Lecture 1-c

Catenary Risers: Global Analysis

Linear Vibration Modes

Analytical approaches

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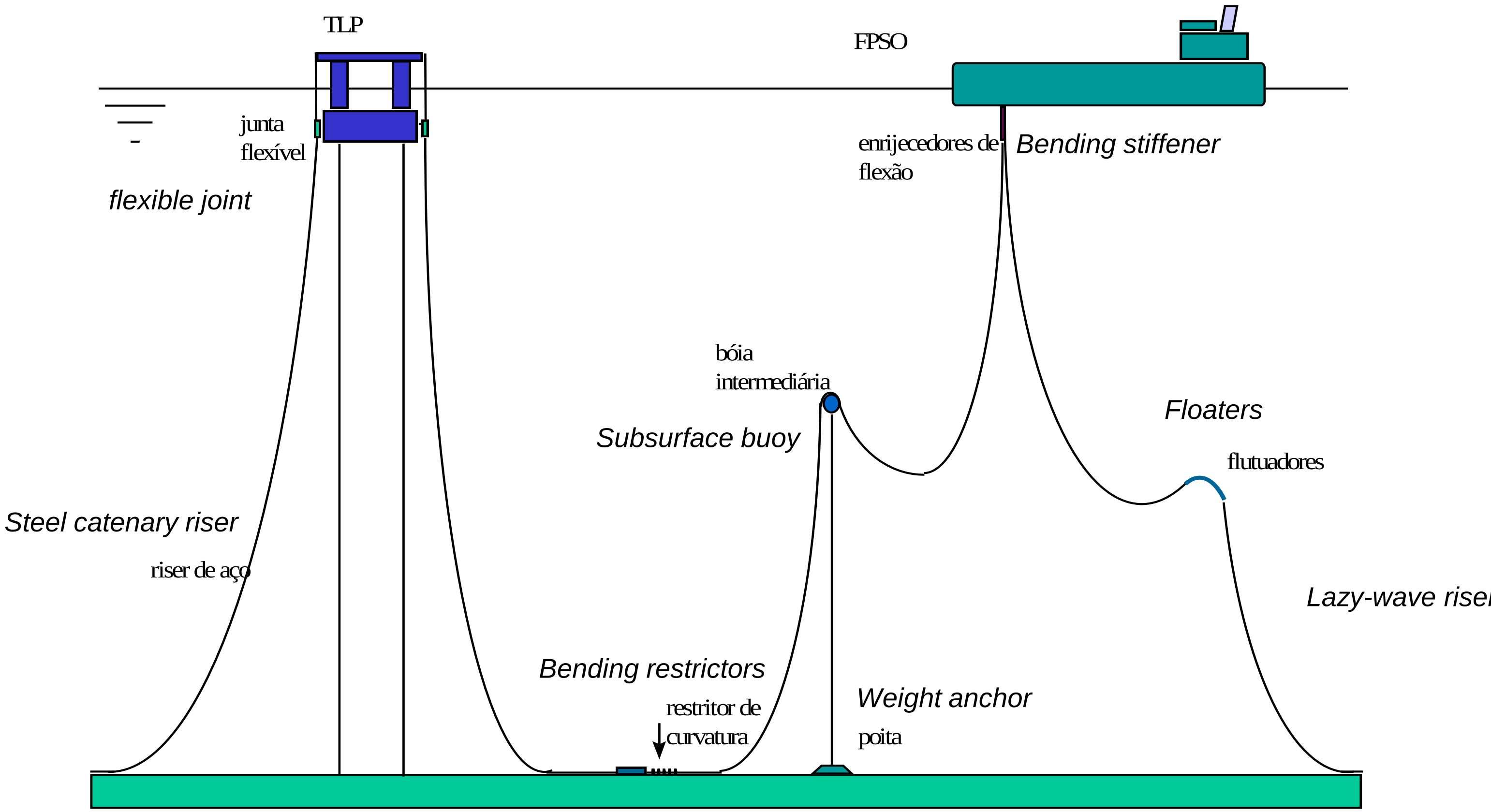
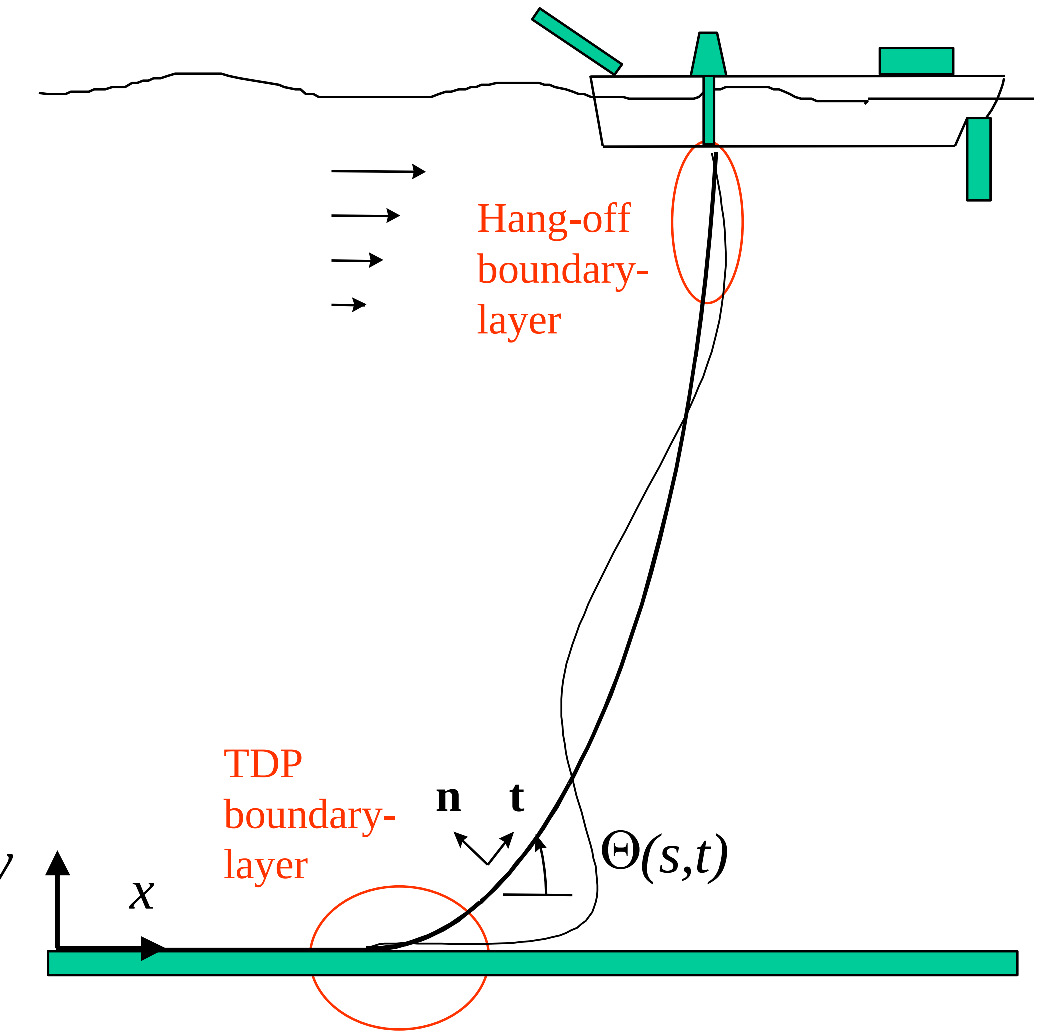
Catenary lines



The dynamic problem



- Global dynamics governed by catenary rigidity.
- Bending stiffness effetcs are importante at the extremities and TDP, or for high-order vibration modes for which the mode vibrattion length is of same order of the local flexural length.
- There are several time scales that govern the overall dynamics of a riser.

Classic dynamic analysis

- Extensible cable static equilibrium solution under current.
- Linearized dynamic solution in frequency domain or in time domain – extensible cable.
- Bending stiffness effects can be accounted for *a posteriori*, through boundary layer techniques at Top and TDZ.
- **Vibration modes may be determined numerically or assessed analytically.**

Recalling.. the bending stiffness effect

Flexural lenght at TDP

Defining:

$$\lambda = \sqrt{\frac{EI}{T_0}}$$

Small length parameter

$$\varepsilon = \frac{\lambda}{L}$$

Static curvature at TDP

$$\chi_0 = \frac{q}{T_0}$$

Transversal wave celerity due to geometric stiffness, at TDP

$$\hat{t} = \frac{c_0}{L} t$$

$$c_0 = \sqrt{\frac{T_0}{m + m_a}}$$

Second-order term

The nondimensional dynamic equation in the normal direction reads:

$$-\varepsilon^2 \frac{\partial^2 \hat{\chi}}{\partial \hat{s}^2} + \hat{T}\hat{\chi} + \left(\frac{\partial \hat{T}}{\partial \hat{s}} - \varepsilon^2 \frac{d\theta}{d\hat{s}} \frac{\partial \hat{\chi}}{\partial \hat{s}} \right) \gamma + \hat{c}_n + \hat{\varpi}_n - \hat{\chi}_0 \cos\theta = \frac{\partial^2 \hat{u}_n}{\partial \hat{t}^2}$$

Recalling.. the bending stiffness effect

Rigid risers (Steel): $\varepsilon \approx O(10^{-2})$

Flexible risers: $\varepsilon \approx O(10^{-3})$

Neglecting secon-order terms (bending stiffness):

$$\hat{\mathrm{T}}\hat{\chi} + \frac{\partial \hat{\mathrm{T}}}{\partial \hat{s}}\gamma + \hat{c}_n + \hat{\varpi}_n - \hat{\chi}_0 \cos\theta = \frac{\partial^2 \hat{u}_n}{\partial \hat{t}^2}(1 + O(\varepsilon^2))$$

Time scales

$$t_1 = \frac{L}{c_g}$$

$$t_2 = \frac{\lambda}{c_g}$$

$$t_3^{(i)} = \frac{L}{c_f^{(i)}}$$

$$t_4^{(i)} = \frac{\lambda}{c_f^{(i)}}$$

$$t_5 = \frac{L}{c_a}$$

Wave celerity associated to geometric rigidity

$$\bar{c}_g = \sqrt{\frac{\bar{T}}{(m+m_a)}}$$

Wave celerity associated to bending stiffness

$$\bar{c}_f^{(i)} = \frac{2\pi}{\bar{\lambda}_f^{(i)}} \sqrt{\frac{EI}{(m+m_a)}}$$

Wave celerity associated to axial stiffness

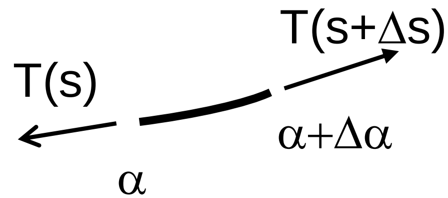
$$c_a = \frac{EA}{m}$$

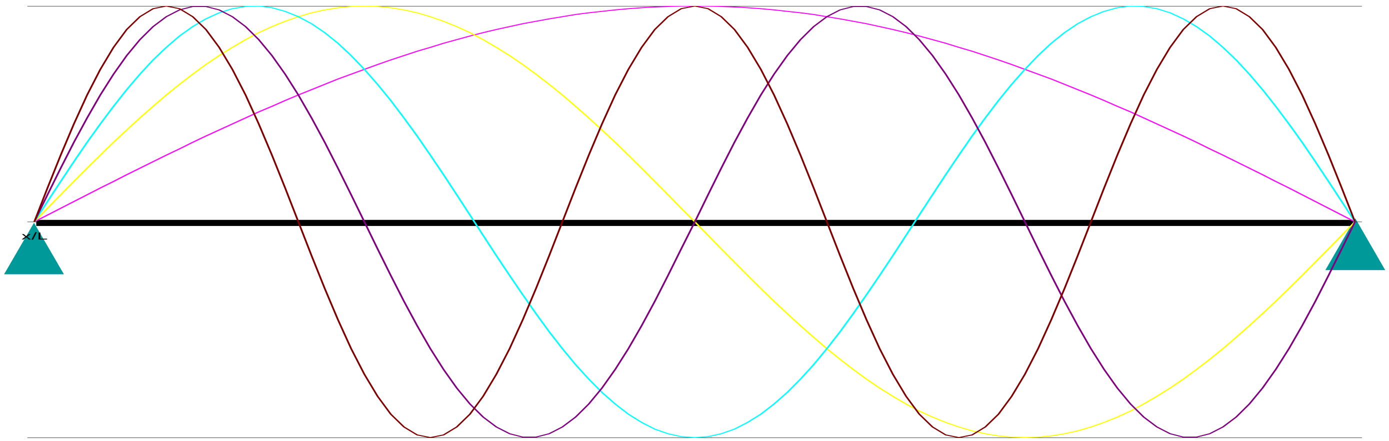
Local flexural length

$$\lambda(s) = \sqrt{\frac{EI}{T(s)}}$$

The classic vibrating string problem

Inextensible string





$$\alpha(x) << 1 \Rightarrow s \approx x \text{ and } \tan\alpha \cong \sin\alpha \cong \frac{dy}{dx}$$

$$\mu(x)\Delta x \frac{\partial^2 y}{\partial t^2} = T(x+\Delta x)\sin(\alpha(x+\Delta x)) - T(x)\sin(\alpha(x))$$

$$\Delta x \to 0 \quad \Rightarrow \quad \boxed{\mu(x)\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}\left(T(x)\frac{\partial y}{\partial x}\right)}$$

The classic vibrating string problem

$$T(x) = T_0 \ \text{ e } \ \mu(x) = \mu = \text{const} \Rightarrow$$

$$\mu \frac{\partial^2 y}{\partial t^2} = T_0 \frac{\partial^2 y}{\partial x^2}$$ or $$\frac{\partial^2 y}{\partial t^2} = c_0^{\ 2} \frac{\partial^2 y}{\partial x^2}$$ with $$c_0 = \sqrt{\frac{T_0}{\mu}}$$

Solution $$y_n(x,t) = \varphi_n(x) e^{i\Omega_n t}$$

$$\frac{\partial^2 y_n}{\partial t^2} = -\Omega_n^{\ 2} \varphi_n(x) e^{i\Omega_n t} \ \text{ e } \ \frac{\partial^2 y_n}{\partial x^2} = \varphi_n'' e^{i\Omega_n t} \Longrightarrow \left(c_0^{\ 2} \varphi_n'' + \Omega_n^{\ 2} \varphi_n(x) \right) e^{i\Omega_n t} = 0$$

Leading to $$\left(c_0^{\ 2} \varphi_n'' + \Omega_n^{\ 2} \varphi_n(x) \right) = 0$$

Solution of the form: $$\varphi_n(x) = A_n e^{ik_n x} \Rightarrow \left(-k_n^{\ 2} + \frac{\Omega_n^{\ 2}}{c_0^{\ 2}} \right) \varphi_n(x) = 0$$

The classic vibrating string problem

Characteristic equation: $$\left(-k_n^{\,2}+\frac{\Omega_n^{\,2}}{c_0^{\,2}}\right)=0$$

Boundary conditions: $$[\varphi_n]_{x=0}=[\varphi_n]_{x=L}\equiv 0 \Rightarrow k_n L = n\pi$$

$$\therefore k_n=\frac{n\pi}{L}$$

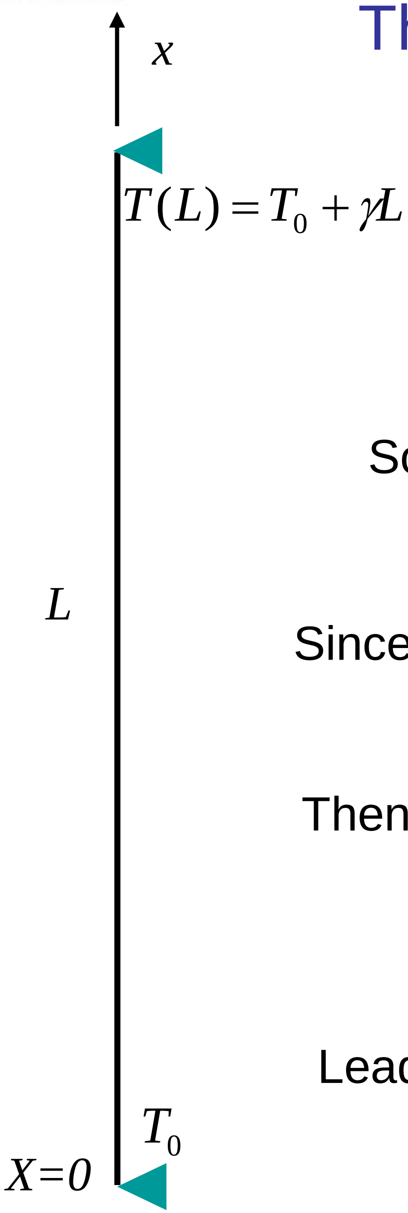
Therefore:

$$\Omega_n=c_0k_n=nc_0\frac{\pi}{L}=n\frac{\pi}{L}\sqrt{\frac{T_0}{\mu}}$$

$$\varphi_n(x)=A_n\sin(k_nx)=A_n\sin(n\pi\frac{x}{L})$$

$$\rightarrow\quad y(x,t)=\sum_{n=1}^{\infty}\left[C_n\varphi_n(x)e^{i(\Omega_nt+\theta_n)}\right]$$

The vertical vibrating string problem



$$\mu(x)\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}\left(T(x)\frac{\partial y}{\partial x}\right)$$

with, $\mu(x) = m(x) + m_a(x)$

$$T(x) = T_0 + \int_0^x \gamma(x)dx$$

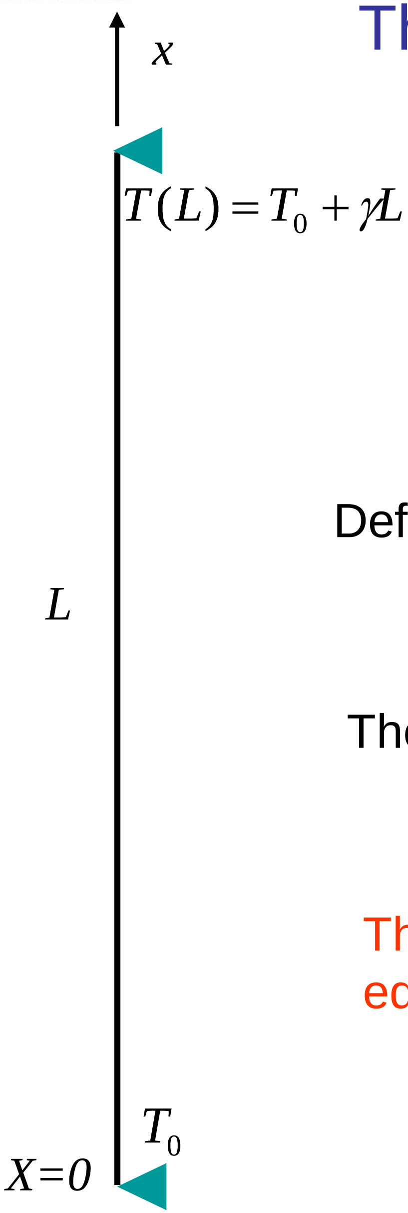
Solution of the form: $y(x,t) = \varphi(x)\eta(t)$

Since: $\frac{\partial^2 y}{\partial t^2} = \varphi\ddot{\eta}$ and $\frac{\partial y}{\partial x} = \varphi'\eta$ $\longrightarrow$ $\frac{\ddot{\eta}}{\eta} = \frac{1}{\mu\varphi}\frac{d}{dx}\left(T(x)\varphi'\right) = -\Omega^2$

Then: $\ddot{\eta} + \Omega^2\eta = 0$ $\longrightarrow$ $\eta_n(t) = e^{i(\Omega_n t + \theta_n)}$

Leading to: $$\frac{d}{dx}\left(T(x)\varphi'\right) + \mu(x)\Omega^2\varphi = 0$$

The vertical vibrating string problem



$$\gamma(x) = \gamma = \text{cte} \Rightarrow T(x) = T_0 + \gamma x$$

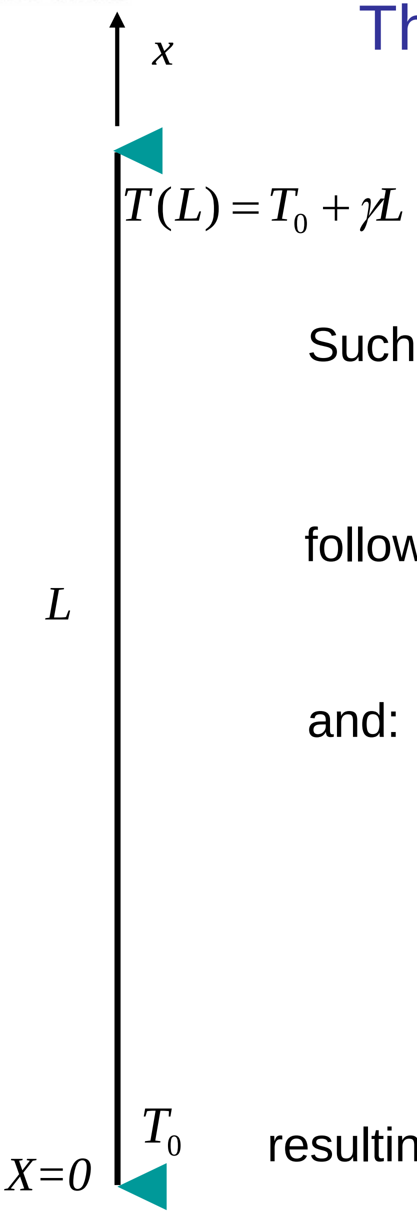
$$\mu(x) = \mu = \text{cte} \Rightarrow \boxed{(T_0 + \gamma x)\varphi'' + \gamma\varphi' + \mu\Omega^2\varphi = 0}$$

Defining: $\xi = \dfrac{T_0}{\gamma} + x$ e $\beta^2 = \dfrac{\mu}{\gamma}\Omega^2$

Then: $\boxed{\xi\varphi'' + \varphi' + \beta^2\varphi = 0}$

This equation can be transformed into a modified Bessel equation, with known solutions.

The vertical vibrating string problem



In fact, defining a new variable: $\xi = \zeta^2$

Such that: $$d\zeta = \frac{1}{2}\xi^{-1/2}d\xi = \frac{1}{2}\frac{1}{\zeta}d\xi$$

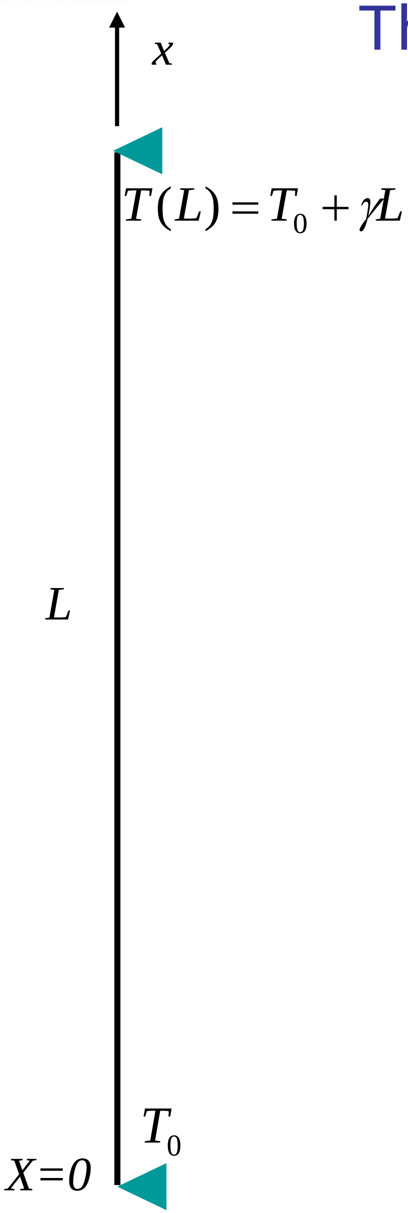
follows: $$\frac{d\varphi}{d\xi} = \frac{d}{d\zeta}[\varphi(\xi(\zeta))]\frac{d\zeta}{d\xi} = \frac{1}{2\zeta}\frac{d\varphi}{d\zeta}$$

and: $$\frac{d^2\varphi}{d\xi^2} = \frac{d}{d\xi}\left[\frac{d\varphi}{d\xi}\right] = \frac{d}{d\zeta}\left[\frac{1}{2\zeta}\frac{d\varphi}{d\zeta}\right]\frac{d\zeta}{d\xi} =$$

$$= \left[\frac{1}{2\zeta}\frac{d^2\varphi}{d\zeta^2} - \frac{1}{2\zeta^2}\frac{d\varphi}{d\zeta}\right]\frac{1}{2\zeta} = \frac{1}{4\zeta^2}\left[\frac{d^2\varphi}{d\zeta^2} - \frac{1}{\zeta}\frac{d\varphi}{d\zeta}\right]$$

resulting: $$\xi\varphi'' = \frac{1}{4}\left[\frac{d^2\varphi}{d\zeta^2} - \frac{1}{\zeta}\frac{d\varphi}{d\zeta}\right]$$

The vertical vibrating string problem



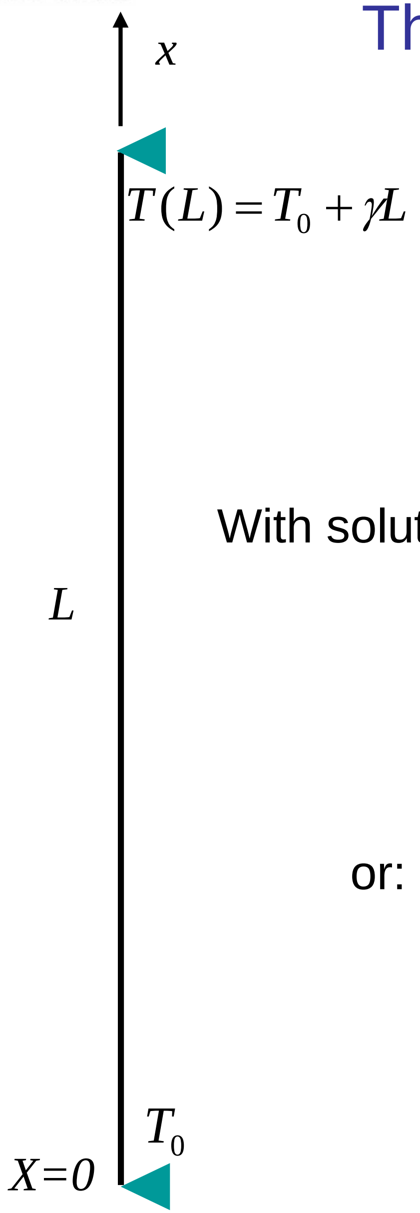
Hence: $$\xi\varphi'' + \varphi' + \beta^2\varphi = 0$$

$$\frac{1}{4}\left[\frac{d^2\varphi}{d\zeta^2} - \frac{1}{\zeta}\frac{d\varphi}{d\zeta}\right] + \frac{1}{2\zeta}\frac{d\varphi}{d\zeta} + \beta^2\varphi = 0$$

$$\frac{d^2\varphi}{d\zeta^2} + \frac{1}{\zeta}\frac{d\varphi}{d\zeta} + 4\beta^2\varphi = 0$$

$$\zeta^2\frac{d^2\varphi}{d\zeta^2} + \frac{d\varphi}{d\zeta} + 4\beta^2\zeta^2\varphi = 0$$

The vertical vibrating string problem



Indicial equation

$$\zeta^2 \frac{d^2\varphi}{d\zeta^2}+\frac{d\varphi}{d\zeta}+4\beta^2\zeta^2\varphi=0$$

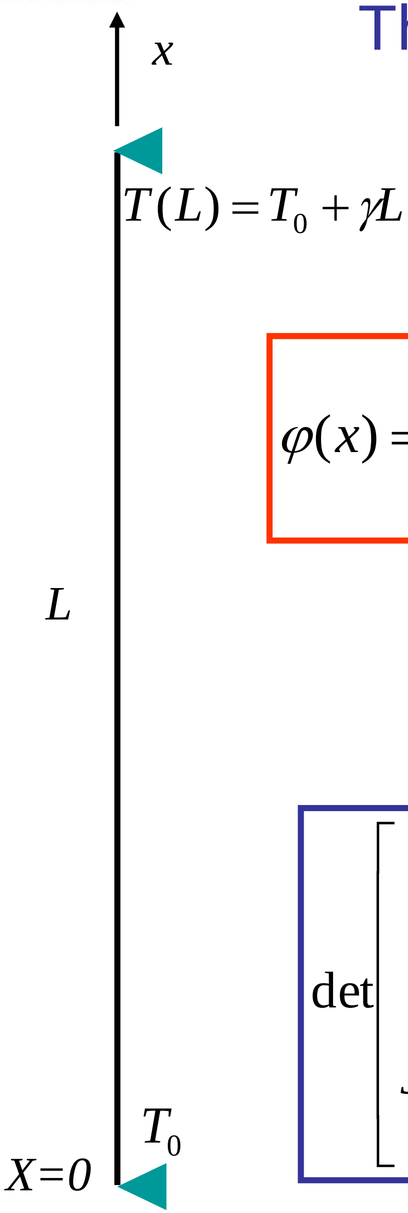
With solutions given in terms of bessel functions of zeroth-order:

$$\varphi(\zeta)=C_1 J_0(2\beta\zeta)+C_2 Y_0(2\beta\zeta)$$

or:

$$\varphi(\xi)=C_1 J_0(2\beta\xi^{1/2})+C_2 Y_0(2\beta\xi^{1/2})$$

The vertical vibrating string problem



Recalling that: $\xi = \dfrac{T_0}{\gamma} + x$ e $\beta^2 = \dfrac{\mu}{\gamma}\Omega^2$

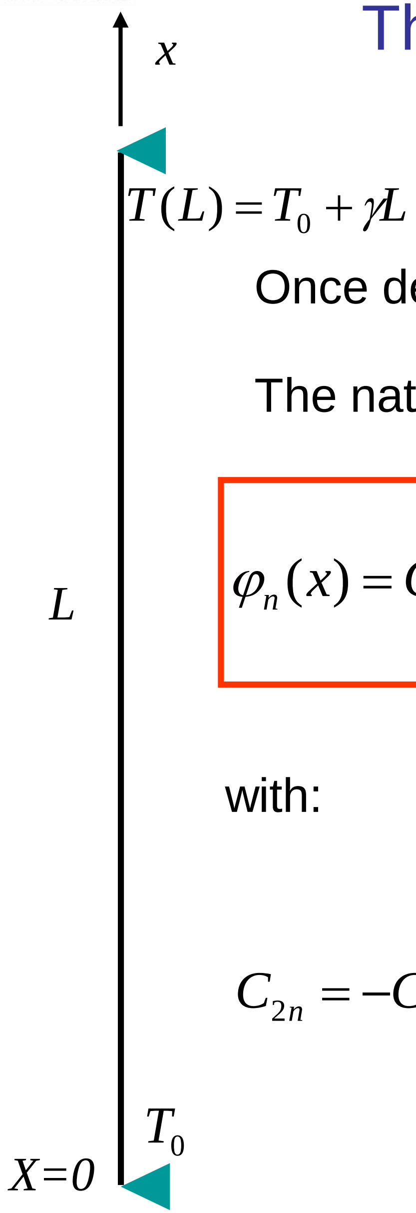
$$\varphi(x) = C_1 J_0\left(2\Omega\left(\frac{\mu}{\gamma}\right)^{1/2}\left(\frac{T_0}{\gamma}+x\right)^{1/2}\right) + C_2 Y_0\left(2\Omega\left(\frac{\mu}{\gamma}\right)^{1/2}\left(\frac{T_0}{\gamma}+x\right)^{1/2}\right)$$

Boundary conditions at extremities: $\varphi_n(0) = \varphi_n(L) = 0$

$$\det\begin{bmatrix} J_0\left(2\Omega_n\left(\frac{\mu}{\gamma}\right)^{1/2}\left(\frac{T_0}{\gamma}\right)^{1/2}\right) & Y_0\left(2\Omega_n\left(\frac{\mu}{\gamma}\right)^{1/2}\left(\frac{T_0}{\gamma}\right)^{1/2}\right) \\ J_0\left(2\Omega_n\left(\frac{\mu}{\gamma}\right)^{1/2}\left(\frac{T_0}{\gamma}+L\right)^{1/2}\right) & Y_0\left(2\Omega_n\left(\frac{\mu}{\gamma}\right)^{1/2}\left(\frac{T_0}{\gamma}+L\right)^{1/2}\right) \end{bmatrix} \equiv 0$$

Characteristic equation

The vertical vibrating string problem



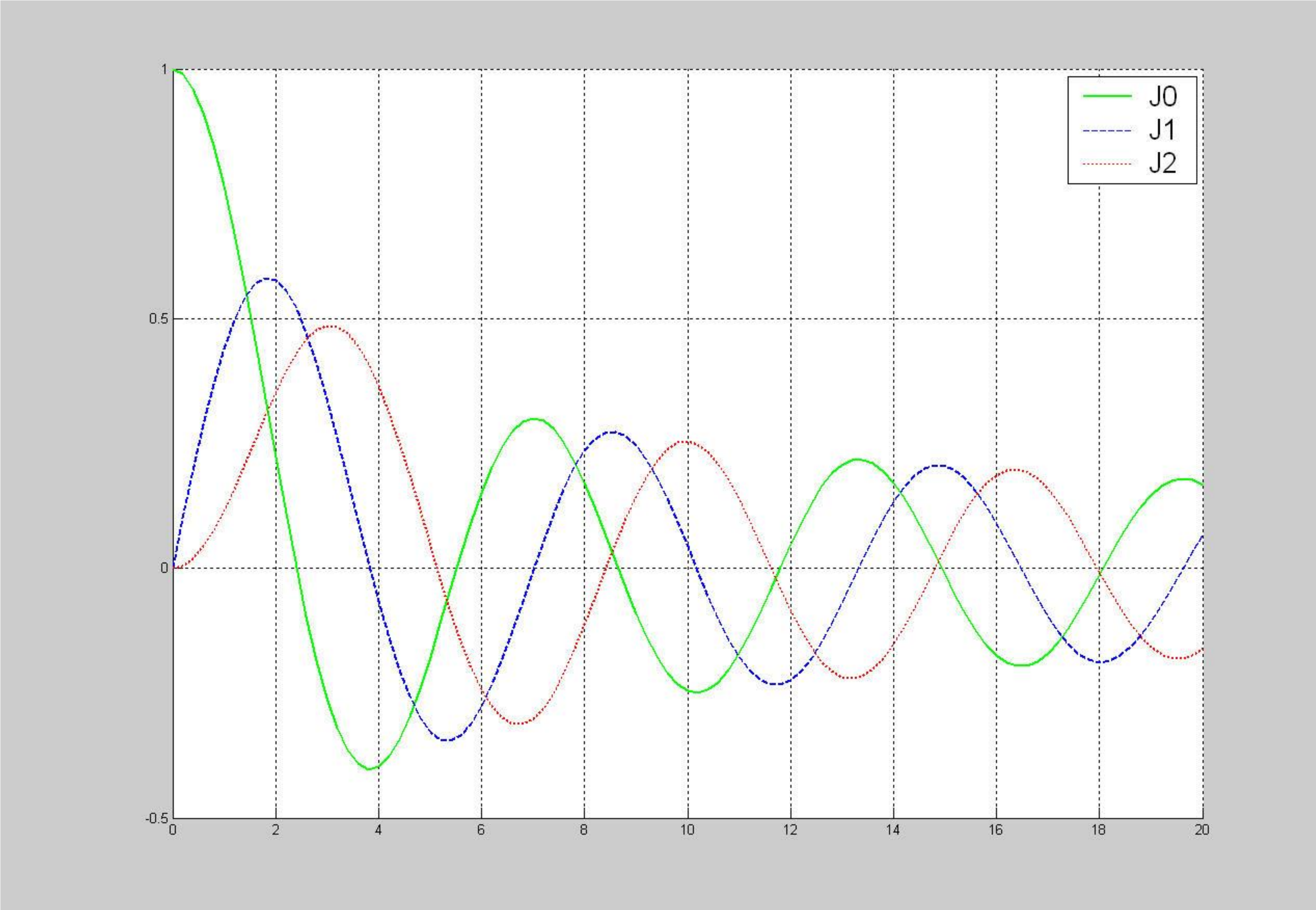
Once determined the eigenvalues $\Omega_n = \Omega_n(T_0, \mu, \gamma, L)$

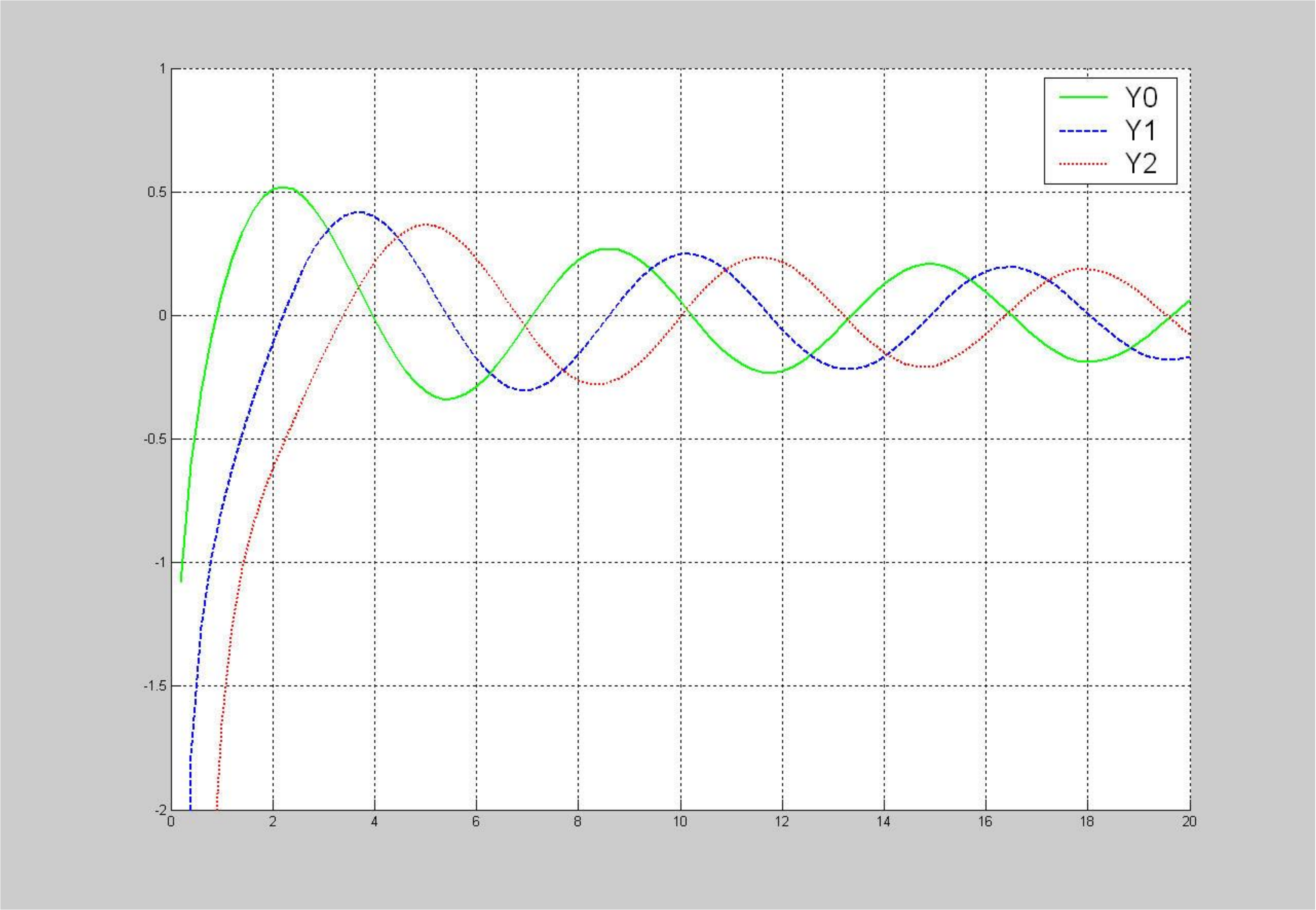
The natural vibration modes are given by:

$$\varphi_n(x) = C_{1n} J_0\left(2\Omega_n \left(\frac{\mu}{\gamma}\right)^{1/2} \left(\frac{T_0}{\gamma} + x\right)^{1/2} \right) + C_{2n} Y_0\left(2\Omega_n \left(\frac{\mu}{\gamma}\right)^{1/2} \left(\frac{T_0}{\gamma} + x\right)^{1/2} \right)$$

with:

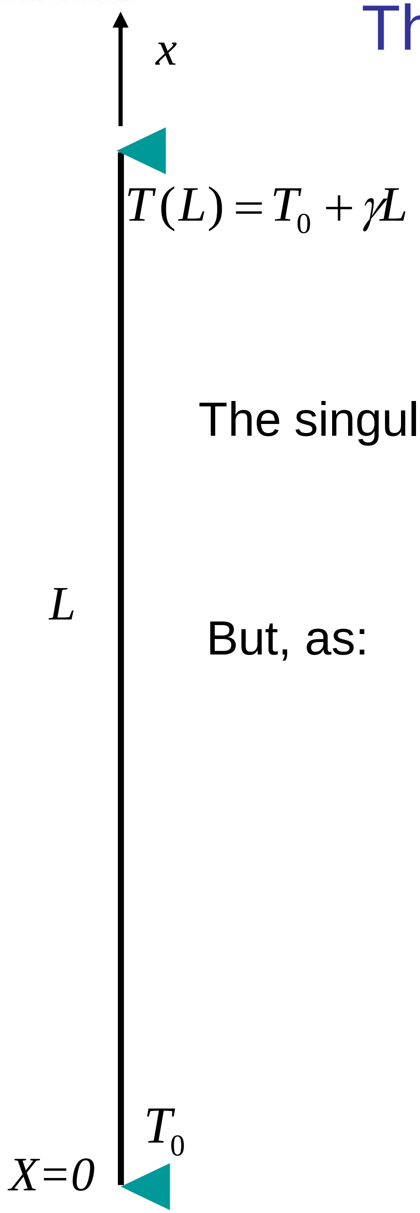
$$C_{2n} = -C_{1n} \frac{J_0\left(2\Omega_n \left(\frac{\mu}{\gamma}\right)^{1/2} \left(\frac{T_0}{\gamma}\right)^{1/2} \right)}{Y_0\left(2\Omega_n \left(\frac{\mu}{\gamma}\right)^{1/2} \left(\frac{T_0}{\gamma}\right)^{1/2} \right)} = -C_{1n} \frac{J_0\left(2\Omega_n \left(\frac{\mu}{\gamma}\right)^{1/2} \left(\frac{T_0}{\gamma} + L\right)^{1/2} \right)}{Y_0\left(2\Omega_n \left(\frac{\mu}{\gamma}\right)^{1/2} \left(\frac{T_0}{\gamma} + L\right)^{1/2} \right)}$$


J0
J1
J2
1
0.5
0
-0.5
0
2
4
6
8
10
12
14
16
18
20



Y0
Y1
Y2
1
0.5
0
-0.5
-1
-1.5
-2
0
2
4
6
8
10
12
14
16
18
20

The vertical vibrating string problem



Singular case: $T_0 = 0$ since $[Y_0(0)] \to \infty$

The singularity could be removed by setting :

$$\boxed{C_{2n} \equiv 0}$$

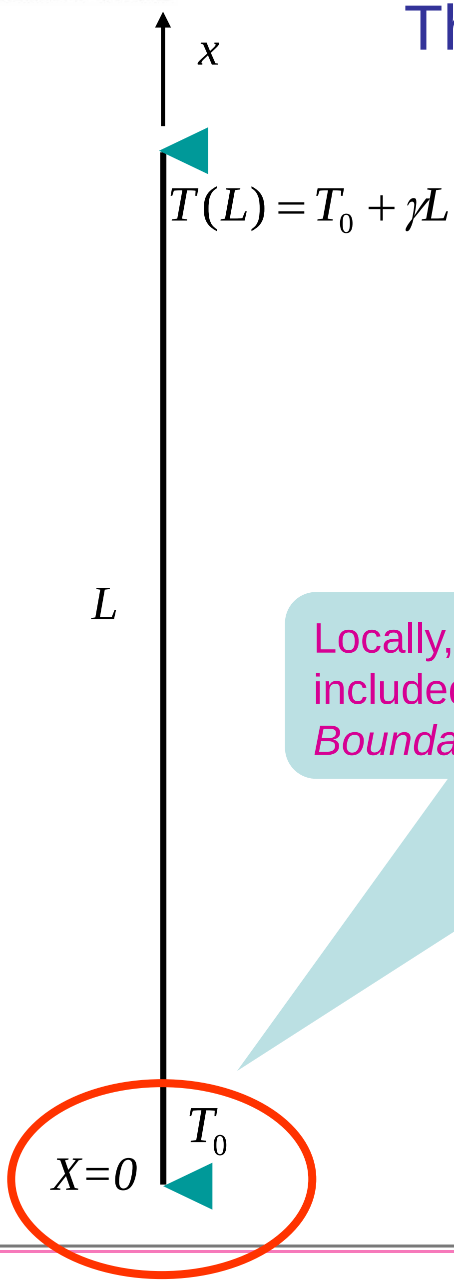
But, as:

$$C_{2n} = -C_{1n} \frac{J_0\left(2\Omega_n\left(\frac{\mu}{\gamma}\right)^{1/2}\left(\frac{T_0}{\gamma}+L\right)^{1/2}\right)}{Y_0\left(2\Omega_n\left(\frac{\mu}{\gamma}\right)^{1/2}\left(\frac{T_0}{\gamma}+L\right)^{1/2}\right)} \quad \longrightarrow$$

$$\boxed{C_{1n} = C_{2n} \equiv 0}$$

Trivial solution

The vertical vibrating string problem



Singular case: $T_0 = 0$

Physical interpretation:

when tension is null the wave celerity is also null:

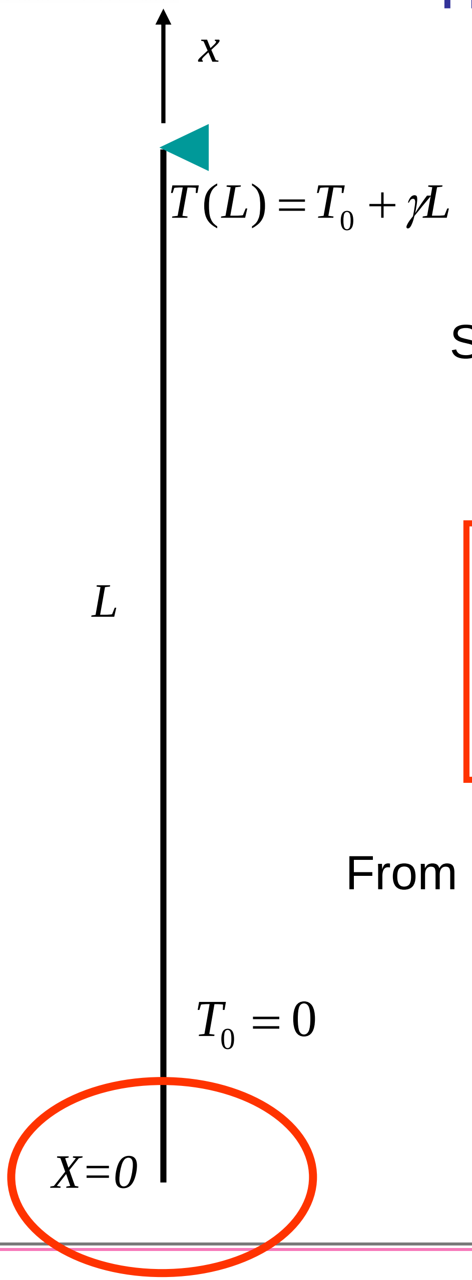
$$c(x) = \sqrt{\frac{T(x)}{\mu}}$$

Locally, the bending stiffness effect has to be included, what cand be done through the *Boundary Layer Technique; Triantafyllou, 1984.*

An analytical solution – Bessel-like modes- has been obtained, by reducing the beam equation to a second-order one, accounting for the bending stiffness effect through an equivalent tension (in an average sense);

see Mazzilli & Lenci, 2014.

The vertically suspended vibrating string problem



Particular case: $T_0 = 0$

Since $[Y_0(0)] \to \infty$ take $C_{2n} \equiv 0$

$$\varphi_n(x) = C_{1n} J_0 \left(2\Omega_n \left(\frac{\mu}{\gamma} \right)^{1/2} (x)^{1/2} \right)$$

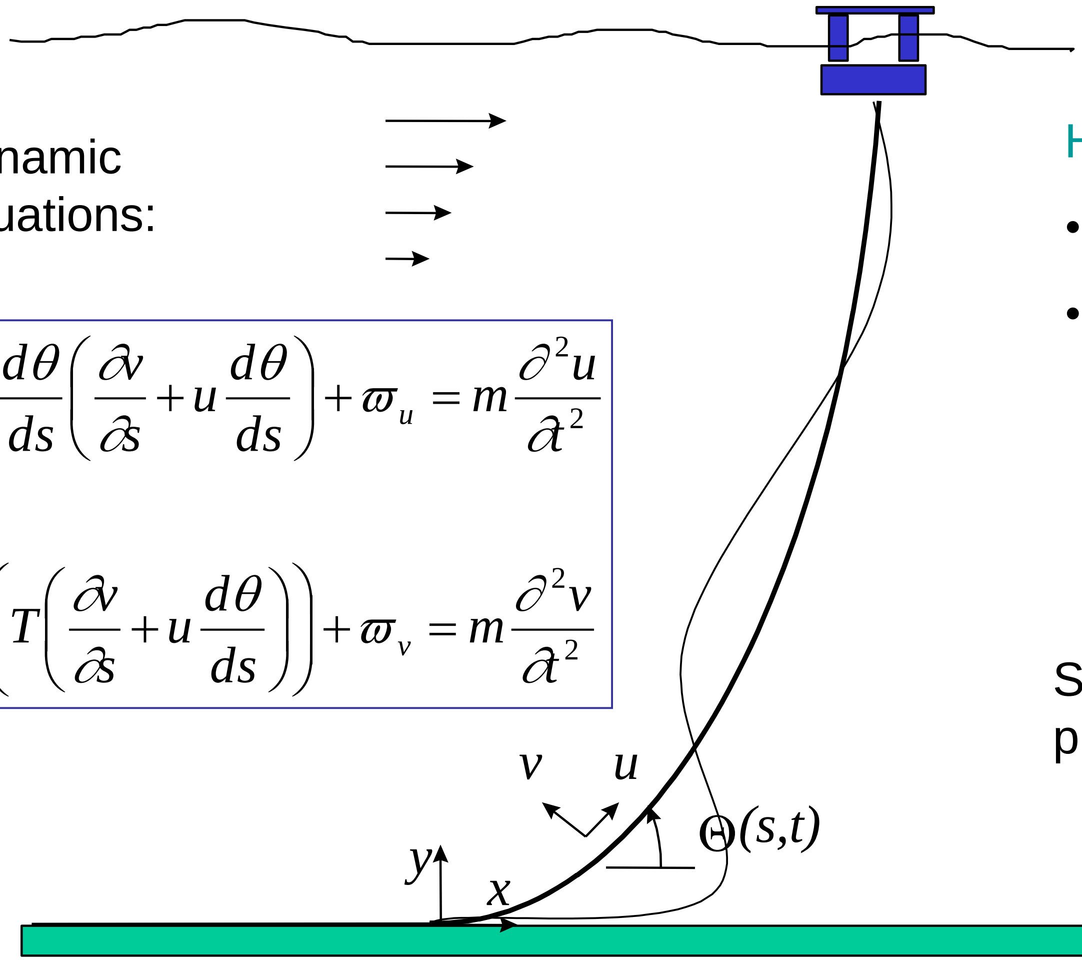
From boundary condition: $\varphi_n(L) \equiv 0$

$$J_0 \left(2\Omega_n \left(\frac{\mu}{\gamma} \right)^{1/2} (L)^{1/2} \right) = 0$$

Characteristic equation

The catenary-like problem under current action; Pesce et al 1999

Dynamic equations:



$$-T\frac{d\theta}{ds}\left(\frac{\partial v}{\partial s}+u\frac{d\theta}{ds}\right)+\varpi_u = m\frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial}{\partial s}\left(T\left(\frac{\partial v}{\partial s}+u\frac{d\theta}{ds}\right)\right)+\varpi_v = m\frac{\partial^2 v}{\partial t^2}$$

Hypotheses:

- planar problem;
- cable:
  - *inextensible;*
  - *perfectly flexible.*

Static solution previously determined:

$$T(s),\ \theta(s)$$

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The catenary-like problem under current action

In the absence of external forces the linearized dynamic equations take the nondimensional form:

$$F(\xi)\left(\frac{\partial \eta}{\partial \xi} + \upsilon \frac{d\theta}{d\xi}\right)\frac{d\theta}{d\xi} + \frac{1}{1+a}\frac{\partial^2 \upsilon}{\partial t^2} = 0$$

$$-\frac{\partial}{\partial \xi}\left(F(\xi)\left(\frac{\partial \eta}{\partial \xi} + \upsilon \frac{d\theta}{d\xi}\right)\right) + \frac{\partial^2 \eta}{\partial t^2} = 0$$

Added mass

where:

$$\xi = s/L; \quad \upsilon = u/L; \quad \eta = v/L$$

$$t = c_0 t/L; \quad c_0 = \sqrt{T_0/(m+m_a)}$$

e $a = \frac{m_a}{m}$

with, $F(\xi) = \frac{T(\xi)}{T_0} = \left(\frac{c(\xi)}{c_0}\right)^2$

Dimensionless tension function

The catenary-like problem under current action

Neglecting second order terms in the static curvature:

Linearly proportional toa φ'

$$\psi(\xi) = \frac{1}{\omega^2}\left[(1+a)F(\xi)\chi(\xi)\right]\varphi'(\xi)$$

$$\left(F(\xi)\varphi'\right)' + \omega^2\varphi = 0$$

Classic Sturm-Liouville problem

$$\omega = \Omega\frac{L}{c_0}$$

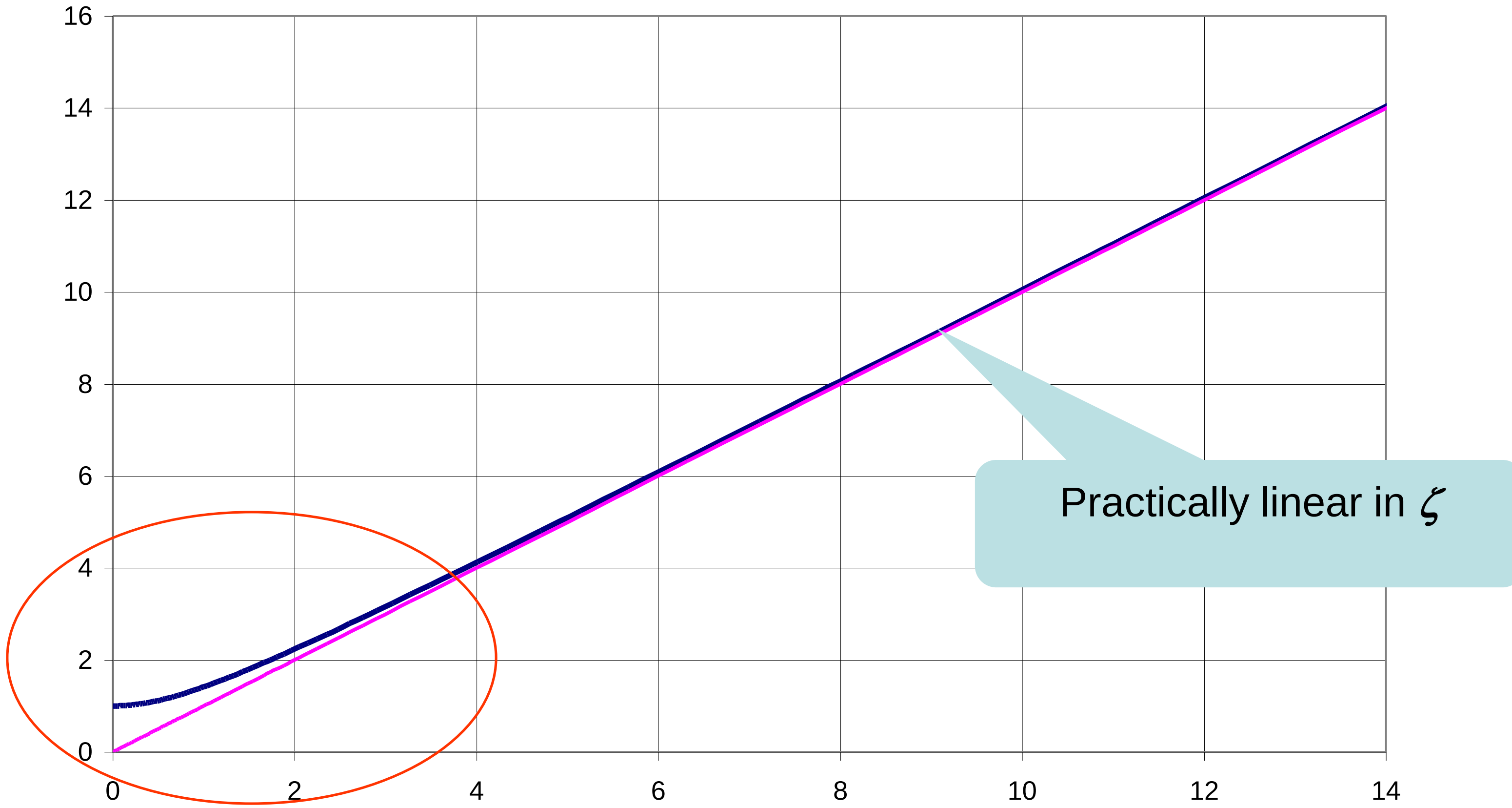
The solution would be a Bessel function if the tension were linear in ξ

$$\upsilon(\xi,t) = \psi(\xi)e^{i\omega t}$$
$$\eta(\xi,t) = \varphi(\xi)e^{i\omega t}$$

Separation of variables technique

The catenary-like problem under current action

$$F_c(\zeta) = \sqrt{1+\zeta^2}$$



$$F_c(\xi) = \sec\theta(\xi) = \sqrt{1+\tan^2\theta(\xi)} \rightarrow \boxed{F_c(\zeta) = \sqrt{1+\zeta^2}}$$ com $\zeta = \tan\theta(\xi)$

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The catenary-like problem under current action: a Bessel equation approximation

Approximating (mimimum quadratic method) as:

$$F(\zeta) \approx \alpha^2 + b\zeta \qquad \text{com} \qquad \beta = b/\alpha^2$$

And defining: $$z^2 = 1 + \beta\zeta = 1 + \left(b/\alpha^2\right)\zeta$$

A modified Bessel equation is obtained in the form:

$$z^2\varphi'' + z\varphi' + 4\mathrm{K}^2 z^2 \varphi = 0$$

$$\mathrm{K} = \frac{\omega}{\alpha\beta \tan\theta_L}$$

With solution:

$$\varphi(z) = C_1 J_0(2\mathrm{K}z) + C_2 Y_0\left(2Kz\right)$$

The catenary-like problem under current action: a Bessel equation approximation

For a ‘catenary’ riser:

$$\varphi_n(\xi) = \\ = J_0\left(2\frac{\omega_n}{\alpha\beta\tan\theta_L}(1+\beta\tan\theta_L\xi)^{1/2}\right) + \frac{J_0\left(2\frac{\omega_n}{\alpha\beta\tan\theta_L}\right)}{Y_0\left(2\frac{\omega_n}{\alpha\beta\tan\theta_L}\right)} Y_0\left(2\frac{\omega_n}{\alpha\beta\tan\theta_L}(1+\beta\tan\theta_L\xi)^{1/2}\right)$$

Satisfying the characteristic equation:

$$J_0\left(2\frac{\omega}{\alpha\beta\tan\theta_L}(1+\beta\tan\theta_L)^{1/2}\right) Y_0\left(2\frac{\omega}{\alpha\beta\tan\theta_L}\right) = \\ = -J_0\left(2\frac{\omega}{\alpha\beta\tan\theta_L}\right) Y_0\left(2\frac{\omega}{\alpha\beta\tan\theta_L}(1+\beta\tan\theta_L)^{1/2}\right)$$

The catenary-like problem under current action: a WKB approximation

$$\psi(\zeta) = \frac{(1+a)}{\Lambda^2} F(\zeta)\chi(\zeta)\varphi'(\zeta)$$

$$\varphi'' + \frac{F'}{F}\varphi' + \frac{\Lambda^2}{F}\varphi = 0$$

$$\Lambda = \frac{\omega}{\tan\theta_L}; \quad \zeta = \tan\theta(\xi)$$

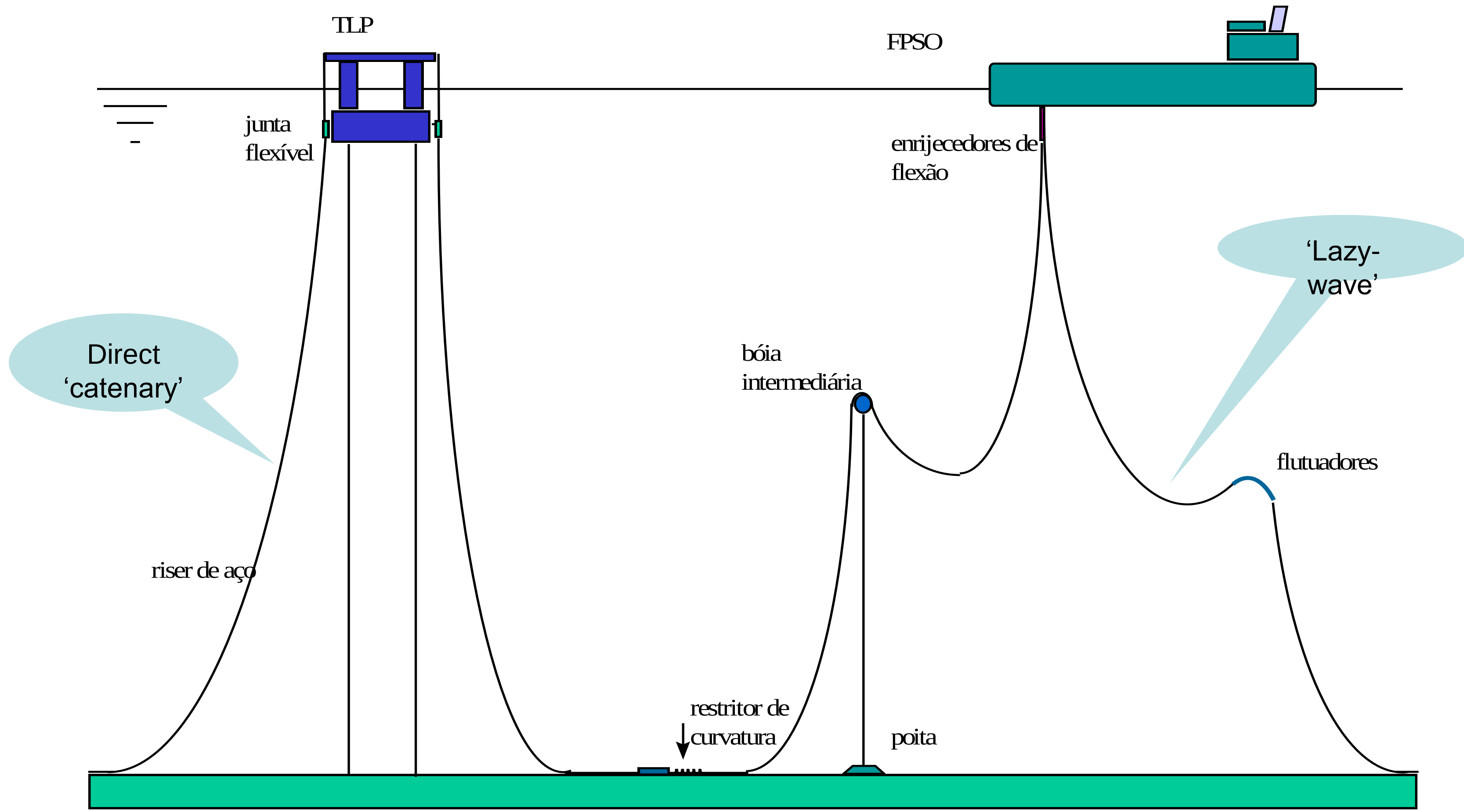
Propitious form for the **WKB method**,

if : $\Lambda^2 >> 1$

Singular perturbation problem if $F(\zeta)=0$

$$F(\zeta) = \left(\frac{c(\zeta)}{c_0}\right)^2 = \frac{T\zeta)}{T_0}$$

WKB: *Wentzel, Kramers, Brillouin* (bem como *Rayleygh* e *Jeffreys*)


TLP
FPSO
junta flexível
enrijecedores de flexão
‘Lazy-wave’
Direct ‘catenary’
bóia intermediária
flutuadores
riser de aço
restritor de curvatura
poita

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The catenary-like problem under current action: a WKB approximation

See, e.g., Bender & Orszag, pg. 490.

$$\varphi(\zeta) \cong$$
$$\cong F^{-1/4}(\zeta)\left[C_1 \sin\left(\Lambda \int^{\zeta} F^{-1/2}(u)du\right) + C_2 \cos\left(\Lambda \int^{\zeta} F^{-1/2}(u)du\right)\right]$$

$$\Lambda^2 >> 1$$

$$\Lambda = \frac{\omega}{\tan \theta_L}$$

'Turning Point' if $\Lambda=0$

Vibration modes are sinusoidal functions, modulated in amplitude and phase, ressembling Bessel functions

$$\phi(\zeta) = \Lambda \int^{\zeta} F^{-1/2} d\zeta$$

Local phase

$$\kappa = \frac{d\phi}{d\zeta} = \frac{\Lambda}{\sqrt{F(\zeta)}}$$

Local wave number

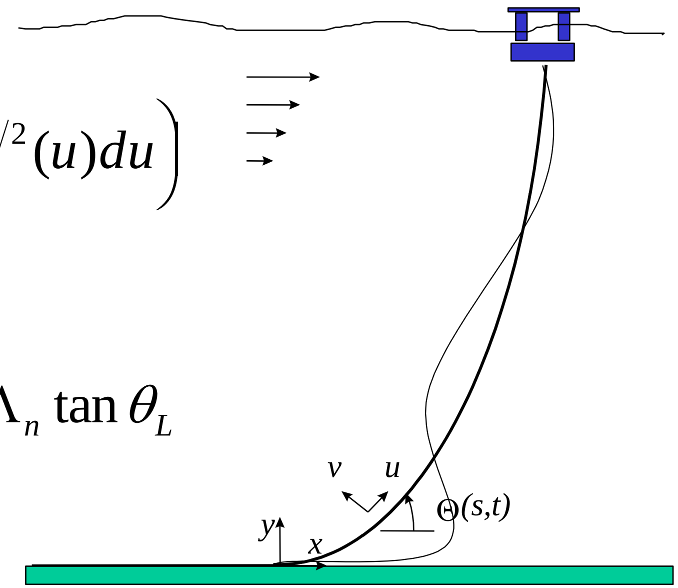
$$\frac{c(\zeta)}{c_0} = \frac{\Lambda}{\kappa(\zeta)}$$

Local wave celerity

The catenary-like problem under current action: a WKB approximation

Eigenvalues are quadratures of the static solution!!!!

Direct 'catenary' riser



$$\varphi_n(\zeta) \cong A_n F^{-1/4}(\zeta)\sin\left(\Lambda_n \int_0^{\zeta} F^{-1/2}(u)\,du\right)$$

$$\Lambda_n \cong n\pi\left(\int_0^{\mu}\frac{d\zeta}{\sqrt{F(\zeta)}}\right)^{-1} \quad ; \quad \omega_n = \Lambda_n \tan\theta_L$$

Linear in n

Vibration modes are sinusoidal functions, modulated in amplitude and phase, ressembling Bessel functions

$$\phi_n(\zeta) = \Lambda_n \int^{\zeta} F^{-1/2}\,d\zeta \qquad \kappa_n = \frac{d\phi}{d\zeta} = \frac{\Lambda_n}{\sqrt{F(\zeta)}} \qquad \frac{c(\zeta)}{c_0} = \frac{\Lambda_n}{\kappa_n(\zeta)}$$

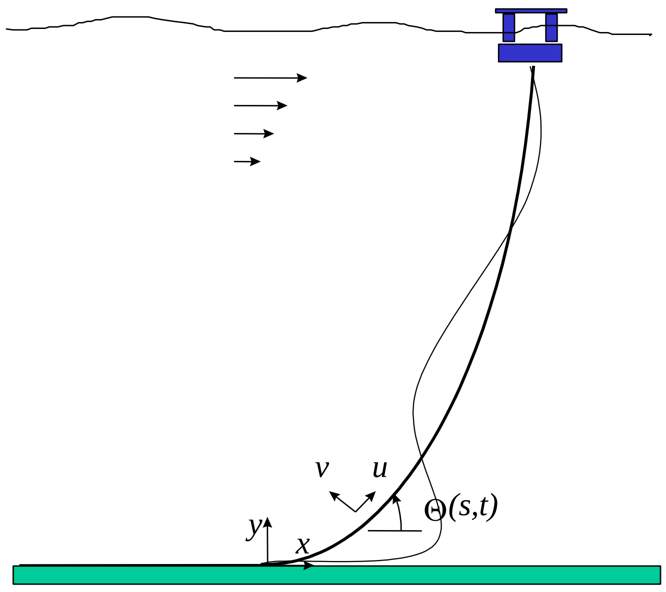
Local phase

Local wave number

Local wave celerity

The catenary-like problem under current action: a WKB approximation

Direct ‘catenary’ riser



Natural frequencies:

$$\Omega_n = \omega_n \frac{c_0}{L} = $$

$$= \Lambda_n \tan\theta_L \frac{1}{L}\sqrt{\frac{T_0}{(m+m_a)}}$$

Analytical solution in closed form

Particular case: NO CURRENT

WKB solution

$$\varphi_n(\theta;\theta_L) \cong A_n(\cos\theta)^{-1/4} \sin\left\{\Lambda_n \int_0^{\theta_L} \frac{d\theta}{(\cos\theta)^{3/2}}\right\}$$

$$\Lambda_n = \Lambda_n(\theta_L) \cong \frac{n\pi}{\int_0^{\theta_L} \frac{d\theta}{(\cos\theta)^{3/2}}}$$

Since $T_0 = \dfrac{qL}{\tan\theta_L}$ $\longrightarrow$ $\Omega_n \cong \Lambda_n \sqrt{\dfrac{q\tan\theta_L}{(m+m_a)L}}$

Particular case: NO CURRENT

WKB solution

Taking $m_a \cong \rho \pi D^2/4 \longrightarrow q \cong (m - m_a)g$

And, with $a = m_a/m \longrightarrow$ $\Omega_n \cong \Lambda_n \sqrt{\tan\theta_L \dfrac{(1-a)}{(1+a)}}\sqrt{\dfrac{g}{L}}$

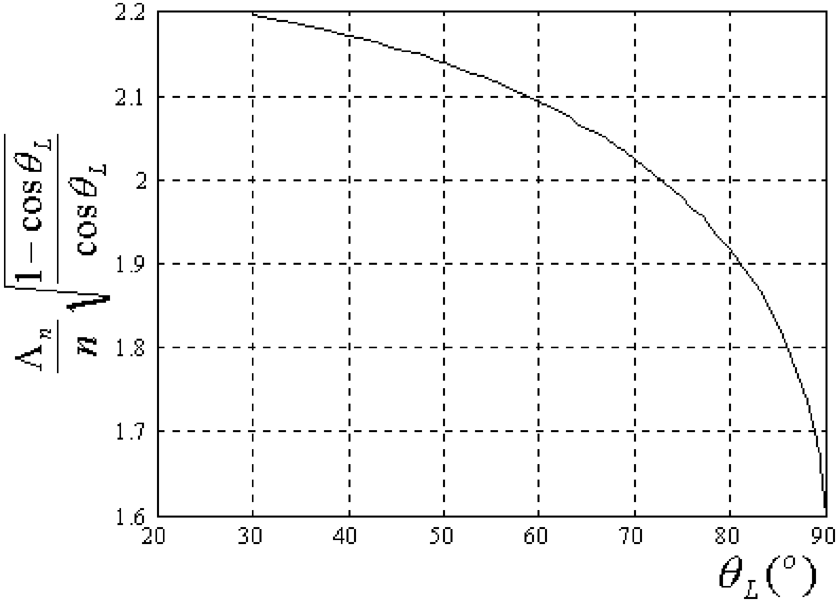
However, $L = H \sin\theta_L / (1 - \cos\theta_L)$

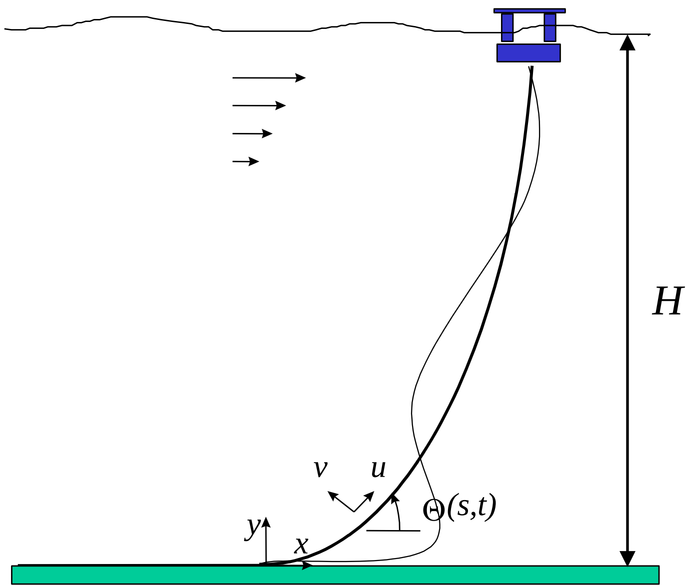
$\downarrow$

$$\Omega_n \cong \Lambda_n \sqrt{\frac{(1-\cos\theta_L)}{\cos\theta_L}}\sqrt{\frac{(1-a)}{(1+a)}}\sqrt{\frac{g}{H}}$$

Particular case: NO CURRENT

WKB solution for a catenary riser

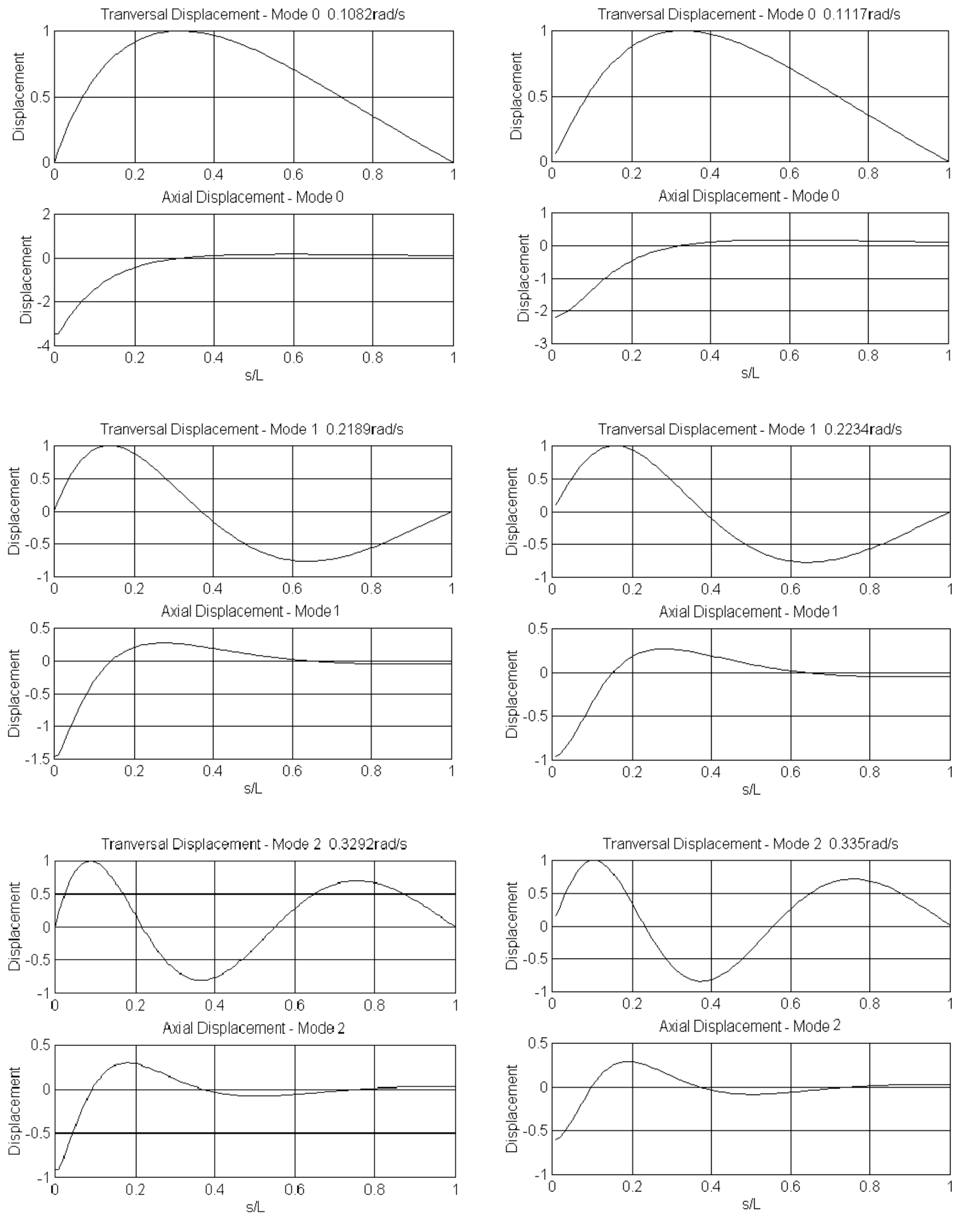




$$\Omega_n \cong \Lambda_n \sqrt{\frac{(1-\cos\theta_L)}{\cos\theta_L}}\sqrt{\frac{(1-a)}{(1+a)}}\sqrt{\frac{g}{H}}$$

Bessel's $\mu=5.7$ WKB

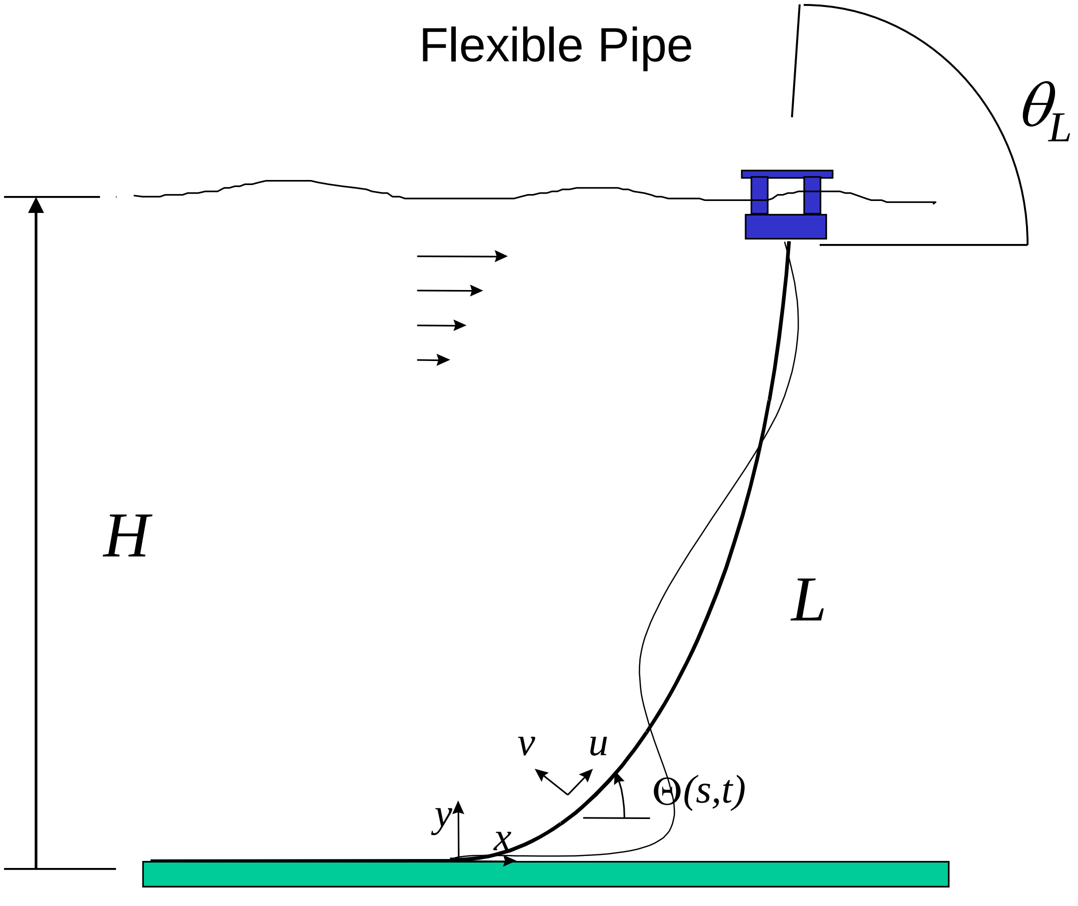
Catenary riser



$$\tan\theta_L = 5.7$$

$$\theta_L = 80^o$$

WKB vs. POLIFLEX



WKB vs. POLIFLEX

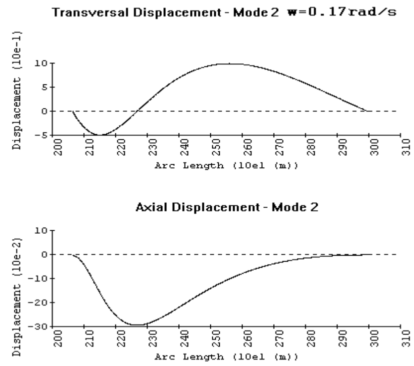
Flexible Pipe

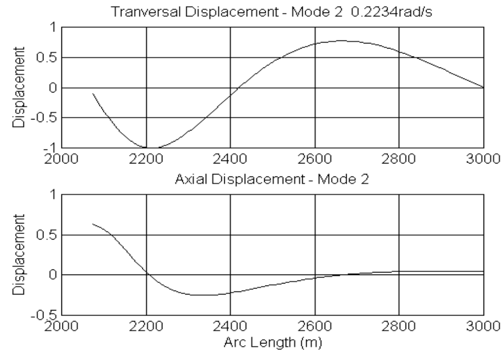
EA (kN)	312500
EJ (kNm^2)	49.61
q (kN/m)	0.914
m (t/m)	0.218
D (m)	0.3934
H (m)	785
Total length (m)	3000
L (m) for $\theta_L = 80^o$	935.5
L (m) for $\theta_L = 60^o$	1359.6

POLIFLEX

WKB

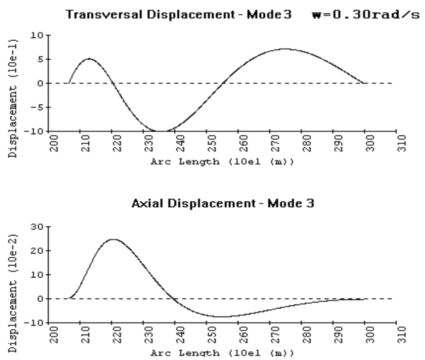
Flexible Pipe

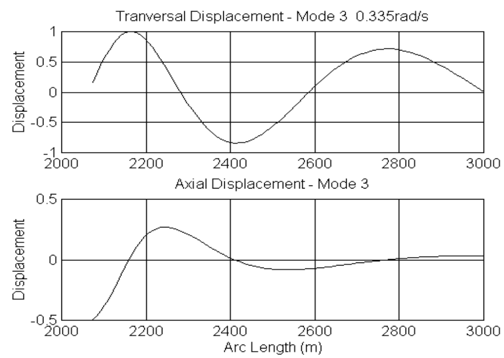


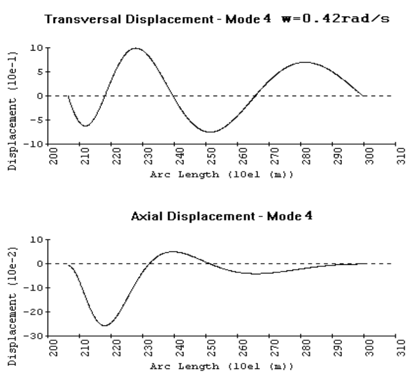


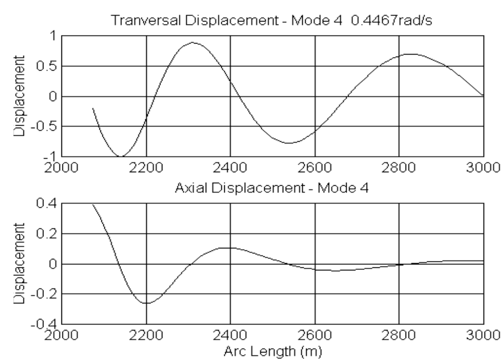
$$\tan\theta_L = 5.7$$

$$\theta_L = 80^o$$





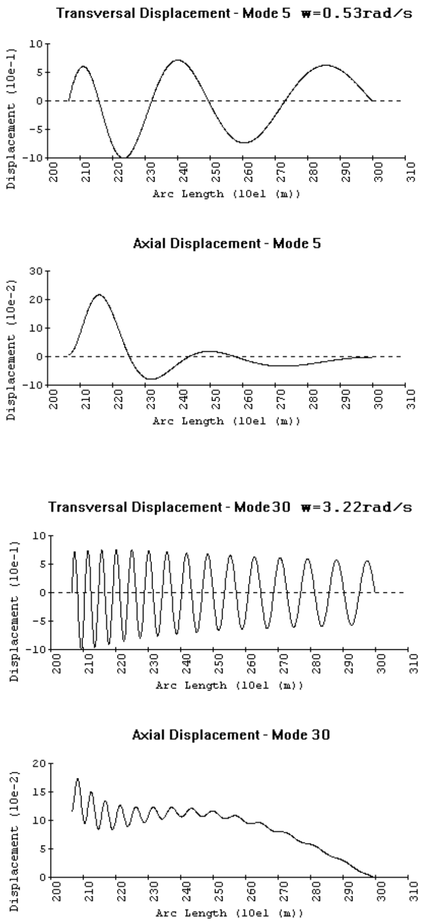


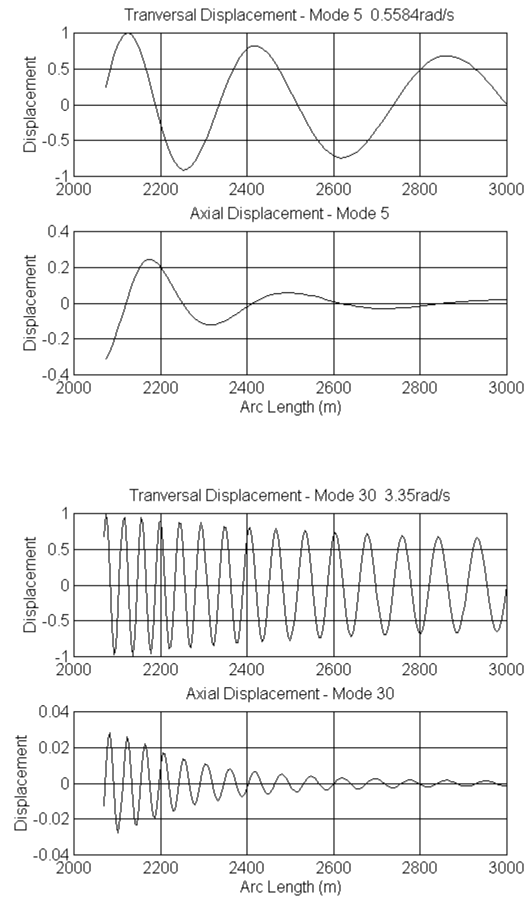


POLIFLEX

WKB

Flexible Pipe





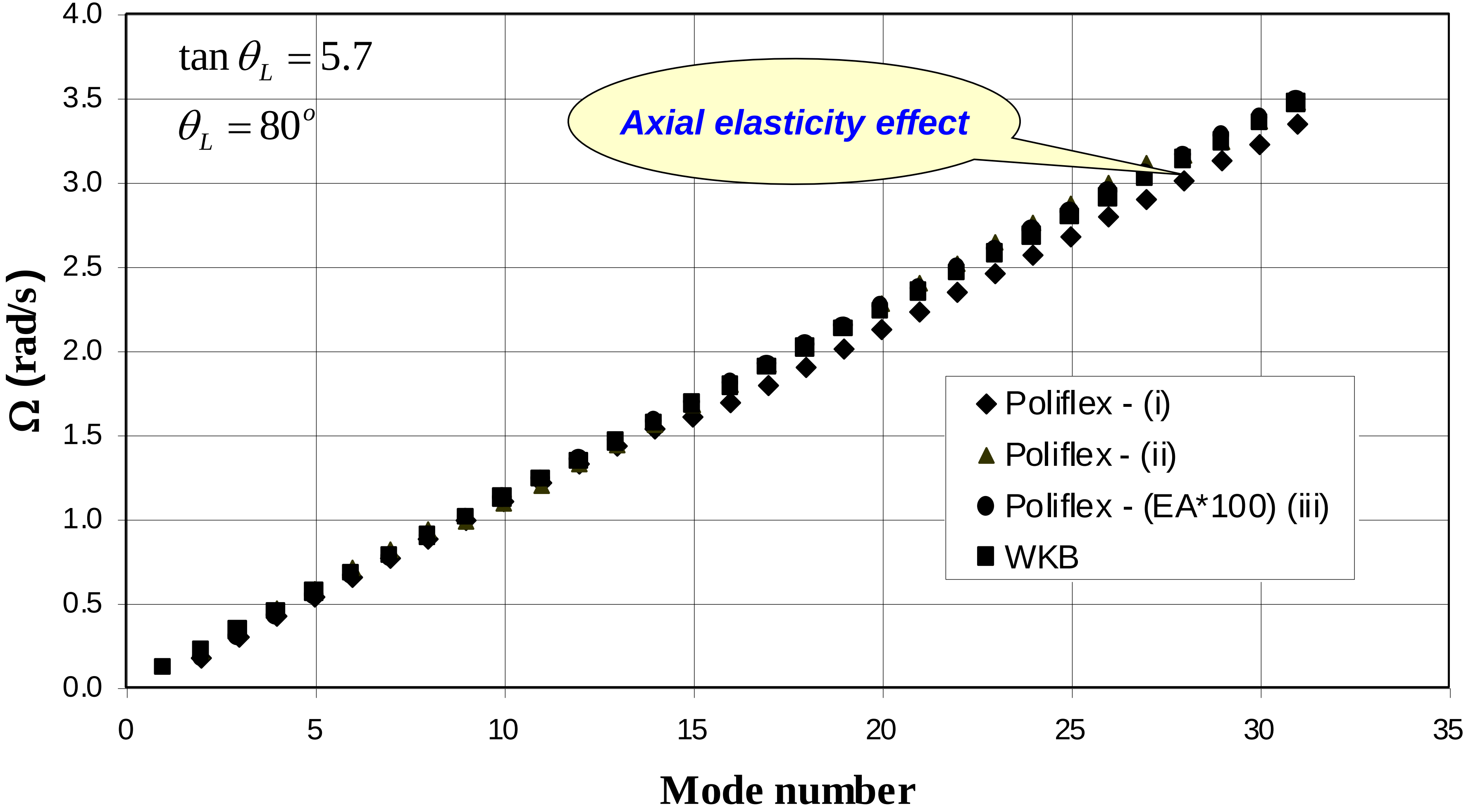
$$\tan\theta_L = 5.7$$

$$\theta_L = 80^o$$

WKB vs. POLIFLEX

Flexible Pipe

$\mu = \tan\theta_L = 5.7$

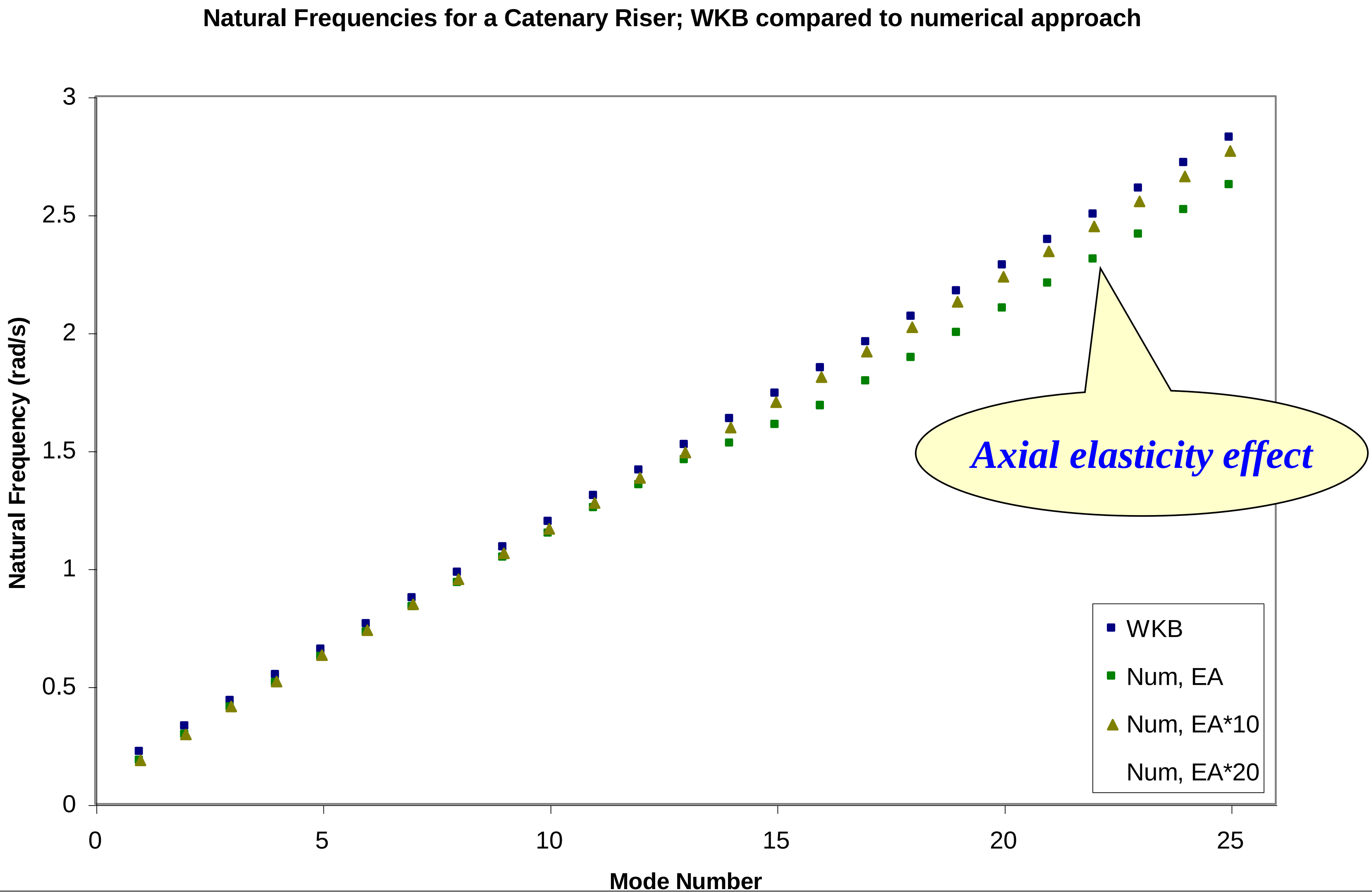


Typical SCR

Axial Rigidity, EA (kN)	2.314 x10^6
Bending Stiffness, EI (kNm^2)	9915
Immersed weight, q (kN/m)	0.727
m (kg/m) (filled with water)	108.0
External diameter, D (m)	0.2032
Thickness (mm)	19.05
Depth H (m)	1800
Total length (m)	5047
Angle at top, $\theta_L (^o)$ (no current)	70 (w.r.t. horizontal)
Soil Rigidity, k (kN/m/m)	466.37
Suspended length, L (m)	2571
Static tension at TDP, T_0 (kN)	680.55
Flexural length, λ(m)	3.82
Curvature at TDP, χ_0 (m^{-1})	1.077E-03
Nondimensional curvature at TDP, $X_0 = \chi_0 \lambda$	4.114E-03
Local scale, $\varepsilon = \lambda / L$	1.486E-03
Nondimensional soil rigidity parameter, $K = kEI / T_0^2$	10

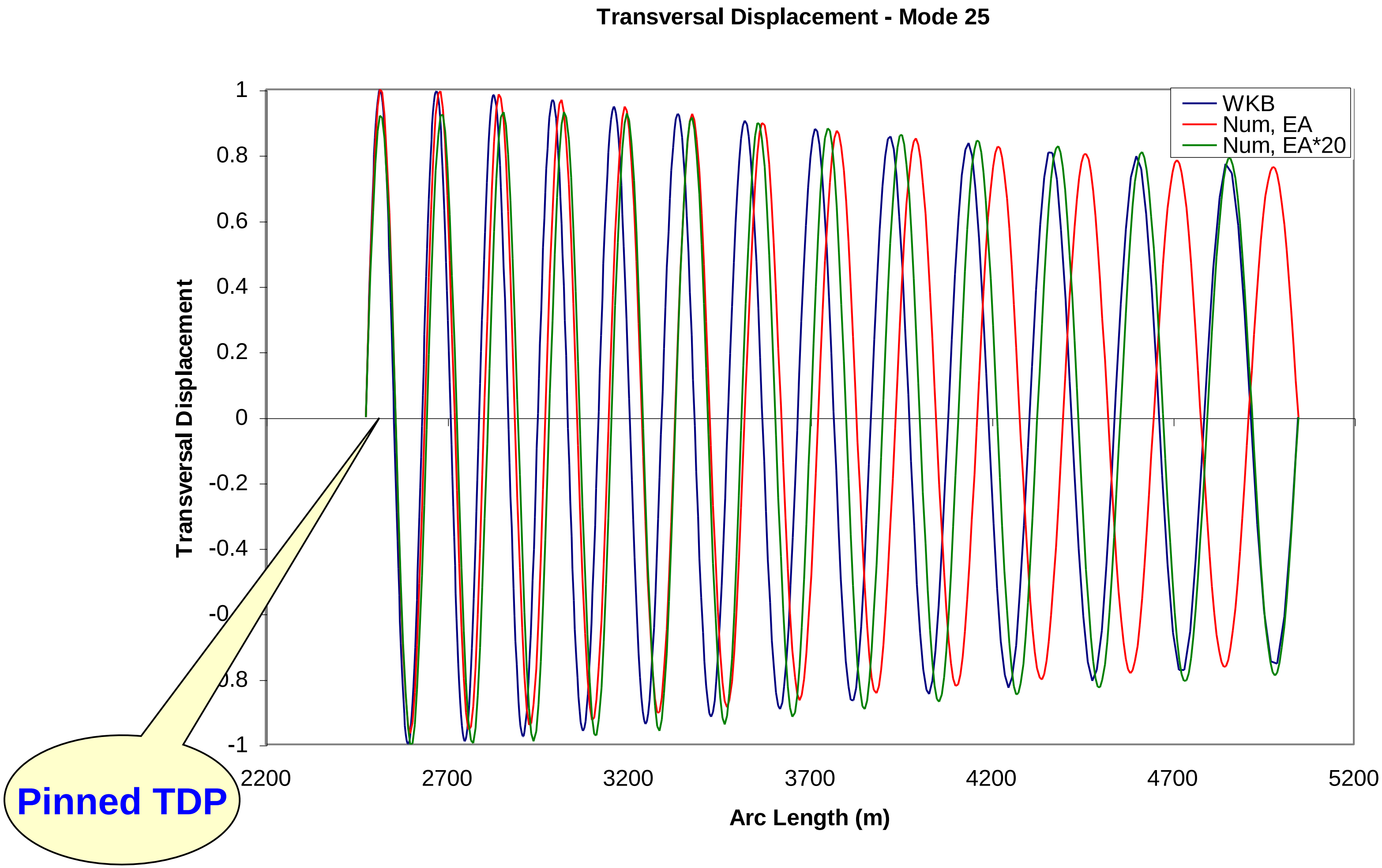
WKB vs. POLIFLEX

SCR



WKB vs. POLIFLEX

SCR: Mode 25

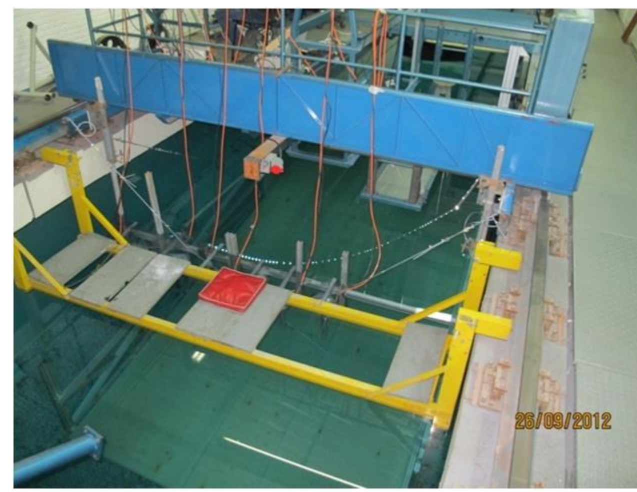


Example: experimental analysis
Small Scale SCR Experiments in a Towing Tank

"Transversal" Catenary model

"Longitudinal" Catenary model

Towed false bottom



Pesce, 2013; Rateiro 2015, Rateiro et al 2016

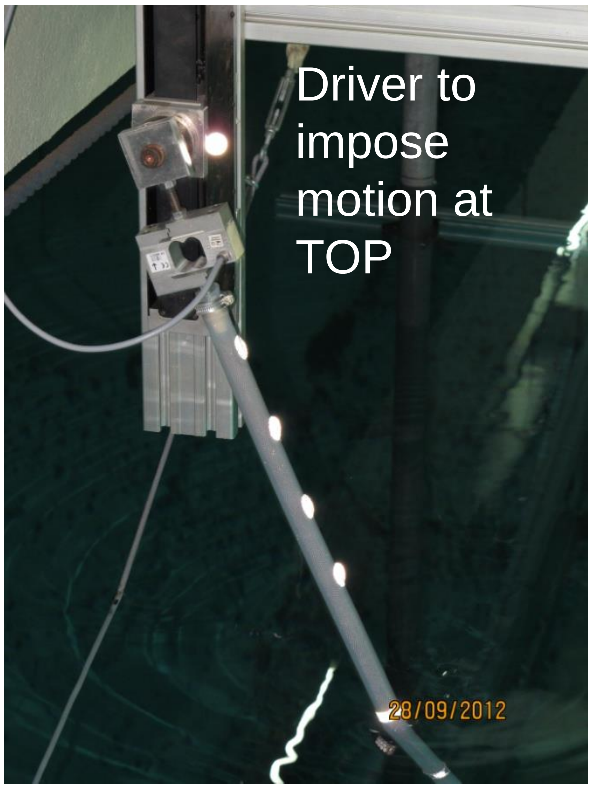
Example: experimental analysis
Small Scale SCR Experiments in a Towing Tank

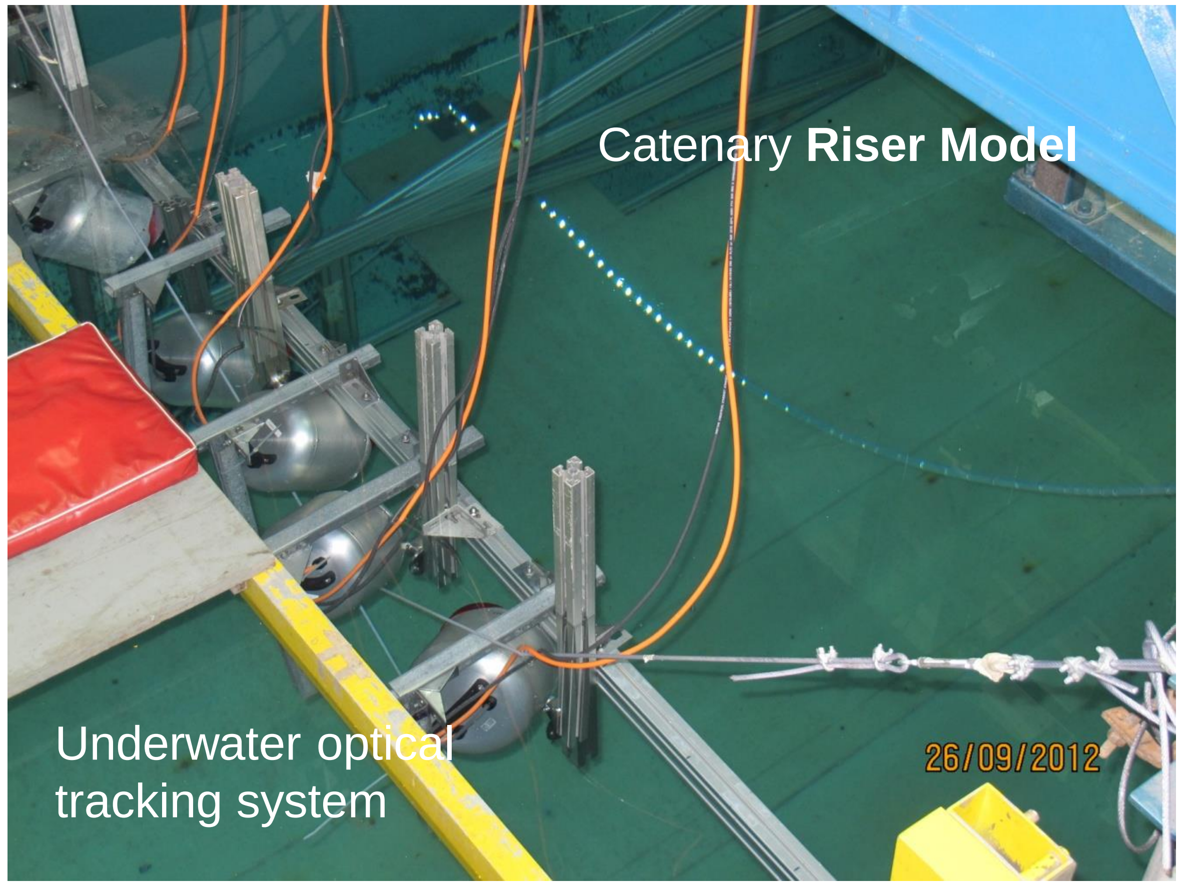
Data	**Scaled (1:100)**	**Designed model**	**As built**
Internal diameter (mm)	1.826	15.800	15.800
External diameter (mm)	2.191	22.200	22.200
Weight in water (N/m)	0.726	7.308	7.308
Axial rigidity, EA (kN)	2.362	1.910	1.0 - 1.6
Bending stiffness, EI (Nm^2)	1.20E-03	8.86E-02	5.60E-02
Flexural length, λ_f (mm)	71.0	61.0	49.0
Added mass, $a=m_a/m$	0.522	0.520	0.520

Pesce, 2013; Rateiro 2015, Rateiro et al 2016

Example: experimental analysis
Small Scale SCR Experiments in a Towing Tank

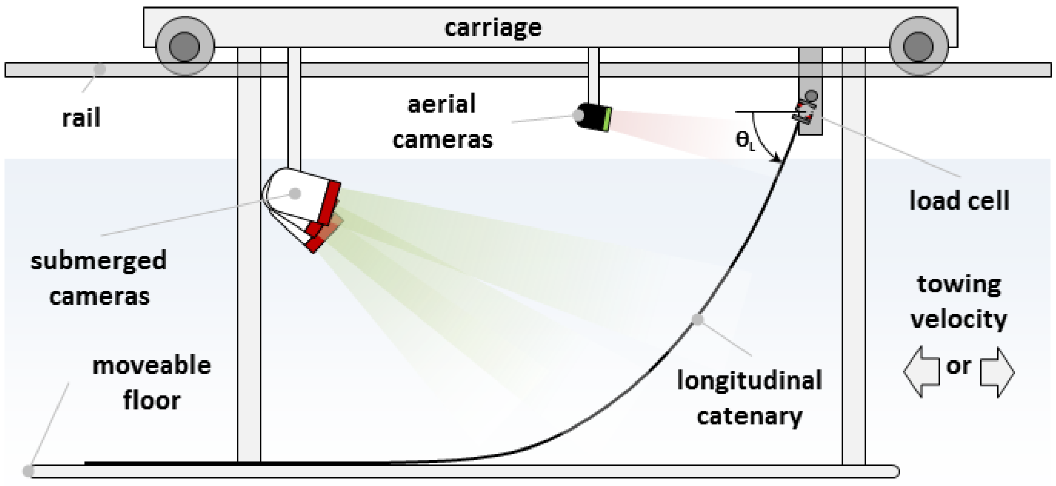
D=22.2 mm
m=3.48*





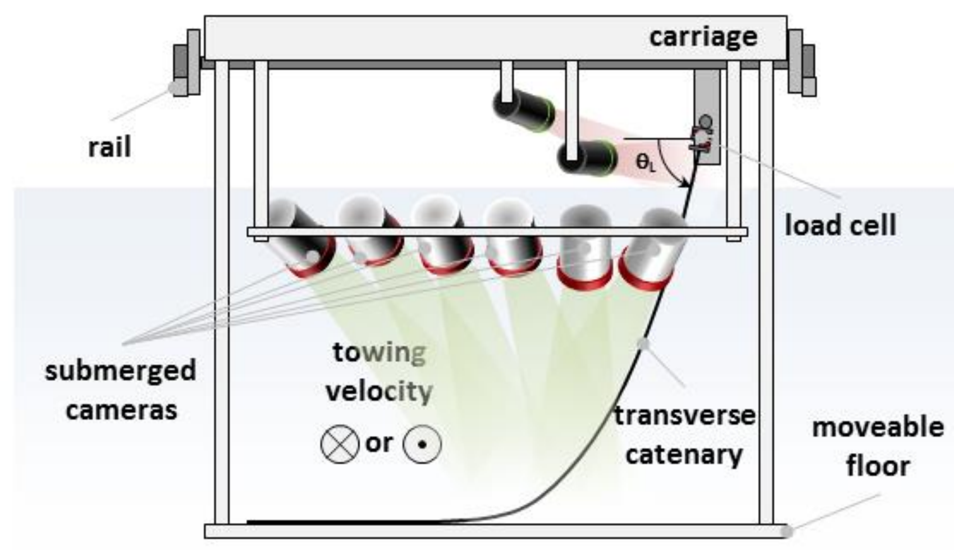
Pesce, 2013; Rateiro 2015, Rateiro et al 2016

Example: experimental analysis
Small Scale SCR Experiments in a Towing Tank



Longitudinal to the flow

Transversal to the flow



$$\mathbf{a}_n(t) = \int_0^L \frac{\langle \mathbf{X}(s,t), \boldsymbol{\varphi}_n(s) \rangle}{\left\| \boldsymbol{\varphi}_n(s) \right\|^2} ds$$

Modal decomposition by Galerkin's projection

Rateiro et al 2016

Example: experimental analysis
Small Scale SCR Experiments in a Towing Tank

1st mode is out-of-plane

Natural Frequencies (Hz)

Mode	POLIFLEX	Experimental	Plane
1	0,42	0,42	out
2	0,70	0,72	in
3	0,83	0,87	out
4	1,10	1,15	in
5	1,25	1,26	out
6	1,52	---	in
7	1,68	1,79	out
8	1,68	1,66	in
9	2,13	2,23	out
10	2,12	2,22	in
11	2,59	2,73	out
12	2,56	2,56	in
13	3,07	3,02	out
14	3,05	3,19	in
15	--	3,80	out

Longitudinal to the flow

Natural Frequencies (Hz)

Mode	POLIFLEX	Experimental	Plane
1	0,42	0,43	out
2	0,72	0,76	in
3	0,84	0,85	out
4	1,09	1,18	in
5	1,26	1,25	out
6	1,44	--	in
7	1,67	1,66	in
8	1,69	1,77	out
9	2,13	2,24	in
10	2,14	2,27	out
11	2,56	2,74	in
12	2,59	2,71	out
13	3,05	3,18	in
14	3,51	--	out
15	--	3,22	in

Transversal to the flow

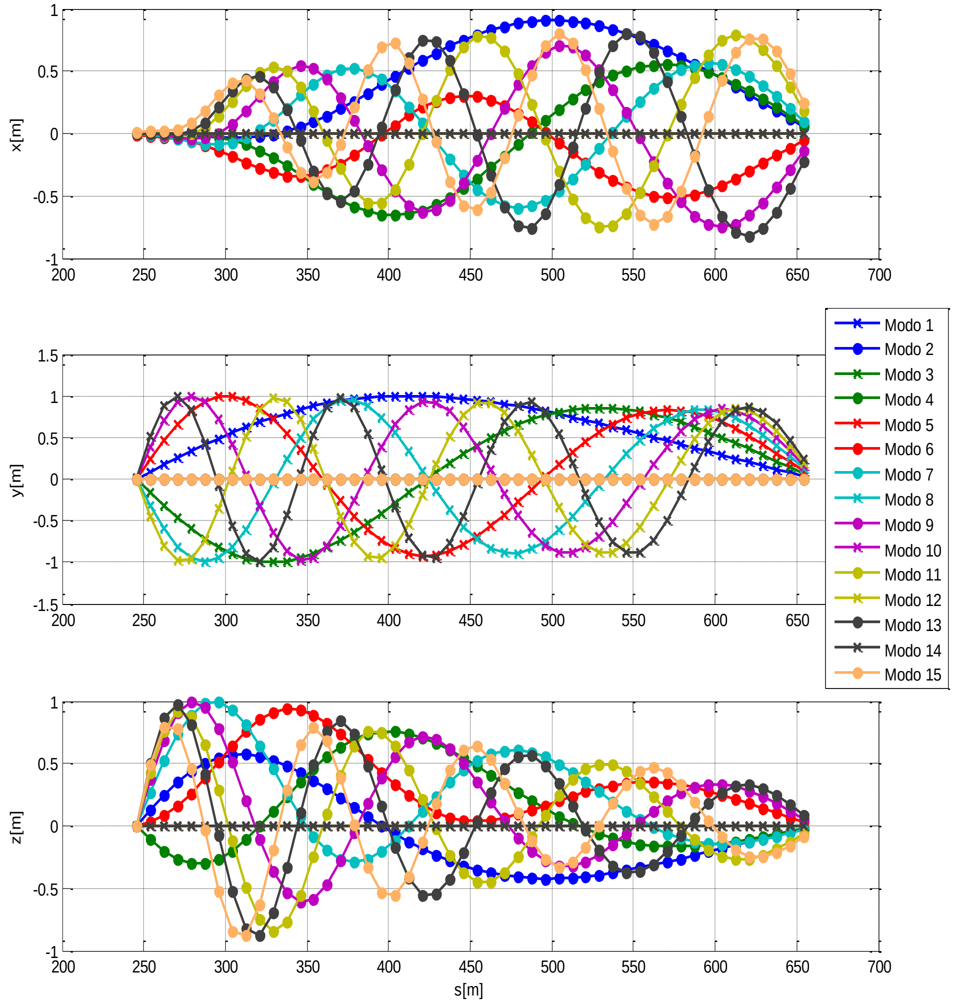
Pesce, 2013; Rateiro 2015, Rateiro et al 2016

Example: experimental analysis
Small Scale SCR Experiments in a Towing Tank

x(m)

y(m)

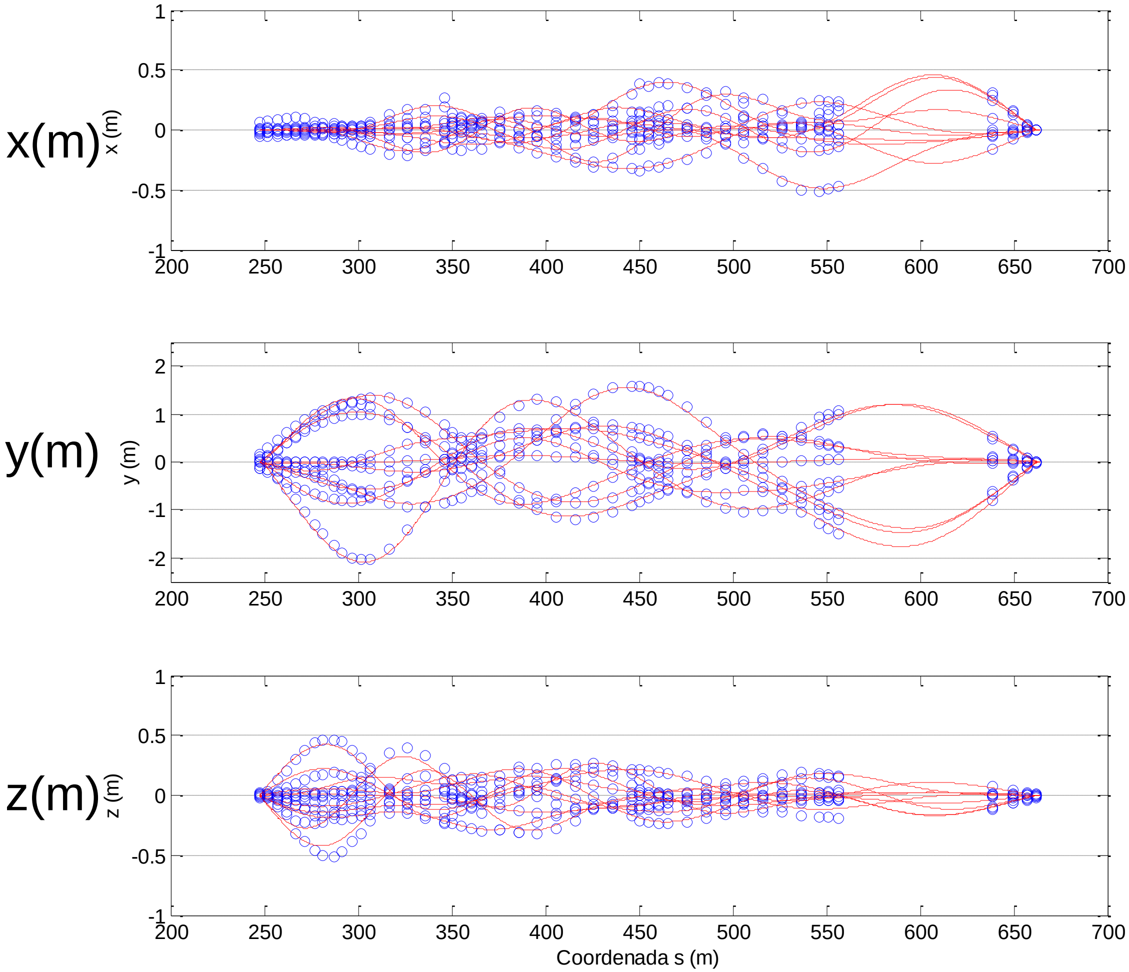
z(m)



Vibration modes: longitudinal configuration

Pesce, 2013; Rateiro 2015, Rateiro et al 2016

Example: experimental analysis
Small Scale SCR Experiments in a Towing Tank

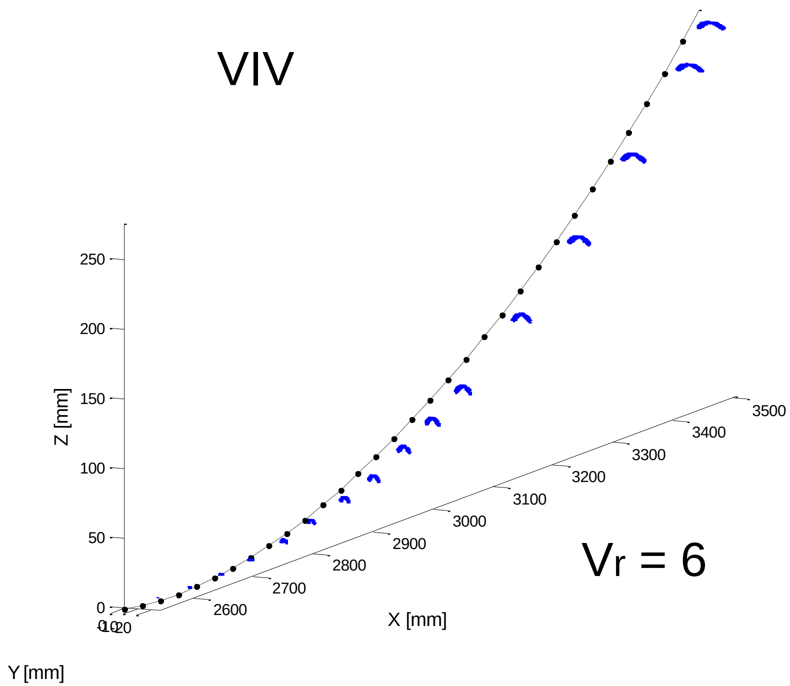


Modal reconstruction: *snapshots*

Pesce, 2013; Rateiro 2015, Rateiro et al 2016

Example: experimental analysis
Small Scale SCR Experiments in a Towing Tank

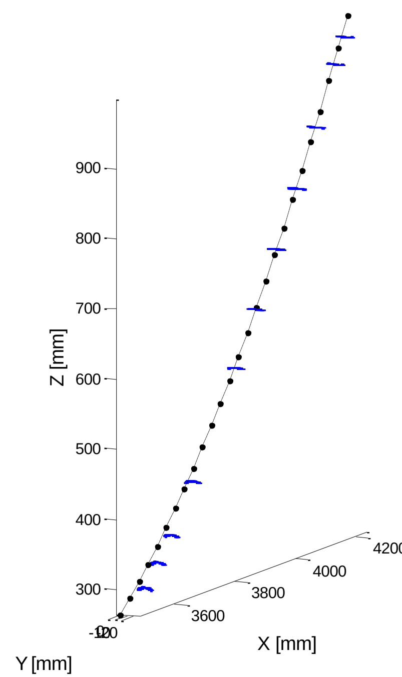
VIV

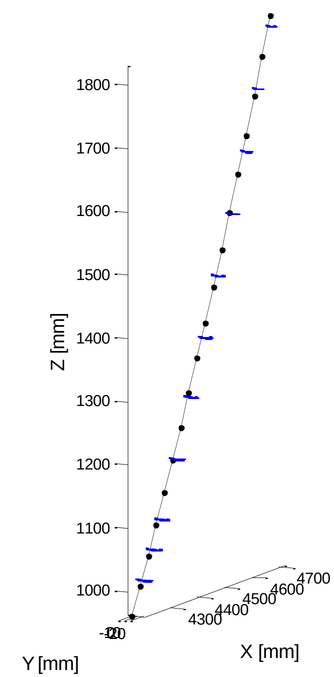


Vr = 6

$$V_{R_1} = \frac{U}{f_1 D} = 6.0$$

Longitudinal Catenary,
Income flow by the dorse

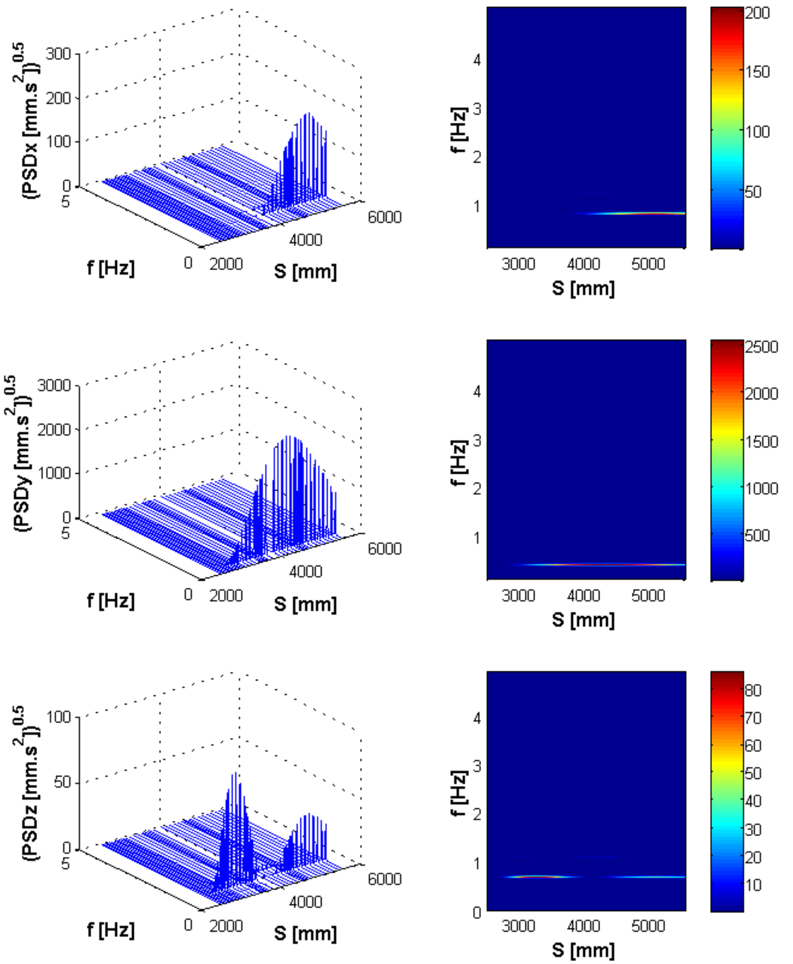


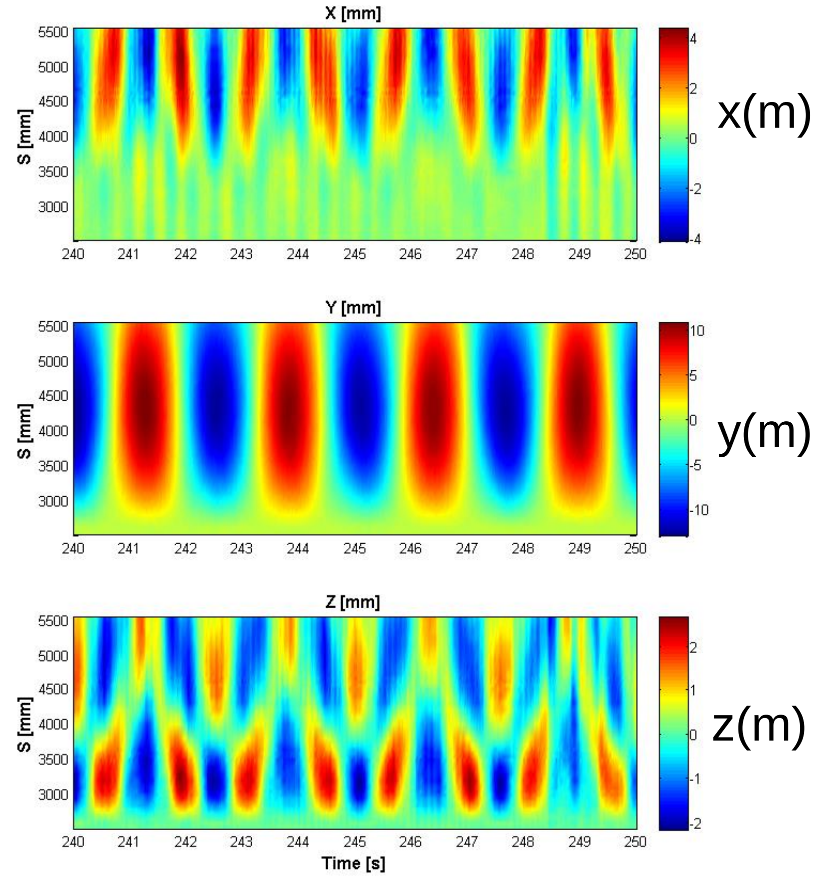


Pesce, 2013; Rateiro 2015, Rateiro et al 2016

Example: experimental analysis
Small Scale SCR Experiments in a Towing Tank

$$V_{R_1} = \frac{U}{f_1 D} = 6.0$$

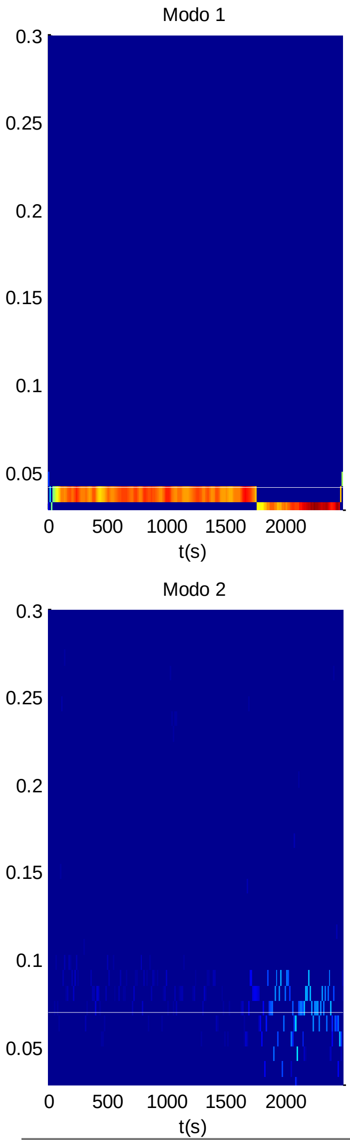




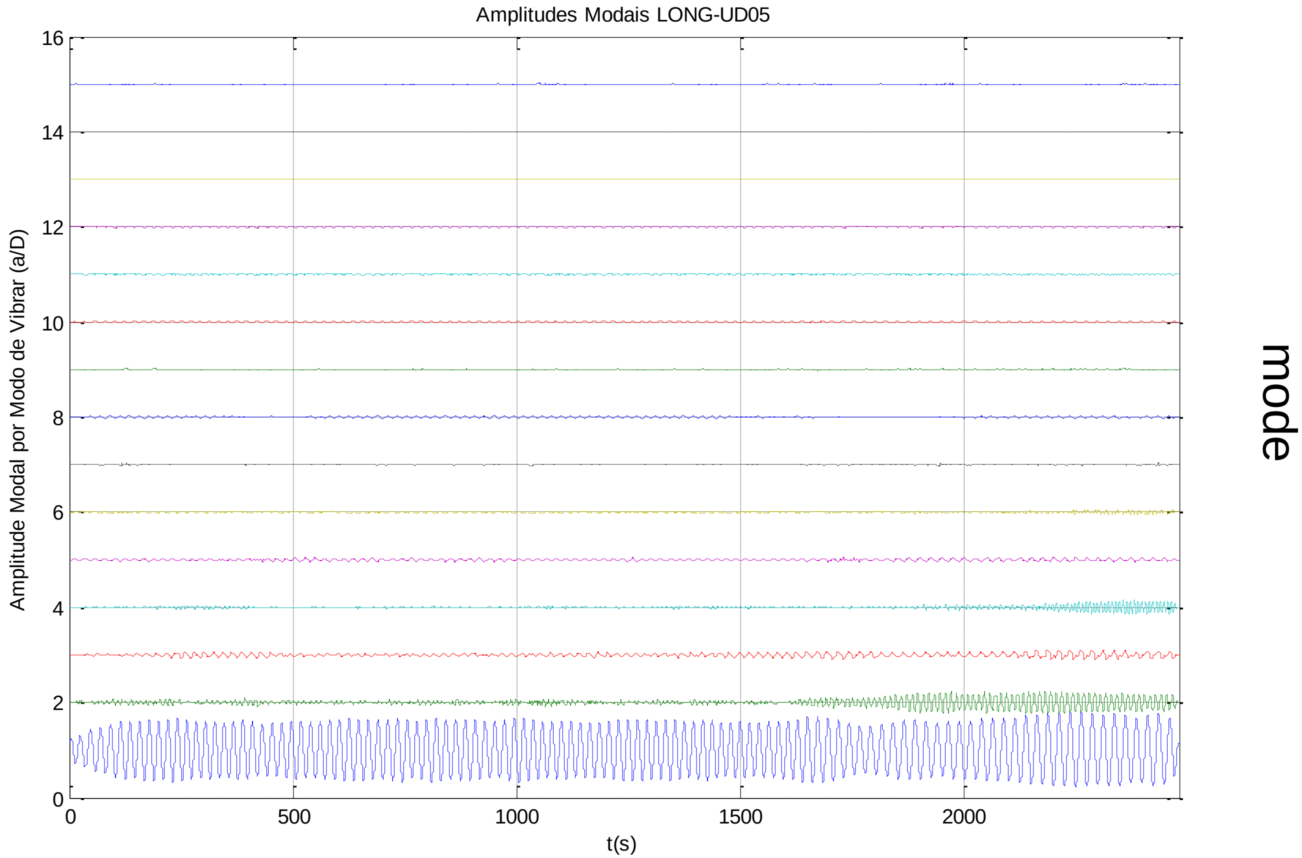
Longitudinal Catenary

Pesce, 2013; Rateiro 2015, Rateiro et al 2016

Example: experimental analysis
Small Scale SCR Experiments in a Towing Tank



$$V_{R_1} = \frac{U}{f_1 D} = 6.0$$

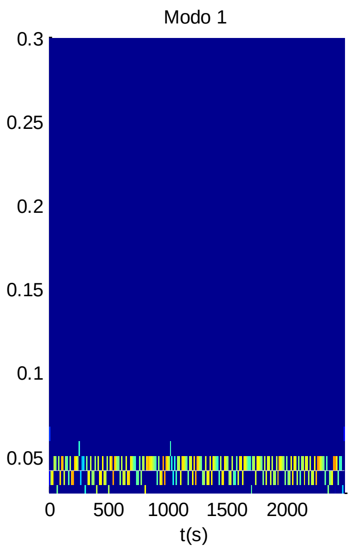


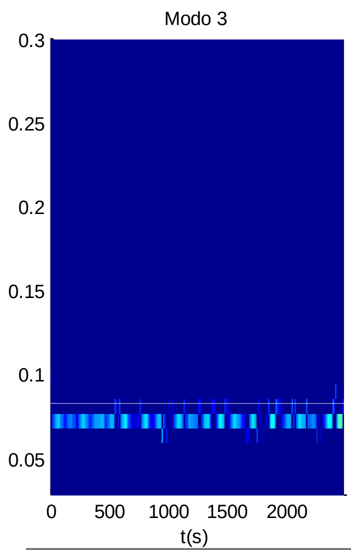
Modal projection in time:
dominance of the first mode at first VIV lock-in peak

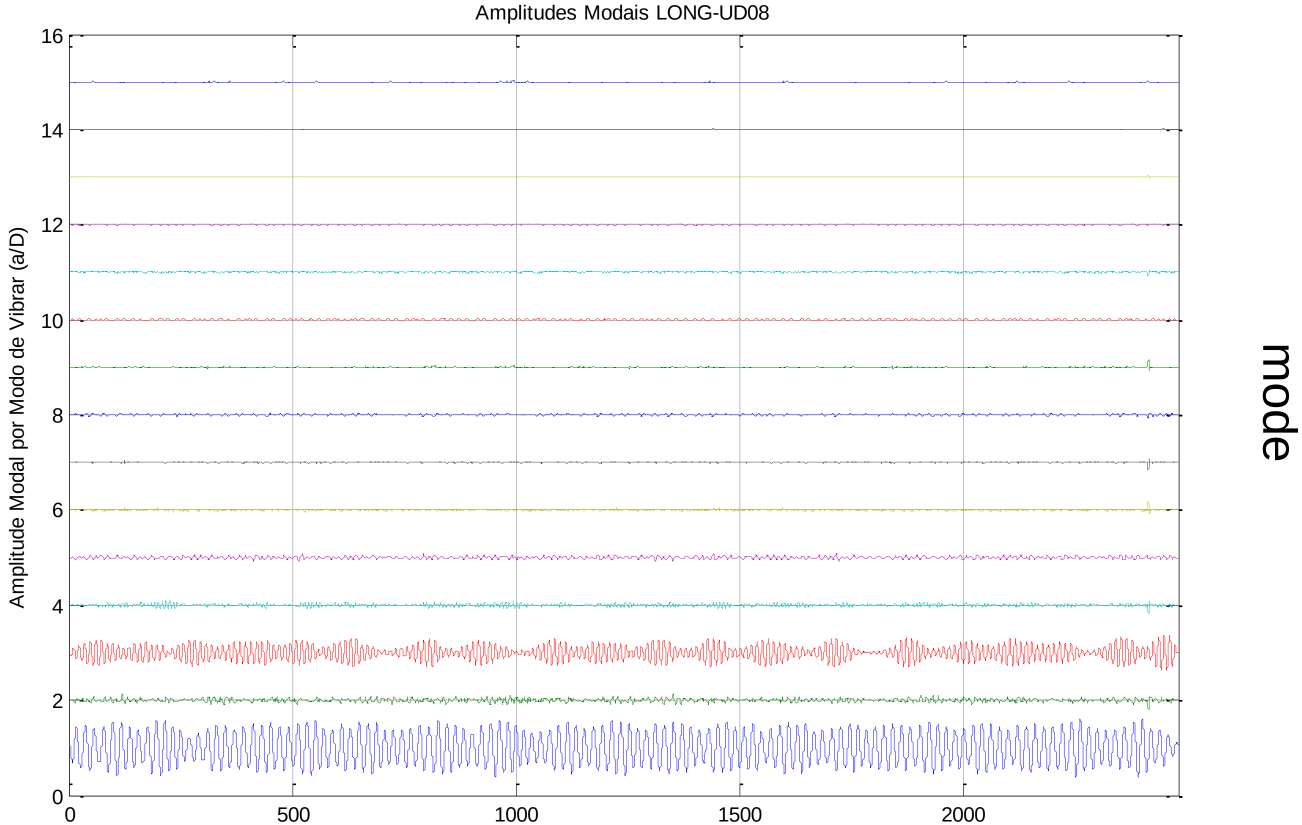
Rateiro 2015, Rateiro et al 2016

Example: experimental analysis
Small Scale SCR Experiments in a Towing Tank

$$V_{R_1} = \frac{U}{f_1 D} = 6.0$$





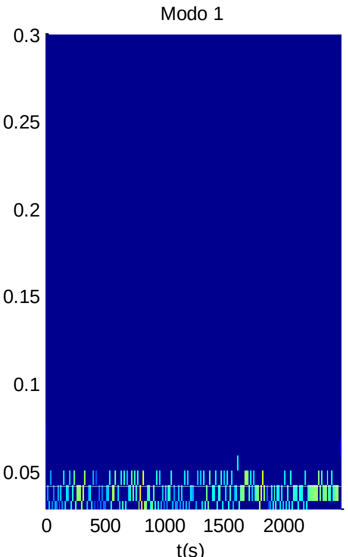


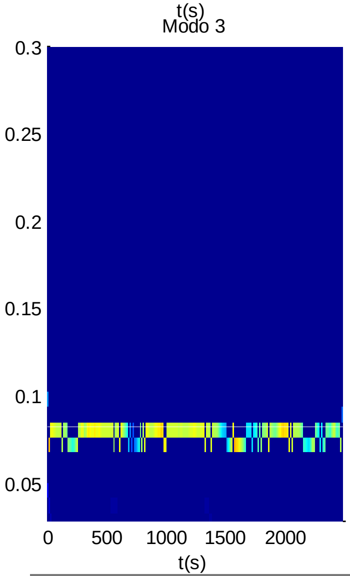
Transition from first to third mode

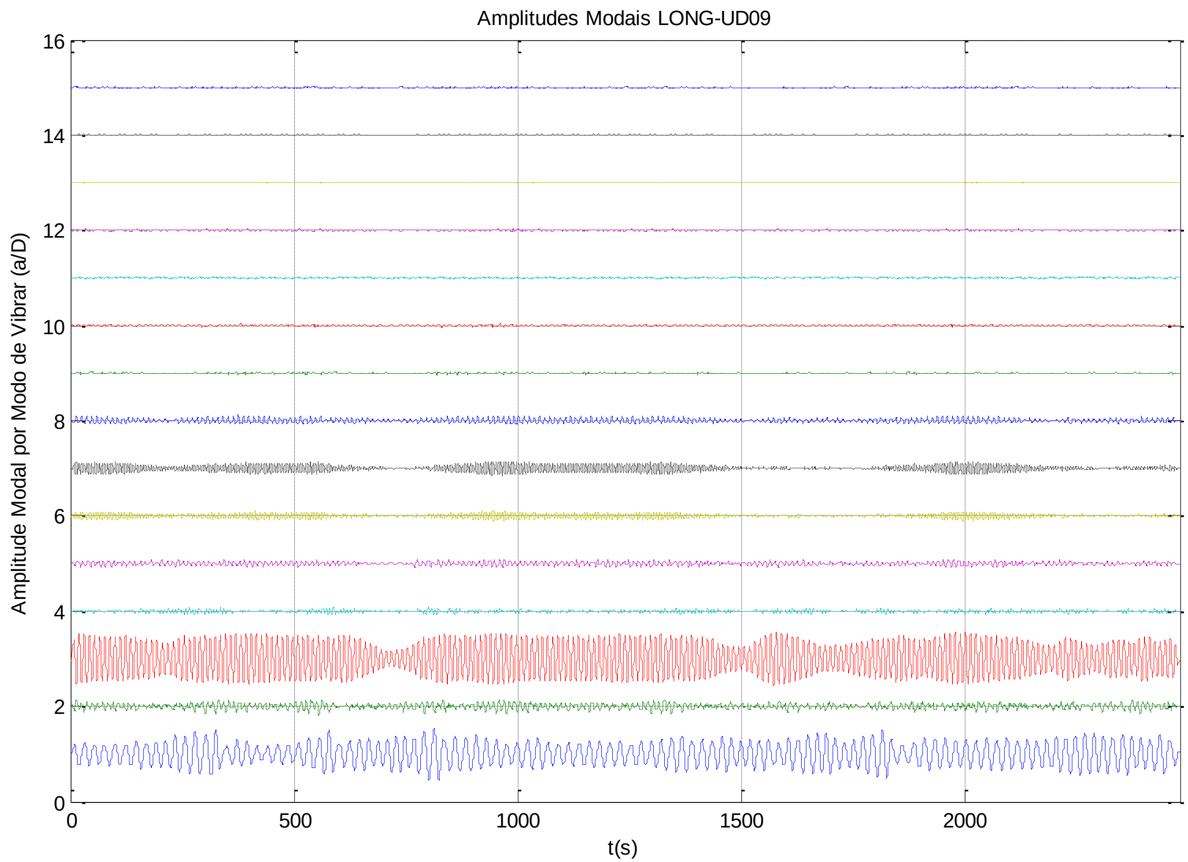
Rateiro 2015, Rateiro et al 2016

Example: experimental analysis
Small Scale SCR Experiments in a Towing Tank

$$V_{R_1} = \frac{U}{f_1 D} = 6.0$$



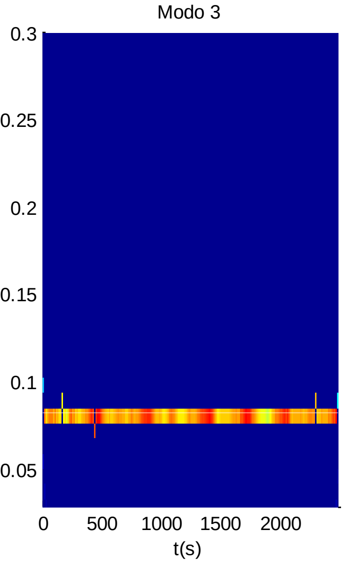


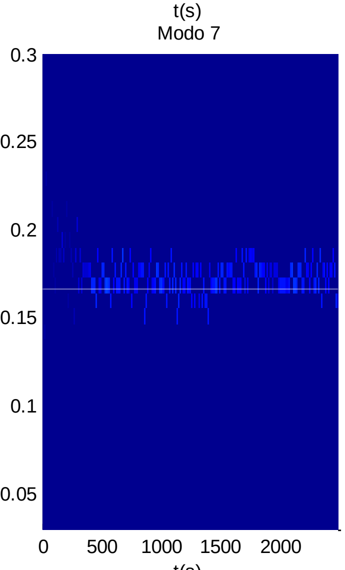


Transition from first to third mode

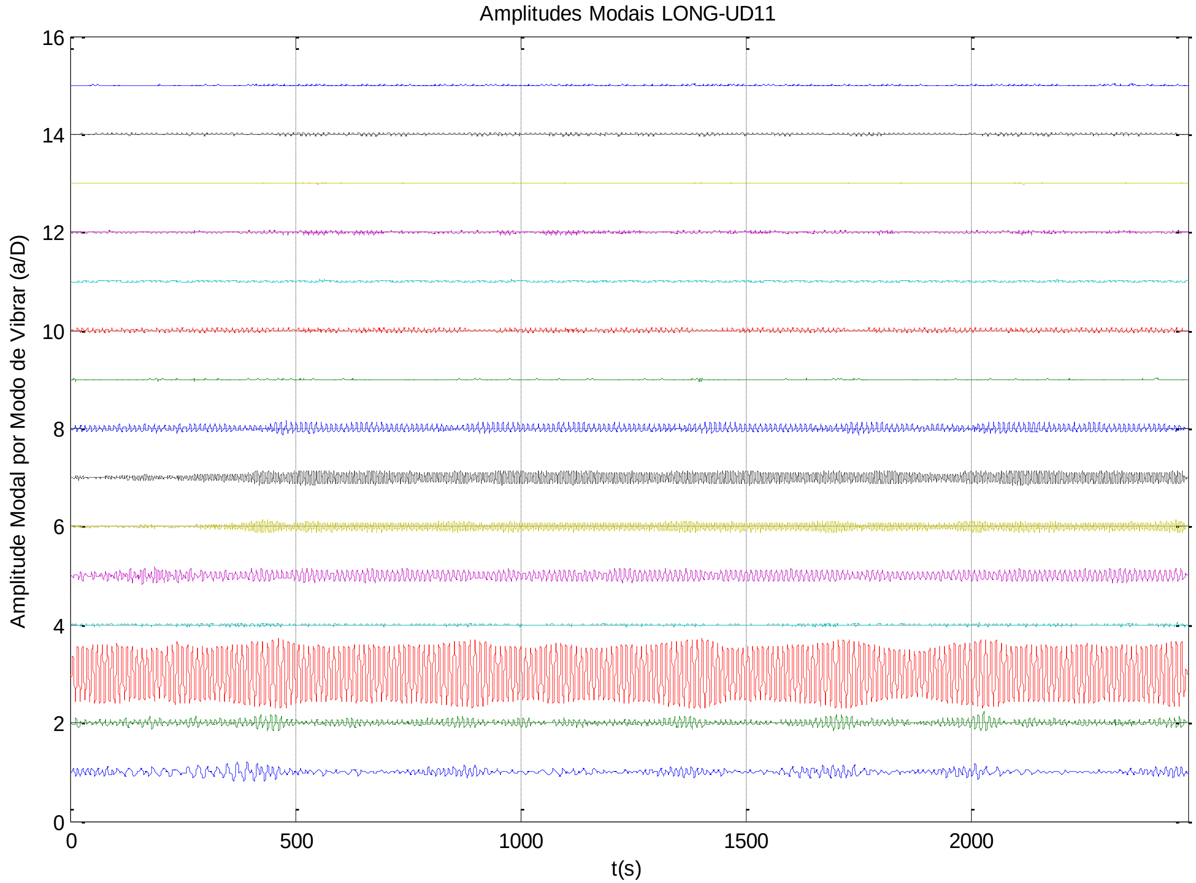
Rateiro 2015, Rateiro et al 2016

Example: experimental analysis
Small Scale SCR Experiments in a Towing Tank





$$V_{R_1} = \frac{U}{f_1 D} = 6.0$$



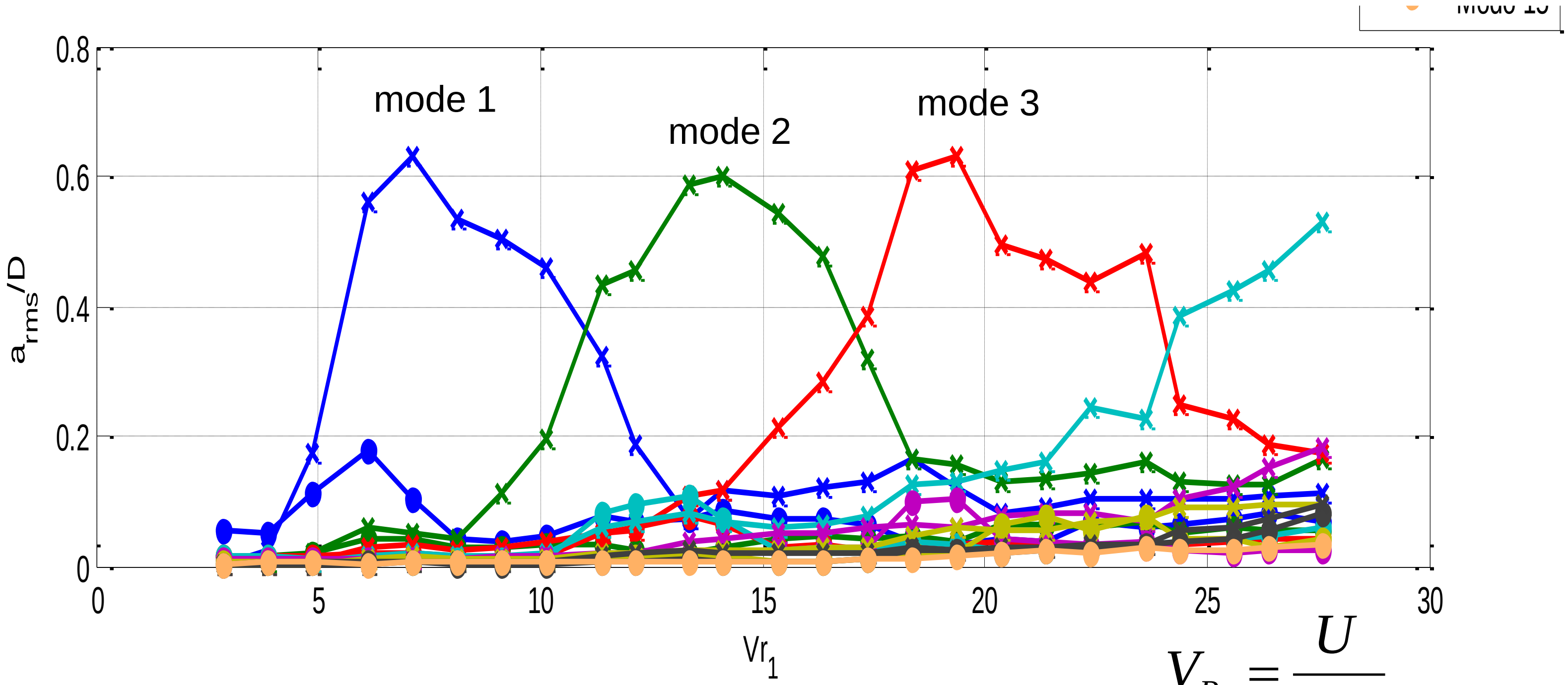
mode

Lock-in at third mode

Rateiro 2015, Rateiro et al 2016

Example: experimental analysis
Small Scale SCR Experiments in a Towing Tank

$a_{n\,RMS}/D$



$$V_{R_1} = \frac{U}{f_1 D}$$

Modal RMS *vs* First Modal Reduced Velocity

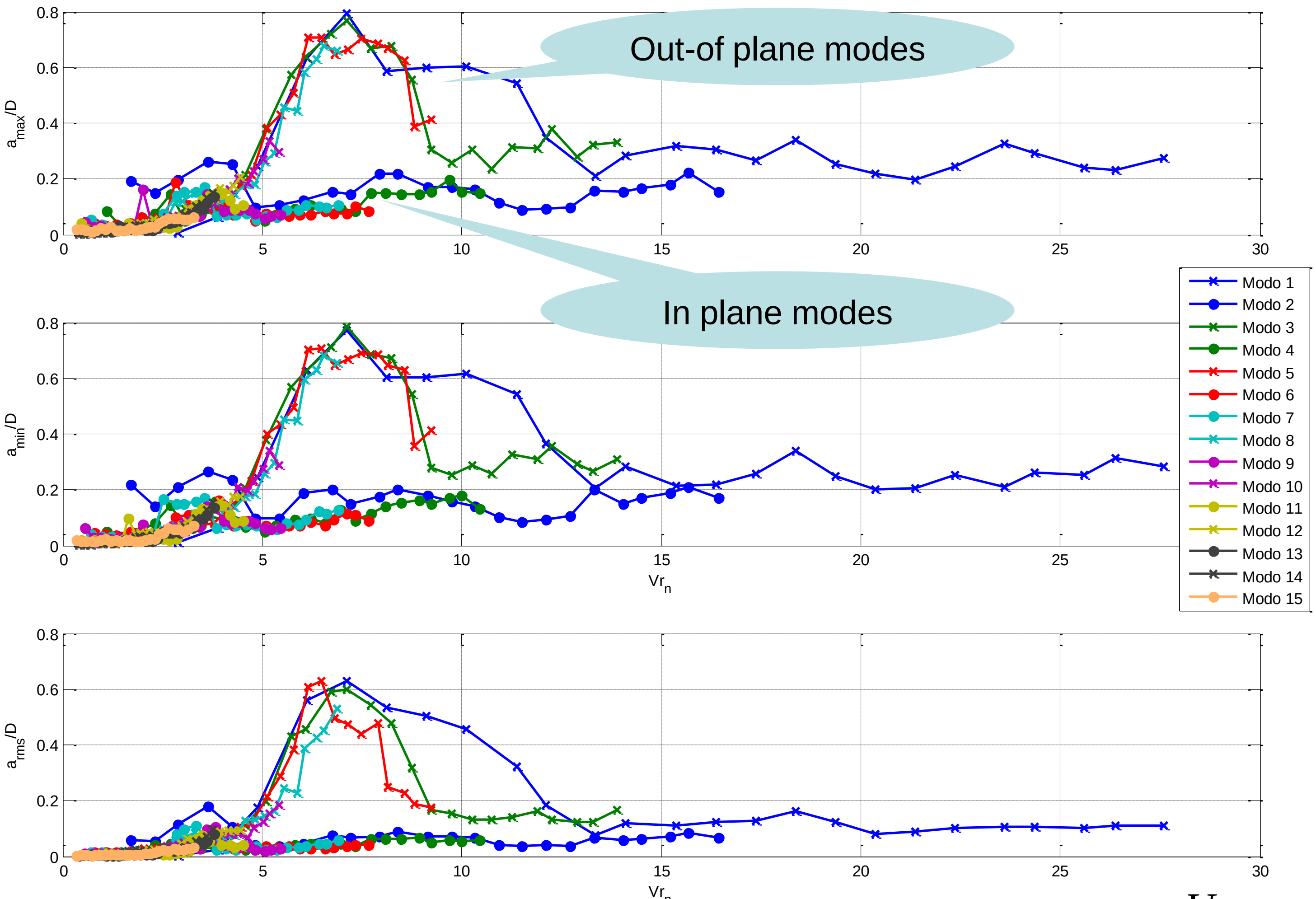
Rateiro 2015, Rateiro et al 2016

Example: experimental analysis
Small Scale SCR Experiments in a Towing Tank

$a_{n\max}/D$

$a_{n\min}/D$

a_{nRMS}/D



Modal RMS vs n-Modal Reduced Velocity

$$V_{R_1} = \frac{U}{f_n D}$$

Rateiro 2015, Rateiro et al 2016

Summary

- It is possible to construct a closed form analytical solution through the WKB asymptotic technique for the planar case of a catenary line under current.
- In the particular case when current is absent the WKB solution is analytical.
- Numerical solutions recover the WKB solution when the line is asynptotically inextensible.
- Experimental assesment with small scale models confirm the power of analytical solutions.

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