be ignored. This is the essence of options: They give the holder the right—but not the obligation—to choose them. Figure 7.3 illustrates this point, showing the effect of adding a third option to the two drawn in Figure 7.2. In the first panel, the person strictly benefits from the third option because there are some states of the world (the highest values of x in the figure) for which it is better than any other alternative, shifting the upper envelope of utilities (the bold curve) up. The third option is worthless in the second panel. Although the third option is not the worst option for many states of the world, it is never the best and so does not improve the upper envelope of utilities relative to Figure 7.2. Still, the addition of the third option is not harmful.

This insight may no longer hold in a strategic setting with multiple decision-makers. In a strategic setting, economic actors may benefit from having some of their options cut off. This may allow a player to commit to a narrower course of action that he or she would not have chosen otherwise, and this commitment may affect the actions of other parties, possibly to the benefit of the party making the commitment. A famous illustration of this point is provided in one of the earliest treatises on military strategy, by Sun Tzu, a Chinese general writing in 400 BC. It seems crazy for an army to destroy all means of retreat, burning bridges behind itself and sinking its own ships, among other measures. Yet this is what Sun Tzu advocated as a military tactic. If the second army observes that the first cannot retreat and will fight to the death, it may retreat itself before engaging the first. We will analyze such strategic issues more formally in the next chapter on game theory.

Computing option value

We can push the analysis further to derive a mathematical expression for the value of a real option. Let F be the fee that has to be paid for the ability to choose the best alternative after x has been realized instead of before. The individual would be willing to pay the fee as long as

$$E\{\max[U(A_1(x)-F),\ldots,[U(A_n(x)-F)]\}\geq\max\{E[U(A_1(x))],\ldots,E[U(A_n(x))]\}. \tag{7.52}$$

The right side is the expected utility from making the choice beforehand, repeated from Equation 7.50. The left side allows for the choice to be made after x has occurred, a benefit, but subtracts the fee for option from every payoff. The fee is naturally assumed to be paid up front, and thus reduces wealth by F whichever option is chosen later. The real option's value is the highest F for which Equation 7.52 is still satisfied, which of course is the F for which the condition holds with equality.

EXAMPLE 7.5 Value of a Flexible-Fuel Car

Let's work out the option value provided by a flexible-fuel car in a numerical example. Let $A_1(x) = 1 - x$ be the payoff from a fossil-fuel–only car and $A_2(x) = x$ be the payoff from a biofuel-only car. The state of the world, x, reflects the relative importance of biofuels compared with fossil fuels over the car's lifespan. Assume x is a random variable that is uniformly distributed between 0 and 1 (the simplest continuous random variable to work with here). The statistics section in Chapter 2 provides some detail on the uniform distribution, showing that the probability density function (PDF) is $f(x) = 1$ in the special case when the uniform random variable ranges between 0 and 1.

Risk neutrality. To make the calculations as easy as possible to start, suppose first that the car buyer is risk neutral, obtaining a utility level equal to the payoff level. Suppose the buyer is forced to choose a biofuel car. This provides an expected utility of

$$E[A_2]=\int_0^1 A_2(x)f(x)\,dx=\int_0^1 x\,dx=\frac{x^2}{2}\bigg|_{x=0}^{x=1}=\frac{1}{2}, \tag{7.53}$$

where the integral simplifies because $f(x) = 1$. Similar calculations show that the expected utility from buying a fossil-fuel car is also 1/2. Therefore, if only single-fuel cars are available, the person is indifferent between them, obtaining expected utility 1/2 from either.

Now suppose that a flexible-fuel car is available, which allows the buyer to obtain either $A_1(x)$ or $A_2(x)$, whichever is higher under the latter circumstances. The buyer's expected utility from this car is

$$\begin{aligned}E[\max(A_1,A_2)]&=\int_0^1\max(1-x,x)f(x)\ dx=\int_0^{\frac{1}{2}}(1-x)\,dx+\int_{\frac{1}{2}}^1 x\,dx\\&=2\int_{\frac{1}{2}}^1 x\ dx=x^2\big|_{x=\frac{1}{2}}^{x=1}=\frac{3}{4}.\end{aligned} \tag{7.54}$$

The second line in Equation 7.54 follows from the fact that the two integrals in the preceding expression are symmetric. Because the buyer's utility exactly equals the payoffs, we can compute the option value of the flexible-fuel car directly by taking the difference between the expected payoffs in Equations 7.53 and 7.54, which equals 1/4. This is the maximum premium the person would pay for the flexible-fuel car over a single-fuel car. Scaling payoffs to more realistic levels by multiplying by, say, \$10,000, the price premium (and the option value) of the flexible-fuel car would be \$2,500.

This calculation demonstrates the general insight that options are a way of dealing with uncertainty that have value even for risk-neutral individuals. The next part of the example investigates whether risk aversion makes options more or less valuable.

Risk aversion. Now suppose the buyer is risk averse, having von Neumann–Morgenstern utility function $U(x) = \sqrt{x}$. The buyer's expected utility from a biofuel car is

$$E[U(A_2)] = \int_0^1 \sqrt{A_2(x)}\, f(x)\, dx = \int_0^1 x^{\frac{1}{2}}\, dx = \frac{2}{3} x^{\frac{3}{2}} \Big|_{x=0}^{x=1} = \frac{2}{3}, \tag{7.55}$$

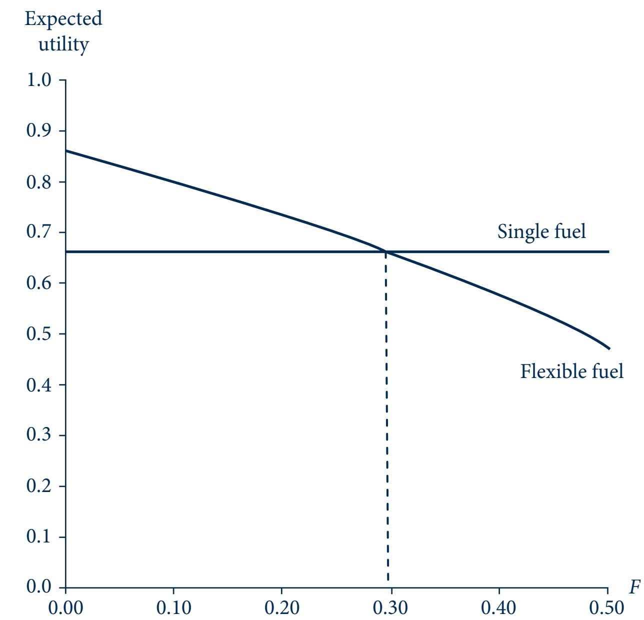
which is the same as from a fossil-fuel car, as similar calculations show. Therefore, a single-fuel car of whatever type provides an expected utility of 2/3.

The expected utility from a flexible-fuel car that costs F more than a single-fuel car is

$$\begin{aligned} E\{\max[U(A_1(x) - F),\, U(A_2(x) - F)]\} &= \int_0^1 \max(\sqrt{1 - x - F},\, \sqrt{x - F}) f(x)\, dx \\ &= \int_0^{\frac{1}{2}} \sqrt{1 - x - F}\, dx + \int_{\frac{1}{2}}^1 \sqrt{x - F}\, dx = 2\int_{\frac{1}{2}}^1 \sqrt{x - F}\, dx \\ &= 2\int_{\frac{1}{2} - F}^{1 - F} u^{\frac{1}{2}}\, du = \frac{4}{3} u^{\frac{3}{2}} \Big|_{u = \frac{1}{2} - F}^{u = 1 - F} \\ &= \frac{4}{3}\left[(1 - F)^{\frac{3}{2}} - \left(\frac{1}{2} - F\right)^{\frac{3}{2}}\right]. \end{aligned} \tag{7.56}$$

FIGURE 7.4 Graphical Method for Computing the Premium for a Flexible-Fuel Car

To find the maximum premium F that the risk-averse buyer would be willing to pay for the flexible-fuel car, we plot the expected utility from a single-fuel car from Equation 7.55 and from the flexible-fuel car from Equation 7.56 and see the value of F where the curves cross.



The calculations involved in Equation 7.56 are somewhat involved and thus require some discussion. The second line relies on the symmetry of the two integrals appearing there, which allows us to collapse them into two times the value of one of them, and we chose the simpler of the two for these purposes. The third line uses the change of variables $u = x - F$ to simplify the integral. (See Equation 2.135 in Chapter 2 for another example of the change-of-variables trick and further discussion.)

To find the maximum premium the buyer would pay for a flexible-fuel car, we can set Equations 7.55 and 7.56 equal and solve for F. Unfortunately the resulting equation is too complicated to be solved analytically. One simple approach is to graph the last line of Equation 7.56 for a range of values of F and eyeball where the graph hits the required value of 2/3 from Equation 7.55. This is done in Figure 7.4, where we see that this value of F is slightly less than 0.3 (0.294 to be more precise). Therefore, the risk-averse buyer is willing to pay a premium of 0.294 for the flexible-fuel car, which is also the option value of this type of car. Scaling up by \$10,000 for more realistic monetary values, the price premium would be \$2,940. This is \$440 more than the risk-neutral buyer was willing to pay. Thus, the option value is greater in this case for the risk-averse buyer.

QUERY: Does risk aversion always increase option value? If so, explain why. If not, modify the example with different shapes to the payoff functions to provide an example where the risk-neutral buyer would pay more.

Option value of delay

Society seems to frown on procrastinators. "Do not put off to tomorrow what you can do today" is a familiar maxim. Yet the existence of real options suggests a possible value in procrastination. There may be a value in delaying big decisions—such as the purchase of a car—that are not easily reversed later. Delaying these big decisions allows the decision-maker to preserve option value and gather more information about the future. To the outside observer, who may not understand all the uncertainties involved in the situation, it may appear that the decision-maker is too inert, failing to make what looks to be the right decision at the time. In fact, delaying may be exactly the right choice to make in the face of uncertainty. Choosing one course of action rules out other courses later. Delay preserves options. If circumstances continue to be favorable or become even more so, the action can still be taken later. But if the future changes and the action is unsuitable, the decision-maker may have saved a lot of trouble by not making it.

The value of delay can be seen by returning to the car application. Suppose for the sake of this example that only single-fuel cars (of either type, fossil fuel or biofuel) are available on the market; flexible-fuel cars have not yet been invented. Even if circumstances start to favor the biofuel car, with the number of filling stations appearing to tip toward offering biofuels, the buyer may want to hold off buying a car until he or she is more sure. This may be true even if the buyer is forgoing considerable consumer surplus from the use of a new car during the period of delay. The problem is that if biofuels do not end up taking over the market, the buyer may be left with a car that is hard to fuel up and hard to trade in for a car burning the other fuel type. The buyer would be willing to experience delay costs up to F to preserve flexibility.

The value of delay hinges on the irreversibility of the underlying decision. If in the car example the buyer manufacturer could recover close to the purchase price by selling it on the used-car market, there would be no reason to delay purchasing. But it is well known that the value of a new car decreases precipitously once it is driven off the car lot (we will discuss reasons for this including the "lemons effect" in Chapter 18); therefore, it may not be so easy to reverse the purchase of a car.

Implications for cost–benefit analysis

To an outside observer, delay may seem like a symptom of irrationality or ignorance. Why is the decision-maker overlooking an opportunity to take a beneficial action? The chapter has now provided several reasons why a rational decision-maker might not want to pursue an action even though the expected benefits from the action outweigh the expected costs. First, a risk-averse individual might avoid a gamble even if it provided a positive expected monetary payoff (because of the decreasing marginal utility from money). And option value provides a further reason for the action not to be undertaken: The decision-maker might be delaying until he or she has more certainty about the potential results of the decision.

Many of us have come across the *cost–benefit rule*, which says that an action should be taken if anticipated costs are less than benefits. This is generally a sensible rule, providing the correct course of action in simple settings without uncertainty. One must be more careful in applying the rule in settings involving uncertainty. The correct decision rule is more complicated because it should account for risk preferences (by converting payoffs into utilities) and for the option value of delay, if present. Failure to apply the simple cost–benefit rule in settings with uncertainty may indicate sophistication rather than irrationality.[20]

INFORMATION

The fourth method of reducing the uncertainty involved in a situation is to acquire better information about the likely outcome that will arise. We have already considered a version of this in the previous section, where we considered the strategy of preserving options while delaying a decision until better information is received. Delay involved some costs, which can be thought of as a sort of "purchase price" for the information acquired. Here, we will be more direct in considering information as a good that can be purchased directly and analyze in greater detail why and how much individuals are willing to pay for it.

Information as a good

By now it should be clear to the reader that information is a valuable economic resource. We have seen an example already: A buyer can make a better decision about which type of car to buy if he or she has better information about the sort of fuels that will be readily available during the life of the car. But the examples do not end there. Shoppers who know where to buy high-quality goods cheaply can make their budgets stretch further than those who do not; doctors can provide better medical care if they are up to date on the latest scientific research.

The study of information economics has become one of the major areas in current research. Several challenges are involved. Unlike the consumer goods we have been studying thus far, information is difficult to quantify. Even if it could be quantified, information has some technical properties that make it an unusual sort of good. Most information is durable and retains value after it has been used. Unlike a hot dog, which is consumed only once, knowledge of a special sale can be used not only by the person who

[20]Economists are puzzled by consumers' reluctance to install efficient appliances even though the savings on energy bills are likely to defray the appliances' purchase price before long. An explanation from behavioral economics is that consumers are too ignorant to perform the cost–benefit calculations or are too impatient to wait for the energy savings to accumulate. K. Hassett and G. Metcalf, in "Energy Conservation Investment: Do Consumers Discount the Future Correctly?" *Energy Policy* (June 1993): 710–16, suggest that consumer inertia may be rational delay in the face of fluctuating energy prices. See Problem 7.9 for a related numerical example.

discovers it but also by anyone else with whom the information is shared. The friends then may gain from this information even though they do not have to spend anything to obtain it. Indeed, in a special case of this situation, information has the characteristic of a pure *public good* (see Chapter 19). That is, the information is both *nonrival*, in that others may use it at zero cost, and *nonexclusive*, in that no individual can prevent others from using the information. The classic example of these properties is a new scientific discovery. When some prehistoric people invented the wheel, others could use it without detracting from the value of the discovery, and everyone who saw the wheel could copy it freely. Information is also difficult to sell because the act of describing the good that is being offered to a potential consumer gives it away to them.

These technical properties of information imply that market mechanisms may often operate imperfectly in allocating resources to information provision and acquisition. After all, why invest in the production of information when one can just acquire it from others at no cost? Therefore, standard models of supply and demand may be of relatively limited use in understanding such activities. At a minimum, models have to be developed that accurately reflect the properties being assumed about the informational environment. Throughout the latter portions of this book, we will describe some of the situations in which such models are called for. Here, however, we will pay relatively little attention to supply–demand equilibria and will instead focus on an example that illustrates the value of information in helping individuals make choices under uncertainty.

Quantifying the value of information

We already have all the tools needed to quantify the value of information from the section on option values. Suppose again that the individual is uncertain about what the state of the world (x) will be in the future. He or she needs to make one of n choices today (this allows us to put aside the option value of delay and other issues we have already studied). As before, $A_i(x)$ represents the payoffs provided by choice i. Now reinterpret F as the fee charged to be told the exact value that x will take on in the future (perhaps this is the salary of the economist hired to make such forecasts).

The same calculations from the option section can be used here to show that the maximum such F is again the value for which Equation 7.52 holds with equality. Just as this was the value of the real option in that section, here it is the value of information. The value of information would be lower than this F if the forecast of future conditions were imperfect rather than perfect as assumed here. Other factors affecting an individual's value of information include the extent of uncertainty before acquiring the information, the number of options he or she can choose between, and his or her risk preferences. The more uncertainty resolved by the new information, the more valuable it is, of course. If the individual does not have much scope to respond to the information because of having only a limited range of choices to make, the information will not be valuable. The degree of risk aversion has ambiguous effects on the value of information (answering the Query in Example 7.5 will provide you with some idea why).

THE STATE-PREFERENCE APPROACH TO CHOICE UNDER UNCERTAINTY

Although our analysis in this chapter has offered insights on a number of issues, it seems rather different from the approach we took in other chapters. The basic model of utility maximization subject to a budget constraint seems to have been lost. To make further progress in the study of behavior under uncertainty, we will develop some new techniques that will permit us to bring the discussion of such behavior back into the standard choice-theoretic framework.

States of the world and contingent commodities

We start by pushing a bit further on an idea already mentioned, thinking about an uncertain future in term of *states of the world.* We cannot predict exactly what will happen, say, tomorrow, but we assume that it is possible to categorize all the possible things that might happen into a fixed number of well-defined *states.* For example, we might make the crude approximation of saying that the world will be in only one of two possible states tomorrow: It will be either “good times” or “bad times.” One could make a much finer gradation of states of the world (involving even millions of possible states), but most of the essentials of the theory can be developed using only two states.

A conceptual idea that can be developed concurrently with the notion of states of the world is that of *contingent commodities.* These are goods delivered only if a particular state of the world occurs. As an example, “\$1 in good times” is a contingent commodity that promises the individual \$1 in good times but nothing should tomorrow turn out to be bad times. It is even possible, by stretching one’s intuitive ability somewhat, to conceive of being able to purchase this commodity: I might be able to buy from someone the promise of \$1 if tomorrow turns out to be good times. Because tomorrow could be bad, this good will probably sell for less than \$1. If someone were also willing to sell me the contingent commodity “\$1 in bad times,” then I could assure myself of having \$1 tomorrow by buying the two contingent commodities “\$1 in good times” and “\$1 in bad times.”

Utility analysis

Examining utility-maximizing choices among contingent commodities proceeds formally in much the same way we analyzed choices previously. The principal difference is that, after the fact, a person will have obtained only one contingent good (depending on whether it turns out to be good or bad times). Before the uncertainty is resolved, however, the individual has two contingent goods from which to choose and will probably buy some of each because he or she does not know which state will occur. We denote these two contingent goods by W_g (wealth in good times) and W_b (wealth in bad times). Assuming that utility is independent of which state occurs[21] and that this individual believes that good times will occur with probability π, the expected utility associated with these two contingent goods is

$$V(W_g,\ W_b) = \pi U(W_g) + (1-\pi)U(W_b). \qquad \textbf{(7.57)}$$

This is the magnitude this individual seeks to maximize given his or her initial wealth, W.

Prices of contingent commodities

Assuming that this person can purchase \$1 of wealth in good times for p_g and \$1 of wealth in bad times for p_b, his or her budget constraint is then

$$W = p_g W_g + p_b W_b. \qquad \textbf{(7.58)}$$

The price ratio p_g/p_b shows how this person can trade dollars of wealth in good times for dollars in bad times. If, for example, $p_g = 0.80$ and $p_b = 0.20$, the sacrifice of \$1 of wealth

[21]This assumption is untenable in circumstances where utility of wealth depends on the state of the world. For example, the utility provided by a given level of wealth may differ depending on whether an individual is “sick” or “healthy.” We will not pursue such complications here, however. For most of our analysis, utility is assumed to be concave in wealth: $U'(W) > 0$, $U''(W) < 0$.

in good times would permit this person to buy contingent claims yielding \$4 of wealth should times turn out to be bad. Whether such a trade would improve utility will, of course, depend on the specifics of the situation. But looking at problems involving uncertainty as situations in which various contingent claims are traded is the key insight offered by the state-preference model.

Fair markets for contingent goods

If markets for contingent wealth claims are well developed and there is general agreement about the likelihood of good times (π), then prices for these claims will be actuarially fair—that is, they will equal the underlying probabilities:

$$\begin{aligned} p_g &= \pi, \\ p_b &= 1 - \pi. \end{aligned} \tag{7.59}$$

Hence the price ratio p_g/p_b will simply reflect the odds in favor of good times:

$$\frac{p_g}{p_b} = \frac{\pi}{1 - \pi}. \tag{7.60}$$

In our previous example, if $p_g = \pi = 0.8$ and $p_b = (1 - \pi) = 0.2$, then $\pi/(1 - \pi) = 4$. In this case the odds in favor of good times would be stated as "4 to 1." Fair markets for contingent claims (such as insurance markets) will also reflect these odds. An analogy is provided by the "odds" quoted in horse races. These odds are "fair" when they reflect the true probabilities that various horses will win.

Risk aversion

We are now in a position to show how risk aversion is manifested in the state-preference model. Specifically, we can show that, if contingent claims markets are fair, then a utility-maximizing individual will opt for a situation in which $W_g = W_b$; that is, he or she will arrange matters so that the wealth ultimately obtained is the same no matter what state occurs.

As in previous chapters, maximization of utility subject to a budget constraint requires that this individual set the *MRS* of W_g for W_b equal to the ratio of these "goods" prices:

$$MRS = \frac{\partial V/\partial W_g}{\partial V/\partial W_b} = \frac{\pi U'(W_g)}{(1 - \pi)U'(W_b)} = \frac{p_g}{p_b}. \tag{7.61}$$

In view of the assumption that markets for contingent claims are fair (Equation 7.60), this first-order condition reduces to

$$\frac{U'(W_g)}{U'(W_b)} = 1$$

or[22]

$$W_g = W_b. \tag{7.62}$$

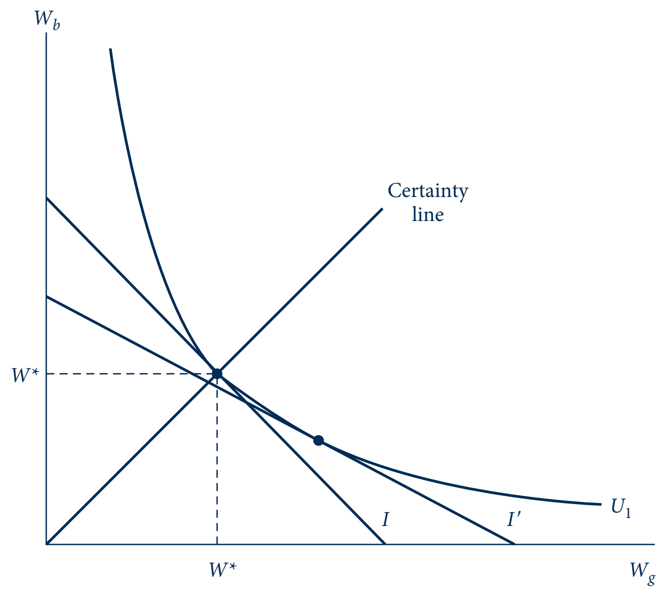
Hence this individual, when faced with fair markets in contingent claims on wealth, will be risk averse and will choose to ensure that he or she has the same level of wealth regardless of which state occurs.

[22]This step requires that utility be state independent and that $U'(W) > 0$.

FIGURE 7.5

Risk Aversions in the State-Preference Model

The line I represents the individual's budget constraint for contingent wealth claims: $W = p_g W_g + p_b W_b$. If the market for contingent claims is actuarially fair [$p_g/p_b = \pi/(1-\pi)$], then utility maximization will occur on the certainty line where $W_g = W_b = W^*$. If prices are not actuarially fair, the budget constraint may resemble I', and utility maximization will occur at a point where $W_g > W_b$.



A graphic analysis

Figure 7.5 illustrates risk aversion with a graph. This individual's budget constraint (I) is shown to be tangent to the U_1 indifference curve where $W_g = W_b$—a point on the "certainty line" where wealth (W^*) is independent of which state of the world occurs. At W^* the slope of the indifference curve [$\pi/(1-\pi)$] is precisely equal to the price ratio p_g/p_b.

If the market for contingent wealth claims were not fair, utility maximization might not occur on the certainty line. Suppose, for example, that $\pi/(1-\pi) = 4$ but that $p_g/p_b = 2$ because ensuring wealth in bad times proves costly. In this case the budget constraint would resemble line I' in Figure 7.5, and utility maximization would occur below the certainty line.[23] In this case this individual would gamble a bit by opting for $W_g > W_b$ because claims on W_b are relatively costly. Example 7.6 shows the usefulness of this approach in evaluating some of the alternatives that might be available.

EXAMPLE 7.6 Insurance in the State-Preference Model

We can illustrate the state-preference approach by recasting the auto insurance illustration from Example 7.2 as a problem involving the two contingent commodities "wealth with no theft" (W_g) and "wealth with a theft" (W_b). If, as before, we assume logarithmic utility and that the probability of a theft (i.e., $1-\pi$) is 0.25, then

[23]Because (as Equation 7.61 shows) the *MRS* on the certainty line is always $\pi/(1-\pi)$, tangencies with a flatter slope than this must occur below the line.

$$\begin{aligned}\text{expected utility} &= 0.75U(W_g) + 0.25U(W_b) \\ &= 0.75 \ln W_g + 0.25 \ln W_b.\end{aligned} \tag{7.63}$$

If the individual takes no action, then utility is determined by the initial wealth endowment, $W_g^* = 100{,}000$ and $W_b^* = 80{,}000$, so

$$\begin{aligned}\text{expected utility} &= 0.75 \ln 100{,}000 + 0.25 \ln 80{,}000 \\ &= 11.45714.\end{aligned} \tag{7.64}$$

To study trades away from these initial endowments, we write the budget constraint in terms of the prices of the contingent commodities, p_g and p_b:

$$p_g W_g^* + p_b W_b^* = p_g W_g + p_b W_b. \tag{7.65}$$

Assuming that these prices equal the probabilities of the two states ($p_g = 0.75$, $p_b = 0.25$), this constraint can be written

$$0.75(100{,}000) + 0.25(80{,}000) = 95{,}000 = 0.75W_g + 0.25W_b; \tag{7.66}$$

that is, the expected value of wealth is \$95,000, and this person can allocate this amount between W_g and W_b. Now maximization of utility with respect to this budget constraint yields $W_g = W_b = 95{,}000$. Consequently, the individual will move to the certainty line and receive an expected utility of

$$\text{expected utility} = \ln 95{,}000 = 11.46163, \tag{7.67}$$

a clear improvement over doing nothing. To obtain this improvement, this person must be able to transfer \$5,000 of wealth in good times (no theft) into \$15,000 of extra wealth in bad times (theft). A fair insurance contract would allow this because it would cost \$5,000 but return \$20,000 should a theft occur (but nothing should no theft occur). Notice here that the wealth changes promised by insurance—$dW_b/dW_g = 15{,}000/-5{,}000 = -3$—exactly equal the negative of the odds ratio $-\pi/(1-\pi) = -0.75/0.25 = -3$.

A policy with a deductible provision. A number of other insurance contracts might be utility improving in this situation, although not all of them would lead to choices that lie on the certainty line. For example, a policy that cost \$5,200 and returned \$20,000 in case of a theft would permit this person to reach the certainty line with $W_g = W_b = 94{,}800$ and

$$\text{expected utility} = \ln 94{,}800 = 11.45953, \tag{7.68}$$

which also exceeds the utility obtainable from the initial endowment. A policy that costs \$4,900 and requires the individual to incur the first \$1,000 of a loss from theft would yield

$$\begin{aligned}W_g &= 100{,}000 - 4{,}900 = 95{,}100, \\ W_b &= 80{,}000 - 4{,}900 + 19{,}000 = 94{,}100;\end{aligned} \tag{7.69}$$

then

$$\begin{aligned}\text{expected utility} &= 0.75 \ln 95{,}100 + 0.25 \ln 94{,}100 \\ &= 11.46004.\end{aligned} \tag{7.70}$$

Although this policy does not permit this person to reach the certainty line, it is utility improving. Insurance need not be complete to offer the promise of higher utility.

QUERY: What is the maximum amount an individual would be willing to pay for an insurance policy under which he or she had to absorb the first \$1,000 of loss?

Risk aversion and risk premiums

The state-preference model is also especially useful for analyzing the relationship between risk aversion and individuals' willingness to pay for risk. Consider two people, each of whom starts with a certain wealth, W^*. Each person seeks to maximize an expected utility function of the form

$$V(W_g, W_b) = \pi \frac{W_g^R}{R} + (1 - \pi)\frac{W_b^R}{R}. \tag{7.71}$$

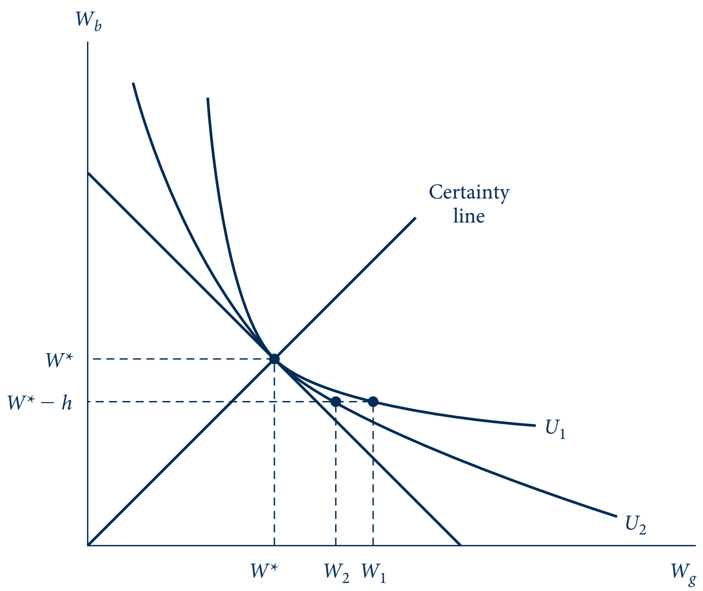
Here the utility function exhibits constant relative risk aversion (see Example 7.4). Notice also that the function closely resembles the CES utility function we examined in Chapter 3 and elsewhere. The parameter R determines both the degree of risk aversion and the degree of curvature of indifference curves implied by the function. A risk-averse individual will have a large negative value for R and have sharply curved indifference curves, such as U_1 shown in Figure 7.6. A person with more tolerance for risk will have a higher value of R and flatter indifference curves (such as U_2).[24]

Suppose now these individuals are faced with the prospect of losing h dollars of wealth in bad times. Such a risk would be acceptable to individual 2 if wealth in good times were to increase from W^* to W_2. For the risk-averse individual 1, however, wealth would have

FIGURE 7.6

Risk Aversion and Risk Premiums

Indifference curve U_1 represents the preferences of a risk-averse person, whereas the person with preferences represented by U_2 is willing to assume more risk. When faced with the risk of losing h in bad times, person 2 will require compensation of $W_2 - W^*$ in good times, whereas person 1 will require a larger amount given by $W_1 - W^*$.



[24]Tangency of U_1 and U_2 at W^* is ensured because the *MRS* along the certainty line is given by $\pi/(1 - \pi)$ regardless of the value of R.

to increase to W_1 to make the risk acceptable. Therefore, the difference between W_1 and W_2 indicates the effect of risk aversion on willingness to assume risk. Some of the problems in this chapter make use of this graphic device for showing the connection between preferences (as reflected by the utility function in Equation 7.71) and behavior in risky situations.

ASYMMETRY OF INFORMATION

One obvious implication of the study of information acquisition is that the level of information that an individual buys will depend on the per-unit price of information messages. Unlike the market price for most goods (which we usually assume to be the same for everyone), there are many reasons to believe that information costs may differ significantly among individuals. Some individuals may possess specific skills relevant to information acquisition (e.g., they may be trained mechanics), whereas others may not possess such skills. Some individuals may have other types of experience that yield valuable information, whereas others may lack that experience. For example, the seller of a product will usually know more about its limitations than will a buyer because the seller will know precisely how the good was made and where possible problems might arise. Similarly, large-scale repeat buyers of a good may have greater access to information about it than would first-time buyers. Finally, some individuals may have invested in some types of information services (e.g., by having a computer link to a brokerage firm or by subscribing to *Consumer Reports*) that make the marginal cost of obtaining additional information lower than for someone without such an investment.

All these factors suggest that the level of information will sometimes differ among the participants in market transactions. Of course, in many instances, information costs may be low and such differences may be minor. Most people can appraise the quality of fresh vegetables fairly well just by looking at them, for example. But when information costs are high and variable across individuals, we would expect them to find it advantageous to acquire different amounts of information. We will postpone a detailed study of such situations until Chapter 18.

SUMMARY

The goal of this chapter was to provide some basic material for the study of individual behavior in uncertain situations. The key concepts covered are listed as follows.

- The most common way to model behavior under uncertainty is to assume that individuals seek to maximize the expected utility of their actions.
- Individuals who exhibit a diminishing marginal utility of wealth are risk averse. That is, they generally refuse fair bets.
- Risk-averse individuals will wish to insure themselves completely against uncertain events if insurance premiums are actuarially fair. They may be willing to pay more than actuarially fair premiums to avoid taking risks.
- Two utility functions have been extensively used in the study of behavior under uncertainty: the constant absolute risk aversion (CARA) function and the constant relative risk aversion (CRRA) function. Neither is completely satisfactory on theoretical grounds.
- Methods for reducing the risk involved in a situation include transferring risk to those who can bear it more effectively through insurance, spreading risk across several activities through diversification, preserving options for dealing with the various outcomes that arise, and acquiring information to determine which outcomes are more likely.
- One of the most extensively studied issues in the economics of uncertainty is the "portfolio problem," which asks how an investor will split his or her wealth among available assets. A simple version of the problem is used to illustrate the value of diversification in the text; the Extensions provide a detailed analysis.

- Information is valuable because it permits individuals to make better decisions in uncertain situations. Information can be most valuable when individuals have some flexibility in their decision making.
- The state-preference approach allows decision making under uncertainty to be approached in a familiar choice-theoretic framework.

Problems

7.1

George is seen to place an even-money \$100,000 bet on the Bulls to win the NBA Finals. If George has a logarithmic utility-of-wealth function and if his current wealth is \$1,000,000, what must he believe is the minimum probability that the Bulls will win?

7.2

Show that if an individual's utility-of-wealth function is convex then he or she will prefer fair gambles to income certainty and may even be willing to accept somewhat unfair gambles. Do you believe this sort of risk-taking behavior is common? What factors might tend to limit its occurrence?

7.3

An individual purchases a dozen eggs and must take them home. Although making trips home is costless, there is a 50 percent chance that all the eggs carried on any one trip will be broken during the trip. The individual considers two strategies: (1) take all 12 eggs in one trip; or (2) take two trips with 6 eggs in each trip.

a. List the possible outcomes of each strategy and the probabilities of these outcomes. Show that, on average, 6 eggs will remain unbroken after the trip home under either strategy.
b. Develop a graph to show the utility obtainable under each strategy. Which strategy will be preferable?
c. Could utility be improved further by taking more than two trips? How would this possibility be affected if additional trips were costly?

7.4

Suppose there is a 50–50 chance that a risk-averse individual with a current wealth of \$20,000 will contract a debilitating disease and suffer a loss of \$10,000.

a. Calculate the cost of actuarially fair insurance in this situation and use a utility-of-wealth graph (such as shown in Figure 7.1) to show that the individual will prefer fair insurance against this loss to accepting the gamble uninsured.
b. Suppose two types of insurance policies were available:
   (1) a fair policy covering the complete loss; and
   (2) a fair policy covering only half of any loss incurred.
   Calculate the cost of the second type of policy and show that the individual will generally regard it as inferior to the first.

7.5

Ms. Fogg is planning an around-the-world trip on which she plans to spend \$10,000. The utility from the trip is a function of how much she actually spends on it (Y), given by

$$U(Y) = \ln Y.$$

a. If there is a 25 percent probability that Ms. Fogg will lose \$1,000 of her cash on the trip, what is the trip's expected utility?
b. Suppose that Ms. Fogg can buy insurance against losing the \$1,000 (say, by purchasing traveler's checks) at an "actuarially fair" premium of \$250. Show that her expected utility is higher if she purchases this insurance than if she faces the chance of losing the \$1,000 without insurance.
c. What is the maximum amount that Ms. Fogg would be willing to pay to insure her \$1,000?