CHAPTER THIRTEEN

General Equilibrium and Welfare

The partial equilibrium models of perfect competition that were introduced in Chapter 12 are clearly inadequate for describing all the effects that occur when changes in one market have repercussions in other markets. Therefore, they are also inadequate for making general welfare statements about how well market economies perform. Instead, what is needed is an economic model that permits us to view many markets simultaneously. In this chapter we will develop a few simple versions of such models. The Extensions to the chapter show how general equilibrium models are applied to the real world.

PERFECTLY COMPETITIVE PRICE SYSTEM

The model we will develop in this chapter is primarily an elaboration of the supply–demand mechanism presented in Chapter 12. Here we will assume that all markets are of the type described in that chapter and refer to such a set of markets as a *perfectly competitive price system.* The assumption is that there is some large number of homogeneous goods in this simple economy. Included in this list of goods are not only consumption items but also factors of production. Each of these goods has an *equilibrium price*, established by the action of supply and demand.[1] At this set of prices, every market is cleared in the sense that suppliers are willing to supply the quantity that is demanded and consumers will demand the quantity that is supplied. We also assume that there are no transaction or transportation charges and that both individuals and firms have perfect knowledge of prevailing market prices.

The law of one price

Because we assume zero transaction cost and perfect information, each good obeys the law of one price: A homogeneous good trades at the same price no matter who buys it or which firm sells it. If one good traded at two different prices, demanders would rush to buy the good where it was cheaper, and firms would try to sell all their output where the good was more expensive. These actions in themselves would tend to equalize the price of the good. In the perfectly competitive market, each good must have only one price. This is why we may speak unambiguously of *the* price of a good.

[1]One aspect of this market interaction should be made clear from the outset. The perfectly competitive market determines only relative (not absolute) prices. In this chapter, we speak only of relative prices. It makes no difference whether the prices of apples and oranges are \$.10 and \$.20, respectively, or \$10 and \$20. The important point in either case is that two apples can be exchanged for one orange in the market. The absolute level of prices is determined mainly by monetary factors—a topic usually covered in macroeconomics.

Behavioral assumptions

The perfectly competitive model assumes that people and firms react to prices in specific ways.

1. There are assumed to be a large number of people buying any one good. Each person takes all prices as given and adjusts his or her behavior to *maximize utility*, given the prices and his or her budget constraint. People may also be suppliers of productive services (e.g., labor), and in such decisions they also regard prices as given.[2]
2. There are assumed to be a large number of firms producing each good, and each firm produces only a small share of the output of any one good. In making input and output choices, firms are assumed to operate to *maximize profits*. The firms treat all prices as given when making these profit-maximizing decisions.

These various assumptions should be familiar because we have been making them throughout this book. Our purpose here is to show how an entire economic system operates when all markets work in this way.

A GRAPHICAL MODEL OF GENERAL EQUILIBRIUM WITH TWO GOODS

We begin our analysis with a graphical model of general equilibrium involving only two goods, which we will call x and y. This model will prove useful because it incorporates many of the features of far more complex general equilibrium representations of the economy.

General equilibrium demand

Ultimately, demand patterns in an economy are determined by individuals' preferences. For our simple model we will assume that all individuals have identical preferences, which can be represented by an indifference curve map[3] defined over quantities of the two goods, x and y. The benefit of this approach for our purposes is that this indifference curve map (which is identical to the ones used in Chapters 3–6) shows how individuals rank consumption bundles containing both goods. These rankings are precisely what we mean by "demand" in a general equilibrium context. Of course, we cannot illustrate which bundles of commodities will be chosen until we know the budget constraints that demanders face. Because incomes are generated as individuals supply labor, capital, and other resources to the production process, we must delay any detailed illustration until we have examined the forces of production and supply in our model.

General equilibrium supply

Developing a notion of general equilibrium supply in this two-good model is a somewhat more complex process than describing the demand side of the market because we have not thus far illustrated production and supply of two goods simultaneously. Our

[2]Hence, unlike our partial equilibrium models, incomes are endogenously determined in general equilibrium models.

[3]There are some technical problems in using a single indifference curve map to represent the preferences of an entire community of individuals. In this case the marginal rate of substitution (i.e., the slope of the community indifference curve) will depend on how the available goods are distributed among individuals: The increase in total y required to compensate for a one-unit reduction in x will depend on which specific individual(s) the x is taken from. Although we will not discuss this issue in detail here, it has been widely examined in the international trade literature.

approach is to use the familiar production possibility curve (see Chapter 1) for this purpose. By detailing the way in which this curve is constructed, we can illustrate, in a simple context, the ways in which markets for outputs and inputs are related.

Edgeworth box diagram for production

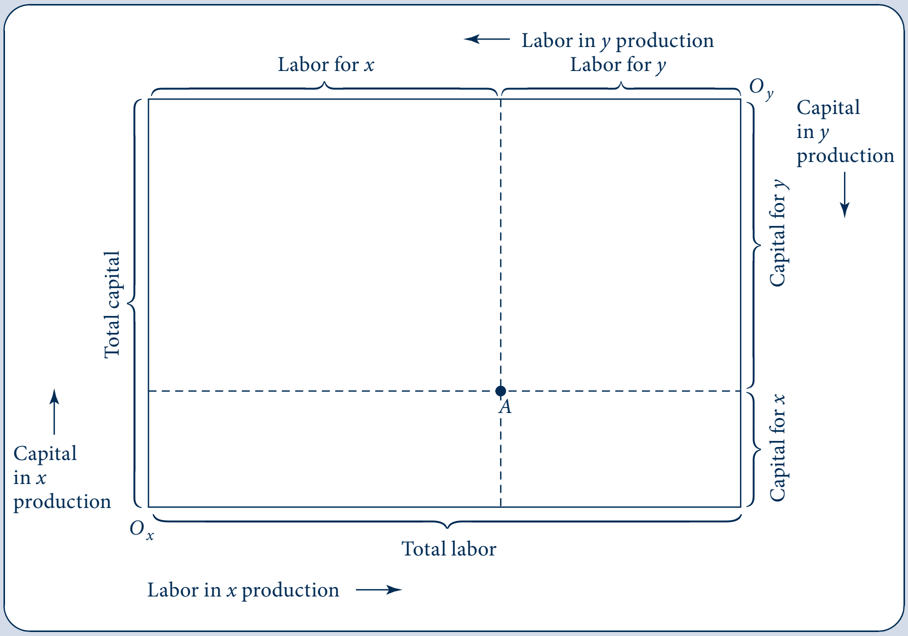
Construction of the production possibility curve for two outputs (x and y) begins with the assumption that there are fixed amounts of capital and labor inputs that must be allocated to the production of the two goods. The possible allocations of these inputs can be illustrated with an Edgeworth box diagram with dimensions given by the total amounts of capital and labor available.

In Figure 13.1, the length of the box represents total labor-hours, and the height of the box represents total capital-hours. The lower left corner of the box represents the "origin" for measuring capital and labor devoted to production of good x. The upper right corner of the box represents the origin for resources devoted to y. Using these conventions, any point in the box can be regarded as a fully employed allocation of the available resources between goods x and y. Point A, for example, represents an allocation in which the indicated number of labor hours are devoted to x production together with a specified number of hours of capital. Production of good y uses whatever labor and capital are "left over." Point A in Figure 13.1, for example, also shows the exact amount of labor and capital used in the production of good y. Any other point in the box has a similar interpretation. Thus, the Edgeworth box shows every possible way the existing capital and labor might be used to produce x and y.

FIGURE 13.1

Construction of an Edgeworth Box Diagram for Production

The dimensions of this diagram are given by the total quantities of labor and capital available. Quantities of these resources devoted to x production are measured from origin O_x; quantities devoted to y are measured from O_y. Any point in the box represents a fully employed allocation of the available resources to the two goods.



Efficient allocations

Many of the allocations shown in Figure 13.1 are technically inefficient in that it is possible to produce both more x and more y by shifting capital and labor around a bit. In our model we assume that competitive markets will not exhibit such inefficient input choices (for reasons we will explore in more detail later in the chapter). Hence we wish to discover the efficient allocations in Figure 13.1 because these illustrate the production outcomes in this model. To do so, we introduce isoquant maps for good x (using O_x as the origin) and good y (using O_y as the origin), as shown in Figure 13.2. In this figure it is clear that the arbitrarily chosen allocation A is inefficient. By reallocating capital and labor, one can produce both more x than x_2 and more y than y_2.

The efficient allocations in Figure 13.2 are those such as P_1, P_2, P_3, and P_4, where the isoquants are tangent to one another. At any other points in the box diagram, the two goods' isoquants will intersect, and we can show inefficiency as we did for point A. At the points of tangency, however, this kind of unambiguous improvement cannot be made. In going from P_2 to P_3, for example, more x is being produced, but at the cost of less y being produced; therefore, P_3 is not "more efficient" than P_2—both of the points are efficient. Tangency of the isoquants for good x and good y implies that their slopes are equal. That is, the *RTS* of capital for labor is equal in x and y production. Later we will show how competitive input markets will lead firms to make such efficient input choices.

Therefore, the curve joining O_x and O_y that includes all these points of tangency shows all the efficient allocations of capital and labor. Points off this curve are inefficient in that unambiguous increases in output can be obtained by reshuffling inputs between the two goods. Points on the curve O_xO_y are all efficient allocations, however, because more x can be produced only by cutting back on y production and vice versa.

FIGURE 13.2

Edgeworth Box Diagram of Efficiency in Production

This diagram adds production isoquants for x and y to Figure 13.1. It then shows technically efficient ways to allocate the fixed amounts of k and l between the production of the two outputs. The line joining O_x and O_y is the locus of these efficient points. Along this line, the *RTS* (of l for k) in the production of good x is equal to the *RTS* in the production of y.

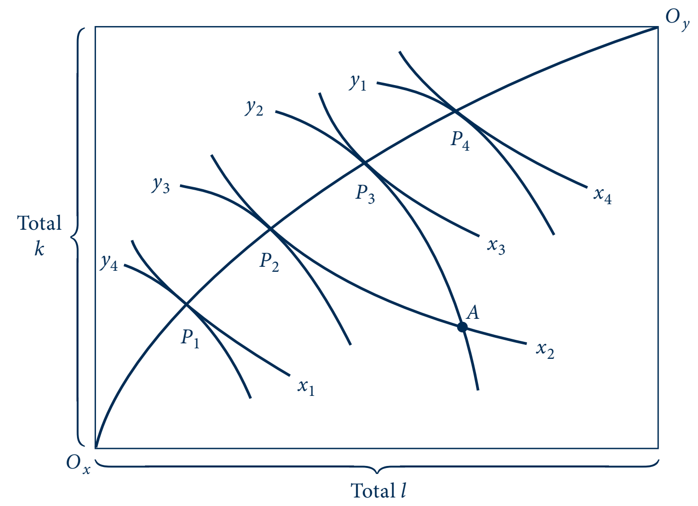
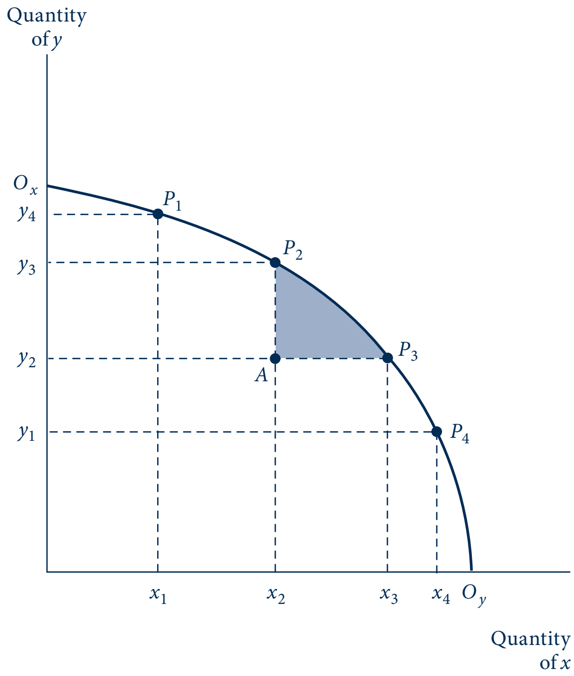


FIGURE 13.3

Production Possibility Frontier

The production possibility frontier shows the alternative combinations of x and y that can be efficiently produced by a firm with fixed resources. The curve can be derived from Figure 13.2 by varying inputs between the production of x and y while maintaining the conditions for efficiency. The negative of the slope of the production possibility curve is called the *rate of product transformation (RPT)*.



Production possibility frontier

The efficiency locus in Figure 13.2 shows the maximum output of y that can be produced for any preassigned output of x. We can use this information to construct a *production possibility frontier*, which shows the alternative outputs of x and y that can be produced with the fixed capital and labor inputs. In Figure 13.3 the O_xO_y locus has been taken from Figure 13.2 and transferred onto a graph with x and y outputs on the axes. At O_x, for example, no resources are devoted to x production; consequently, y output is as large as is possible with the existing resources. Similarly, at O_y, the output of x is as large as possible. The other points on the production possibility frontier (say, P_1, P_2, P_3, and P_4) are derived from the efficiency locus in an identical way. Hence we have derived the following definition.

DEFINITION

Production possibility frontier. The *production possibility frontier* shows the alternative combinations of two outputs that can be produced with fixed quantities of inputs if those inputs are employed efficiently.

Rate of product transformation

The slope of the production possibility frontier shows how x output can be substituted for y output when total resources are held constant. For example, for points near O_x on the production possibility frontier, the slope is a small negative number—say, $-1/4$; this implies that, by reducing y output by 1 unit, x output could be increased by 4. Near O_y, on the other hand, the slope is a large negative number (say, -5), implying that y output must be reduced by 5 units to permit the production of one more x. The slope of the production possibility frontier clearly shows the possibilities that exist for trading y for x in production. The negative of this slope is called the *rate of product transformation (RPT)*.

DEFINITION

Rate of product transformation. The *rate of product transformation* (*RPT*) between two outputs is the negative of the slope of the production possibility frontier for those outputs. Mathematically,

$$\begin{aligned} RPT \text{ (of } x \text{ for } y) &= -[\text{slope of production possibility frontier}] \\ &= -\frac{dy}{dx}(\text{along } O_xO_y), \end{aligned} \qquad \textbf{(13.1)}$$

The *RPT* records how x can be technically traded for y while continuing to keep the available productive inputs efficiently employed.

Shape of the production possibility frontier

The production possibility frontier illustrated in Figure 13.3 exhibits an increasing *RPT*. For output levels near O_x, relatively little y must be sacrificed to obtain one more x ($-dy/dx$ is small). Near O_y, on the other hand, additional x may be obtained only by substantial reductions in y output ($-dy/dx$ is large). In this section we will show why this concave shape might be expected to characterize most production situations.

A first step in that analysis is to recognize that *RPT* is equal to the ratio of the marginal cost of x (MC_x) to the marginal cost of y (MC_y). Intuitively, this result is obvious. Suppose, for example, that x and y are produced only with labor. If it takes two labor hours to produce one more x, we might say that MC_x is equal to 2. Similarly, if it takes only one labor hour to produce an extra y, then MC_y is equal to 1. But in this situation it is clear that the *RPT* is 2: two y must be forgone to provide enough labor so that x may be increased by one unit. Hence the *RPT* is equal to the ratio of the marginal costs of the two goods.

More formally, suppose that the costs (say, in terms of the "disutility" experienced by factor suppliers) of any output combination are denoted by $C(x, y)$. Along the production possibility frontier, $C(x, y)$ will be constant because the inputs are in fixed supply. If we call this constant level of costs $\overline{C}$, we can write $C(x, y) - \overline{C} = 0$. It is this implicit function that underlies the production possibility frontier. Applying the results from Chapter 2 for such a function yields:

$$RPT = \frac{dy}{dx}\Big|_{C(x,\ y)\ -\ \overline{C}=0} = -\frac{C_x}{C_y} = -\frac{MC_x}{MC_y}. \qquad \textbf{(13.2)}$$

To demonstrate reasons why the *RPT* might be expected to increase for clockwise movements along the production possibility frontier, we can proceed by showing why the ratio of MC_x to MC_y should increase as x output expands and y output contracts. We first present two relatively simple arguments that apply only to special cases; then we turn to a more sophisticated general argument.

Diminishing returns

The most common rationale offered for the concave shape of the production possibility frontier is the assumption that both goods are produced under conditions of diminishing returns. Hence increasing the output of good x will raise its marginal cost, whereas decreasing the output of y will reduce its marginal cost. Equation 13.2 then shows that the *RPT* will increase for movements along the production possibility frontier from O_x to O_y. A problem with this explanation, of course, is that it applies only to cases in which both goods exhibit diminishing returns to scale, and that assumption is at variance with the theoretical reasons for preferring the assumption of constant or even increasing returns to scale as mentioned elsewhere in this book.

Specialized inputs

If some inputs were "more suited" for x production than for y production (and vice versa), the concave shape of the production frontier also could be explained. In that case, increases in x output would require drawing progressively less suitable inputs into the production of that good. Therefore, marginal costs of x would increase. Marginal costs for y, on the other hand, would decrease because smaller output levels for y would permit the use of only those inputs most suited for y production. Such an argument might apply, for example, to a farmer with a variety of types of land under cultivation in different crops. In trying to increase the production of any one crop, the farmer would be forced to grow it on increasingly unsuitable parcels of land. Although this type of specialized input assumption has considerable importance in explaining a variety of real-world phenomena, it is nonetheless at variance with our general assumption of homogeneous factors of production. Hence it cannot serve as a fundamental explanation for concavity.

Differing factor intensities

Even if inputs are homogeneous and production functions exhibit constant returns to scale, the production possibility frontier will be concave if goods x and y use inputs in different proportions.[4] In the production box diagram of Figure 13.2, for example, good x is *capital intensive* relative to good y. That is, at every point along the O_xO_y contract curve, the ratio of k to l in x production exceeds the ratio of k to l in y production: The bowed curve O_xO_y is always above the main diagonal of the Edgeworth box. If, on the other hand, good y had been relatively capital intensive, the O_xO_y contract curve would have been bowed downward below the diagonal. Although a formal proof that unequal factor intensities result in a concave production possibility frontier will not be presented here, it is possible to suggest intuitively why that occurs. Consider any two points on the frontier O_xO_y in Figure 13.3—say, P_1 (with coordinates x_1, y_4) and P_3 (with coordinates x_3, y_2). One way of producing an output combination "between" P_1 and P_3 would be to produce the combination

$$\frac{x_1 + x_3}{2}, \frac{y_4 + y_2}{2}.$$

Because of the constant returns-to-scale assumption, that combination would be feasible and would fully use both factors of production. The combination would lie at the midpoint of a straight-line chord joining points P_1 and P_3. Although such a point is feasible, it is not efficient, as can be seen by examining points P_1 and P_3 in the box diagram of Figure 13.2. Because of the bowed nature of the contract curve, production at a point midway between P_1 and P_3 would be off the contract curve: Producing at a point such as P_2 would provide more of both goods. Therefore, the production possibility frontier in Figure 13.3 must "bulge out" beyond the straight line P_1P_3. Because such a proof could be constructed for any two points on O_xO_y, we have shown that the frontier is concave; that is, the *RPT* increases as the output of good X increases. When production is reallocated in a northeast

[4]If, in addition to homogeneous factors and constant returns to scale, each good also used k and l in the same proportions under optimal allocations, then the production possibility frontier would be a straight line.

direction along the O_xO_y contract curve (in Figure 13.3), the capital–labor ratio decreases in the production of *both* x and y. Because good x is capital intensive, this change increases MC_x. On the other hand, because good y is labor intensive, MC_y decreases. Hence the relative marginal cost of x (as represented by the RPT) increases.

Opportunity cost and supply

The production possibility curve demonstrates that there are many possible efficient combinations of the two goods and that producing more of one good necessitates cutting back on the production of some other good. This is precisely what economists mean by the term *opportunity cost*. The cost of producing more x can be most readily measured by the reduction in y output that this entails. Therefore, the cost of one more unit of x is best measured as the RPT (of x for y) at the prevailing point on the production possibility frontier. The fact that this cost increases as more x is produced represents the formulation of supply in a general equilibrium context.

EXAMPLE 13.1 Concavity of the Production Possibility Frontier

In this example we look at two characteristics of production functions that may cause the production possibility frontier to be concave.

Diminishing returns. Suppose that the production of both x and y depends only on labor input and that the production functions for these goods are

$$\begin{aligned} x &= f(l_x) = l_x^{0.5}, \\ y &= f(l_y) = l_y^{0.5}. \end{aligned} \tag{13.3}$$

Hence production of each of these goods exhibits diminishing returns to scale. If total labor supply is limited by

$$l_x + l_y = 100, \tag{13.4}$$

then simple substitution shows that the production possibility frontier is given by

$$x^2 + y^2 = 100 \quad \text{for} \quad x, y \geq 0. \tag{13.5}$$

In this case, the frontier is a quarter-circle and is concave. The RPT can now be computed directly from the equation for the production possibility frontier (written in implicit form as $f(x, y) = x^2 + y^2 - 100 = 0$):

$$RPT = -\frac{dy}{dx} = -\left(-\frac{f_x}{f_y}\right) = \frac{2x}{2y} = \frac{x}{y}, \tag{13.6}$$

and this slope increases as x output increases. A numerical illustration of concavity starts by noting that the points (10, 0) and (0, 10) both lie on the frontier. A straight line joining these two points would also include the point (5, 5), but that point lies below the frontier. If equal amounts of labor are devoted to both goods, then production is $x = y = \sqrt{50}$, which yields more of both goods than the midpoint.

Factor intensity. To show how differing factor intensities yield a concave production possibility frontier, suppose that the two goods are produced under constant returns to scale but with different Cobb–Douglas production functions:

$$\begin{aligned} x &= f(k, l) = k_x^{0.5} l_x^{0.5}, \\ y &= g(k, l) = k_y^{0.25} l_y^{0.75}. \end{aligned} \tag{13.7}$$

Suppose also that total capital and labor are constrained by

$$k_x + k_y = 100, \qquad l_x + l_y = 100. \tag{13.8}$$

It is easy to show that

$$RTS_x = \frac{k_x}{l_x} = \kappa_x, \qquad RTS_y = \frac{3k_y}{l_y} = 3\kappa_y, \tag{13.9}$$

where $\kappa_i = k_i/l_i$. Being located on the production possibility frontier requires $RTS_x = RTS_y$ or $\kappa_x = 3\kappa_y$. That is, no matter how total resources are allocated to production, being on the production possibility frontier requires that x be the capital-intensive good (because, in some sense, capital is more productive in x production than in y production). The capital–labor ratios in the production of the two goods are also constrained by the available resources:

$$\frac{k_x + k_y}{l_x + l_y} = \frac{k_x}{l_x + l_y} + \frac{k_y}{l_x + l_y} = \alpha\kappa_x + (1-\alpha)\kappa_y = \frac{100}{100} = 1, \tag{13.10}$$

where $\alpha = l_x/(l_x + l_y)$—that is, α is the share of total labor devoted to x production. Using the condition that $\kappa_x = 3\kappa_y$, we can find the input ratios of the two goods in terms of the overall allocation of labor:

$$\kappa_y = \frac{1}{1+2\alpha}, \qquad \kappa_x = \frac{3}{1+2\alpha}. \tag{13.11}$$

Now we are in a position to phrase the production possibility frontier in terms of the share of labor devoted to x production:

$$\begin{aligned} x &= \kappa_x^{0.5} l_x = \kappa_x^{0.5}\alpha(100) = 100\alpha\left(\frac{3}{1+2\alpha}\right)^{0.5}, \\ y &= \kappa_y^{0.25} l_y = \kappa_y^{0.25}(1-\alpha)(100) = 100(1-\alpha)\left(\frac{1}{1+2\alpha}\right)^{0.25}. \end{aligned} \tag{13.12}$$

We could push this algebra even further to eliminate α from these two equations to get an explicit functional form for the production possibility frontier that involves only x and y, but we can show concavity with what we already have. First, notice that if $\alpha = 0$ (x production gets no labor or capital inputs), then $x = 0$, $y = 100$. With $\alpha = 1$, we have $x = 100$, $y = 0$. Hence a linear production possibility frontier would include the point (50, 50). But if $\alpha = 0.39$, say, then

$$\begin{aligned} x &= 100\alpha\left(\frac{3}{1+2\alpha}\right)^{0.5} = 39\left(\frac{3}{1.78}\right)^{0.5} = 50.6, \\ y &= 100(1-\alpha)\left(\frac{1}{1+2\alpha}\right)^{0.25} = 61\left(\frac{1}{1.78}\right)^{0.25} = 52.8, \end{aligned} \tag{13.13}$$

which shows that the actual frontier is bowed outward beyond a linear frontier. It is worth repeating that both of the goods in this example are produced under constant returns to scale and that the two inputs are fully homogeneous. It is only the differing input intensities involved in the production of the two goods that yields the concave production possibility frontier.

QUERY: How would an increase in the total amount of labor available shift the production possibility frontiers in these examples?

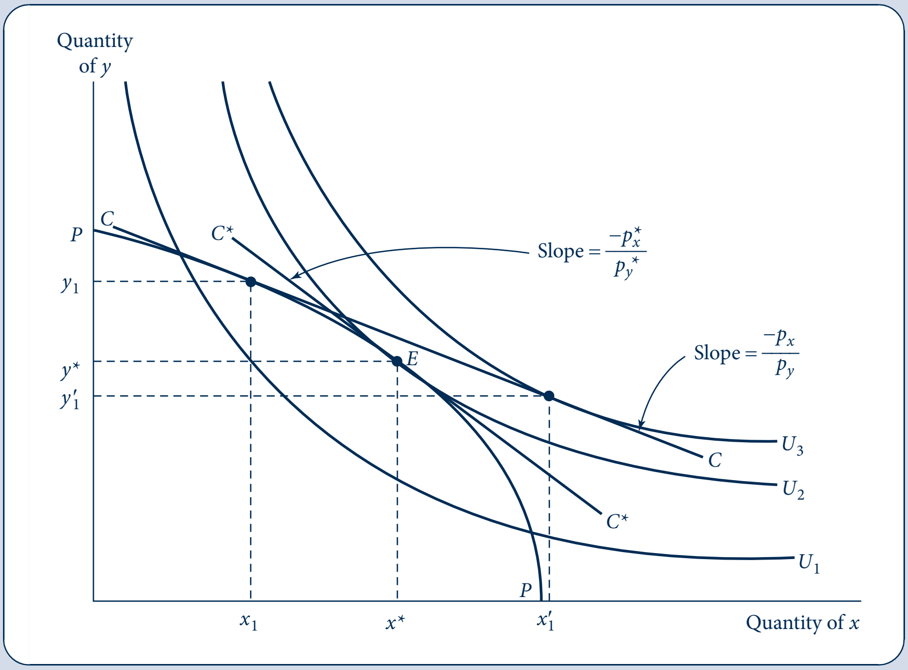
Determination of equilibrium prices

Given these notions of demand and supply in our simple two-good economy, we can now illustrate how equilibrium prices are determined. Figure 13.4 shows PP, the

FIGURE 13.4

Determination of Equilibrium Prices

With a price ratio given by p_x/p_y, firms will produce x_1, y_1; society's budget constraint will be given by line C. With this budget constraint, individuals demand x_1' and y_1'; that is, there is an excess demand for good x and an excess supply of good y. The workings of the market will move these prices toward their equilibrium levels p_x^*, p_y^*. At those prices, society's budget constraint will be given by line C^*, and supply and demand will be in equilibrium. The combination x^*, y^* of goods will be chosen.



production possibility frontier for the economy, and the set of indifference curves represents individuals' preferences for these goods. First, consider the price ratio p_x/p_y. At this price ratio, firms will choose to produce the output combination x_1, y_1. Profit-maximizing firms will choose the more profitable point on PP. At x_1, y_1 the ratio of the two goods' prices (p_x/p_y) is equal to the ratio of the goods' marginal costs (the RPT); thus, profits are maximized there. On the other hand, given this budget constraint (line C),[5] individuals will demand x_1', y_1'. Consequently, with these prices, there is an excess demand for good x (individuals demand more than is being produced) but an excess supply of good y. The workings of the marketplace will cause p_x to increase and p_y to decrease. The price ratio p_x/p_y will increase; the price line will take on a steeper slope. Firms will respond to these price changes by moving clockwise along the production possibility frontier; that is, they will increase their production of good x and decrease their production of good y. Similarly, individuals will respond to the changing prices by substituting y for x in their consumption choices. These actions of both firms and individuals serve to eliminate the excess demand for x and the excess supply of y as market prices change.

[5]It is important to recognize why the budget constraint has this location. Because p_x and p_y are given, the value of total production is $p_x \cdot x_1 + p_y \cdot y_1$. This is the value of "GDP" in the simple economy pictured in Figure 13.4. It is also, therefore, the total income accruing to people in society. Society's budget constraint therefore passes through x_1, y_1 and has a slope of $-p_x/p_y$. This is precisely the budget constraint labeled C in the figure.

Equilibrium is reached at x^*, y^* with a price ratio of p_x^*/p_y^*. With this price ratio,[6] supply and demand are equilibrated for both good x and good y. Given p_x and p_y, firms will produce x^* and y^* in maximizing their profits. Similarly, with a budget constraint given by C^*, individuals will demand x^* and y^*. The operation of the price system has cleared the markets for both x and y simultaneously. Therefore, this figure provides a "general equilibrium" view of the supply–demand process for two markets working together. For this reason we will make considerable use of this figure in our subsequent analysis.

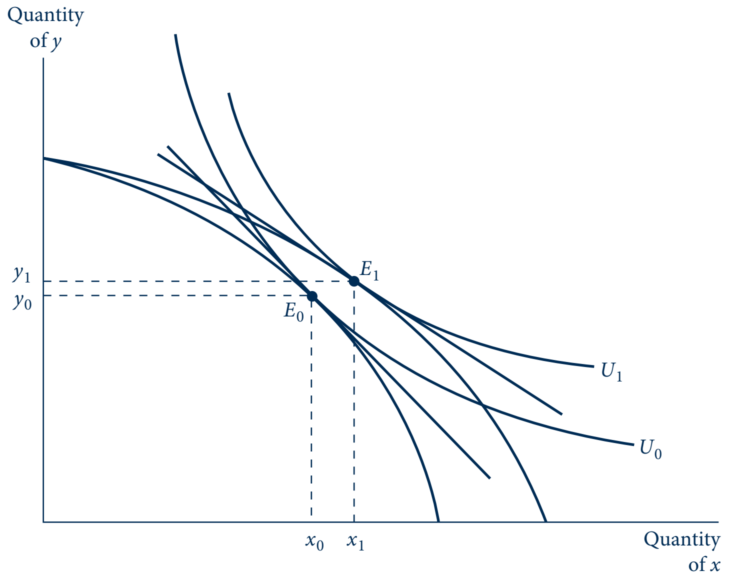
COMPARATIVE STATICS ANALYSIS

As in our partial equilibrium analysis, the equilibrium price ratio p_x^*/p_y^* illustrated in Figure 13.4 will tend to persist until either preferences or production technologies change. This competitively determined price ratio reflects these two basic economic forces. If preferences were to shift, say, toward good x, then p_x/p_y would increase and a new equilibrium would be established by a clockwise move along the production possibility curve. More x and less y would be produced to meet these changed preferences. Similarly, technical progress in the production of good x would shift the production possibility curve outward, as illustrated in Figure 13.5. This would tend to decrease the relative price of x and increase the quantity of x consumed (assuming x is a normal good). In the figure the quantity of y

FIGURE 13.5

Effects of Technical Progress in x Production

Technical advances that lower marginal costs of x production will shift the production possibility frontier. This will generally create income and substitution effects that cause the quantity of x produced to increase (assuming x is a normal good). Effects on the production of y are ambiguous because income and substitution effects work in opposite directions.



[6]Notice again that competitive markets determine only equilibrium relative prices. Determination of the absolute price level requires the introduction of money into this barter model.

consumed also increases as a result of the income effect arising from the technical advance; however, a slightly different drawing of the figure could have reversed that result if the substitution effect had been dominant. Example 13.2 looks at a few such effects.

EXAMPLE 13.2 Comparative Statics in a General Equilibrium Model

To explore how general equilibrium models work, let's start with a simple example based on the production possibility frontier in Example 13.1. In that example we assumed that production of both goods was characterized by decreasing returns $x = l_x^{0.5}$ and $y = l_y^{0.5}$ and also that total labor available was given by $l_x + l_y = 100$. The resulting production possibility frontier was given by $x^2 + y^2 = 100$, and $RPT = x/y$. To complete this model we assume that the typical individual's utility function is given by $U(x, y) = x^{0.5}y^{0.5}$, so the demand functions for the two goods are

$$\begin{aligned} x &= x(p_x, p_y, I) = \frac{0.5I}{p_x}, \\ y &= y(p_x, p_y, I) = \frac{0.5I}{p_y}. \end{aligned} \tag{13.14}$$

Base-case equilibrium. Profit maximization by firms requires that $p_x/p_y = MC_x/MC_y = RPT = x/y$, and utility-maximizing demand requires that $p_x/p_y = y/x$. Thus, equilibrium requires that $x/y = y/x$, or $x = y$. Inserting this result into the equation for the production possibility frontier shows that

$$x^* = y^* = \sqrt{50} = 7.07 \quad \text{and} \quad \frac{p_x}{p_y} = 1. \tag{13.15}$$

This is the equilibrium for our base case with this model.

The budget constraint. The budget constraint that faces individuals is not especially transparent in this illustration; therefore, it may be useful to discuss it explicitly. To bring some degree of absolute pricing into the model, let's consider all prices in terms of the wage rate, w. Because total labor supply is 100, it follows that total labor income is $100w$. However, because of the diminishing returns assumed for production, each firm also earns profits. For firm x, say, the total cost function is $C(w, x) = wl_x = wx^2$, so $p_x = MC_x = 2wx = 2w\sqrt{50}$. Therefore, the profits for firm x are $\pi_x = (p_x - AC_x)x = (p_x - wx)x = wx^2 = 50w$. A similar computation shows that profits for firm y are also given by $50w$. Because general equilibrium models must obey the national income identity, we assume that consumers are also shareholders in the two firms and treat these profits also as part of their spendable incomes. Hence total consumer income is

$$\begin{aligned} \text{total income} &= \text{labor income} + \text{profits} \\ &= 100w + 2(50w) = 200w. \end{aligned} \tag{13.16}$$

This income will just permit consumers to spend $100w$ on each good by buying $\sqrt{50}$ units at a price of $2w\sqrt{50}$, so the model is internally consistent.

A shift in supply. There are only two ways in which this base-case equilibrium can be disturbed: (1) by changes in "supply"—that is, by changes in the underlying technology of this economy; or (2) by changes in "demand"—that is, by changes in preferences. Let's first consider changes in technology. Suppose that there is technical improvement in x production so that the production function is $x = 2l_x^{0.5}$. Now the production possibility frontier is given by $x^2/4 + y^2 = 100$, and $RPT = x/4y$. Proceeding as before to find the equilibrium in this model:

$$\frac{p_x}{p_y} = \frac{x}{4y} \quad \text{(supply)},$$
$$\frac{p_x}{p_y} = \frac{y}{x} \quad \text{(demand)}, \tag{13.17}$$

so $x^2 = 4y^2$ and the equilibrium is

$$x^* = 2\sqrt{50}, \quad y^* = \sqrt{50} \quad \text{and} \quad \frac{p_x}{p_y} = \frac{1}{2}. \tag{13.18}$$

Technical improvements in x production have caused its relative price to decrease and the consumption of this good to increase. As in many examples with Cobb–Douglas utility, the income and substitution effects of this price decrease on y demand are precisely offsetting. Technical improvements clearly make consumers better off, however. Whereas utility was previously given by $U(x, y) = x^{0.5}y^{0.5} = \sqrt{50} = 7.07$, now it has increased to $U(x, y) = x^{0.5}y^{0.5} = (2\sqrt{50})^{0.5}(\sqrt{50})^{0.5} = \sqrt{2} \cdot \sqrt{50} = 10$. Technical change has increased consumer welfare substantially.

A shift in demand. If consumer preferences were to switch to favor good y as $U(x, y) = x^{0.1}y^{0.9}$, then demand functions would be given by $x = 0.1I/p_x$ and $y = 0.9I/p_y$, and demand equilibrium would require $p_x/p_y = y/9x$. Returning to the original production possibility frontier to arrive at an overall equilibrium, we have

$$\frac{p_x}{p_y} = \frac{x}{y} \quad \text{(supply)},$$
$$\frac{p_x}{p_y} = \frac{y}{9x} \quad \text{(demand)}, \tag{13.19}$$

so $9x^2 = y^2$ and the equilibrium is given by

$$x^* = \sqrt{10}, \quad y^* = 3\sqrt{10} \quad \text{and} \quad \frac{p_x}{p_y} = \frac{1}{3} \tag{13.20}$$

Hence the decrease in demand for x has significantly reduced its relative price. Observe that in this case, however, we cannot make a welfare comparison to the previous cases because the utility function has changed.

QUERY: What are the budget constraints in these two alternative scenarios? How is income distributed between wages and profits in each case? Explain the differences intuitively.

GENERAL EQUILIBRIUM MODELING AND FACTOR PRICES

This simple general equilibrium model reinforces Marshall's observations about the importance of both supply and demand forces in the price determination process. By providing an explicit connection between the markets for all goods, the general equilibrium model makes it possible to examine more complex questions about market relationships than is possible by looking at only one market at a time. General equilibrium modeling also permits an examination of the connections between goods and factor markets; we can illustrate that with an important historical case.

The Corn Laws debate

High tariffs on grain imports were imposed by the British government following the Napoleonic wars. Debate over the effects of these Corn Laws dominated the analytical efforts of economists between the years 1829 and 1845. A principal focus of the debate concerned the effect that elimination of the tariffs would have on factor prices—a question that continues to have relevance today, as we will see.

The production possibility frontier in Figure 13.6 shows those combinations of grain (x) and manufactured goods (y) that could be produced by British factors of production. Assuming (somewhat contrary to actuality) that the Corn Laws completely prevented trade, market equilibrium would be at E with the domestic price ratio given by p_x^*/p_y^*. Removal of the tariffs would reduce this price ratio to p_x'/p_y'. Given that new ratio, Britain would produce combination A and consume combination B. Grain imports would amount to $x_B - x_A$, and these would be financed by export of manufactured goods equal to $y_A - y_B$. Overall utility for the typical British consumer would be increased by the opening of trade. Therefore, use of the production possibility diagram demonstrates the implications that relaxing the tariffs would have for the production of both goods.

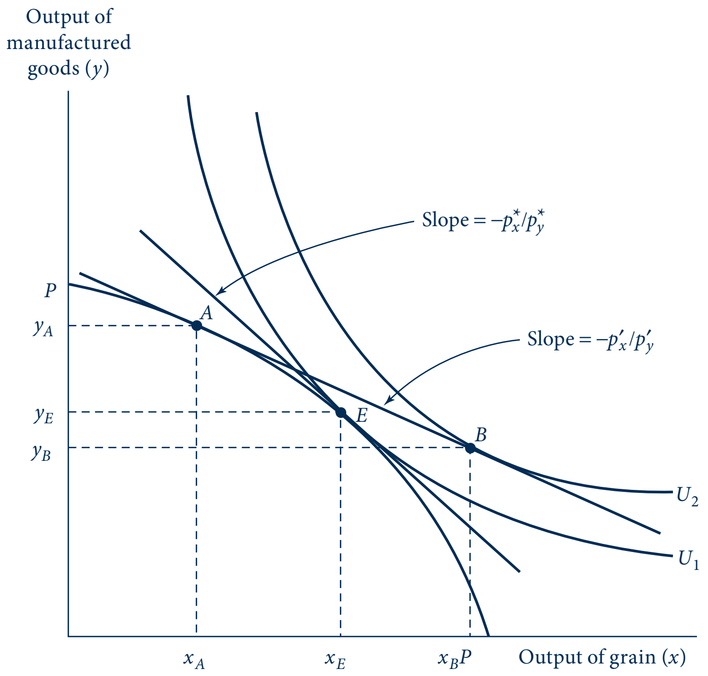
Trade and factor prices

By referring to the Edgeworth production box diagram (Figure 13.2) that lies behind the production possibility frontier (Figure 13.3), it is also possible to analyze the effect of

FIGURE 13.6

Analysis of the Corn Laws Debate

Reduction of tariff barriers on grain would cause production to be reallocated from point E to point A; consumption would be reallocated from E to B. If grain production is relatively capital intensive, the relative price of capital would decrease as a result of these reallocations.



tariff reductions on factor prices. The movement from point E to point A in Figure 13.6 is similar to a movement from P_3 to P_1 in Figure 13.2, where production of x is decreased and production of y is increased.

This figure also records the reallocation of capital and labor made necessary by such a move. If we assume that grain production is relatively capital intensive, then the movement from P_3 to P_1 causes the ratio of k to l to increase in both industries.[7] This in turn will cause the relative price of capital to decrease (and the relative price of labor to increase). Hence we conclude that repeal of the Corn Laws would be harmful to capital owners (i.e., landlords) and helpful to laborers. It is not surprising that landed interests fought repeal of the laws.

Political support for trade policies

The possibility that trade policies may affect the relative incomes of various factors of production continues to exert a major influence on political debates about such policies. In the United States, for example, exports tend to be intensive in their use of skilled labor, whereas imports tend to be intensive in unskilled labor input. By analogy to our discussion of the Corn Laws, it might thus be expected that further movements toward free trade policies would result in increasing relative wages for skilled workers and in decreasing relative wages for unskilled workers. Therefore, it is not surprising that unions representing skilled workers (the machinists or aircraft workers) tend to favor free trade, whereas unions of unskilled workers (those in textiles, shoes, and related businesses) tend to oppose it.[8]

A MATHEMATICAL MODEL OF EXCHANGE

Although the previous graphical model of general equilibrium with two goods is fairly instructive, it cannot reflect all the features of general equilibrium modeling with an arbitrary number of goods and productive inputs. In the remainder of this chapter we will illustrate how such a more general model can be constructed, and we will look at some of the insights that such a model can provide. For most of our presentation we will look only at a model of exchange—quantities of various goods already exist and are merely traded among individuals. In such a model there is no production. Later in the chapter we will look briefly at how production can be incorporated into the general model we have constructed.

Vector notation

Most general equilibrium modeling is conducted using vector notation. This provides great flexibility in specifying an arbitrary number of goods or individuals in the models. Consequently, this seems to be a good place to offer a brief introduction to such notation. A *vector* is simply an ordered array of variables (which each may take on specific values). Here we will usually adopt the convention that the vectors we use are column vectors. Hence we will write an $n \times 1$ column vector as:

[7] In the Corn Laws debate, attention centered on the factors of land and labor.

[8] The finding that the opening of trade will raise the relative price of the abundant factor is called the Stolper–Samuelson theorem after the economists who rigorously proved it in the 1950s.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}, \qquad \textbf{(13.21)}$$

where each x_i is a variable that can take on any value. If $\mathbf{x}$ and $\mathbf{y}$ are two $n \times 1$ column vectors, then the (vector) sum of them is defined as:

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n + y_n \end{bmatrix}. \qquad \textbf{(13.22)}$$

Notice that this sum only is defined if the two vectors are of equal length. In fact, checking the length of vectors is one good way of deciding whether one has written a meaningful vector equation.

The (dot) product of two vectors is defined as the sum of the component-by-component product of the elements in the two vectors. That is:

$$\mathbf{xy} = x_1y_1 + x_2y_2 + \cdots + x_ny_n. \qquad \textbf{(13.23)}$$

Notice again that this operation is only defined if the vectors are of the same length. With these few concepts we are now ready to illustrate the general equilibrium model of exchange.

Utility, initial endowments, and budget constraints

In our model of exchange there are assumed to be n goods and m individuals. Each individual gains utility from the vector of goods he or she consumes $u^i(\mathbf{x}^i)$ where $i = 1 \ldots m$. Individuals also possess initial endowments of the goods given by $\bar{\mathbf{x}}^i$. Individuals are free to exchange their initial endowments with other individuals or to keep some or all the endowment for themselves. In their trading individuals are assumed to be price-takers—that is, they face a price vector ($\mathbf{p}$) that specifies the market price for each of the n goods. Each individual seeks to maximize utility and is bound by a budget constraint that requires that the total amount spent on consumption equals the total value of his or her endowment:

$$\mathbf{px}^i = \mathbf{p}\bar{\mathbf{x}}^i. \qquad \textbf{(13.24)}$$

Although this budget constraint has a simple form, it may be worth contemplating it for a minute. The right side of Equation 13.24 is the market value of this individual's endowment (sometimes referred to as his or her *full income*). He or she could "afford" to consume this endowment (and only this endowment) if he or she wished to be self-sufficient. But the endowment can also be spent on some other consumption bundle (which, presumably, provides more utility). Because consuming items in one's own endowment has an opportunity cost, the terms on the left of Equation 13.24 consider the costs of all items that enter into the final consumption bundle, including endowment goods that are retained.

Demand functions and homogeneity

The utility maximization problem outlined in the previous section is identical to the one we studied in detail in Part 2 of this book. As we showed in Chapter 4, one outcome of this process is a set of n individual demand functions (one for each good) in which quantities demanded depend on all prices and income. Here we can denote these in vector form as $\mathbf{x}^i(\mathbf{p}, \mathbf{p}\overline{\mathbf{x}}^i)$. These demand functions are continuous, and, as we showed in Chapter 4, they are homogeneous of degree 0 in all prices and income. This latter property can be indicated in vector notation by

$$\mathbf{x}^i(t\mathbf{p}, t\mathbf{p}\overline{\mathbf{x}}^i) = \mathbf{x}^i(\mathbf{p}, \mathbf{p}\overline{\mathbf{x}}^i) \tag{13.25}$$

for any $t > 0$. This property will be useful because it will permit us to adopt a convenient normalization scheme for prices, which, because it does not alter relative prices, leaves quantities demanded unchanged.

Equilibrium and Walras' law

Equilibrium in this simple model of exchange requires that the total quantities of each good demanded be equal to the total endowment of each good available (remember, there is no production in this model). Because the model used is similar to the one originally developed by Leon Walras,[9] this equilibrium concept is customarily attributed to him.

DEFINITION

Walrasian equilibrium. *Walrasian equilibrium* is an allocation of resources and an associated price vector, $\mathbf{p}^*$, such that

$$\sum_{i=1}^{m} \mathbf{x}^i(\mathbf{p}^*, \mathbf{p}^*\overline{\mathbf{x}}^i) = \sum_{i=1}^{m} \overline{\mathbf{x}}^i, \tag{13.26}$$

where the summation is taken over the m individuals in this exchange economy.

The n equations in Equation 13.26 state that in equilibrium demand equals supply in each market. This is the multimarket analog of the single market equilibria examined in the previous chapter. Because there are n prices to be determined, a simple counting of equations and unknowns might suggest that the existence of such a set of prices is guaranteed by the simultaneous equation solution procedures studied in elementary algebra. Such a supposition would be incorrect for two reasons. First, the algebraic theorem about simultaneous equation systems applies only to linear equations. Nothing suggests that the demand equations in this problem will be linear—in fact, most examples of demand equations we encountered in Part 2 were definitely nonlinear.

A second problem with Equation 13.26 is that the equations are not independent of one another—they are related by what is known as *Walras' law*. Because each individual in this exchange economy is bound by a budget constraint of the form given in Equation 13.24, we can sum over all individuals to obtain

$$\sum_{i=1}^{m} \mathbf{p}\mathbf{x}^i = \sum_{i=1}^{m} \mathbf{p}\overline{\mathbf{x}}^i \quad \text{or} \quad \sum_{i=1}^{m} \mathbf{p}(\mathbf{x}^i - \overline{\mathbf{x}}^i) = 0. \tag{13.27}$$

In words, Walras' law states that the value of all quantities demanded must equal the value of all endowments. This result holds for any set of prices, not just for equilibrium

[9] The concept is named for the nineteenth century French/Swiss economist Leon Walras, who pioneered the development of general equilibrium models. Models of the type discussed in this chapter are often referred to as models of *Walrasian equilibrium*, primarily because of the price-taking assumptions inherent in them.

prices.[10] The general lesson is that the logic of individual budget constraints necessarily creates a relationship among the prices in any economy. It is this connection that helps to ensure that a demand–supply equilibrium exists, as we now show.

Existence of equilibrium in the exchange model

The question of whether all markets can reach equilibrium together has fascinated economists for nearly 200 years. Although intuitive evidence from the real world suggests that this must indeed be possible (market prices do not tend to fluctuate wildly from one day to the next), proving the result mathematically proved to be rather difficult. Walras himself thought he had a good proof that relied on evidence from the market to adjust prices toward equilibrium. The price would increase for any good for which demand exceeded supply and decrease when supply exceeded demand. Walras believed that if this process continued long enough, a full set of equilibrium prices would eventually be found. Unfortunately, the pure mathematics of Walras' solution were difficult to state, and ultimately there was no guarantee that a solution would be found. But Walras' idea of adjusting prices toward equilibrium using market forces provided a starting point for the modern proofs, which were largely developed during the 1950s.

A key aspect of the modern proofs of the existence of equilibrium prices is the choice of a good normalization rule. Homogeneity of demand functions makes it possible to use any absolute scale for prices, providing that relative prices are unaffected by this choice. Such an especially convenient scale is to normalize prices so that they sum to one. Consider an arbitrary set of n non-negative prices $p_1, p_2 \ldots p_n$. We can normalize[11] these to form a new set of prices

$$p_i' = \frac{p_i}{\sum_{k=1}^{n} p_k}. \tag{13.28}$$

These new prices will have the properties that $\sum_{k=1}^{n} p_k' = 1$ and that relative price ratios are maintained:

$$\frac{p_i'}{p_j'} = \frac{p_i/\sum p_k}{p_j/\sum p_k} = \frac{p_i}{p_j}. \tag{13.29}$$

Because this sort of mathematical process can always be done, we will assume, without loss of generality, that the price vectors we use ($\mathbf{p}$) have all been normalized in this way.

Therefore, proving the existence of equilibrium prices in our model of exchange amounts to showing that there will always exist a price vector $\mathbf{p}^*$ that achieves equilibrium in all markets. That is,

$$\sum_{i=1}^{m} \mathbf{x}^i(\mathbf{p}^*, \mathbf{p}^*\bar{\mathbf{x}}^i) = \sum_{i=1}^{m} \bar{\mathbf{x}}^i \quad \text{or} \quad \sum_{i=1}^{m} \mathbf{x}^i(\mathbf{p}^*, \mathbf{p}^*\bar{\mathbf{x}}^i) - \sum_{i=1}^{m} \bar{\mathbf{x}}^i = 0 \quad \text{or} \quad \mathbf{z}(\mathbf{p}^*) = 0, \tag{13.30}$$

where we use $\mathbf{z}(\mathbf{p})$ as a shorthand way of recording the "excess demands" for goods at a particular set of prices. In equilibrium, excess demand is zero in all markets.[12]

[10] Walras' law holds trivially for equilibrium prices as multiplication of Equation 13.26 by $\mathbf{p}$ shows.

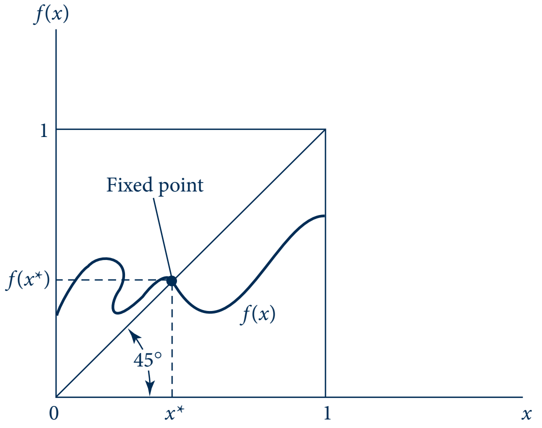
[11] This is possible only if at least one of the prices is nonzero. Throughout our discussion we will assume that not all equilibrium prices can be zero.

[12] Goods that are in excess supply at equilibrium will have a zero price. We will not be concerned with such "free goods" here.

FIGURE 13.7

A Graphical Illustration of Brouwer's Fixed Point Theorem

Because any continuous function must cross the 45° line somewhere in the unit square, this function must have a point for which $f(x^*) = x^*$. This point is called a *fixed point*.



Now consider the following way of implementing Walras' idea that goods in excess demand should have their prices increased, whereas those in excess supply should have their prices reduced.[13] Starting from any arbitrary set of prices, $\mathbf{p}_0$, we define a new set, $\mathbf{p}_1$, as

$$\mathbf{p}_1 = f(\mathbf{p}_0) = \mathbf{p}_0 + k\,\mathbf{z}(\mathbf{p}_0), \tag{13.31}$$

where k is a positive constant. This function will be continuous (because demand functions are continuous), and it will map one set of normalized prices into another (because of our assumption that all prices are normalized). Hence it will meet the conditions of the Brouwer's fixed point theorem, which states that any continuous function from a closed compact set onto itself (in the present case, from the "unit simplex" onto itself) will have a "fixed point" such that $\mathbf{x} = f(\mathbf{x})$. The theorem is illustrated for a single dimension in Figure 13.7. There, no matter what shape the function $f(x)$ takes, as long as it is continuous, it must somewhere cross the 45° line and at that point $x = f(x)$.

If we let $\mathbf{p}^*$ represent the fixed point identified by Brouwer's theorem for Equation 13.31, we have:

$$\mathbf{p}^* = f(\mathbf{p}^*) = \mathbf{p}^* + k\,\mathbf{z}(\mathbf{p}^*). \tag{13.32}$$

Hence at this point $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$; thus, $\mathbf{p}^*$ is an equilibrium price vector. The proof that Walras sought is easily accomplished using an important mathematical result developed a few years after his death. The elegance of the proof may obscure the fact that it uses a number of assumptions about economic behavior such as: (1) price-taking by all parties; (2) homogeneity of demand functions; (3) continuity of demand functions; and (4) presence of budget constraints and Walras' law. All these play important roles in showing that a system of simple markets can indeed achieve a multimarket equilibrium.

[13]What follows is an extremely simplified version of the proof of the existence of equilibrium prices. In particular, problems of free goods and appropriate normalizations have been largely assumed away. For a mathematically correct proof, see, for example, G. Debreu, *Theory of Value* (New York: John Wiley & Sons, 1959).

First theorem of welfare economics

Given that the forces of supply and demand can establish equilibrium prices in the general equilibrium model of exchange we have developed, it is natural to ask what are the welfare consequences of this finding. Adam Smith[14] hypothesized that market forces provide an "invisible hand" that leads each market participant to "promote an end [social welfare] which was no part of his intention." Modern welfare economics seeks to understand the extent to which Smith was correct.

Perhaps the most important welfare result that can be derived from the exchange model is that the resulting Walrasian equilibrium is "efficient" in the sense that it is not possible to devise some alternative allocation of resources in which at least some people are better off and no one is worse off. This definition of efficiency was originally developed by Italian economist Vilfredo Pareto in the early 1900s. Understanding the definition is easiest if we consider what an "inefficient" allocation might be. The total quantities of goods included in initial endowments would be allocated inefficiently if it were possible, by shifting goods around among individuals, to make at least one person better off (i.e., receive a higher utility) and no one worse off. Clearly, if individuals' preferences are to count, such a situation would be undesirable. Hence we have a formal definition.

DEFINITION

Pareto efficient allocation. An allocation of the available goods in an exchange economy is efficient if it is not possible to devise an alternative allocation in which at least one person is better off and no one is worse off.

A proof that all Walrasian equilibria are Pareto efficient proceeds indirectly. Suppose that $\mathbf{p}^*$ generates a Walrasian equilibrium in which the quantity of goods consumed by each person is denoted by ${}^*\mathbf{x}^k (k = 1 \ \ldots \ m)$. Now assume that there is some alternative allocation of the available goods ${}'\mathbf{x}^k (k = 1 \ \ldots \ m)$ such that, for at least one person, say, person i, it is that case that ${}'\mathbf{x}^i$ is preferred to ${}^*\mathbf{x}^i$. For this person, it must be the case that

$$\mathbf{p}^* \, {}'\mathbf{x}^i > \mathbf{p}^* \, {}^*\mathbf{x}^i \qquad \textbf{(13.33)}$$

because otherwise this person would have bought the preferred bundle in the first place. If all other individuals are to be equally well off under this new proposed allocation, it must be the case for them that

$$\mathbf{p}^* \, {}'\mathbf{x}^k = \mathbf{p}^* \, {}^*\mathbf{x}^k \quad k = 1 \ \ldots \ m, \ \ k \neq i. \qquad \textbf{(13.34)}$$

If the new bundle were less expensive, such individuals could not have been minimizing expenditures at $\mathbf{p}^*$. Finally, to be feasible, the new allocation must obey the quantity constraints

$$\sum_{i=1}^{m} {}'\mathbf{x}^i = \sum_{i=1}^{m} \overline{\mathbf{x}}^i. \qquad \textbf{(13.35)}$$

Multiplying Equation 13.35 by $\mathbf{p}^*$yields

$$\sum_{i=1}^{m} \mathbf{p}^* \, {}'\mathbf{x}^i = \sum_{i=1}^{m} \mathbf{p}^* \, \overline{\mathbf{x}}^i, \qquad \textbf{(13.36)}$$

[14] Adam Smith, *The Wealth of Nations* (New York: Modern Library, 1937) p. 423.

but Equations 13.33 and 13.34 together with Walras' law applied to the original equilibrium imply that

$$\sum_{i=1}^{m} \mathbf{p}^{*\prime}\mathbf{x}^i > \sum_{i=1}^{m} \mathbf{p}^{*}\,{}^{*}\mathbf{x}^i = \sum_{i=1}^{m} \mathbf{p}^{*}\,\overline{\mathbf{x}}^i. \qquad \textbf{(13.37)}$$

Hence we have a contradiction and must conclude that no such alternative allocation can exist. Therefore, we can summarize our analysis with the following definition.

DEFINITION

First theorem of welfare economics. Every Walrasian equilibrium is Pareto efficient.

The significance of this "theorem" should not be overstated. The theorem does not say that every Walrasian equilibrium is in some sense socially desirable. Walrasian equilibria can, for example, exhibit vast inequalities among individuals arising in part from inequalities in their initial endowments (see the discussion in the next section). The theorem also assumes price-taking behavior and full information about prices—assumptions that need not hold in other models. Finally, the theorem does not consider possible effects of one individual's consumption on another. In the presence of such externalities even a perfect competitive price system may not yield Pareto optimal results (see Chapter 19).

Still, the theorem does show that Smith's "invisible hand" conjecture has some validity. The simple markets in this exchange world can find equilibrium prices, and at those equilibrium prices the resulting allocation of resources will be efficient in the Pareto sense. Developing this proof is one of the key achievements of welfare economics.

A graphic illustration of the first theorem

In Figure 13.8 we again use the Edgeworth box diagram, this time to illustrate an exchange economy. In this economy there are only two goods (x and y) and two individuals (A and B). The total dimensions of the Edgeworth box are determined by the total quantities of the two goods available ($\overline{x}$ and $\overline{y}$). Goods allocated to individual A are recorded using 0_A as an origin. Individual B gets those quantities of the two goods that are "left over" and can be measured using 0_B as an origin. Individual A's indifference curve map is drawn in the usual way, whereas individual B's map is drawn from the perspective of 0_B. Point E in the Edgeworth box represents the initial endowments of these two individuals. Individual A starts with $\overline{x}^A$ and $\overline{y}^A$. Individual B starts with $\overline{x}^B = \overline{x} - \overline{x}^A$ and $\overline{y}^B = \overline{y} - \overline{y}^A$.

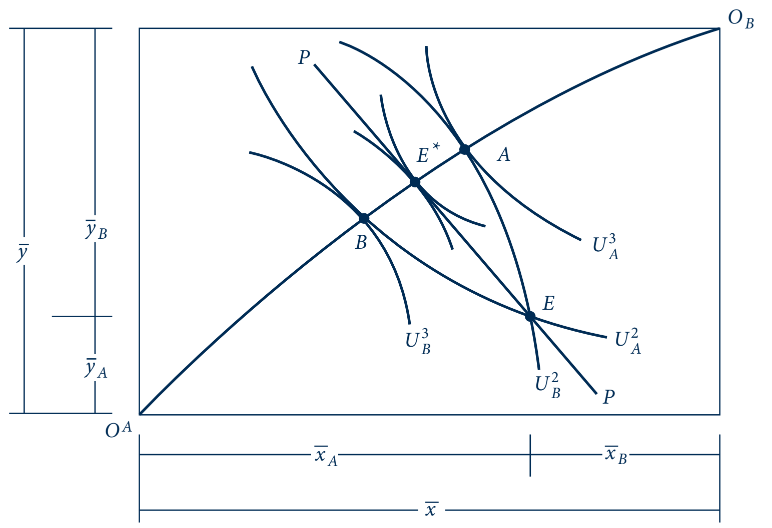
The initial endowments provide a utility level of U_A^2 for person A and U_B^2 for person B. These levels are clearly inefficient in the Pareto sense. For example, we could, by reallocating the available goods,[15] increase person B's utility to U_B^3 while holding person A's utility constant at U_A^2 (point B). Or we could increase person A's utility to U_A^3 while keeping person B on the U_B^2 indifference curve (point A). Allocations A and B are Pareto efficient, however, because at these allocations it is not possible to make either person better off without making the other worse off. There are many other efficient allocations in the Edgeworth box diagram. These are identified by the tangencies of the two individuals' indifference curves. The set of all such efficient points is shown by the line joining O_A to O_B. This line is sometimes called the "contract curve" because it represents all the Pareto-efficient contracts that might be reached by these two individuals. Notice, however, that (assuming that no individual would voluntarily opt for a contract that made him or her

[15] This point could in principle be found by solving the following constrained optimization problem: Maximize $U_B(x_B, y_B)$ subject to the constraint $U_A(x_A, y_A) = U_A^2$. See Example 13.3.

FIGURE 13.8

The First Theorem of Welfare Economics

With initial endowments at point E, individuals trade along the price line PP until they reach point E^*. This equilibrium is Pareto efficient.



worse off) only contracts between points B and A are viable with initial endowments given by point E.

The line PP in Figure 13.8 shows the competitively established price ratio that is guaranteed by our earlier existence proof. The line passes through the initial endowments (E) and shows the terms at which these two individuals can trade away from these initial positions. Notice that such trading is beneficial to both parties—that is, it allows them to get a higher utility level than is provided by their initial endowments. Such trading will continue until all such mutual beneficial trades have been completed. That will occur at allocation E^* on the contract curve. Because the individuals' indifference curves are tangent at this point, no further trading would yield gains to both parties. Therefore, the competitive allocation E^* meets the Pareto criterion for efficiency, as we showed mathematically earlier.

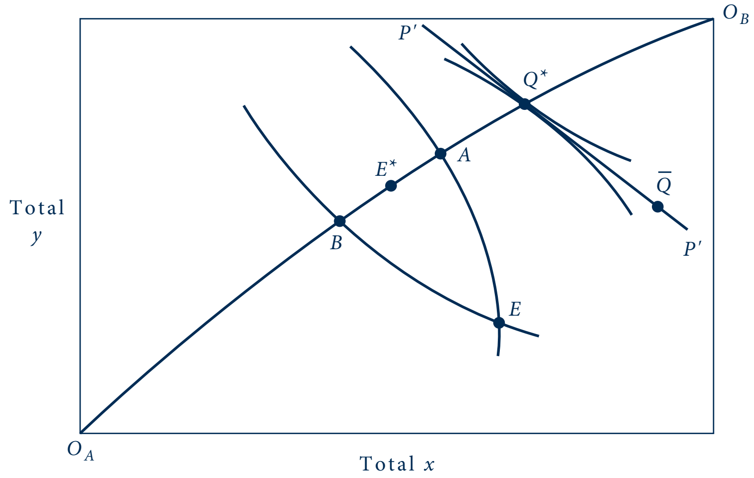
Second theorem of welfare economics

The first theorem of welfare economics shows that a Walrasian equilibrium is Pareto efficient, but the social welfare consequences of this result are limited because of the role played by initial endowments in the demonstration. The location of the Walrasian equilibrium at E^* in Figure 13.8 was significantly influenced by the designation of E as the starting point for trading. Points on the contract curve outside the range of AB are not attainable through voluntary transactions, even though these may in fact be more socially desirable than E^* (perhaps because utilities are more equal). The second theorem of welfare economics addresses this issue. It states that for any Pareto optimal allocation of resources there exists a set of initial endowments and a related price vector such that this allocation is also a Walrasian equilibrium. Phrased another way, any Pareto optimal allocation of resources can also be a Walrasian equilibrium, providing that initial endowments are adjusted accordingly.

FIGURE 13.9

The Second Theorem of Welfare Economics

If allocation Q^* is regarded as socially optimal, this allocation can be supported by any initial endowments on the price line $P'P'$. To move from E to, say, $\overline{Q}$ would require transfers of initial endowments.



A graphical proof of the second theorem should suffice. Figure 13.9 repeats the key aspects of the exchange economy pictures in Figure 13.8. Given the initial endowments at point E, all voluntary Walrasian equilibrium must lie between points A and B on the contract curve. Suppose, however, that these allocations were thought to be undesirable—perhaps because they involve too much inequality of utility. Assume that the Pareto optimal allocation Q^* is believed to be socially preferable, but it is not attainable from the initial endowments at point E. The second theorem states that one can draw a price line through Q^* that is tangent to both individuals' respective indifference curves. This line is denoted by $P'P'$ in Figure 13.9. Because the slope of this line shows potential trades these individuals are willing to make, any point on the line can serve as an initial endowment from which trades lead to Q^*. One such point is denoted by $\overline{Q}$. If a benevolent government wished to ensure that Q^* would emerge as a Walrasian equilibrium, it would have to transfer initial endowments of the goods from E to $\overline{Q}$ (making person A better off and person B worse off in the process).

EXAMPLE 13.3 A Two-Person Exchange Economy

To illustrate these various principles, consider a simple two-person, two-good exchange economy. Suppose that total quantities of the goods are fixed at $\bar{x} = \bar{y} = 1{,}000$. Person A's utility takes the Cobb–Douglas form:

$$U_A(x_A, y_A) = x_A^{2/3} y_A^{1/3}, \tag{13.38}$$

and person B's preferences are given by:

$$U_B(x_B, y_B) = x_B^{1/3} y_B^{2/3}. \tag{13.39}$$

Notice that person A has a relative preference for good x and person B has a relative preference for good y. Hence you might expect that the Pareto-efficient allocations in this model would have the property that person A would consume relatively more x and person B would consume relatively more y. To find these allocations explicitly, we need to find a way of dividing the available goods in such a way that the utility of person A is maximized for any preassigned utility level for person B. Setting up the Lagrangian expression for this problem, we have:

$$\mathcal{L}(x_A, y_A) = U_A(x_A, y_A) + \lambda[U_B(1{,}000 - x_A,\ 1{,}000 - y_A) - \overline{U}_B]. \tag{13.40}$$

Substituting for the explicit utility functions assumed here yields

$$\mathcal{L}(x_A, y_A) = x_A^{2/3} y_A^{1/3} + \lambda[(1{,}000 - x_A)^{1/3}(1{,}000 - y_A)^{2/3} - \overline{U}_B], \tag{13.41}$$

and the first-order conditions for a maximum are

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x_A} &= \frac{2}{3}\left(\frac{y_A}{x_A}\right)^{1/3} - \frac{\lambda}{3}\left(\frac{1{,}000 - y_A}{1{,}000 - x_A}\right)^{2/3} = 0,\\ \frac{\partial \mathcal{L}}{\partial y_A} &= \frac{1}{3}\left(\frac{x_A}{y_A}\right)^{2/3} - \frac{2\lambda}{3}\left(\frac{1{,}000 - x_A}{1{,}000 - y_A}\right)^{1/3} = 0.\end{aligned} \tag{13.42}$$

Moving the terms in λ to the right and dividing the top equation by the bottom gives

$$2\left(\frac{y_A}{x_A}\right) = \frac{1}{2}\left(\frac{1{,}000 - y_A}{1{,}000 - x_A}\right)$$

or

$$\frac{x_A}{1{,}000 - x_A} = \frac{4y_A}{1{,}000 - y_A}. \tag{13.43}$$

This equation allows us to identify all the Pareto optimal allocations in this exchange economy. For example, if we were to arbitrarily choose $x_A = x_B = 500$, Equation 13.43 would become

$$\frac{4y_A}{1{,}000 - y_A} = 1 \quad \text{so} \quad y_A = 200,\ y_B = 800. \tag{13.44}$$

This allocation is relatively favorable to person B. At this point on the contract curve $U_A = 500^{2/3}200^{1/3} = 369$, $U_B = 500^{1/3}800^{2/3} = 683$. Notice that although the available quantity of x is divided evenly (by assumption), most of good y goes to person B as efficiency requires.

Equilibrium price ratio. To calculate the equilibrium price ratio at this point on the contract curve, we need to know the two individuals' marginal rates of substitution. For person A,

$$MRS = \frac{\partial U_A/\partial x_A}{\partial U_A/\partial y_A} = 2\frac{y_A}{x_A} = 2\frac{200}{500} = 0.8 \tag{13.45}$$

and for person B

$$MRS = \frac{\partial U_B/\partial x_B}{\partial U_B/\partial y_B} = 0.5\frac{y_A}{x_A} = 0.5\frac{800}{500} = 0.8. \tag{13.46}$$

Hence the marginal rates of substitution are indeed equal (as they should be), and they imply a price ratio of $p_x/p_y = 0.8$.

Initial endowments. Because this equilibrium price ratio will permit these individuals to trade 8 units of y for each 10 units of x, it is a simple matter to devise initial endowments consistent with this Pareto optimum. Consider, for example, the endowment $\overline{x}_A = 350$, $\overline{y}_A = 320$; $\overline{x}_B = 650$, $\overline{y}_B = 680$. If $p_x = 0.8$, $p_y = 1$, the value of person A's initial endowment is 600. If he or she spends two thirds of this amount on good x, it is possible to purchase 500 units of good x and 200 units of good y. This would increase utility from $U_A = 350^{2/3}\,320^{1/3} = 340$ to 369. Similarly, the value of person B's endowment is 1,200. If he or she spends one third of this on good x, 500 units can be bought. With the remaining two thirds of the value of the endowment being spent on good y, 800 units can be bought. In the process, B's utility increases from 670 to 683. Thus, trading from the proposed initial endowment to the contract curve is indeed mutually beneficial (as shown in Figure 13.8).

QUERY: Why did starting with the assumption that good x would be divided equally on the contract curve result in a situation favoring person B throughout this problem? What point on the contract curve would provide equal utility to persons A and B? What would the price ratio of the two goods be at this point?

Social welfare functions

Figure 13.9 shows that there are many Pareto-efficient allocations of the available goods in an exchange economy. We are assured by the second theorem of welfare economics that any of these can be supported by a Walrasian system of competitively determined prices, providing that initial endowments are adjusted accordingly. A major question for welfare economics is how (if at all) to develop criteria for choosing among all these allocations. In this section we look briefly at one strand of this large topic—the study of *social welfare functions*. Simply put, a social welfare function is a hypothetical scheme for ranking potential allocations of resources based on the utility they provide to individuals. In mathematical terms:

$$Social\ Welfare = SW[U_1(\mathbf{x}^1),\ U_2(\mathbf{x}^2),\ \ldots,\ U_m(\mathbf{x}^m)]. \tag{13.47}$$

The "social planner's" goal then is to choose allocations of goods among the m individuals in the economy in a way that maximizes SW. Of course, this exercise is a purely conceptual one—in reality there are no clearly articulated social welfare functions in any economy, and there are serious doubts about whether such a function could ever arise from some type of democratic process.[16] Still, assuming the existence of such a function can help to illuminate many of the thorniest problems in welfare economics.

A first observation that might be made about the social welfare function in Equation 13.47 is that any welfare maximum must also be Pareto efficient. If we assume that every individual's utility is to "count," it seems clear that any allocation that permits further Pareto improvements (that make one person better off and no one else worse off) cannot be a welfare maximum. Hence achieving a welfare maximum is a problem in choosing among Pareto-efficient allocations and their related Walrasian price systems.

We can make further progress in examining the idea of social welfare maximization by considering the precise functional form that SW might take. Specifically, if we assume utility is measurable, using the CES form can be particularly instructive:

$$SW(U_1,\ U_2,\ \ldots,\ U_m) = \frac{U_1^R}{R} + \frac{U_2^R}{R} + \ldots + \frac{U_m^R}{R} \quad R \le 1. \tag{13.48}$$

Because we have used this functional form many times before in this book, its properties should by now be familiar. Specifically, if $R = 1$, the function becomes:

$$SW(U_1,\ U_2,\ \ldots,\ U_m) = U_1 + U_2 + \ \ldots\ + U_m. \tag{13.49}$$

[16]The "impossibility" of developing a social welfare function from the underlying preferences of people in society was first studied by K. Arrow in *Social Choice and Individual Values*, 2nd ed. (New York: Wiley, 1963). There is a large body of literature stemming from Arrow's initial discovery.

Thus, utility is a simple sum of the utility of every person in the economy. Such a social welfare function is sometimes called a *utilitarian* function. With such a function, social welfare is judged by the aggregate sum of utility (or perhaps even income) with no regard for how utility (income) is distributed among the members of society.

At the other extreme, consider the case $R = -\infty$. In this case, social welfare has a "fixed proportions" character and (as we have seen in many other applications),

$$SW(U_1, U_2, \ldots, U_m) = Min\,[U_1, U_2, \ldots, U_m]. \tag{13.50}$$

Therefore, this function focuses on the worse-off person in any allocation and chooses that allocation for which this person has the highest utility. Such a social welfare function is called a *maximin* function. It was made popular by the philosopher John Rawls, who argued that if individuals did not know which position they would ultimately have in society (i.e., they operate under a "veil of ignorance"), they would opt for this sort of social welfare function to guard against being the worse-off person.[17] Our analysis in Chapter 7 suggests that people may not be this risk averse in choosing social arrangements. However, Rawls' focus on the bottom of the utility distribution is probably a good antidote to thinking about social welfare in purely utilitarian terms.

It is possible to explore many other potential functional forms for a hypothetical welfare function. Problem 13.14 looks at some connections between social welfare functions and the income distribution, for example. But such illustrations largely miss a crucial point if they focus only on an exchange economy. Because the quantities of goods in such an economy are fixed, issues related to production incentives do not arise when evaluating social welfare alternatives. In actuality, however, any attempt to redistribute income (or utility) through taxes and transfers will necessarily affect production incentives and therefore affect the size of the Edgeworth box. Therefore, assessing social welfare will involve studying the trade-off between achieving distributional goals and maintaining levels of production. To examine such possibilities we must introduce production into our general equilibrium framework.

A MATHEMATICAL MODEL OF PRODUCTION AND EXCHANGE

Adding production to the model of exchange developed in the previous section is a relatively simple process. First, the notion of a "good" needs to be expanded to include factors of production. Therefore, we will assume that our list of n goods now includes inputs whose prices also will be determined within the general equilibrium model. Some inputs for one firm in a general equilibrium model are produced by other firms. Some of these goods may also be consumed by individuals (cars are used by both firms and final consumers), and some of these may be used only as intermediate goods (steel sheets are used only to make cars and are not bought by consumers). Other inputs may be part of individuals' initial endowments. Most importantly, this is the way labor supply is treated in general equilibrium models. Individuals are endowed with a certain number of potential labor hours. They may sell these to firms by taking jobs at competitively determined wages, or they may choose to consume the hours themselves in the form of "leisure," In making such choices we continue to assume that individuals maximize utility.[18]

We will assume that there are r firms involved in production. Each of these firms is bound by a production function that describes the physical constraints on the ways the

[17] J. Rawls, *A Theory of Justice* (Cambridge, MA: Harvard University Press, 1971).

[18] A detailed study of labor supply theory is presented in Chapter 16.

firm can turn inputs into outputs. By convention, outputs of the firm take a positive sign, whereas inputs take a negative sign. Using this convention, each firm's production plan can be described by an $n \times 1$ column vector, $\mathbf{y}^j (j = 1 \ldots r)$, which contains both positive and negative entries. The only vectors that the firm may consider are those that are feasible given the current state of technology. Sometimes it is convenient to assume each firm produces only one output. But that is not necessary for a more general treatment of production.

Firms are assumed to maximize profits. Production functions are assumed to be sufficiently convex to ensure a unique profit maximum for any set of output and input prices. This rules out both increasing returns to scale technologies and constant returns because neither yields a unique maxima. Many general equilibrium models can handle such possibilities, but there is no need to introduce such complexities here. Given these assumptions, the profits for any firm can be written as:

$$\begin{aligned} &\pi_j(\mathbf{p}) = \mathbf{p}\mathbf{y}^j \text{ if } \pi_j(\mathbf{p}) \geq 0 \quad \text{and} \\ &\mathbf{y}^j = 0 \text{ if } \pi_j(\mathbf{p}) < 0. \end{aligned} \tag{13.51}$$

Hence this model has a "long run" orientation in which firms that lose money (at a particular price configuration) hire no inputs and produce no output. Notice how the convention that outputs have a positive sign and inputs a negative sign makes it possible to phrase profits in a compact way.[19]

Budget constraints and Walras' law

In an exchange model, individuals' purchasing power is determined by the values of their initial endowments. Once firms are introduced, we must also consider the income stream that may flow from ownership of these firms. To do so, we adopt the simplifying assumption that each individual owns a predefined share, s_i (where $\sum_{i=1}^{m} s_i = 1$) of the profits of all firms. That is, each person owns an "index fund" that can claim a proportionate share of all firms' profits. We can now rewrite each individual's budget constraint (from Equation 13.24) as:

$$\mathbf{p}\mathbf{x}^i = s_i \sum_{j=1}^{r} \mathbf{p}\mathbf{y}^j + \mathbf{p}\overline{\mathbf{x}}^i \quad i = 1 \ldots m. \tag{13.52}$$

Of course, if all firms were in long-run equilibrium in perfectly competitive industries, all profits would be zero and the budget constraint in Equation 13.52 would revert to that in Equation 13.24. But allowing for long-term profits does not greatly complicate our model; therefore, we might as well consider the possibility.

As in the exchange model, the existence of these m budget constraints implies a constraint of the prices that are possible—a generalization of Walras' law. Summing the budget constraints in Equation 13.52 over all individuals yields:

$$\mathbf{p} \sum_{i=1}^{m} \mathbf{x}^i(\mathbf{p}) = \mathbf{p} \sum_{j=1}^{r} \mathbf{y}^j(\mathbf{p}) + \mathbf{p} \sum_{i=1}^{m} \overline{\mathbf{x}}^i, \tag{13.53}$$

and letting $\mathbf{x}(\mathbf{p}) = \sum \mathbf{x}^i(\mathbf{p})$, $\mathbf{y}(\mathbf{p}) = \sum \mathbf{y}^j(\mathbf{p})$, $\overline{\mathbf{x}} = \sum \overline{\mathbf{x}}^i$ provides a simple statement of Walras' law:

$$\mathbf{p}\mathbf{x}(\mathbf{p}) = \mathbf{p}\mathbf{y}(\mathbf{p}) + \mathbf{p}\overline{\mathbf{x}}. \tag{13.54}$$

[19] As we saw in Chapter 11, profit functions are homogeneous of degree 1 in all prices. Hence both output supply functions and input demand functions are homogeneous of degree 0 in all prices because they are derivatives of the profit function.

Notice again that Walras' law holds for any set of prices because it is based on individuals' budget constraints.

Walrasian equilibrium

As before, we define a Walrasian equilibrium price vector ($\mathbf{p}^*$) as a set of prices at which demand equals supply in all markets simultaneously. In mathematical terms this means that:

$$\mathbf{x}(\mathbf{p}^*) = \mathbf{y}(\mathbf{p}^*) + \bar{\mathbf{x}}. \tag{13.55}$$

Initial endowments continue to play an important role in this equilibrium. For example, it is individuals' endowments of potential labor time that provide the most important input for firms' production processes. Therefore, determination of equilibrium wage rates is a major output of general equilibrium models operating under Walrasian conditions. Examining changes in wage rates that result from changes in exogenous influences is perhaps the most important practical use of such models.

As in the study of an exchange economy, it is possible to use some form of fixed point theorem[20] to show that there exists a set of equilibrium prices that satisfy the n equations in Equation 13.55. Because of the constraint of Walras' law, such an equilibrium price vector will be unique only up to a scalar multiple—that is, any absolute price level that preserves relative prices can also achieve equilibrium in all markets. Technically, excess demand functions

$$\mathbf{z}(\mathbf{p}) = \mathbf{x}(\mathbf{p}) - \mathbf{y}(\mathbf{p}) - \bar{\mathbf{x}} \tag{13.56}$$

are homogeneous of degree 0 in prices; therefore, any price vector for which $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$ will also have the property that $\mathbf{z}(t\mathbf{p}^*) = \mathbf{0}$ and $t > 0$. Frequently it is convenient to normalize prices so that they sum to one. But many other normalization rules can also be used. In macroeconomic versions of general equilibrium models it is usually the case that the absolute level of prices is determined by monetary factors.

Welfare economics in the Walrasian model with production

Adding production to the model of an exchange economy greatly expands the number of feasible allocations of resources. One way to visualize this is shown in Figure 13.10. There PP represents that production possibility frontier for a two-good economy with a fixed endowment of primary factors of production. Any point on this frontier is feasible. Consider one such allocation, say, allocation A. If this economy were to produce x_A and y_A, we could use these amounts for the dimensions of the Edgeworth exchange box shown inside the frontier. Any point within this box would also be a feasible allocation of the available goods between the two people whose preferences are shown. Clearly a similar argument could be made for any other point on the production possibility frontier.

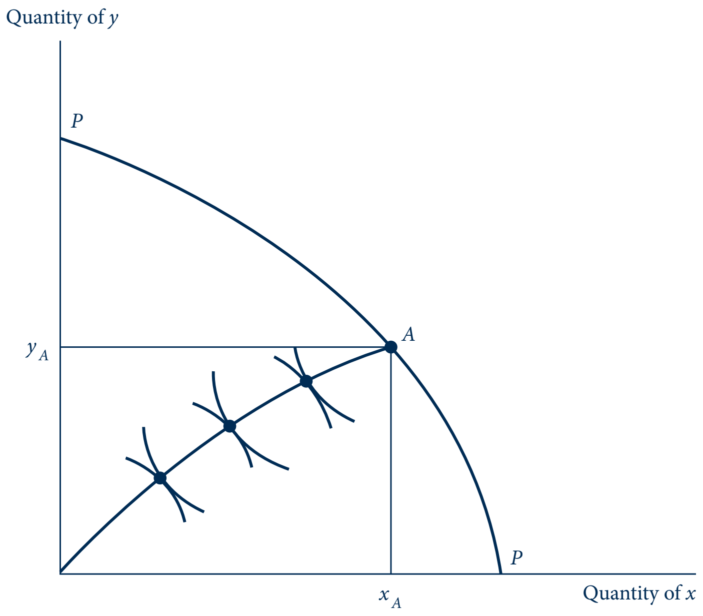
Despite these complications, the first theorem of welfare economics continues to hold in a general equilibrium model with production. At a Walrasian equilibrium there are no further market opportunities (either by producing something else or by reallocating the available goods among individuals) that would make some one individual (or group of individuals) better off without making other individuals worse off. Adam Smith's "invisible hand" continues to exert its logic to ensure that all such mutually beneficial opportunities are exploited (in part because transaction costs are assumed to be zero).

[20]For some illustrative proofs, see K. J. Arrow and F. H. Hahn, *General Competitive Analysis* (San Francisco: Holden-Day, 1971) chap. 5.

FIGURE 13.10

Production Increases the Number of Feasible Allocations

Any point on the production possibility frontier *PP* can serve as the dimensions of an Edgeworth exchange box.



Again, the general social welfare implications of the first theorem of welfare economics are far from clear. There is, of course, a second theorem, which shows that practically any Walrasian equilibrium can be supported by suitable changes in initial endowments. One also could hypothesize a social welfare function to choose among these. But most such exercises are rather uninformative about actual policy issues.

More interesting is the use of the Walrasian mechanism to judge the hypothetical impact of various tax and transfer policies that seek to achieve specific social welfare criteria. In this case (as we shall see) the fact that Walrasian models stress interconnections among markets, especially among product and input markets, can yield important and often surprising results. In the next section we look at a few of these.

COMPUTABLE GENERAL EQUILIBRIUM MODELS

Two advances have spurred the rapid development of general equilibrium models in recent years. First, the theory of general equilibrium itself has been expanded to include many features of real-world markets such as imperfect competition, environmental externalities, and complex tax systems. Models that involve uncertainty and that have a dynamic structure also have been devised, most importantly in the field of macroeconomics. A second related trend has been the rapid development of computer power and the associated software for solving general equilibrium models. This has made it possible to study models with virtually any number of goods and types of households. In this section we will briefly explore some conceptual aspects of these models.[21] The Extensions to the chapter describe a few important applications.

Structure of general equilibrium models

Specification of any general equilibrium model begins by defining the number of goods to be included in the model. These "goods" include not only consumption goods but also intermediate goods that are used in the production of other goods (e.g., capital equipment), productive inputs such as labor or natural resources, and goods that are to be produced by the government (public goods). The goal of the model is then to solve for equilibrium prices for all these goods and to study how these prices change when conditions change.

Some of the goods in a general equilibrium model are produced by firms. The technology of this production must be specified by production functions. The most common such specification is to use the types of CES production functions that we studied in Chapters 9 and 10 because these can yield some important insights about the ways in which inputs are substituted in the face of changing prices. In general, firms are assumed to maximize their profits given their production functions and given the input and output prices they face.

Demand is specified in general equilibrium models by defining utility functions for various types of households. Utility is treated as a function both of goods that are consumed and of inputs that are not supplied to the marketplace (e.g., available labor that is not supplied to the market is consumed as leisure). Households are assumed to maximize utility. Their incomes are determined by the amounts of inputs they "sell" in the market and by the net result of any taxes they pay or transfers they receive.

Finally, a full general equilibrium model must specify how the government operates. If there are taxes in the model, how those taxes are to be spent on transfers or on public goods (which provide utility to consumers) must be modeled. If government borrowing is allowed, the bond market must be explicitly modeled. In short, the model must fully specify the flow of both sources and uses of income that characterize the economy being modeled.

Solving general equilibrium models

Once technology (supply) and preferences (demand) have been specified, a general equilibrium model must be solved for equilibrium prices and quantities. The proof earlier in this chapter shows that such a model will generally have such a solution, but actually finding that solution can sometimes be difficult—especially when the number of goods and households is large. General equilibrium models are usually solved on computers via modifications of an algorithm originally developed by Herbert Scarf in the 1970s.[22] This algorithm (or more modern versions of it) searches for market equilibria by mimicking the way markets work. That is, an initial solution is specified and then prices are raised in markets with excess demand and lowered in markets with excess supply until an equilibrium is found in which all excess demands are zero. Sometimes multiple equilibria will occur, but usually economic models have sufficient curvature in the underlying production and utility functions that the equilibrium found by the Scarf algorithm will be unique.

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[21]For more detail on the issues discussed here, see W. Nicholson and F. Westhoff, "General Equilibrium Models: Improving the Microeconomics Classroom," *Journal of Economic Education* (Summer 2009): 297–314.

[22]Herbert Scarf with Terje Hansen, *On the Computation of Economic Equilibria* (New Haven, CT: Yale University Press, 1973).

Economic insights from general equilibrium models

General equilibrium models provide a number of insights about how economies operate that cannot be obtained from the types of partial equilibrium models studied in Chapter 12. Some of the most important of these are:

- All prices are endogenous in economic models. The exogenous elements of models are preferences and productive technologies.
- All firms and productive inputs are owned by households. All income ultimately accrues to households.
- Any model with a government sector is incomplete if it does not specify how tax receipts are used.
- The "bottom line" in any policy evaluation is the utility of households. Firms and governments are only intermediaries in getting to this final accounting.
- All taxes distort economic decisions along some dimension. The welfare costs of such distortions must always be weighed against the benefits of such taxes (in terms of public good production or equity-enhancing transfers).

Some of these insights are illustrated in the next two examples. In later chapters we will return to general equilibrium modeling whenever such a perspective seems necessary to gain a more complete understanding of the topic being covered.

EXAMPLE 13.4 A Simple General Equilibrium Model

Let's look at a simple general equilibrium model with only two households, two consumer goods (x and y), and two inputs (capital k and labor l). Each household has an "endowment" of capital and labor that it can choose to retain or sell in the market. These endowments are denoted by $\bar{k}_1, \bar{l}_1$ and $\bar{k}_2, \bar{l}_2$, respectively. Households obtain utility from the amounts of the consumer goods they purchase and from the amount of labor they do not sell into the market (i.e., leisure $= \bar{l}_i - l_i$). The households have simple Cobb–Douglas utility functions:

$$U_1 = x_1^{0.5} y_1^{0.3} (\bar{l}_1 - l_1)^{0.2}, \qquad U_2 = x_2^{0.4} y_2^{0.4} (\bar{l}_2 - l_2)^{0.2}. \tag{13.57}$$

Hence household 1 has a relatively greater preference for good x than does household 2. Notice that capital does not enter into these utility functions directly. Consequently, each household will provide its entire endowment of capital to the marketplace. Households will retain some labor, however, because leisure provides utility directly.

Production of goods x and y is characterized by simple Cobb–Douglas technologies:

$$x = k_x^{0.2} l_x^{0.8}, \qquad y = k_y^{0.8} l_y^{0.2}. \tag{13.58}$$

Thus, in this example, production of x is relatively labor intensive, whereas production of y is relatively capital intensive.

To complete this model we must specify initial endowments of capital and labor. Here we assume that

$$\bar{k}_1 = 40, \bar{l}_1 = 24 \quad \text{and} \quad \bar{k}_2 = 10, \bar{l}_2 = 24. \tag{13.59}$$

Although the households have equal labor endowments (i.e., 24 "hours"), household 1 has significantly more capital than does household 2.

Base-case simulation. Equations 13.57–13.59 specify our complete general equilibrium model in the absence of a government. A solution to this model will consist of four equilibrium prices (for x, y, k, and l) at which households maximize utility and firms maximize profits.[23]

[23]Because firms' production functions are characterized by constant returns to scale, in equilibrium each earns zero profits; therefore, there is no need to specify firm ownership in this model.

Because any general equilibrium model can compute only relative prices, we are free to impose a price-normalization scheme. Here we assume that the prices will always sum to unity. That is,

$$p_x + p_y + p_k + p_l = 1. \tag{13.60}$$

Solving[24] for these prices yields

$$p_x = 0.363, \quad p_y = 0.253, \quad p_k = 0.136, \quad p_l = 0.248. \tag{13.61}$$

At these prices, total production of x is 23.7 and production of y is 25.1. The utility-maximizing choices for household 1 are

$$x_1 = 15.7, \quad y_1 = 8.1, \quad \bar{l}_1 - l_1 = 24 - 14.8 = 9.2, \quad U_1 = 13.5; \tag{13.62}$$

for household 2, these choices are

$$x_2 = 8.1, \quad y_2 = 11.6, \quad \bar{l}_2 - l_2 = 24 - 18.1 = 5.9, \quad U_2 = 8.75. \tag{13.63}$$

Observe that household 1 consumes quite a bit of good x but provides less in labor supply than does household 2. This reflects the greater capital endowment of household 1 in this base-case simulation. We will return to this base case in several later simulations.

QUERY: How would you show that each household obeys its budget constraint in this simulation? Does the budgetary allocation of each household exhibit the budget shares that are implied by the form of its utility function?

EXAMPLE 13.5 The Excess Burden of a Tax

In Chapter 12 we showed that taxation may impose an excess burden in addition to the tax revenues collected because of the incentive effects of the tax. With a general equilibrium model we can show much more about this effect. Specifically, assume that the government in the economy of Example 13.4 imposes an ad valorem tax of 0.4 on good x. This introduces a wedge between what demanders pay for this good x (p_x) and what suppliers receive for the good ($p'_x = (1 - t)p_x = 0.6p_x$). To complete the model we must specify what happens to the revenues generated by this tax. For simplicity we assume that these revenues are rebated to the households in a 50–50 split. In all other respects the economy remains as described in Example 13.4.

Solving for the new equilibrium prices in this model yields

$$p_x = 0.472, \; p_y = 0.218, \; p_k = 0.121, \; p_l = 0.188. \tag{13.64}$$

At these prices, total production of x is 17.9, and total production of y is 28.8. Hence the allocation of resources has shifted significantly toward y production. Even though the relative price of x experienced by consumers ($= p_x/p_y = 0.472/0.218 = 2.17$) has increased significantly from its value (of 1.43) in Example 13.4, the price ratio experienced by firms ($0.6p_x/p_y = 1.30$) has decreased somewhat from this prior value. Therefore, one might expect, based on a partial equilibrium analysis, that consumers would demand less of good x and likewise that firms would similarly produce less of that good. Partial equilibrium analysis would not, however, allow us to predict the increased production of y (which comes about because the relative price

[24] The computer program used to find these solutions is accessible at www.amherst.edu/~fwesthoff/compequ/FixedPoints CompEquApplet.html.

of y has decreased for consumers but has increased for firms) nor the reduction in relative input prices (because there is less being produced overall). A more complete picture of all these effects can be obtained by looking at the final equilibrium positions of the two households. The post-tax allocation for household 1 is

$$x_1 = 11.6,\ y_1 = 15.2,\ \bar{l}_1 - l_1 = 11.8,\ U_1 = 12.7; \tag{13.65}$$

for household 2,

$$x_2 = 6.3,\ y_2 = 13.6,\ \bar{l}_2 - l_2 = 7.9,\ U_2 = 8.96. \tag{13.66}$$

Hence imposition of the tax has made household 1 considerably worse off: utility decreases from 13.5 to 12.7. Household 2 is made slightly better off by this tax and transfer scheme, primarily because it receives a relatively large share of the tax proceeds that come mainly from household 1. Although total utility has decreased (as predicted by the simple partial equilibrium analysis of excess burden), general equilibrium analysis gives a more complete picture of the distributional consequences of the tax. Notice also that the total amount of labor supplied decreases as a result of the tax: total leisure increases from 15.1 (hours) to 19.7. Therefore, imposition of a tax on good x has had a relatively substantial labor supply effect that is completely invisible in a partial equilibrium model.

QUERY: Would it be possible to make both households better off (relative to Example 13.4) in this taxation scenario by changing how the tax revenues are redistributed?

Summary

This chapter has provided a general exploration of Adam Smith's conjectures about the efficiency properties of competitive markets. We began with a description of how to model many competitive markets simultaneously and then used that model to make a few statements about welfare. Some highlights of this chapter are listed here.

- Preferences and production technologies provide the building blocks on which all general equilibrium models are based. One particularly simple version of such a model uses individual preferences for two goods together with a concave production possibility frontier for those two goods.
- Competitive markets can establish equilibrium prices by making marginal adjustments in prices in response to information about the demand and supply for individual goods. Walras' law ties markets together so that such a solution is assured (in most cases).
- General equilibrium models can usually be solved by using computer algorithms. The resulting solutions yield many insights about the economy that are not obtainable from partial equilibrium analysis of single markets.
- Competitive prices will result in a Pareto-efficient allocation of resources. This is the first theorem of welfare economics.
- Factors that interfere with competitive markets' abilities to achieve efficiency include (1) market power, (2) externalities, (3) existence of public goods, and (4) imperfect information. We explore all these issues in detail in later chapters.
- Competitive markets need not yield equitable distributions of resources, especially when initial endowments are highly skewed. In theory, any desired distribution can be attained through competitive markets accompanied by appropriate transfers of initial endowments (the second theorem of welfare economics). But there are many practical problems in implementing such transfers.