PEF-5737
Non-Linear Dynamics and Stability

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Equations of motion for systems of material points

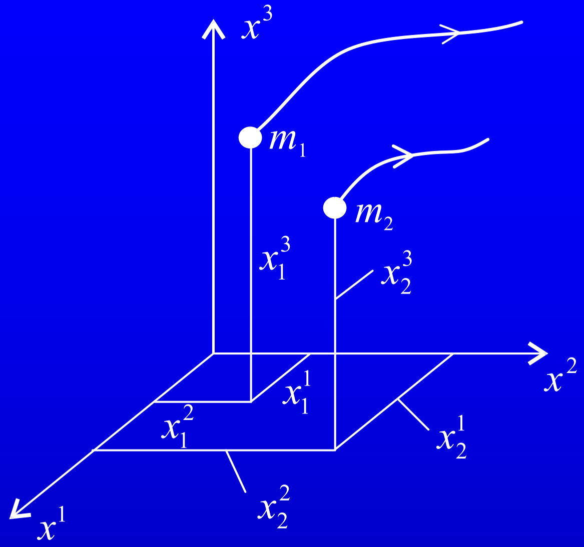
- Newton's laws

1st law (inertia): there are special observers (called inertial observers) with respect to whom isolated material points (that is, those subjected to null resultant force) are at rest or perform URM (uniform rectilinear motion).

2nd law (fundamental): the resultant force acting on a material point is proportional to its acceleration with respect to an inertial observer ($\vec{F} = m\,\vec{a}$). The constant of proportionality is a material point property called mass, $m>0$

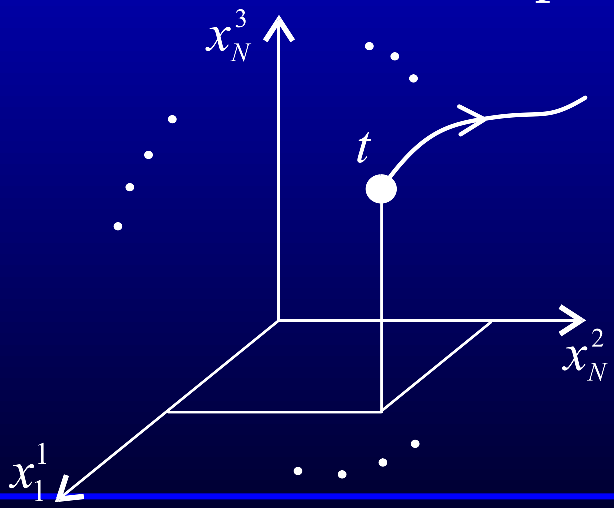
3rd law (action and reaction): to every action of a material point over another one it corresponds a reaction of same intensity and direction, although in opposite sense.

Physical space: afine Euclidian space of dimension 3.



- N material points m_i
- position of m_i caracterized by 3 coordinates: x_i^1, x_i^2, x_i^3

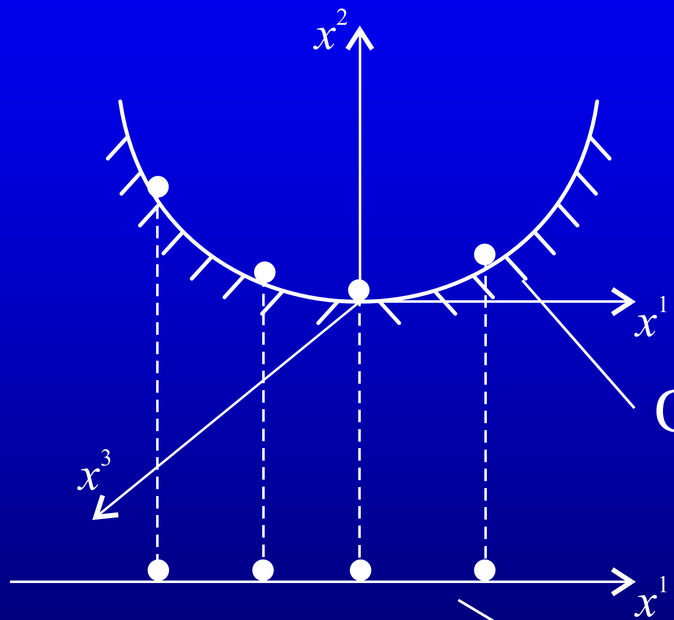
Configuration space: afine Euclidian space of dimension 3N (if the 3N coordinates are independent).



- a “point” in this space fully characterizes the material point system in time t (coordinates of material points obtained by “projections”)

• if there are <u>c</u> constraint equations relating the coordinates it is possible to use a configuration space os dimension $n = 3N\text{-}c$ (called number of "degrees of freedom")

Example: a material point moving on a parabole



$$\left.\begin{aligned} x^3 &= 0 \\ x^2 &= \beta\,(x^1)^2 \end{aligned}\right\}$$

$c = 2$ constraint equations

Configuration space of dim $3N = 3$

Configuration space of dim $n = 3N\text{-}c$

$n = 1$

- Generalized coordinates $Q_1(t)$, $Q_2(t)$, ... , $Q_n(t)$, where n = number of degrees of freedom, are conveniently chosen scalar quantities that allow for stablishing a bi-univocal relationship with the $3N$ coordinates of the material point system.

$$\left.\begin{aligned}
x_1^1 &= x_1^1\left(Q_1, Q_2, \ldots, Q_n, t\right) \\
x_1^2 &= x_1^2\left(Q_1, Q_2, \ldots, Q_n, t\right) \\
&\cdots \\
x_N^3 &= x_N^3\left(Q_1, Q_2, \ldots, Q_n, t\right)
\end{aligned}\right\}$$

$3N$ holonomic (reonomic) constraint equations

- the functions $x_\alpha^i\left(Q_1, Q_2, \ldots, Q_n, t\right)$ are finite of class C^1
- Transformation matrix
- Non-null Jacobian

- Particular case of holonomic constraint: scleronomic constraint

$$x_\alpha^i = x_\alpha^i\left(Q_1, Q_2, \ldots, Q_n\right)$$

Example: a material point moving on a parabole

$$x^1 = Q$$

$$x^2 = \beta\left(x^1\right)^2 = \beta\, Q^2$$

$$x^3 = 0$$

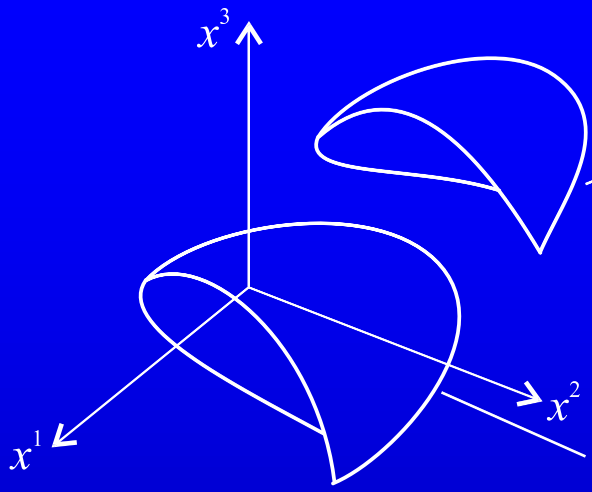
$$\frac{\partial x^1}{\partial Q} = 1$$

∃ a transformation "matrix" T (of order $n = 1$) with det $T \neq 0$

Let it be $T = \left[\dfrac{\partial x^1}{\partial Q}\right]$

J= det $T = 1$

Virtual displacements – holonomic constraint

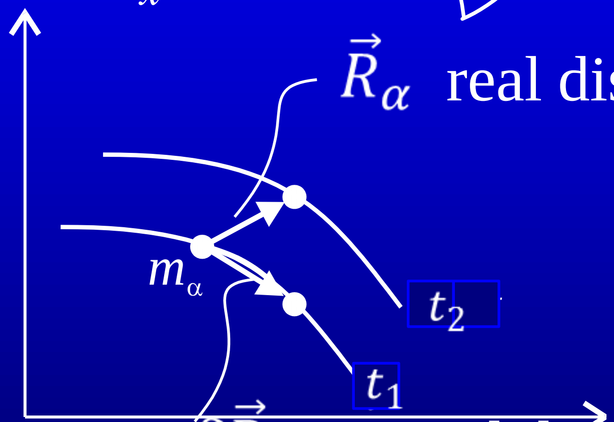


$$f_2\left(x^1, x^2, x^3, t_2\right) = 0$$

constraint equation in t_2

$$f_1\left(x^1, x^2, x^3, t_1\right) = 0$$

constraint equation in t_1



$\vec{R}_\alpha$ real displacement

$\delta\vec{R}_\alpha$ virtual displacement

$$\Updownarrow$$

$$\delta x^i_\alpha$$

$$\Updownarrow$$

$$\delta Q_j$$

- Virtual displacements are kinematically admissible displacements in a fixed time <u>t</u>, that is, satisfy the constraint equations for that time <u>t</u>.

- The class of real displacements do not necessarily coincide with that of the virtual displacements for holonomic constraints.

- However, for scleronomic constraints, since the constraint equations are independent of <u>t</u>, the class of real displacements coincides with that of virtual displacements, that is, the real displacements are a particular case of the virtual displacements.

- Ideal constraint reactions are orthogonal to the respective virtual displacements. Hence, ideal constraint reactions do not give rise to virtual work:

$$\delta W = \vec{F}_\alpha^{\,vi} \,.\, \delta \vec{R}_\alpha = 0$$

D'Alembert's Principle

2nd Newton's law $\vec{F}_\alpha = m_\alpha \dfrac{d^2\vec{R}_\alpha}{dt^2} \quad \alpha = 1 \text{ a } N$

$$\vec{F}_\alpha - m_\alpha \frac{d^2\vec{R}_\alpha}{dt^2} = \vec{0} \quad \alpha = 1 \text{ a } N$$

$$\vec{F}_\alpha = \underbrace{\vec{F}_\alpha^{\,a}}_{\text{active}} + \underbrace{\vec{F}_\alpha^{\,vi}}_{\text{ideal constraint}} + \underbrace{\vec{F}_\alpha^{\,vn}}_{\text{non-ideal constraint}} = \text{ resultant force}$$

$$\vec{F}_\alpha^{\,I} = -m_\alpha \frac{d^2\vec{R}_\alpha}{dt^2} = \text{ inertial force}$$

The sum of the resultant and the inertial forces is the null vector

("closing" of force polygon, as in statics)

$$\sum_{\alpha=1}^{N} \left(\vec{F}_\alpha^{\,a} + \vec{F}_\alpha^{\,vi} + \vec{F}_\alpha^{\,vn} + \vec{F}_\alpha^{\,I} \right) . \delta\vec{R}_\alpha = \sum_{\alpha=1}^{N} \vec{0} \,.\, \delta\vec{R}_\alpha = 0$$

$$\boxed{\sum_{\alpha=1}^{N} \left(\vec{F}_\alpha^{\,a} + \vec{F}_\alpha^{\,vn} + \vec{F}_\alpha^{\,I} \right) . \delta\vec{R}_\alpha = 0}$$

Generalized D'Alembert's Principle

- Remark 1 Effective force $\vec{F}_{\alpha}^{e} = \vec{F}_{\alpha}^{a} + \vec{F}_{\alpha}^{vn}$

- Remark 2 System of ideal constraints, only:

$$\sum_{\alpha=1}^{N} \left(\vec{F}_{\alpha}^{a} + \vec{F}_{\alpha}^{I}\right).\delta\vec{R}_{\alpha} = 0$$

(it is not necessary to know the constraint forces to formulate the equations of statics/dynamics)

- Remark 3 Principle of virtual displacements in statics is a particular case:

$$\sum_{\alpha=1}^{N} \vec{F}_{\alpha}^{a}.\delta\vec{R}_{\alpha} = 0 \Longleftrightarrow \text{equilibrium}$$

Hamilton's Principle

2nd Newton's law $\Longleftrightarrow$ D'Alembert's Principle $\Longleftrightarrow$ Hamilton's Principle

$$\int_{t_1}^{t_2}\left(\delta T - \delta V + \delta W^{nc}\right)dt = 0$$

δT = variation of kinetic energy

δV = variation of potential energy

δW^{nc} = virtual work of non-conservative forces

$$T = \text{kinetic energy} = \frac{1}{2}\sum_{\alpha=1}^{N} m_\alpha \left(\frac{d\vec{R}_\alpha}{dt}\cdot\frac{d\vec{R}_\alpha}{dt}\right)$$

$$\delta T = \sum_{\alpha=1}^{N} m_\alpha \left(\dot{\vec{R}}_\alpha . \delta \dot{\vec{R}}_\alpha\right)$$

notation $\dot{(\ )} = \frac{d}{dt}(\ )$

$$\delta V = -\sum_{\alpha=1}^{N} \vec{F}^{\,c}_\alpha . \delta \vec{R}_\alpha = - \boxed{\text{virtual work of conservative forces}}$$

$$\delta W^{nc} = \sum_{\alpha=1}^{N} \vec{F}^{\,nc}_\alpha . \delta \vec{R}_\alpha = \boxed{\text{virtual work of non-conservative forces}}$$

Lagrange's Equations

Hamilton's Principle $\Longleftrightarrow$ Lagrange's Equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{Q}_i}\right)-\frac{\partial T}{\partial Q_i}=-\frac{\partial V}{\partial Q_i}+N_i$$

$T=T\left(Q_1,Q_2,\ldots,Q_n,\dot{Q}_1,\dot{Q}_2,\ldots,\dot{Q}_n,t\right)$ kinetic energy

$V=V\left(Q_1,Q_2,\ldots,Q_n,t\right)$ potential energy

$N_i =$ generalized non-conservative force $=\sum_{\alpha=1}^{N}\vec{F}_\alpha^{\,nc}\cdot\frac{\partial \vec{R}_\alpha}{\partial Q_i}$

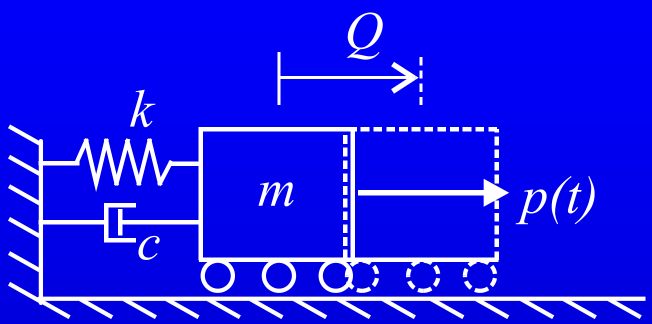
- Remark

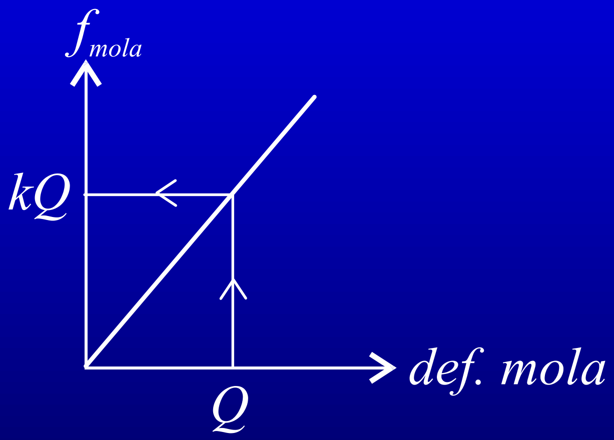
$$\sum_{i=1}^{n} N_i\delta Q_i=\sum_{\alpha=1}^{N}\vec{F}_\alpha^{\,nc}\cdot\delta\vec{R}_\alpha=\delta W^{nc}$$

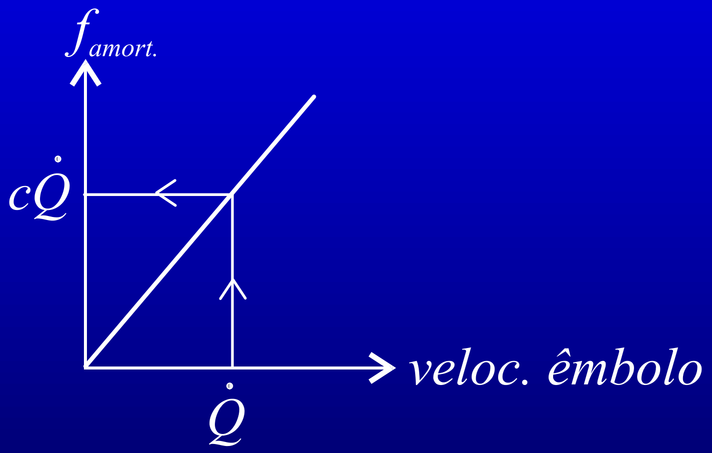
virtual work of non-conservative forces

Formulation of equations of motion

Example 1: One-degree-of-freedom linear oscillator







2ª Newton's law:

$$p(t) - kQ - c\dot{Q} = m\ddot{Q}$$

D'Alembert's Principle:

$$p(t) - kQ - c\dot{Q} - m\ddot{Q} = 0$$

$$\boxed{m\ddot{Q} + c\dot{Q} + kQ = p(t)}$$

Generalized D'Alembert's Principle:

$$\left[p(t) - kQ - c\dot{Q} - m\ddot{Q}\right]\delta Q = 0 \ \forall \ \delta Q$$

Hamilton's Principle:

$$\int_{t_1}^{t_2} \left(\delta T - \delta V + \delta W^{nc}\right) dt = 0$$

$$T = \frac{1}{2} m\dot{Q}^2 \quad \Longrightarrow \quad \delta T = m\dot{Q}\,\delta\dot{Q}$$

$$V = \frac{1}{2} kQ^2 \quad \Longrightarrow \quad \delta V = kQ\,\delta Q$$

$$\delta W^{nc} = N\,\delta Q = \left(p(t) - c\dot{Q}\right)\delta Q$$

Substituting: $$\int_{t_1}^{t_2} m\dot{Q}\,\delta\dot{Q}\,dt + \int_{t_1}^{t_2} \left[-kQ + p(t) - c\dot{Q}\right]\delta Q\,dt = 0$$

$$\Big\downarrow \text{ integrating by parts}$$

$$\delta Q(t_1) = \delta Q(t_2) = 0 \quad \Longrightarrow \quad m\dot{Q}\,\underbrace{\cancel{\delta Q}\Big|_{t_1}^{t_2}}_{0} - \int_{t_1}^{t_2} \left[m\ddot{Q} + c\dot{Q} + kQ - p(t)\right]\delta Q\,dt = 0$$

Hence, $$\int_{t_1}^{t_2} \left[m\ddot{Q} + c\dot{Q} + kQ - p(t)\right]\delta Q\,dt = 0 \quad \forall\,\delta Q$$

$$\boxed{m\ddot{Q} + c\dot{Q} + kQ = p(t)}$$

Lagrange's equation:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{Q}}\right)-\frac{\partial T}{\partial Q}=-\frac{\partial V}{\partial Q}+N$$

$$T=\frac{1}{2}m\dot{Q}^2 \quad \Longrightarrow \quad \frac{\partial T}{\partial \dot{Q}}=m\dot{Q} \quad ; \quad \frac{\partial T}{\partial Q}=0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{Q}}\right)=m\ddot{Q}$$

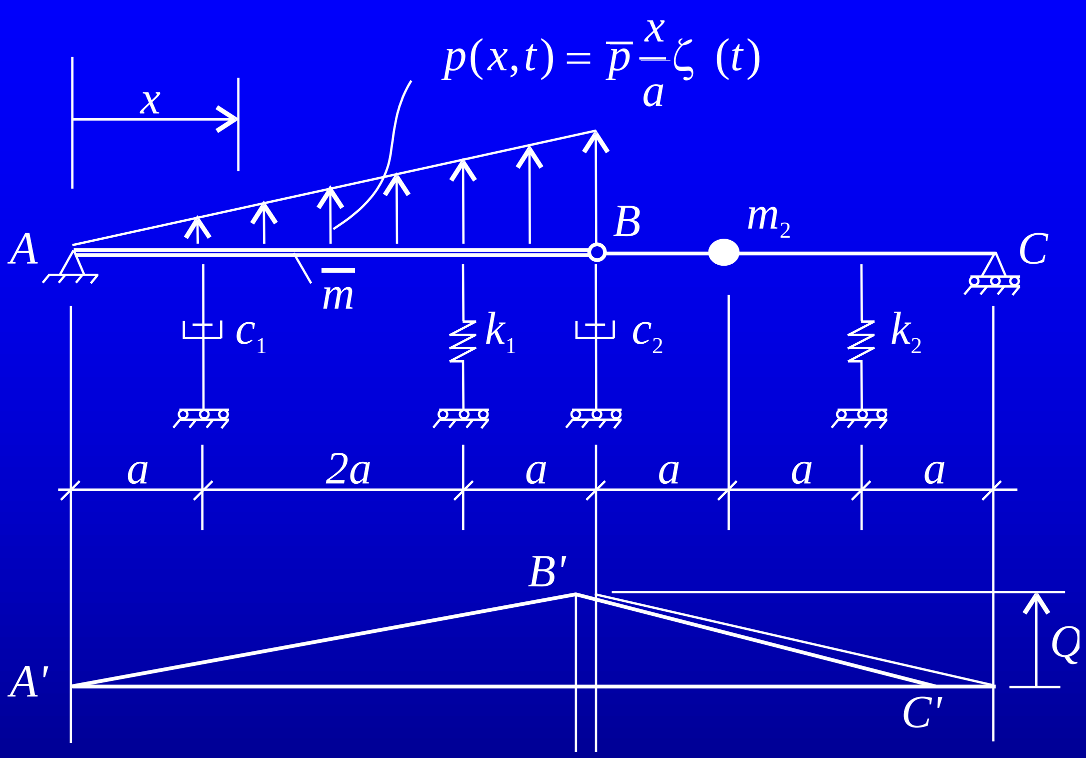
$$V=\frac{1}{2}kQ^2 \quad \Longrightarrow \quad \frac{\partial V}{\partial Q}=kQ$$

$$\delta W^{nc}=N\delta Q=\left(p(t)-c\dot{Q}\right)\delta Q \quad \Longrightarrow \quad N=p(t)-c\dot{Q}$$

Substituting: $m\ddot{Q}=-kQ+p(t)-c\dot{Q}$

$$\boxed{m\ddot{Q}+c\dot{Q}+kQ=p(t)}$$

Example 2



AB e *BC* rigid bars

BC massless bar

Linear dynamics: horizontal displacements at B and C are neglected, which is reasonable for small Q.

$$T=\frac{1}{2}m_2\left(\frac{2}{3}\dot{Q}\right)^2+\int_0^{4a}\frac{1}{2}\overline{m}\left(\frac{x}{4a}\dot{Q}\right)^2dx=\frac{1}{2}m^*\dot{Q}^2$$

with $m^*=\frac{4}{9}m_2+\frac{4}{3}\overline{m}a$

$$V=\frac{1}{2}k_1\left(\frac{3}{4}Q\right)^2+\frac{1}{2}k_2\left(\frac{1}{3}Q\right)^2+\int_0^{4a}\overline{m}g\left(\frac{x}{4a}Q\right)dx+m_2g\left(\frac{2}{3}Q\right)$$

$$V=\frac{1}{2}k^*Q^2-p_0^*Q$$

with $k^*=\frac{9}{16}k_1+\frac{1}{9}k_2$ e $p_0^*=-\left(2\overline{m}a+\frac{2}{3}m_2\right)g$

$$\delta W^{nc} = -c_1 \frac{\dot{Q}}{4}\frac{\delta Q}{4} - c_2 \dot{Q}\,\delta Q + \int_0^{4a} \bar{p}\,\frac{x}{a}\,\zeta(t)\left(\frac{x}{4a}\delta Q\right)dx = N\delta Q$$

$$N = -c^{*}\dot{Q} + p^{*}(t)$$

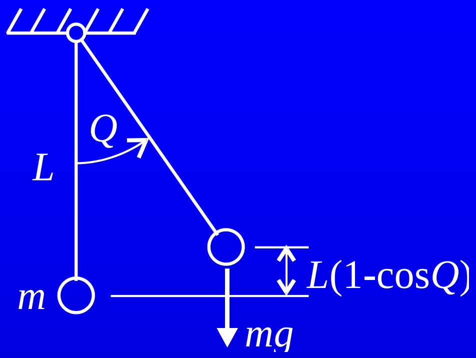
with $c^{*} = \dfrac{c_1}{16} + c_2$ and $p^{*}(t) = \dfrac{16}{3}\bar{p}\,a\,\zeta(t)$

Lagrange's equation:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{Q}}\right) - \frac{\partial T}{\partial Q} = -\frac{\partial V}{\partial Q} + N$$

$$m^{*}\ddot{Q} + c^{*}\dot{Q} + k^{*}Q = p_0^{*} + p^{*}(t)$$

Example 3: Simple pendulum



$$T = \frac{1}{2} m \left(L\dot{Q} \right)^2$$

$$V = +mgL\left(1 - \cos Q\right)$$

Lagrange's equation:

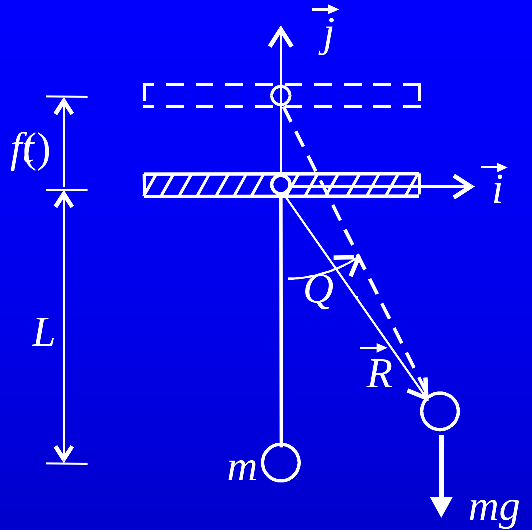
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{Q}}\right) - \underbrace{\cancel{\frac{\partial T}{\partial Q}}}_{0} = -\frac{\partial V}{\partial Q} + \underbrace{\cancel{N}}_{0}$$

$$mL^2\ddot{Q} = -mgL \operatorname{sen} Q$$

$$\ddot{Q} + \frac{g}{L} \operatorname{sen} Q = 0$$ (non-linear)

$$\ddot{Q} + \frac{g}{L} Q = 0$$ (linear)

Example 4: Simple pendulum subjected to support excitation



$$\vec{R} = L \operatorname{sen} Q\,\vec{i} + (f - L\cos Q)\,\vec{j}$$

$$\dot{\vec{R}} = L\dot{Q}\cos Q\,\vec{i} + (\dot{f} + L\dot{Q}\operatorname{sen} Q)\,\vec{j}$$

$$V = mg\left[f + L\left(1 - \cos Q\right)\right]$$

$$T = \frac{1}{2}m\left(L^2\dot{Q}^2\cos^2 Q + L^2\dot{Q}^2\operatorname{sen}^2 Q + 2L\dot{f}\dot{Q}\operatorname{sen} Q + \dot{f}^2\right)$$

$$T = \frac{1}{2}mL^2\dot{Q}^2 + \frac{1}{2}m\dot{f}^2 + mL\dot{f}\dot{Q}\operatorname{sen} Q$$

Lagrange’s equation:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{Q}}\right)-\frac{\partial T}{\partial Q}=-\frac{\partial V}{\partial Q}+\cancelto{0}{N}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{Q}}\right)= mL^2\ddot{Q}+mL\ddot{f}\,\mathrm{sen}\,Q+mL\dot{f}\,\dot{Q}\cos Q$$

$$\frac{\partial T}{\partial Q}= mL\dot{f}\,\dot{Q}\cos Q \qquad\qquad \frac{\partial V}{\partial Q}= mgL\,\mathrm{sen}\,Q$$

Substituting:

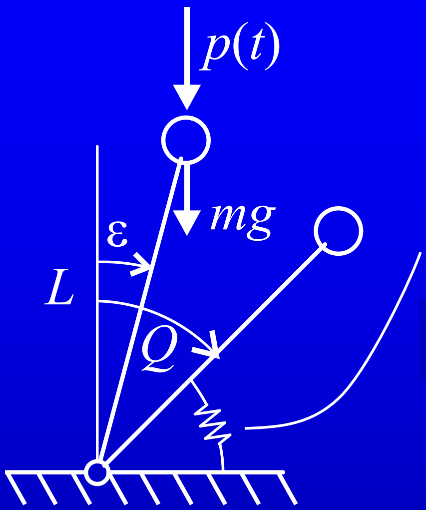
$$mL^2\ddot{Q}+mL\ddot{f}\,\mathrm{sen}\,Q+mL\dot{f}\cancel{\dot{Q}}\cos Q-mL\dot{f}\cancel{\dot{Q}}\cos Q=-mgL\,\mathrm{sen}\,Q$$

$$mL^2\ddot{Q}+mL(g+\ddot{f})\,\mathrm{sen}\,Q=0$$

ou $$\ddot{Q}+\frac{1}{L}(g+\ddot{f})\,\mathrm{sen}\,Q=0$$ (non-linear)

$$\ddot{Q}+\frac{1}{L}(g+\ddot{f})Q=0$$ (linear)

Example 5: Rigid bar with non-linear rotational spring and imperfection, subjected to dynamical load $p(t)$ and static pre-loading mg



$$M(Q) = K(Q-\varepsilon)\left[1-(Q-\varepsilon)^2\right]$$ Non-linear "constitutive" law

imperfection $\varepsilon <<< 1$

$$T = \frac{1}{2} m L^2 \dot{Q}^2$$

$$V = \int_0^{Q-\varepsilon} K\theta\left[1-\theta^2\right] d\theta - mgL(\cos\varepsilon - \cos Q) = K\left[\frac{(Q-\varepsilon)^2}{2} - \frac{(Q-\varepsilon)^4}{4}\right] - mgL(\cos\varepsilon - \cos Q)$$

$$\delta W^{nc} = N\,\delta Q = P(t)L\,\mathrm{sen}\,Q\,\delta Q \Longrightarrow N = P(t)L\,\mathrm{sen}\,Q$$

Lagrange's equation:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{Q}}\right) - \frac{\partial T}{\partial Q} = -\frac{\partial V}{\partial Q} + N$$

$$mL^2\ddot{Q} = -K(Q-\varepsilon)\left[1-(Q-\varepsilon)^2\right] + \left[mg + P(t)\right]L\,\mathrm{sen}\,Q$$

$$mL^2\ddot{Q} + K(Q-\varepsilon)\left[1-(Q-\varepsilon)^2\right] = \left[mg + P(t)\right]L\,\mathrm{sen}\,Q$$