PEF-5737
Non-linear Dynamics and Stability

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Lagrangian formulation

Holonomic constraints $\mathbf{R} = \mathbf{R}\left(q_1, q_2, ..., q_n, t\right)$ $\Rightarrow$ $\dot{\mathbf{R}} = \dfrac{\partial \mathbf{R}}{\partial q_i}\dot{q}_i + \dfrac{\partial \mathbf{R}}{\partial t}$

Kinetic energy

$$T = \frac{1}{2}\int_{\Omega} \rho\left(\dot{\mathbf{R}} \cdot \dot{\mathbf{R}}\right)d\Omega = \frac{1}{2}A^{ij}\dot{q}_i\dot{q}_j + B^i\dot{q}_i + \frac{1}{2}C$$

$$A^{ij}\left(q_1, q_2, ..., q_n, t\right) = \int_{\Omega} \rho\left(\frac{\partial \mathbf{R}}{\partial q_i} \cdot \frac{\partial \mathbf{R}}{\partial q_j}\right)d\Omega$$

$$B^i\left(q_1, q_2, ..., q_n, t\right) = \int_{\Omega} \rho\left(\frac{\partial \mathbf{R}}{\partial q_i} \cdot \frac{\partial \mathbf{R}}{\partial t}\right)d\Omega$$
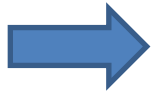
$$C\left(q_1, q_2, ..., q_n, t\right) = \int_{\Omega} \rho\left(\frac{\partial \mathbf{R}}{\partial t} \cdot \frac{\partial \mathbf{R}}{\partial t}\right)d\Omega$$

Total potential energy $V\left(q_1, q_2, ..., q_n, t\right)$

Lagrangiana formulation

Lagrange's equation
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = N_i$$

$$L = T\left(q_1, q_2, ..., q_n, \dot{q}_1, \dot{q}_2, ..., \dot{q}_n, t\right) - V\left(q_1, q_2, ..., q_n, t\right)$$

$$A^{ij}\ddot{q}_j + \left(\frac{\partial A^{ij}}{\partial q_k} - \frac{1}{2}\frac{\partial A^{jk}}{\partial q_i}\right)\dot{q}_j\dot{q}_k + \frac{\partial A^{ij}}{\partial t}\dot{q}_j$$

$$+ \left(\frac{\partial B^i}{\partial q_j} - \frac{\partial B^j}{\partial q_i}\right)\dot{q}_j + \frac{\partial B^i}{\partial t} - \frac{1}{2}\frac{\partial C}{\partial q_i} + \frac{\partial V}{\partial q_i} = N_i$$

Lagrangian formulation

Gyroscopic (Coriolis) $$\left(\frac{\partial B^i}{\partial q_j} - \frac{\partial B^j}{\partial q_i}\right)\dot{q}_j$$

$\downarrow$

"Damping" antisymmetric matrix

Holonomic scleronomic systems $$\mathbf{R} = \mathbf{R}\left(q_1, q_2, ..., q_n\right)$$

$$\frac{\partial A^{ij}}{\partial t} = 0; \quad B^i = 0; \quad C = 0 \implies A^{ij}\ddot{q}_j + \left(\frac{\partial A^{ij}}{\partial q_k} - \frac{1}{2}\frac{\partial A^{jk}}{\partial q_i}\right)\dot{q}_j\dot{q}_k + \frac{\partial V}{\partial q_i} = N_i$$

If the mass matrix is constant $$\frac{\partial A^{ij}}{\partial q_k} = 0 \implies A^{ij}\ddot{q}_j + \frac{\partial V}{\partial q_i} = N_i$$

Hamiltonian formulation

Legendre's transform $\quad f(x,y) \Rightarrow g(u,y) = ux - f(x,y)$

$$\text{with} \quad u = \frac{\partial f}{\partial x}$$

Remark $\quad df = \dfrac{\partial f}{\partial x} dx + \dfrac{\partial f}{\partial y} dy = u\, dx + \dfrac{\partial f}{\partial y} dy$

$$dg = \frac{\partial g}{\partial u} du + \frac{\partial g}{\partial y} dy$$

$$= x\,du + \overbrace{u\,dx - \frac{\partial f}{\partial x} dx}^{0} - \frac{\partial f}{\partial y} dy$$

$$= x\,du - \frac{\partial f}{\partial y} dy \Rightarrow x = \frac{\partial g}{\partial u} \quad \text{e} \quad \frac{\partial g}{\partial y} = -\frac{\partial f}{\partial y}$$

Hamiltonian formulation

Duality

"Old"

$$f(x,y) = ux - g(u,y)$$

$$u = \frac{\partial f}{\partial x}$$

"New"

$$g(u,y) = ux - f(x,y)$$

$$x = \frac{\partial g}{\partial u}$$

Application to the Lagrangian

$$L(\mathbf{q}, \dot{\mathbf{q}}, t) \implies H(\mathbf{q}, \mathbf{p}, t) = \mathbf{p}^{\mathrm{T}} \dot{\mathbf{q}} - L(\mathbf{q}, \dot{\mathbf{q}}, t)$$

$$\text{with} \quad p_i = \frac{\partial L}{\partial \dot{q}_i}$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \impliedby$$

Hamiltonian formulation

$$H(\mathbf{q},\mathbf{p},t) = \mathbf{p}^{\mathrm{T}}\dot{\mathbf{q}} - L\big(\mathbf{q},\dot{\mathbf{q}},t\big)$$

$$dH = \frac{\partial H}{\partial q_i}dq_i + \frac{\partial H}{\partial p_i}dp_i + \frac{\partial H}{\partial t}dt =$$

$$= \dot{q}_i dp_i + \underbrace{p_i d\dot{q}_i - \frac{\partial L}{\partial \dot{q}_i}d\dot{q}_i}_{0} - \frac{\partial L}{\partial q_i}dq_i - \frac{\partial L}{\partial t}dt$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

$$\frac{\partial H}{\partial q_i} = -\frac{\partial L}{\partial q_i}$$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

Hamiltonian formulation

Lagrange's equation with $\qquad N_i = 0$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \implies \dot{p}_i = \frac{\partial L}{\partial q_i} = -\frac{\partial H}{\partial q_i}$$

First-order system

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

Hamilton's canonical equations

$$\dot{H} = \frac{dH}{dt} = \frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial H}{\partial q_i} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$$

Hamiltonian formulation

Lagrange's equation with $\qquad N_i \neq 0$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = N_i \implies \dot{p}_i = N_i + \frac{\partial L}{\partial q_i} = N_i - \frac{\partial H}{\partial q_i}$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

Hamilton's canonical equations

$$\dot{H} = \frac{dH}{dt} = \frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} + \frac{\partial H}{\partial p_i}\left(N_i - \frac{\partial H}{\partial q_i}\right) + \frac{\partial H}{\partial t} = N_i\frac{\partial H}{\partial p_i} + \frac{\partial H}{\partial t}$$

Hamiltonian formulation

Conservative scleronomic system $\quad \mathbf{R} = \mathbf{R}(\mathbf{q}) \quad$ e $\quad N_i = 0$

$$H = H(\mathbf{p}, \mathbf{q}) \therefore \frac{\partial H}{\partial t} = 0 \Rightarrow \dot{H} = 0 \Rightarrow H = const.$$

$$L = T - V = \frac{1}{2} A^{ij} \dot{q}_i \dot{q}_j - V(\mathbf{q}) \Rightarrow \frac{\partial L}{\partial \dot{q}_j} = A^{ij} \dot{q}_i$$

$$H(\mathbf{q}, \mathbf{p}) = \mathbf{p}^{\mathrm{T}} \dot{\mathbf{q}} - L(\mathbf{q}, \dot{\mathbf{q}}) = p_j \dot{q}_j - L = \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j - L$$

$$= A^{ij} \dot{q}_i \dot{q}_j - \frac{1}{2} A^{ij} \dot{q}_i \dot{q}_j + V = \frac{1}{2} A^{ij} \dot{q}_i \dot{q}_j + V$$

$$= T + V$$

Hamiltonian formulation

Reonomic system $\quad \mathbf{R} = \mathbf{R}(\mathbf{q}, t) \quad$ with $\quad N_i \neq 0$

$$L = T - V = \frac{1}{2} A^{ij} \dot{q}_i \dot{q}_j + B^i \dot{q}_i + \frac{1}{2} C - V(\mathbf{q}) \Rightarrow \frac{\partial L}{\partial \dot{q}_j} = A^{ij} \dot{q}_i + B^j$$

$$H(\mathbf{q}, \mathbf{p}) = \mathbf{p}^{\mathbf{T}} \dot{\mathbf{q}} - L(\mathbf{q}, \dot{\mathbf{q}}) = p_j \dot{q}_j - L = \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j - L$$

$$= A^{ij} \dot{q}_i \dot{q}_j + B^j \dot{q}_j - \frac{1}{2} A^{ij} \dot{q}_i \dot{q}_j - B^i \dot{q}_i - \frac{1}{2} C + V =$$

$$= \quad T + V \quad - \quad C \quad - \quad B^j \dot{q}_j$$

$\quad$ energia mecânica $\quad$ energia pseudo-potencial $\quad$ 'energia' Coriolis